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Nonlinear axisymmetric resistive magnetohydrodynamic equilibria with toroidal flow

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The equilibrium of a resistive axisymmetric plasma with purely toroidal flow surrounded by a conductor is investigated within the framework of nonlinear magnetohydrodynamic theory. It is proved that (*a*) the poloidal current-density vanishes and (*b*) apart from an idealized case, the pressure profile should vanish on the plasma boundary. For the cases of isothermal magnetic surfaces, isentropic magnetic surfaces and magnetic surfaces with constant density, the equilibrium states obey an elliptic partial differential equation for the poloidal magnetic flux function, which is identical in form to the corresponding equation governing ideal equilibria. The conductivity, which can be neither uniform nor a surface quantity, results, however, in a restriction of the possible classes of equilibrium solutions; for example for the cases considered, the only possible equilibria with Spitzer conductivity are of cylindrical shape.

1. Introduction

Understanding the equilibrium properties of a magnetically confined plasma is one of the basic objectives of fusion research. Most studies to date have been devoted to static magnetohydrodynamic (MHD) equilibria of an axisymmetric plasma. These are governed by a partial elliptic differential equation for the poloidal magnetic flux function ψ, which contains two surface quantities (namely the pressure $P(\psi)$ and the poloidal current $I(\psi)$), known as the Grad–Schlüter–Shafranov equation (Lüst and Schlüter 1957; Shafranov 1957; Grad and Rubin 1958).

In the presence of flow, the equilibrium becomes more complicated. Although equilibrium studies of flowing plasmas began in the mid 1950s (see e.g. Grad 1960), since the early 1980s there has been increasing interest in the investigation of the equilibrium properties of plasmas with mass flow (Zehrfeld and Green 1972; Maschke and Perrin 1980; Morozov and Solovév 1980; Hameiri 1983; Clemente and Farengo 1984; Semenzato *et al.* 1984; Kerner and Tokuda 1987; Throumoulopoulos and Pantis 1989, 1996; Avinash *et al.* 1992; Żelazny *et al.* 1993; Throumoulopoulos and Tasso 1997), which was motivated by the observation of plasma rotation in many tokamaks heated by neutral beams (Suckewer *et al.* 1979; Brau *et al.* 1983; Core *et al.* 1987; Scott *et al.* 1987; Tammen *et al.* 1994) With the adoption of a specific equation of state, e.g. isentropic magnetic surfaces (Morozov and Solovév 1980), the axisymmetric ideal MHD equilibrium states obey a partial differential equation

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for ψ, containing five surface quantities, in conjunction with a nonlinear algebraic Bernoulli equation. Unlike the case of static equilibria, the above-mentioned differential equation is not always elliptic: there are three critical values of the poloidal flow at which the type of this equation changes, i.e. it becomes alternately elliptic and hyperbolic. To solve the equilibrium problem in the two elliptic regions, several computer codes have been developed (Semenzato *et al.* 1984; Kerner and Tokuda 1987; Żelazny *et al.* 1993). There is experimental evidence, however, that either the flow is purely toroidal (Suckewer *et al.* 1979; Core *et al.* 1987; Scott *et al.* 1987; Tammen *et al.* 1994) or its poloidal component is one order of magnitude lower than the toroidal one (Brau *et al.* 1983). It was also found that the poloidal flow is efficiently damped by magnetic pumping (Hassam and Kulsrud 1978). In addition, toroidal rotation has been suggested as a possible means of creating a magnetic field in the Sun and stars (Plumpton and Ferraro 1955) and of stabilizing the tilting (Hayashi and Sato 1987), ballooning (Miller *et al.* 1995) and drift-like (Sen and Rusbridge 1995) modes. Therefore toroidal flow may be considered a dominant effect. For a purely toroidal velocity component, the partial differential equation becomes elliptic and can be solved analytically (Maschke and Perrin 1980; Clemente and Farengo 1984; Throumoulopoulos and Pantis 1989).

The investigation of the general MHD equilibrium, involving plasma flow, finite conductivity and viscosity, is a very difficult problem, particularly in light of the uncertainties in the viscous stress tensor. For example, for the case in which the mean free path is greater than a gyroradius, as it is in the case of a tokamak, the 'ion parallel viscosity' (Braginskii 1965; Balescu 1988) takes an extraordinary large value – so large as to make the applicability of either version of the MHD viscosity stress tensor to tokamak dynamics questionable. It may be noted, however, that the ion parallel viscosity first increases strongly with the temperature until the mean free path reaches the order of magnitude of the machine, then decreases for larger values of the mean free path. In particular, in the high-viscosity regime resistive equilibria with flows originated from paired toroidal vortices, which are very likely created in a toroidal magnetofluid, were investigated recently (Montgomery *et al.* 1997*a*); the existence of such vortices is, however, conjectured to characterize magnetofluids beyond the high-viscosity limit. Thus the form and the magnitude of the appropriate viscous stress tensor to be used in tokamak MHD has been the subject of considerable discussion, and at present these discussions show no signs of converging.

For an inviscid plasma, Pfirsch and Schlüter (1962; see also Wesson 1987) showed a long time ago that in the collisional regime the toroidal curvature gives rise to an enhanced diffusion, which is related to the conductivity parallel to the magnetic field. Since then, there have been attempts to examine the effects of finite conductivity (see e.g. Grad and Hogan 1970), but these attempts appear not to have led to a satisfactory resolution; too many possibilities are raised to be able to deal with any one of them conclusively. In the above mentioned studies the inertial-force flow term $\rho(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}$ is neglected in the equation of momentum conservation. For ion flow velocities of the order of 10^7 cm s^{-1}, which have been observed in neutral-beam-heating experiments (Suckewer *et al.* 1979; Brau *et al.* 1983; Tammen *et al.* 1994), however, the term $\rho(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}$ cannot be considered negligible. Therefore it is worthwhile investigating the nonlinear resistive equilibrium. Such an investigation is also encouraged by a proof (Tasso 1979) according to which a conductivity profile remaining constant on a magnetic surface is not compatible with the Grad–

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Schlüter–Shafranov equation. The non-existence of static axisymmetric resistive equilibria with a uniform conductivity was also suggested recently (Montgomery and Shan 1994; Bates and Lewis 1996; Montgomery *et al.* 1997*b*).

In the present paper we study the MHD equilibrium of an axisymmetric plasma with scalar conductivity and purely toroidal flow surrounded by a conductor, including the term $\rho(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}$ in the momentum-conservation equation. It is found that the presence of flow does not remove the above-mentioned incompatibilities associated with the conductivity. Also, the conductivity has an impact on the direction of the current density and on the boundary conditions, and restricts the possible classes of equilibrium solutions.

The paper is organized as follows. For the system under consideration equilibrium equations independent of the plasma equation of state are derived in Sec. 2. The cases of equilibria with (*a*) isothermal, (*b*) isentropic and (*c*) constant-density magnetic surfaces are examined in Sec. 3. Section 4 summarizes our conclusions.

2. Equilibrium equations

The MHD equilibrium states of an inviscid plasma with scalar conductivity are governed by the following set of equations, written in standard notation and convenient units:

$$\boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0, \tag{1}$$

$$\rho(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = \mathbf{j}\times\mathbf{B} - \boldsymbol{\nabla}P, \tag{2}$$

$$\boldsymbol{\nabla}\times\mathbf{E} = 0, \tag{3}$$

$$\boldsymbol{\nabla}\times\mathbf{B} = \mathbf{j}, \tag{4}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0, \tag{5}$$

$$\mathbf{E} + \mathbf{v}\times\mathbf{B} = \frac{\mathbf{j}}{\sigma}. \tag{6}$$

It should be pointed out that, unlike the usual procedure followed in equilibrium studies with flow (Zehrfeld and Green 1972; Morozov and Solovév 1980; Hameiri 1983; Semenzato *et al.* 1984; Kerner and Tokuda 1987; Żelazny *et al.* 1993), in the present work an equation of state is not included in the above set of equations from the outset, and therefore equations (17) and (18) below are first derived, independently of the equation of state. This alternative procedure is convenient because the equilibrium problem is then further reduced for specific cases associated with several equations of state.

The system under consideration is a toroidal axisymmetric magnetically confined plasma, which is surrounded by a conductor (Fig. 1). Cylindrical coordinates (R, ϕ, z) are employed, with unit basis vectors $\mathbf{e}_R, \mathbf{e}_\phi, \mathbf{e}_z$, where $\mathbf{e}_z$ is parallel to the axis of symmetry. The position of the surface of the conductor is specified by some boundary curve in the (R, z) plane. The equilibrium quantities do not depend on the azimuthal coordinate ϕ. Also, it is assumed that the plasma elements flow solely along $\mathbf{e}_\phi$. Equation (1) is then satisfied identically.

For the above-mentioned system, we first prove that the equilibrium poloidal current density must vanish. Inserting the expression

$$\mathbf{B} = B_\phi\mathbf{e}_\phi + \mathbf{B}_{\rm pol} = B_\phi\mathbf{e}_\phi + \frac{1}{R}\boldsymbol{\nabla}\psi\times\mathbf{e}_\phi \tag{7}$$

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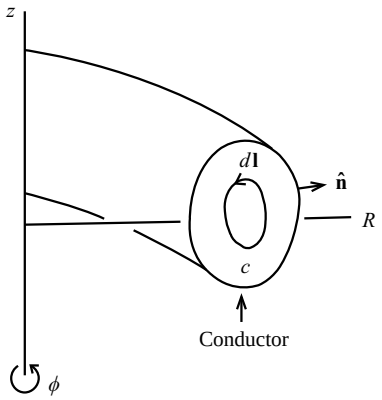


Figure 1. The geometry of an axisymmetric toroidal plasma surrounded by a conductor.

for the magnetic field into Ampère's law (4) yields for the current density

$$\mathbf{j} = -\frac{1}{R}\Delta^\star\psi\mathbf{e}_\phi + \frac{1}{R}\boldsymbol{\nabla}(RB_\phi)\times\mathbf{e}_\phi. \tag{8}$$

Here ψ is the poloidal magnetic flux function, which labels the magnetic surfaces, and $\Delta^\star$ is the elliptic operator defined by $\Delta^\star\psi = R^2\boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}\psi/R^2\right)$. With the aid of (7) and (8), the component of the force-balance equation (2) along $\boldsymbol{e}_\phi$ is written in the form

$$\boldsymbol{e}_\phi\cdot\boldsymbol{\nabla}(RB_\phi)\times\boldsymbol{\nabla}\psi = 0. \tag{9}$$

Equation (9) implies that RB_ϕ is a surface quantity:

$$RB_\phi \equiv I(\psi). \tag{10}$$

Integration of Ohm's law (6) along a contour c defined by the cut of an arbitrary closed magnetic surface with the poloidal plane (Fig. 1) leads to the equation

$$\int_c \mathbf{E}\cdot d\mathbf{l} + \int_c (\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l} = \int_c \frac{1}{\sigma}(\mathbf{j}\cdot d\mathbf{l}). \tag{11}$$

Since in equilibrium it holds that $\partial\mathbf{B}/\partial t = 0$, the first integral vanishes by Stokes' theorem. Also, the second integral vanishes because $\mathbf{v}$ is purely toroidal. Consequently, the term on the right-hand side must vanish, implying that $\mathbf{j}_{\rm pol} \equiv 0$ and therefore I = constant.

The ϕ component of the electric field associated with the current density can be obtained from Faraday's law (3):

$$E_\phi = \frac{j_\phi}{\sigma} = E_0\frac{R_0}{R}, \tag{12}$$

where E_0 is the value of the electric field on the cylindrical surface $R = R_0$. An electric field of this kind can be created by a linear change in time of the magnetic flux confined to a very long axisymmetric iron core, which is directed along $\boldsymbol{e}_z$ and is located within the hole of the torus. This can be considered as a highly simplified representation of a tokamak.

Equation (12) has an impact on the boundary conditions, i.e. the component of $\mathbf{E}$ tangential to the plasma–conductor interface does not vanish. Therefore, apart from a special idealized situation in which the surface of a perfectly conducting

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container is coated by a very thin layer of insulating dielectric (Bates and Lewis 1996; Montgomery *et al.* 1997*b*), the container cannot be considered perfectly conducting. Accordingly, Ohm's law with finite conductivity applied in the vicinity of the plasma–conductor interface does not permit the existence of a surface layer of current. It is now assumed that the position of the conductor is such that its surface coincides with the outermost of the closed magnetic surfaces. Thus the condition $\mathbf{B}\cdot\mathbf{n}=0$, where $\mathbf{n}$ is an outward unit vector normal to the plasma surface (Fig. 1), holds on the plasma–conductor interface. It should be noted that this is possible only in equilibrium, because in the context of resistive MHD time-dependent equations the magnetic flux is not conserved. Integration of the momentum conservation equation across the plasma–conductor interface and projection of the resulting equation parallel to $\mathbf{n}$ yields (with $\mathbf{B}$ being continuous on the plasma–conductor interface)

$$P+\rho(\mathbf{n}\cdot\mathbf{v})^2=0. \tag{13}$$

Since all the terms of (13) are non-negative and the flow is purely toroidal, it follows that the pressure P must vanish at the plasma boundary.

To reduce the equilibrium equations (1)–(6), a second surface quantity is identified by projecting the equation

$$\boldsymbol{\nabla}\times\left(\mathbf{v}\times\mathbf{B}-\frac{1}{\sigma}j_\phi\mathbf{e}_\phi\right)=0 \tag{14}$$

(following from a combination of (3) and (6)) along $\boldsymbol{e}_\phi$. This yields

$$\boldsymbol{e}_\phi\cdot\boldsymbol{\nabla}\left(\frac{v_\phi}{R}\right)\times\boldsymbol{\nabla}\psi=0. \tag{15}$$

Equation (15) implies that the angular frequency of the flow shares the same surfaces with ψ:

$$\omega\equiv\frac{v_\phi}{R}=\omega(\psi). \tag{16}$$

With the aid of (7), (8), (9) with $I=\text{const}$, (10) and (16), the components of (2) along $\boldsymbol{B}_{\rm pol}$ and perpendicular to a magnetic surface are put in the respective forms,

$$\mathbf{B}_{\rm pol}\cdot\left[\frac{\boldsymbol{\nabla}P}{\rho}-\boldsymbol{\nabla}\left(\frac{\omega^2R^2}{2}\right)\right]=0 \tag{17}$$

and

$$\Delta^\star\psi+R^2\frac{\boldsymbol{\nabla}\psi}{|\boldsymbol{\nabla}\psi|^2}\cdot\left[\boldsymbol{\nabla}P-\rho\omega^2\boldsymbol{\nabla}\left(\frac{R^2}{2}\right)\right]=0. \tag{18}$$

It should be noted here that (9), (17) and (18) are orthogonal components of (2), and are therefore independent.

Summarizing, the resistive MHD equilibrium of an axisymmetric toroidal plasma with purely toroidal flow is governed by the set of equations (17), (18) and

$$\Delta^\star\psi+R_0E_0\sigma=0, \tag{19}$$

following from (8) and (12). Owing to the purely toroidal direction of the flow, which leads to $\mathbf{j}_{\rm pol}=0$ and $\omega=\omega(\psi)$, (17) and (18) *do not contain the conductivity* and are identical in form to the corresponding equations governing ideal equilibria. Therefore, on the one hand, several properties of the ideal equilibria, e.g. the Shafranov shift of the magnetic surfaces and the detachment of the isobaric surfaces from the

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magnetic surfaces (see the discussion following (22) below) remain valid. On the other hand, (19) can be employed as an algebraic equation for the conductivity, i.e. the solutions of the 'ideal MHD' equations (17) and (18) can determine the spatial dependence of σ by means of the formula

$$\sigma = -\frac{1}{R_0 E_0}\Delta\psi^\star. \tag{20}$$

Equation (20) implies that the conductivity cannot be uniform, as the case in static equilibria (Tasso 1979; Montgomery and Shan 1994; Bates and Lewis 1996; Montgomery *et al.* 1997*b*).

It may be noted here that the equilibrium problem can be treated in a different way, suggested for the case of static resistive equilibria (Bates and Lewis 1996), namely (19) is employed as an elliptic partial differential equation for ψ, which is solved after the spatial dependence of σ is assigned. In our opinion, this method has two drawbacks: (*a*) a physically unjustifiable assignment of conductivity is required from the outset; (*b*) the conductivity profile must be compatible with (18) – a requirement that (at least in the presence of flow) makes the assignment difficult. This becomes clearer in next section (see (23), (29) and (33)).

3. Magnetic surfaces with special properties

In order to further reduce the equilibrium equations (17) and (18) and find their influence on the conductivity profile, the starting set of equations (1)–(6) must be supplemented by an equation of state, e.g. $P = P(\rho, T)$, along with an equation determining the transport of internal energy such as

$$C_v\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\right)T + P\boldsymbol{\nabla}\cdot\mathbf{v} + \kappa\nabla^2 T = 0, \tag{21}$$

the energy transport equation for a perfect gas, where C_v is the volume specific heat, κ the thermal conductivity and $\partial T/\partial t = 0$. Such a rigorous treatment, however, makes the equilibrium problem very cumbersome. Alternatively, since for fusion plasmas the thermal conduction along $\mathbf{B}$ is expected to be fast in relation to the heat transport perpendicular to a magnetic surface, the assumption of equilibria with isothermal magnetic surfaces is a reasonable approximation (see e.g. Mashke and Perrin 1980; Clemente and Farengo 1984; Throumoulopoulos and Pantis 1989). In particular, the even simpler case of isothermal resistive equilibria (T = const) has also been considered (Grad and Hogan 1970). Equilibria with isothermal magnetic surfaces are examined as follows.

3.1. Equilibria with isothermal magnetic surfaces

In addition to $T = T(\psi)$, the plasma is assumed to obey the perfect-gas law $P = \lambda\rho T$. For this kind of equilibria, (17) can be integrated, yielding an expression for the pressure:

$$P = P_T(\psi)\exp\left(\frac{R^2\omega^2}{2\lambda T}\right). \tag{22}$$

We note here that, unlike in static equilibria, in the presence of flow magnetic surfaces do not coincide with isobaric surfaces, because (2) implies that $\mathbf{B}\cdot\boldsymbol{\nabla}P$ in general differs from zero. In this respect, the term $P_T(\psi)$ is the static part of the

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pressure ($\omega = 0$). Inserting (22) into (18), the latter takes the form

$$\Delta^\star\psi + R^2\left[P_T' + P_T\frac{R^2}{2}\left(\frac{\omega^2}{\lambda T}\right)'\right]\exp\left(\frac{R^2\omega^2}{2\lambda T}\right) = 0, \tag{23}$$

where the prime denotes differentiation with respect to ψ. This is the equilibrium equation for a resistive plasma with purely toroidal flow and isothermal magnetic surfaces. It contains the three arbitrary surface quantities, $T(\psi)$, $\omega(\psi)$ and $P_T(\psi)$, which must be found from other physical considerations. For $\omega \equiv 0$, it reduces to the Grad–Schlüter–Shafranov equation for vanishing poloidal current. For ideal equilibria, (23) was obtained by Maschke and Perrin (1980). For special choices of the surface quantities, which must lead to physically reasonable profiles for the equilibrium quantities, (23) can be linearized and solved analytically. In particular, with the ansatz $\omega^2/(2\lambda T) \equiv \omega_T = \text{const}$, it takes the simpler form

$$\Delta^\star\psi + R^2 P_T(\psi)' \exp(R^2\omega_T). \tag{24}$$

Solutions of (24) have been obtained for $P_T \propto \psi$ (Maschke and Perrin 1980) and for $P_T \propto \psi^2$ (Clemente and Farengo 1984; Throumoulopoulos and Pantis 1989), and the detachment of the isobaric surfaces from the magnetic surfaces has been evaluated.

Equations (20) and (23) yield

$$E_0 R_0 \sigma = R^2\left[P_T' + P_T\frac{R^2}{2}\left(\frac{\omega^2}{\lambda T}\right)'\right]\exp\left(\frac{R^2\omega^2}{2\lambda T}\right). \tag{25}$$

As is the case in static equilibria (Tasso 1970; Montgomery and Shan 1994), toroidal equilibria with the conductivity remaining constant in a magnetic surface are not possible. Indeed, if $\sigma = \sigma(\psi)$ then the left-hand side of (25) becomes a surface quantity, while the right-hand side (in addition to surface quantities) involves an explicit dependence of R. Considering the variables R and ψ, instead of R and z, as independent (then $z = z(R, \psi)$), (25), after expanding the exponential term in a Taylor series, can be written in the form $\sum_{n=0}^{\infty} a_n(\psi)R^n = 0$. The latter requires that the relation $a_n(\psi) \equiv 0$ be satisfied for all n. This is impossible because $a_0 = E_0 R_0 \sigma(\psi)$, and therefore it follows that $\psi = \psi(R)$. Thus the only possible $\sigma = \sigma(\psi)$ equilibria are of cylindrical shape. It may be noted here that the non-existence of another class of resistive toroidal MHD equilibria, namely axisymmetric incompressible $\beta_p = 1$ equilibria with purely poloidal flow, was proved by Tasso (1970).

To find out whether the above incompatibilities associated with the conductivity profile arise from the choice $T = T(\psi)$, one can alternatively consider the cases of isentropic processes and incompressible flows, which have been the subjects of extensive ideal MHD studies, and are respectively associated with isentropic magnetic surfaces (Maschke and Perrin 1980; Morozov and Solovév 1980; Hameiri 1983; Semenzato *et al.* 1984; Kerner and Tokuda 1987; Throumoulopoulos and Pantis 1989; Żelazny *et al.* 1993) and constant-density magnetic surfaces (Avinash *et al.* 1992; Throumoulopoulos and Pantis 1996; Throumoulopoulos and Tasso 1997).

3.2. Equilibria with isentropic magnetic surfaces

We consider a plasma with large but finite conductivity such that for times short compared with the diffusion time scale, the dissipative term $\approx j^2/\sigma$ can be neglected. This permits one to assume that the magnetic surfaces are isentropic, i.e $S = S(\psi)$ where S is the plasma specific entropy. It may be noted that $S = S(\psi)$ can be

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proved for the stationary states of an ideal plasma with arbitrary flows on the basis of entropy conservation (Morozov and Solovév 1980; Hameiri 1983). This case was also studied for ideal equilibria with purely toroidal flow (Maschke and Perrin 1980; Throumoulopoulos and Pantis 1989). In addition, the plasma is assumed to be a perfect gas whose internal energy density E is simply proportional to the temperature. Then the equations for the thermodynamic potentials lead to (Maschke and Perrin 1980)

$$P = A(S)\rho^{\gamma}, \tag{26}$$

$$E = \frac{A(S)}{\gamma - 1}\rho^{\gamma-1} = \frac{H}{\gamma}. \tag{27}$$

Here $A = A(S)$ is an arbitrary function of the specific entropy $S \equiv S(\psi)$, $H = E + P/\rho$ is the specific enthalpy and γ is the ratio of specific heats. For simplicity and without loss of generality, we choose the function A to be identical with S. Consequently, integration of (17) yields

$$H - \frac{R^2\omega^2}{2} \equiv \theta(\psi), \tag{28}$$

where θ is a surface quantity. Equation (18) then reduces to

$$\Delta^{\star}\psi + R^2\left(1 + \frac{R^2\omega^2}{2\theta}\right)^{\eta-1}\left\{\left(\frac{\theta}{\eta}S^{1-\eta}\right)' + R^2\left[\left\{\left(\frac{\theta}{\eta}\right)^{\eta}\right\}' S^{1-\eta}\frac{\omega\omega'}{\theta'} + \left(\frac{\theta}{\eta}\right)^{\eta}(S^{1-\eta})'\frac{\omega^2}{2\theta}\right]\right\} = 0, \tag{29}$$

where $\eta = \gamma/(\gamma - 1)$. This is the resistive equilibrium equation for isentropic magnetic surfaces, which is the same as the ideal equation (Maschke and Perrin 1980; Throumoulopoulos and Pantis 1989) with $j_{\mathrm{pol}} = 0$. The three surface quantities are now $\omega(\psi)$, $S(\psi)$ and $\theta(\psi)$. With the ansatz $\omega^2/(2\theta) = \omega_s = \text{const}$, (29) is put into the simpler form

$$\Delta^{\star}\psi + R^2(1 + \omega_s R^2)^{\eta}(P_s)' = 0, \tag{30}$$

where the function

$$P_s(\psi) \equiv \left(\frac{\theta}{\eta}\right)^{\eta} S^{1-\eta}$$

is the static equilibrium pressure. Solutions of (30) are available in the literature for $P_s \propto \psi$ (Maschke and Perrin 1980) and for $P_s \propto \psi^2$ (Throumoulopoulos and Pantis 1989).

Equations (20) and (29) imply that toroidal isentropic equilibria are not compatible with $\sigma = \sigma(\psi)$. In addition, equilibria with Spitzer conductivity $\sigma = aT^{3/2}$ satisfy the equation

$$R_0 E_0 a\left[\frac{1}{\gamma S}\left(\theta + \frac{R^2\omega^2}{2}\right)\right]^{3/2} = R^2\left(1 + \frac{R^2\omega^2}{2\theta}\right)^{\eta-1} \times\left\{\left(\frac{\theta}{\eta}S^{1-\eta}\right)' + R^2\left[\left\{\left(\frac{\theta}{\eta}\right)^{\eta}\right\}' S^{1-\eta}\frac{\omega\omega'}{\theta'} + \left(\frac{\theta}{\eta}\right)^{\eta}(S^{1-\eta})'\frac{\omega^2}{2\theta}\right]\right\} = 0, \tag{31}$$

following from (20) and (26)–(29). Equation (31) contains, in addition to surface quantities, the coordinate R explicitly. Therefore, as in the case of isothermal mag-

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netic surfaces and $\sigma = \sigma(\psi)$, equilibria with isentropic magnetic surfaces and Spitzer conductivity must have cylindrical shape.

3.3. Equilibria with magnetic surfaces of constant density

Equilibria with incompressible flows are usually associated with constant-density magnetic surfaces (Avinash *et al.* 1992; Throumoulopoulos and Pantis 1996; Throumoulopoulos and Tasso 1997). For axisymmetric equilibria with purely toroidal flow, (1) and $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$ are satisfied identically and provide no information about ρ. Nevertheless, $\rho = \rho(\psi)$ is a physically possible equation of state. For this case, integration of (17) leads to

$$P = P_\rho(\psi) + \frac{\rho R^2 \omega^2}{2}, \tag{32}$$

where $P_\rho(\psi)$ is the static equilibrium pressure. Consequently, (18) reduces to

$$\Delta^\star \psi + R^2 \left[P_\rho' + \frac{R^2}{2} (\rho \omega^2)' \right] = 0. \tag{33}$$

As in the cases of isothermal and isentropic magnetic surfaces, (20) and (33) imply that in toroidal equilibria the conductivity cannot be a surface quantity. Also, equilibria with Spitzer conductivity satisfy the relation

$$E_0 R_0 a \left(\frac{P_\rho}{\lambda \rho} + \frac{\omega^2 R^2}{2\lambda} \right)^{3/2} = R^2 \left[P_\rho' + \frac{R^2}{2} (\rho \omega^2)' \right], \tag{34}$$

which implies that magnetic surfaces must have cylindrical shape.

Concluding this subsection, we derive two classes of exact solutions of (33). With the use of the ansatz

$$\rho \omega^2 = \text{const}, \tag{35}$$

(33) reduces to

$$\Delta^\star \psi + R^2 P_\rho' = 0, \tag{36}$$

which is identical in form to the the Grad–Schlüter–Shafranov equation for vanishing poloidal current. A simple solution of (36) can be derived for $P_\rho \propto \psi$ (Shafranov 1966):

$$\psi = \psi_c \frac{R^2}{R_c^4} (2R_c^2 - R^2 - 4d^2 z^2). \tag{37}$$

Here ψ_c is the value of the flux function at the position of the magnetic axis ($z = 0$, $R = R_c$) and d^2 is a parameter that determines the shape of the flux surfaces; for example, for $d^2 = 1$, (37) describes a compact torus contained within a spherical conductor. The current density, however, is flat, and resistive equilibria of this kind are associated with the idealized boundary. For $P_\rho \propto \psi^2$, solutions of (36) separable in the coordinates R and z can be expressed in terms of Coulomb wave functions, and describe equilibria of either a tokamak with rectangular cross-section (Hernegger 1972; Maschke 1973) or a compact torus contained within a rectangular conductor (Berk *at al.* 1981). These solutions can yield a peaked current-density profile vanishing on the boundary. Owing to the presence of the flow term in (32), however, the pressure does not vanish on the boundary, as it must for resistive equilibria. Thus solutions of this kind are not valid in the vicinity of the boundary. The drawbacks of the solutions presented above in this paragraph arise from

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the choices (35) and $P_\rho \propto \psi$. They can be eliminated if, alternatively, the ansatz $\rho\omega^2 = k\psi^2$ and $P_\rho = \frac{1}{2}\mu\psi^2$ is employed, where k and μ are constants. Then (33) reduces to

$$\Delta^\star\psi + R^2(\mu + kR^2)\psi = 0. \tag{38}$$

The last equation cannot be solved analytically, and numerical solutions will not be derived here.

4. Conclusions

The equilibrium of an axisymmetric plasma with scalar conductivity and purely toroidal flow surrounded by a conductor has been investigated within the framework of nonlinear MHD theory. It has been found that (*a*) the poloidal current density $\mathbf{j}_{\rm pol}$ must vanish and (*b*) apart from an idealized case, perfectly conducting-boundary conditions are not relevant. The equilibrium states are governed by a set of two differential equations (17) and (18), which do not contain the conductivity and are identical in form to the corresponding equations for ideal stationary equilibria. The conductivity, however, restricts the allowed equilibria to those that satisfy $\mathbf{j}_{\rm pol} = 0$ and have pressure profile vanishing on the boundary. With the assumption of a specific equation of state, the above-mentioned set of equations reduces to a single elliptic partial differential equation for the poloidal magnetic flux function that contains three surface quantities (e.g. (23), (29), (33)). The cases of isothermal magnetic surfaces, isentropic magnetic surfaces and constant-density magnetic surfaces have been examined, and, for specific choices of the surface quantities involved in each case, equilibrium solutions have been presented and discussed.

As is the case in static equilibria, the conductivity cannot be uniform. In addition, for all three cases of the equation of state examined, the conductivity cannot be a surface quantity – the only possible equilibria with conductivity remaining constant in a magnetic surface have cylindrical shape. Also, for isentropic magnetic surfaces and constant-density magnetic surfaces, the non-existence of toroidal equilibria with Spitzer conductivity has been shown.

It is interesting to investigate whether these unusual properties of the conductivity persist when additional physical input (e.g. a general flow with non-vanishing poloidal and toroidal components) is included. Of course, in pursuing such of investigations, one gets a step closer to the cumbersome problem of MHD equilibrium with flow, finite conductivity and viscosity. The results of the few studies that have been conducted to date indicate, however, that this objective may be of crucial importance for fusion plasmas.

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