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RESEARCH ARTICLE

2-Gorenstein stable surfaces with $K_X^2 = 1$ and $\chi(X) = 3$

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Abstract

The compactification $\overline{\mathfrak{M}}_{1,3}$ of the Gieseker moduli space of surfaces of general type with $K_X^2 = 1$ and $\chi(X) = 3$ in the moduli space of stable surfaces parametrises the so-called stable I-surfaces.

We classify all such surfaces which are 2-Gorenstein into four types using a mix of algebraic and geometric techniques. We find a new divisor in the closure of the Gieseker component and a new irreducible component of the moduli space.

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1. Introduction

The Gieseker moduli space $\mathfrak{M}_{1,3}$ of canonical models of surfaces of general type with $K_X^2 = 1$ and $\chi(X) = 3$ is a rational variety of dimension 28 and it was classically known that the surfaces it parametrises are hypersurfaces of degree 10 contained in the smooth locus of $\mathbb{P}(1,1,2,5)$. More geometrically speaking, the bicanonical map realises these surfaces as double covers of the quadric cone in $\mathbb{P}^3$ branched over the vertex and a sufficiently general quintic section.

Nowadays, the Gieseker moduli space $\mathfrak{M}_{a,b}$ is known to admit a modular compactification $\overline{\mathfrak{M}}_{a,b}$, the moduli space of stable surfaces with $K_X^2 = a$ and $\chi(X) = b$, sometimes called KSBA-moduli space after Kollár, Shepherd-Barron and Alexeev (compare [Kol23]). For brevity we call the surfaces parametrised by $\overline{\mathfrak{M}}_{1,3}$ stable I-surfaces (see Definition 2.2 for a precise definition).

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Table 1. *Known irreducible strata in the moduli space $\overline{\mathfrak{M}}_{1,3}$ of stable I-surfaces.*

Name	Dim.	index	closure of	reference
$\overline{\mathfrak{M}}^{\text{class}}$	28	1	Gieseker component	[BHPV04, FPR17] Theorem 4.1
$\overline{\mathfrak{M}}^{\text{RU}}$	28	5	Rana–Urzúa component	[FPRR22]
$\overline{\mathfrak{M}}^{DE}$	30	2	DE-component	Theorem 8.1
$\mathfrak{D}_A$	27	2	type A and not Gorenstein resp. the $\frac{1}{4}(1,1)$ divisor	Theorem 4.1 [FPRR22]
$\mathfrak{D}_B$	27	2	type B	Theorem 5.1
$\mathfrak{D}^{\text{RU}}_{\text{cusp}}$	27	5	cuspidal RU surfaces	[CFP+23]
$\mathfrak{D}^{\text{RU}}_{1/9}$	27	15	(nodal) RU surfaces with a $\frac{1}{9}(1,5)$ singularity	[CFP+23]
$\mathfrak{D}_{1/18}$	27	3	surfaces with a $\frac{1}{18}(1,5)$ singularity	[CFP+23]
$\mathfrak{D}^{\text{GPSZ}}_i$	27	>2	8 divisors from exceptional unimodal singularities	[GPSZ24, RT24]
$\mathfrak{R}_{DD}$	26	2	type DD	Theorem 7.1
$\mathfrak{R}_{1/4,1/18}$	26	6	surfaces with one $\frac{1}{4}(1,1)$ and one $\frac{1}{18}(1,5)$ singularity	[CFP+23]
$\mathfrak{R}^{\text{RU}}_{1/18}$	26	15	RU-surfaces with one $\frac{1}{18}(1,5)$ singularity	[CFP+23]
$\mathfrak{R}_{AB}$	26	2	type B, in the closure of type A	Theorem 5.1

Our detailed understanding of the Gieseker moduli space, or classical component, and the small values of the invariants make $\overline{\mathfrak{M}}_{1,3}$ into a fertile testing ground to explore stable surfaces and their moduli.

So far a full picture seems out of reach, and the current approaches aim for classification under some extra conditions on the singularities. The first result in this direction was the extension of the classical description to Gorenstein stable I-surfaces [FPR17], which was refined and explored further in [CFPR22] from a Hodge theoretic perspective. In [FPRR22, CFP+23] we explored surfaces with few T-singularities, finding several divisors and an additional component. Meanwhile in [GPSZ24] Gallardo, Pearlstein, Schaffler and Zhang found eight more divisors by considering the stable replacement of Gorenstein degenerations with an exceptional unimodal singularity; in [RT24] it was shown that there are not more divisors of this kind.

In the present paper we completely classify 2-Gorenstein stable I-surfaces, that is, those with $2K_X$ Cartier, dividing them into four types:

Theorem 1.1. *Let X be a 2-Gorenstein stable I-surface. Then X is one of the following:*

type A *Semi-log-canonical hypersurfaces in $\mathbb{P}(1,1,2,5)$ not passing through the point $(0:0:0:1)$. This includes all Gorenstein stable I-surfaces and the divisor whose general element is a surface with one singularity of type $\frac{1}{4}(1,1)$.*

type B *These are reducible surfaces $X = X_1 \cup X_2$, where X_1 is a singular Enriques surface and X_2 is a singular K3 surface. They form a family of dimension 27 in the closure of the Gieseker component, thus are smoothable, but no longer complete intersections.*

type DD *These are reducible complete intersections of bidegree $(2,10)$ in weighted projective space $\mathbb{P}(1,1,2,2,5)$ and the bicanonical map realises them as double covers of the union of two planes. Each component is a singular K3 surface and they form a subset of codimension two in the closure of the Gieseker component.*

type DE *These are reducible surfaces $X = X_1 \cup X_2$, where both X_i are singular K3 surfaces of a special kind. They form a 30-dimensional family and none of them is smoothable, that is, they form a new irreducible component. Their canonical ring is quite complicated.*

We give an overview of the known strata in $\overline{\mathfrak{M}}_{1,3}$ in Table 1, including dimension of the stratum, index of the general surface and a reference to precise information and proofs.

Our current knowledge of the locus of 2-Gorenstein stable I-surfaces in relation to the rest of $\overline{\mathfrak{M}}_{1,3}$ is shown in Figure 1. We have not studied possible degenerations of the surfaces in the previously

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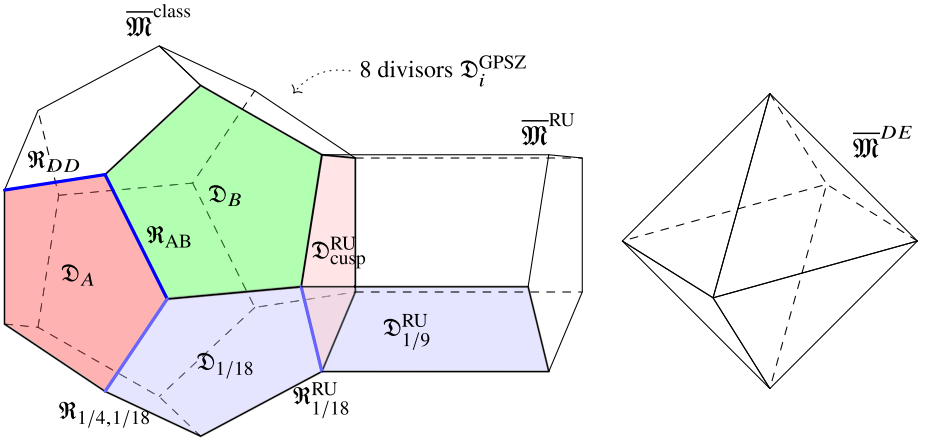


Figure 1. *Known strata in* $\overline{\mathfrak{M}}_{1,3}$ *(compare Table 1 for notation).*

unknown component $\mathfrak{M}^{DE}$ and we would not dare to speculate if it is a connected component without further evidence.

Outline of the paper and proof of Theorem 1.1

Let X be a 2-Gorenstein stable I-surface as defined in Definition 2.2. Then we show in Proposition 3.3 that the general section of ω_X does not vanish on any component and thus defines a canonical curve C.

If C can be chosen reduced, then we use the hyperplane section principle [Rei90] and our work on generalised Gorenstein spin structures on reduced curves of genus two [CFPR23] (summarised in Theorem 3.8) to compute the canonical ring. This gives the surfaces of type A and type B, classified in Sections 4 and 5.

We were unable to argue along the same lines when the general canonical curve is nonreduced. However, in this case the surface is reducible and we succeed in giving a geometric description of the components and the possible glueings, resulting in two cases (Lemma 6.3).

Type DD is identified in Section 7 with a smoothable example already considered in [FPR17]. Type DE is shown to give a new irreducible component in Section 8; its canonical ring is computed a posteriori from the geometric description.

2. Notation and preliminary results

2.1. Set-up

We work with schemes of finite type over the complex numbers.

Given a sheaf $\mathcal{F}$ on a scheme X, we denote by $\mathcal{F}^\vee = \mathscr{H}om(\mathcal{F}, \mathscr{O}_X)$ its dual and by $\mathcal{F}^{[m]}$ the m-th reflexive power, i.e., the double dual of $\mathcal{F}^{\otimes m}$.

A curve C is a Cohen–Macaulay projective scheme of pure dimension 1 (possibly nonreduced or reducible). It has a dualising sheaf ω_C and we denote by $p_a(C)$ the arithmetic genus of C, that is, $p_a(C) = 1 - \chi(\mathscr{O}_C)$. The curve C is Gorenstein if ω_C is invertible; if this is the case K_C denotes a canonical divisor such that $\mathscr{O}_C(K_C) \cong \omega_C$.

Our standard reference for stable surfaces is [Kol13]. A stable surface X has semi-log-canonical singularities and ample canonical divisor. In particular, it is by definition Cohen–Macaulay and Gorenstein in codimension one, so the canonical sheaf ω_X exists and is reflexive. We call X m-Gorenstein if the m-th reflexive power of the canonical sheaf $\omega_X^{[m]}$ is invertible. A canonical divisor is a Weil divisor K_X such that $\omega_X \cong \mathscr{O}_X(K_X)$. We use the notations $\chi(X) = \chi(\mathscr{O}_X)$, $p_g(X) = h^0(X, \omega_X)$ and $q(X) = h^1(X, \mathscr{O}_X)$.

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If X is a non-normal stable surface, then we denote by $\pi\colon \bar{X} \to X$ the normalisation of X and by $D \subset X$ and $\bar{D} \subset \bar{X}$ the curves defined by the conductor ideal $\mathscr{H}om_{\mathscr{O}_X}(\pi_*\mathscr{O}_{\bar{X}}, \mathscr{O}_X)$. The invariants are related by $K_X^2 = (K_{\bar{X}} + \bar{D})^2$ and $\chi(X) = \chi(\bar{X}) + \chi(D) - \chi(\bar{D})$ (see also [FPR15, Prop. 3.3]).

The map $\pi\colon \bar{D} \to D$ on the conductor divisors is generically a double cover and thus induces a rational involution on $\bar{D}$. Normalising the conductor loci we get an honest involution $\tau\colon \bar{D}^\nu \to \bar{D}^\nu$ such that $D^\nu = \bar{D}^\nu/\tau$ and such that the different $\mathsf{Diff}_{\bar{D}^\nu}(0)$ is τ-invariant, which fits in the following pushout diagram:

$$\begin{array}{ccccc} \bar{X} & \xleftarrow{\bar{\iota}} & \bar{D} & \xleftarrow{\bar{\nu}} & \bar{D}^\nu \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle /\tau} \\ X & \xleftarrow{\iota} & D & \xleftarrow{\nu} & D^\nu \end{array}. \tag{2.1}$$

By Kollár's glueing construction [Kol13, Thm. 5.13], the surface X can be reconstructed uniquely from the triple $(\bar{X}, \bar{D}, \tau)$.

2.2. *I-surfaces*

Our main interest stems from the Gieseker moduli space $\mathfrak{M}_{1,3}$ and its closure in the moduli space of stable surfaces.

For canonical or just Gorenstein stable surfaces X with fixed K_X^2 and $\chi(X)$ the Riemann–Roch formula and Kodaira vanishing determine all higher plurigenera as $P_m(X) = \chi(X) + \frac{1}{2}m(m-1)K_X^2$ for $m \geq 2$. Together with the geometric genus p_g or equivalently the irregularity q this determines the Hilbert function of the canonical ring $R(X, K_X)$.

By Noether's inequality or [FPR17] a Gorenstein stable surface with $K_X^2 = 1$ and $\chi(X) = 3$ satisfies $q(X) = 0$ and thus has Hilbert series[1]

$$\frac{1 - t^{10}}{(1-t)^2(1-t^2)(1-t^5)}. \tag{2.2}$$

We will show below in Proposition 3.3 that this continues to hold if X is only 2-Gorenstein.

If the Cartier index of X is larger, this is no longer true:

Example 2.1. Let $X = X_9 \subset \mathbb{P}(1,1,3,3)$ a general hypersurface of degree 9. Then X has three singularities of type $\frac{1}{3}(1,1)$ and $\omega_X = \mathscr{O}_X(1)$. Thus $3K_X$ is Cartier and ample, so X is stable. The invariants are $K_X^2 = 1$, $\chi(X) = 3$ with $p_g(X) = 2$. However, $P_2(X) = H^0(X, \mathscr{O}_X(2)) = 3 \neq \chi(X) + K_X^2$.

This example and more general instances of this phenomenon were considered in [Rol23].

To exclude such pathological examples that are unrelated to the study of canonical surfaces, we specify the Hilbert series in our definition.

Definition 2.2. A stable I-surface is a stable surface with $K_X^2 = 1$, $\chi(X) = 3$ and Hilbert series of the canonical ring as in (2.2), in particular $p_g(X) = 2$ and $q(X) = 0$.

We denote the moduli space of stable I-surfaces by $\overline{\mathfrak{M}}_{1,3}$, the closure of the Gieseker component inside it by $\overline{\mathfrak{M}}_{1,3}^{\mathrm{class}}$.

3. Existence of canonical curves and consequences

If X is a stable surface, we say that X contains a canonical curve if there is a section $x_0 \in H^0(X, \omega_X)$ which is nonzero at each generic point. In the irreducible case this just means that $x_0 \neq 0$, in general it can happen that a section vanishes identically on some but not every component.

[1] As suggested by the Hilbert series these surfaces are canonically embedded as hypersurfaces of degree 10 in $\mathbb{P}(1,1,2,5)$ (see Theorem 4.1).

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Example 3.1. Let C be a smooth plane quartic, $X_1 = S^2C$ and D_1 a coordinate curve. We have $p_g(X_1) = q(X_1) = 3$ and $K^2_{X_1} = 6$ (see [HP02, Example 1]). Since D_1 is ample, the Riemann–Roch formula gives $h^0(K + D_1) = 3 = h^0(K)$, so that D_1 is in the fixed part of $|K + D_1|$.

Now take $(X_2, D_2) = (\mathbb{P}^2, C)$ and glue $D_1 \cong C \cong D_2$ to get a stable surface X, which consists of the two components intersecting with normal crossings in the curve C. One can check that $K^2_X = 13$, $p_g(X) = 3$ and $q(X) = 0$.

By the above, all canonical sections vanish on the intersection curve, because their pullback to the normalisation vanishes on D_1, thus they vanish identically on the component X_2 and there is no canonical curve.

3.1. Existence of canonical curves

In this section we prove the existence of canonical curves on any 2-Gorenstein stable I-surface. We start with slightly weaker hypotheses, which we then prove to imply that we have a stable I-surface.

Lemma 3.2. *Let X be a reducible 2-Gorenstein stable surface with $K^2_X = 1$ and $\chi(X) = 3$. Then:*

1. *the normalisation $(\bar{X}, \bar{D})$ of $\bar{X}$ is equal to $(\bar{X}_1, \bar{D}_1) \sqcup (\bar{X}_2, \bar{D}_2)$, with $\bar{X}_i$ irreducible and $\bar{D}_i > 0$, $i = 1, 2$;*
2. *for $i = 1, 2$ the divisor $K_{\bar{X}_i} + \bar{D}_i$ is 2-Cartier and ample with $(K_{\bar{X}_i} + \bar{D}_i)^2 = \frac{1}{2}$.*

Proof. Let $\pi\colon \bar{X} = \bigsqcup_{i=1}^r \bar{X}_i \to X$ be the normalisation. Since $2K_X$ is an ample Cartier divisor, its pullback $\pi^* 2K_X|_{\bar{X}_i} = 2(K_{\bar{X}_i} + \bar{D}_i)$ is an ample Cartier divisor as well, and

$$2 = 2K^2_X = \sum_{i=1}^{r} \big(2(K_{\bar{X}_i} + \bar{D}_i)\big)(K_{\bar{X}_i} + \bar{D}_i).$$

Because the intersection of an ample Cartier divisor with a Weil divisor is a positive integer and because by assumption we have more than one component, we get $r = 2$ and $(K_{\bar{X}_i} + \bar{D}_i)^2 = \frac{1}{2}$. The divisors $\bar{D}_i$ cannot be zero, because they contain the preimage of the intersection of the two components, which is a curve because X is S_2 and connected. □

If X is a reducible 2-Gorenstein stable surface with $K^2_X = 1$ and $\chi(X) = 3$ and normalisation $(\bar{X}_1, \bar{D}_1) \sqcup (\bar{X}_2, \bar{D}_2)$, we write $\bar{D}_i = \bar{Z}_i + \bar{\Gamma}_i$, where $\bar{\Gamma}_1$ is glued to $\bar{\Gamma}_2$ by π, while $\bar{Z}_i$ is glued to itself for $i = 1, 2$. Note that $\bar{\Gamma}_i > 0$ since X is connected and S_2, while $\bar{Z}_i$ may be zero.

With this notation in place we can state the main result of this section:

Proposition 3.3. *Let X be a 2-Gorenstein stable surface with $K^2_X = 1$ and $\chi(X) = 3$. Then:*

1. *the zero locus of a general section of $H^0(K_X)$ is one-dimensional;*
2. *$q(X) = 0$ and X is a stable I-surface;*
3. *all canonical curves are nonreduced if and only if $X = X_1 \cup X_2$ is reducible and $K_{\bar{X}_i} + \bar{Z}_i = 0$ for $i = 1, 2$. In this case, every canonical curve is supported on $X_1 \cap X_2$.*

Before proving Proposition 3.3 we give some auxiliary results.

Lemma 3.4. *Let Y be a smooth projective surface and let M be a nef line bundle. If $M^2 = 2$, then $h^0(M) \leq 4$.*

Proof. We can of course assume that $r := h^0(M) - 1 \geq 1$, otherwise the statement is empty. Write $|M| = Z + |D|$, where Z is the fixed part and D the moving part, denote by $h\colon X \to \mathbb{P}^r$ the map defined by $|D|$ and let d be the degree of the image Σ of h. Assume first that Σ is a surface: then since M and D are nef we have $2 = M^2 \geq MD \geq D^2 \geq d \geq r - 1$, hence $r \leq 3$. If instead Σ is a curve, then D is numerically equivalent to dG, where G is irreducible and $2 = M^2 \geq MD = dMG \geq d \geq r$, so $r \leq 2$ in this case. □

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If $X = X_1 \cup X_2$ is a reducible stable 2-Gorenstein stable surface with $K_X^2 = 1$ and $\chi(X) = 3$, then we denote by

$$\rho = (\rho_1, \rho_2) \colon H^0(K_X) \to H^0(K_{\bar{X}_1} + \bar{D}_1) \oplus H^0(K_{\bar{X}_2} + \bar{D}_2)$$

the pullback map.

Lemma 3.5. *In the above set-up and notation, assume that ρ_1 is not injective: then* $\dim \ker \rho_1 = 1$ *and* $K_{\bar{X}_2} + \bar{Z}_2 = 0$.

Proof. Set $L_i := 2(K_{\bar{X}_i} + \bar{D}_i)$ for $i = 1, 2$. Recall that by assumption L_i is an ample line bundle with $L_i^2 = 2$. Consider $0 \neq \sigma \in \ker \rho_1$: since ρ is injective, $\rho_2(\sigma)$ is a nonzero section of $K_{\bar{X}_2} + \bar{D}_2$ that vanishes on $\bar{\Gamma}_2$, hence $K_{\bar{X}_2} + \bar{Z}_2 \geq 0$. Since L_2 is ample, $1 = L_2(K_{\bar{X}_2} + \bar{D}_2) = L_2(K_{\bar{X}_2} + \bar{Z}_2) + L_2\bar{\Gamma}_2$ and $\bar{\Gamma}_2 > 0$ give $L_2\bar{\Gamma}_2 = 1$, $L_2(K_{\bar{X}_2} + \bar{Z}_2) = 0$, therefore $K_{\bar{X}_2} + \bar{Z}_2 = 0$ and $\bar{\Gamma}_2$ is irreducible. Note that $\bar{\Gamma}_2$ is the divisor of $\rho_2(\sigma)$. If $0 \neq \tau$ is another element of $\ker \rho_1$, then the divisor of $\rho_2(\tau)$ is also equal to $\bar{\Gamma}_2$ and therefore, $\rho_2(\tau)$ and $\rho_2(\sigma)$ are linearly dependent. Since ρ is injective, it follows that $\ker \rho_1$ has dimension 1. □

Proof of Proposition 3.3.

1. Assume for contradiction that all sections of K_X have a two-dimensional zero locus. Then $X = X_1 \cup X_2$ is reducible and we may assume that all sections of K_X vanish on X_1. By Lemma 3.5 we get $p_g(X) \leq 1$, contradicting $\chi(X) = 3$.
2. Assume for contradiction $q(X) > 0$, namely $p_g(X) \geq 3$. If X is irreducible, then $4 = K_X^2 + \chi(X) = h^0(2K_X)$ by [LR14, Prop. 16], because $2K_X$ is Cartier. On the other hand the Hopf lemma (see [ACGH85, p. 108]) gives $h^0(2K_X) \geq 2p_g(X) - 1 \geq 5$, a contradiction. So $X = X_1 \cup X_2$ is reducible. If ρ_i is injective, then $h^0(K_{\bar{X}_i} + \bar{D}_i) \geq 3$. Then the Hopf lemma gives $h^0(2(K_{\bar{X}_i} + \bar{D}_i)) \geq 2h^0(K_{\bar{X}_i} + \bar{D}_i) - 1 \geq 5$. Pulling back $2(K_{\bar{X}_i} + \bar{D}_i)$ to a line bundle M on a desingularisation Y of $\bar{X}_i$ we obtain a contradiction to Lemma 3.4. So ρ_1 and ρ_2 are not injective. Identifying $H^0(K_X)$ with its image via ρ, we can find three independent sections of the form: $(\sigma_1, 0)$, $(0, \tau_1)$, (σ_2, τ_2). Note that σ_1 and σ_2 are independent because $\ker \rho_2$ is one-dimensional by Lemma 3.5 and by the same argument τ_1 and τ_2 are also independent. Now the following correspond to linearly independent sections of $2K_X$:

$$(\sigma_1^2, 0),\ (\sigma_1\sigma_2, 0),\ (0, \tau_1^2),\ (0, \tau_1\tau_2),\ (\sigma_2^2, \tau_2^2),$$

contradicting again $h^0(2K_X) = 4$. We have proved that $q(X) = 0$.

It remains to control the plurigenera of X. Since even multiples of the canonical divisor are Cartier, the Riemann–Roch formula applies, so we only need to control the odd plurigenera.

Let C be a canonical curve defined by a section x_0. As explained in Proposition 3.6 below the reflexive restriction sequence (3.2) defines a torsion free sheaf of rank one $\mathcal{L} = \omega_X|_C^{[1]}$ on C with $\chi(\mathcal{L}) = \chi(\mathcal{O}_X) + \chi(\omega_X) = 0$. By definition we have $\deg \mathcal{L} = 1$, see [Har86, CFHR99]. We now twist (3.2) with the line bundle $\omega_X^{[2m]}$ with $m \geq 1$ and compute using generalised Kodaira vanishing and the Riemann–Roch formula on singular curves

$$\begin{aligned} P_{2m+1}(X) &= \chi(\omega_X^{[2m+1]}) = \chi(\omega_X^{[2m]}) + \chi(\mathcal{L} \otimes \omega_X^{[2m]}|_C) \\ &= \chi(X) + m(2m-1) + \chi(C) + 2m + 1 = \chi(X) + m(2m+1). \end{aligned}$$

This is the correct plurigenus for a stable I-surface.

3. If X is irreducible, then pulling back the canonical system to a desingularisation Y of X and considering its moving part we see that the general canonical curve C has at least a reduced component. Since $2K_X C = 2$ and $2K_X$ is an ample line bundle, it follows that C is reduced. Therefore $X = X_1 \cup X_2$. Consider the pullback map $\rho = \rho_1 \oplus \rho_2$: if, say, ρ_1 is injective, then a similar argument shows that the restriction to X_1 of a general $C \in |K_X|$ is reduced and not contained in $\bar{\Gamma}_1$, hence C is reduced.

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So it follows that ρ_1 and ρ_2 are not injective and, by Lemma 3.5 $K_{\bar{X}_i} + \bar{Z}_i = 0$, $i = 1, 2$, and all canonical curves are supported on $X_1 \cap X_2$.

Conversely, assume that $X = X_1 \cup X_2$ is reducible and $K_{\bar{X}_i} + \bar{Z}_i = 0$. Then $K_{\bar{X}_i} + \bar{D}_i = \bar{\Gamma}_i > 0$ for $i = 1, 2$. Take $\sigma_i \in H^0(K_{\bar{X}_i} + \bar{D}_i)$ a nonzero section that vanishes on $\bar{\Gamma}_i$: then $(\sigma_1, 0)$ and $(0, \sigma_2)$ correspond to independent canonical sections. Since $p_g(X) = 2$ by 2., these sections generate $H^0(K_X)$ and therefore every canonical curve is supported on $X_1 \cap X_2$ and is not reduced. □

3.2. Restriction of the canonical ring to a canonical curve

In order to apply Reid's hyperplane section principle, we need to describe the restriction of the canonical ring to a canonical curve carefully. In the end, this will be useful only if the general canonical curve is reduced.

Let X be a stable I-surface and fix C a canonical curve on X defined by $x_0 \in H^0(X, \omega_X)$. Noting that $\mathscr{O}_X(C) = \omega_X$ is the canonical bundle we get three exact sequences

$$0 \longrightarrow \mathscr{O}_X(-C) \xrightarrow{\cdot x_0} \mathscr{O}_X \longrightarrow \mathscr{O}_C \longrightarrow 0\ , \tag{3.1}$$

$$0 \longrightarrow \mathscr{O}_X \xrightarrow{\cdot x_0} \omega_X \longrightarrow \mathcal{L} \longrightarrow 0\ , \tag{3.2}$$

$$0 \longrightarrow \omega_X \xrightarrow{\cdot x_0} \omega_X(C) = \omega_X^{[2]} \longrightarrow \omega_C \longrightarrow 0\ . \tag{3.3}$$

Indeed, (3.3) arises by applying $\mathscr{H}om_{\mathscr{O}_X}(-, \omega_X)$ to the restriction sequence (3.1) to get the exact sequence $0 \to \omega_X \to \omega_X(C) \to \mathscr{E}xt^1_{\mathscr{O}_X}(\mathscr{O}_C, \omega_X) \to 0$. The third sheaf in this sequence may be identified with ω_C by definition of the dualising sheaf (see [Rei94, Theorem 2.12]).

Proposition 3.6. *Let X be a 2-Gorenstein stable I-surface with canonical curve C defined by $x_0 \in H^0(X, \omega_X)$. Let $\mathcal{L}$ be the sheaf on C defined by (3.2). Then*

1. *The curve C is a Gorenstein curve of arithmetic genus 2 with ample canonical bundle ω_C and $h^0(C, \mathscr{O}_C) = 1$.*
2. *The sheaf $\mathcal{L} \cong \omega_X|_C^{[1]}$ is a torsion-free sheaf with $\chi(\mathcal{L}) = 0$, $h^0(\mathcal{L}) = 1$ and the map μ defined by the diagram*

$$\begin{array}{ccc} \omega_X \otimes_{\mathscr{O}_X} \omega_X & \longrightarrow & \omega_X^{[2]} \\ \downarrow & & \downarrow \\ \mathcal{L} \otimes_{\mathscr{O}_C} \mathcal{L} & \xrightarrow{\mu} & \omega_C \end{array}$$

   is an isomorphism in codimension zero.
3. *The multiplication on global sections induced by μ, namely*

$$H^0(\mathcal{L} \otimes \omega_C^{\otimes m}) \times H^0(\mathcal{L} \otimes \omega_C^{\otimes n}) \to H^0(\mathcal{L} \otimes \mathcal{L} \otimes \omega_C^{\otimes (m+n)}) \to H^0(\omega_C^{\otimes (m+n+1)}),$$

   makes the natural restriction map

$$\phi\colon R(X, K_X) \twoheadrightarrow \bigoplus_{n \geq 0} \left(H^0(C, \omega_C^{\otimes n}) \oplus H^0(C, \mathcal{L} \otimes \omega_C^{\otimes n}) \right) =: R(C, \{\mathcal{L}, \omega_C\})$$

   into a surjective ring homomorphism with kernel generated by x_0.

In the notation of [CFPR23] the pair $(\mathcal{L}, \mu)$ is a ggs (generalised Gorenstein spin) structure on C and $R(C, \{\mathcal{L}, \omega_C\})$ is the associated half-canonical ring.

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Proof. Several times we make use of the long exact cohomology sequences of (3.1), (3.2), (3.3), possibly twisted with $\omega_X^{[2k]}$, the fact that $q(X) = 0$ by Proposition 3.3 and generalised Kodaira vanishing [LR14, Prop. 21])

To prove the first point, note that by assumption $\omega_X^{[2]}$ is a line bundle and from (3.3) we get

- ω_C is the restriction of an ample line bundle, thus ample and C is Gorenstein,
- $h^0(C, \mathscr{O}_C) = h^1(C, \omega_C) = h^2(X, \omega_X) = 1$ by the long exact cohomology sequence and Kodaira vanishing and duality on C,
- $\deg \omega_C = C \cdot 2K_X = 2$ and thus the arithmetic genus $p_a(C) = 2$.

We now prove the second point. By the depth-Lemma [Eis95, Cor. 18.6] and the fact that $\omega_X^{[m]}$ is S_2 for every m, the sheaf $\mathcal{L}$ is the quotient of $\omega_X \otimes_{\mathscr{O}_X} \mathscr{O}_C$ by the torsion submodule, $\mathcal{L}$ is torsion-free on C and coincides with $\omega_X|_C^{[1]}$ (see [Har94, Proposition 1.6]). The values $\chi(\mathcal{L}) = 0$, $h^0(\mathcal{L}) = 1$ follow from (3.2).

The map μ exists because every torsion sheaf maps to zero in the line bundle ω_C. Outside the codimension two subset of X where ω_X is not locally free, that is, outside a finite number of points, the map μ is an isomorphism.

For the last point, the existence of the ring structure on $R(C, \{\mathcal{L}, \omega_C\})$ and the ring homomorphism ϕ is clear by construction. The surjectivity follows from twists of (3.2) in odd degrees and twists of (3.3) in even degrees. □

Remark 3.7. It is only slightly more tedious to work out the nature of the restriction of a section ring of any $\mathbb{Q}$-Cartier divisor to a curve in a similar fashion, but we don't need this here.

In the case where the canonical curve is reduced, we can use the results of [CFPR23] to describe the restricted canonical ring $R(C, \{\mathcal{L}, \omega_C\})$.

Theorem 3.8. *Let $(C, \mathcal{L}, \mu)$ be as in Proposition 3.6, that is, C is a Gorenstein curve of arithmetic genus two with ample canonical bundle and $(\mathcal{L}, \mu)$ is a ggs structure on C with $h^0(C, \mathcal{L}) = 1$. If C is reduced, then the following two cases are possible*

type A *The curve C is a flat double cover of $\mathbb{P}^1$, so in particular it is either irreducible or the union of two smooth rational curves. The half-canonical ring is*

$$R(C, \{\mathcal{L}, \omega_C\}) = \mathbb{C}[x, y, z]/(z^2 - f_{10}(x, y)),$$

where $\deg(x, y, z) = (1, 2, 5)$ and $f_{10} \neq 0$ is weighted homogeneous of degree 10.

type B *The curve C is the union of two irreducible curves C_i with $p_a(C_i) = 1$, $i = 1, 2$, that meet transversely at a single point p that is smooth for both. The half-canonical ring is*

$$R(C, \{\mathcal{L}, \omega_C\}) = \mathbb{C}[x, y, w, v, z, u]/I,$$

with $\deg(x, y, w, v, z, u) = (1, 2, 3, 4, 5, 6)$ and the ideal I is generated by the following equations

$$\operatorname{rk}\begin{pmatrix} 0 & y & w & z \\ x & w & v & u \end{pmatrix} \leq 1, \quad \begin{aligned} z^2 &= y g_8(y, v) \\ zu &= w g_8(y, v) \\ u^2 &= v g_8(y, v) + x^4 h_8(x, v) \end{aligned}$$

where g_8 and h_8 are weighted homogeneous of degree 8 and v^2 appears in g_8 with nonzero coefficient.

Proof. The distinction and description of the cases comes from Proposition 3.3 and Corollary 3.9 in [CFPR23], while the half-canonical rings are the cases $A(1)$ and $B(1)$ of [CFPR23, Theorem 5.2]. □

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4. Surfaces with reduced canonical curve of type A

It turns out that these are well known to us.

Theorem 4.1. *Let X be a 2-Gorenstein I-surface containing a reduced canonical curve C of type A and let*

$$\mathfrak{D}_A = \overline{\left\{ [X] \in \overline{\mathfrak{M}}_{1,3} \,\middle|\, \begin{array}{l} X \text{ is 2-Gorenstein but not Gorenstein with reduced} \\ \text{canonical curve of type A} \end{array} \right\}}$$

where the closure is taken in $\overline{\mathfrak{M}}_{1,3}$.

1. *The surface X is canonically embedded as a hypersurface of degree 10 in $\mathbb{P}(1,1,2,5)$ not passing through $(0:0:0:1)$ and X is $\mathbb{Q}$-Gorenstein smoothable. It is Gorenstein if and only if X does not contain the point $(0:0:1:0)$.*
   Conversely, any such hypersurface with slc singularities is a stable I-surface.
2. *The set $\mathfrak{D}_A$ is an irreducible divisor in the closure of the Gieseker component. It coincides with the closure of the divisor of surfaces with one singularity of type $\frac{1}{4}(1,1)$ considered in [FPRR22, CFP+23].*

Proof.

1. The description as a hypersurface follows directly via the hyperplane section principle [Rei90] from the description of the ring $R(C, \{\mathcal{L}, \omega_C\})$ given in Theorem 3.8, type A. The 'converse' statement follows by using adjunction for hypersurfaces in $\mathbb{P}(1,1,2,5)$, while the criterion for X being Gorenstein was proved in [FPR17, Theorem 3.3, Proposition 4.1] (see also [CFP+23, Remark 3.2]).
2. The set $\mathfrak{D}_A$ is the closure of the image of an open subset of the linear system of hypersurfaces of degree 10 containing the point $(0:0:1:0)$ by 1. and as such it is an irreducible divisor. Its general element has a unique singular point of type $\frac{1}{4}(1,1)$ by the discussion in [FPRR22, Section 3.A]. □

Example 4.2. If we take X to be the hypersurface in $\mathbb{P}(1_{x_1}, 1_{x_2}, 2_y, 5_z)$ defined by the equation $z^2 - x_1 x_2 f_4(x_1, x_2, y)^2 = 0$, with f_4 a general polynomial of degree 4, then X is irreducible with normalisation $\mathbb{P}(1,1,4)$, and the canonical system has only one base point, occurring above the vertex of the quadric cone. Nevertheless, all the canonical curves are reducible.

5. Surfaces with reduced canonical curve of type B

In this section we describe 2-Gorenstein stable I-surface with reduced canonical curve of type B (in the notation of Theorem 3.8).

Theorem 5.1. *Let X be a 2-Gorenstein stable I-surface with reduced canonical curve of type B. Then the canonical ring of X is as in Proposition 5.6 and if X is general, it is glued from an Enriques surface and a K3 surface as in Proposition 5.4.*

The closure of the set of such surfaces,

$$\mathfrak{D}_B = \overline{\left\{ [X] \in \overline{\mathfrak{M}}_{1,3} \,\middle|\, \begin{array}{l} X \text{ 2-Gorenstein with reduced} \\ \text{canonical curve of type B} \end{array} \right\}} \subset \overline{\mathfrak{M}}_{1,3}.$$

is an irreducible divisor in the closure of the Gieseker moduli space.

The intersection of divisors $\mathfrak{D}_A \cap \mathfrak{D}_B$ contains the irreducible codimension two subset

$$\mathfrak{R}_{AB} = \{[X] \in \mathfrak{D}_B \mid \tilde{g}_8 \text{ does not contain } y^4 \text{ in } (5.2)\}.$$

We will start with the geometric description, via which the surfaces were initially found, and then prove Theorem 5.1 in Proposition 5.6, Corollary 5.7 and Proposition 5.8.

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5.1. A geometric construction

We will start with a geometric description of 2-Gorenstein stable surfaces with reduced canonical curve of type B. Let us fix some notation: we denote by $\mathbb{F}_n$ the Hirzebruch surface with negative section s_∞ of square $s_\infty^2 = -n$. We call a section s_0 disjoint from s_∞ a positive section.

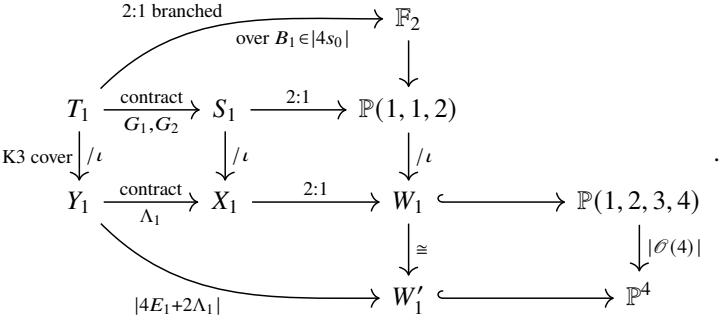
Construction 5.2. Let Y_1 be an Enriques surface containing a half-pencil E_1, that is, an elliptic curve E_1 such that $|2E_1|$ is an elliptic pencil. Assume further that Y_1 contains a (-2)-curve which is a bisection Λ_1 of $|2E_1|$.

Contracting Λ_1 to an A_1 point p, we get a singular Enriques surface X_1 containing an elliptic curve E_1 through the point p with $E_1^2 = \frac{1}{2}$.

The K3 cover $T_1 \to Y_1$ is an elliptic K3 surface and contains two disjoint (-2)-curves G_1, G_2, which realise T_1 as a double cover of $\mathbb{F}_2$ branched over a (symmetric) divisor $B_1 \in |4s_0|$ and one can construct explicit examples in this way.

Such Enriques surfaces have been studied in [CD89, Ch.4, §7] where it is shown that the so-called superelliptic linear system $|4E_1 + 2\Lambda_1|$ on Y_1 realises X_1 as the double cover of a degenerate symmetric del Pezzo surface W_1' of degree 4 in $\mathbb{P}^4$.

One can check that $W_1' \cong W_1 = \{w^2 - yv = 0\} \subset \mathbb{P}(1_{x_0}, 2_y, 3_w, 4_v)$ fitting in the following commutative diagram:

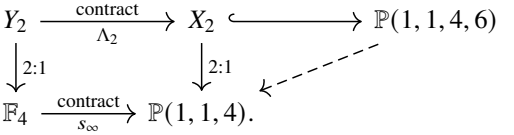


(5.1)

Finally, we count the number of moduli of this construction. The K3 cover S_1 of X_1 is a hypersurface of degree 8 in $\mathbb{P}(1_{a_0}, 1_{a_1}, 2_b, 4_c)$ with equation $c^2 = g_8(a_0, a_1^2, a_1 b, b^2)$. In addition, g_8 is invariant under the involution ι of $\mathbb{P}(1,1,2,4)$ defined by $(a_0, a_1, b, c) \mapsto (a_0, -a_1, -b, -c)$ (cf. [BHPV04, Thm. VIII.18.2]). An explicit computation shows that g_8 varies in a linear system of dimension 12. Since the subgroup of automorphisms of $\mathbb{P}(1,1,2,4)$ that commute with ι has dimension 3, we get 9 parameters for (X_1, E_1).

Construction 5.3. Let $B_2 \subset \mathbb{F}_4$ be a general divisor in $|s_\infty + 3s_0| = s_\infty + |3s_0|$ and let Y_2 be the double cover branched over B_2. By the Hurwitz formula Y_2 is a smooth K3 surface with an elliptic fibration induced by the fibration on $\mathbb{F}_4$.

The preimage of s_∞ is a (-2)-curve $\Lambda_2 \subset Y_2$ and contracting it we get a singular K3 surface X_2 with one A_1 singularity at a point p. It can be viewed as a double cover of $\mathbb{P}(1,1,4)$ or as a hypersurface of degree 12 in $\mathbb{P}(1,1,4,6)$, resulting in a diagram



The image of any fibre of the elliptic fibration is a curve E_2 of arithmetic genus one in X_2 containing the point p and satisfying $E_2^2 = \frac{1}{2}$. Note that a general Y_2 is not isotrivial, so every isomorphism class of elliptic curves occurs a finite number of times as a fibre.

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To count parameters for Y_2, we observe that the linear system $|\sigma_\infty + 3\sigma_0|$ on $\mathbb{F}_4$ has dimension 27, while the automorphism group of $\mathbb{F}_4$ has dimension 9, so we get 18 parameters.

Proposition 5.4. *Let (X_1, E_1) be as in Construction 5.2 and (X_2, E_2) be as in Construction 5.3 with $E_1 \cong E_2$ a smooth elliptic curve. Then the two pairs can be glued to a 2-Gorenstein stable I-surface $X = X_1 \cup_{E_1 \cong E_2} X_2$ such that the general canonical curve is of type B.*

The construction depends on 27 parameters.

Proof. Note that (X_i, E_i) are stable pairs with volume $\frac{1}{2}$. By Kollár's glueing construction [Kol13, Thm. 5.13], the two components can be glued to a stable surface if the glueing isomorphism $E_1 \cong E_2$ preserves the different, that is, if it preserves the point p.

It is easy to check from the classification of slc singularities that X is 2-Gorenstein. The two components of a general canonical curve C are defined by sections in $H^0(X_i, K_{X_i} + E_i)$ and are thus two curves of arithmetic genus one, intersecting in the point p.

The pair (X_1, E_1) depends on 9 parameters (see Construction 5.2) and X_2 depends on 18 parameters (see Construction 5.3), which leaves us with finitely many choices of a fibre E_2 of the elliptic fibration on Y_2 which is isomorphic to E_1. □

Remark 5.5. The canonical system of a surface as in Proposition 5.4 has a fixed part: on the Enriques surface X_1 there is a unique section of $K_{X_1} + E_1$, corresponding to the other half-pencil in the elliptic fibration on Y_1.

5.2. Classification via the canonical ring

To completely classify 2-Gorenstein stable I-surfaces with reduced canonical curve of type B, we again rely on a description of the canonical ring and show that this coincides with the geometric description given in Section 5.1.

Proposition 5.6. *Let X be a 2-Gorenstein I-surface with reduced canonical curve of type B. Then the surface X is canonically embedded in $\mathbb{P}(1_{x_0}, 1_x, 2_y, 3_w, 4_v, 5_z, 6_u)$ with weighted homogeneous equations:*

$$\bigwedge^2 \begin{pmatrix} 0 & y & w & z \\ x & w & v & u \end{pmatrix} = 0, \quad \begin{aligned} z^2 &= y\tilde{g}_8(x_0, y, w, v) \\ zu &= w\tilde{g}_8(x_0, y, w, v) \\ u^2 &= v\tilde{g}_8(x_0, y, w, v) + x^2\tilde{k}_{10}(x_0, x, v) \end{aligned} \tag{5.2}$$

where v^2 appears in $\tilde{g}_8$ with nonzero coefficient, which we fix to be 1.

Conversely, if X is defined by general equations of this form, then $\omega_X = \mathcal{O}_X(1)$ and X is a 2-Gorenstein stable I-surface.

Proof. The equations from Theorem 3.8 which define the canonical curve of type B, can be expressed in the following format:

$$\bigwedge^2 \begin{pmatrix} 0 & y & w & z \\ x & w & v & u \end{pmatrix} = 0, \quad \begin{aligned} z^2 &= yg_8(y, v) \\ zu &= wg_8(y, v) \\ u^2 &= vg_8(y, v) + x^4 h_8(x, v) \end{aligned}$$

This format is useful for studying the equations of the surface X which contains C as canonical hyperplane section.

We reconstruct X from C by applying the hyperplane section principle [Rei90]. The ideal of relations has 9 generators and 16 syzygies, so it is possible to work out the equations of X by hand après Reid [Rei90]. We also used the computer package of Ilten [Ilt12] which in turn follows an algorithm laid out by Stevens [Ste01]. We highlight the two most important features of the result.

The 2×4 matrix for C is weighted homogeneous with weights $\left(\begin{smallmatrix} 0 & 2 & 3 & 5 \\ 1 & 3 & 4 & 6 \end{smallmatrix}\right)$ and generic entries, except that w appears twice. Thus (after relabelling the entries) the only possible change to the matrix for X

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would be to one of the w-entries. In fact, it is impossible to change one w-entry without violating some of the 16 syzygies yoking the relations together, hence the matrix for X is unchanged.

The last relation output by the algorithm has the unsatisfactory form

$$u^2 = v\tilde{g}_8(x_0, y, w, v) + x^4\tilde{h}_8 + x_0x^3a_8 + x_0^2x^2b_8 + x_0^3xc_8$$

where $\tilde{h}_8(x_0, x, v)|_{x_0=0} = h_8$ and a_8, b_8, c_8 are general forms of degree 8 in (x_0, x, v). We write $\tilde{k}_{10} = x^2\tilde{h}_8 + x_0xa_8 + x_0^2b_8$ and absorb those terms of c_8 that are divisible by x or v into $x^2\tilde{k}_{10}$ or $v\tilde{g}_8$ respectively. This leaves

$$u^2 = v\tilde{g}_8(x_0, y, w, v) + x^2\tilde{k}_{10}(x_0, x, v) + cx_0^{11}x,$$

where c is now a constant. Using the coordinate transformations $x \mapsto x + \alpha x_0$ and $v \mapsto v + \beta x_0^4$ for appropriate choices of parameters α, β, we eliminate c to obtain the stated equation. Moreover, the new coordinates do not affect the original choice of curve section C. Note that $\tilde{g}_8(x_0, y, w, v)|_{x_0=0} = g_8(y, v)$ because after utilising the relation $w^2 = yv$, all monomials of $\tilde{g}_8$ involving w also involve x_0, for degree reasons.

To find ω_X, we follow the same procedure as in the proof of [CFP+23, Cor. 3.12]. That is, we use computer algebra to get the minimal free resolution of $\mathscr{O}_X$ as an $\mathscr{O}_{\mathbb{P}}$-module. This is a self-dual complex of length four because the homogeneous coordinate ring of X is Gorenstein, and the last term of the complex is $\mathscr{O}_{\mathbb{P}}(-23)$. Since the complex is self-dual, and $\omega_{\mathbb{P}} = \mathscr{O}_{\mathbb{P}}(-1-1-2-3-4-5-6) = \mathscr{O}_{\mathbb{P}}(-22)$, when we apply $\mathcal{H}om(-, \omega_{\mathbb{P}})$, we can read off $\omega_X = \mathcal{E}xt^4(\mathscr{O}_X, \omega_{\mathbb{P}}) = \mathscr{O}_X(-22+23) = \mathscr{O}_X(1)$.

To show that $\omega_X^{[2]}$ is invertible, it suffices to show that X does not intersect the loci $\mathbb{P}(3_w, 6_u)$, $\mathbb{P}(4_v)$, $\mathbb{P}(5_z)$ where $\mathscr{O}_{\mathbb{P}}(2)$ is not invertible. This can be deduced from the equations (5.2): the monomials v^3 and z^2 appear in some equation, so X is disjoint from $\mathbb{P}(4)$ and from $\mathbb{P}(5)$, and since w^2 and u^2 appear in separate equations, it follows that X is disjoint from $\mathbb{P}(3, 6)$ as well. Hence $\mathscr{O}_X(2)$ is invertible and X is 2-Gorenstein. □

Corollary 5.7. *Let X be a 2-Gorenstein I-surface with reduced canonical curve of type B. Then $X = X_1 \cup X_2$ is reducible and fits in a diagram*

$$\begin{array}{ccc}
X_1 \cup X_2 & \xrightarrow{\quad 2:1 \quad} & W_1 \cup W_2 \\
\downarrow & & \downarrow \\
\mathbb{P}(1_{x_0}, 2_y, 3_w, 4_v, 5_z, 6_u) \cup \mathbb{P}(1_{x_0}, 1_x, 4_v, 6_u) & \dashrightarrow & \mathbb{P}(1_{x_0}, 2_y, 3_w, 4_v) \cup \mathbb{P}(1_{x_0}, 1_x, 4_v) \\
\downarrow & & \downarrow \\
\mathbb{P}(1_{x_0}, 1_x, 2_y, 3_w, 4_v, 5_z, 6_u) & \dashrightarrow & \mathbb{P}(1_{x_0}, 1_x, 2_y, 3_w, 4_v)
\end{array}$$

where $W_1 = \{w^2 - yv = 0\} \subset \mathbb{P}(1_{x_0}, 2_y, 3_w, 4_v)$ and $W_2 = \mathbb{P}(1_{x_0}, 1_x, 4_v)$.

1. *The intersection*

$$X_1 \cap X_2 = E \hookrightarrow \mathbb{P}(1_{x_0}, 4_v, 6_u) \cong \mathbb{P}(1_{x_0^2}, 2_v, 3_u)$$

is a curve of arithmetic genus one embedded via the section ring of $\mathscr{O}_E(p)$, where $p = (0 : 1_v : 1_u)$.

If the defining equations are sufficiently general, then:

2. *The pair (X_1, E) is a singular Enriques surface as described in Construction 5.2.*
3. *The pair (X_2, E) is a singular K3 surface as described in Construction 5.3.*

In particular, X arises as in Proposition 5.4.

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Proof. The defining ideal of X contains the three reducible equations $xy = xw = xz = 0$ cutting out the two (weighted) linear subspaces such that X is contained in their union. Since X cannot be contained in one of them, $X = X_1 \cup X_2$ has two irreducible components; it cannot have more by Lemma 3.2.

The linear projection corresponds to the subring generated by x_0, x, y, w, v and the three equations on the right hand side of (5.2) show that the map is of degree two on both components of X.

We translate the equations into the geometric descriptions.

1. The equation for E,
$$u^2 - v\tilde{g}_8(x_0, 0, 0, v) = u^2 - v^3 + \cdots = 0,$$
is obtained by substituting $x = y = w = z = 0$ and is exactly the Weierstrass model we get from a divisor of degree one on a curve of arithmetic genus one.
2. The component $X_1 = X \cap (x = 0)$ in $\mathbb{P}(1, 2, 3, 4, 5, 6)$ has equations
$$\bigwedge^2 \begin{pmatrix} y & w & z \\ w & v & u \\ z & u & \tilde{g}_8 \end{pmatrix} = 0 \tag{5.3}$$
For $\tilde{g}_8$ general, X_1 has one A_1 singularity at $(0 : 0 : 0 : 1_v : 0 : 1_u)$. We are going to show that X_1 is a singular Enriques surface by describing explicitly the singular K3 cover $S_1 \to X_1$, compare (5.1). Consider the weighted projective space $\mathbb{P}(1_{a_0}, 1_{a_1}, 2_b, 4_c)$ and let S_1 be defined by the equation
$$c^2 = g_8(a_0, a_1^2, a_1 b, b^2). \tag{5.4}$$
The surface S_1 has A_1 singularities at the two points $(0 : 0 : 1 : \pm 1)$. By adjunction for weighted projective hypersurfaces S_1 is a singular K3 surface.

Let ι be the involution on $\mathbb{P}(1, 1, 2, 4)$ given by $(a_0 : a_1 : b : c) \mapsto (a_0 : -a_1 : -b : -c)$. One can check that ι induces a fixed point free involution on S_1 exchanging the two A_1 singularities.
We compute the equations of the quotient surface S_1/ι. The ι-invariants are
$$x_0 = a_0,\ y = a_1^2,\ w = a_1 b,\ v = b^2,\ z = a_1 c,\ u = bc,\ c^2$$
with respective degrees $1, 2, 3, 4, 5, 6, 8$. There are several monomial relations between these generators which are analogous to those for the affine cone on the Veronese embedding of $\mathbb{P}^2$ with coordinates a_1, b, c, hence they may be expressed as the 2×2 minors of a 3×3 matrix as in equation (5.3), thus S_1/ι is isomorphic to X_1.

The projection $\mathbb{P}(1, 1, 2, 4) \dashrightarrow \mathbb{P}(1, 1)$ induces an elliptic pencil on S_1 that descends to an elliptic pencil on X_1. The image of the ι-invariant elliptic curve $S_1 \cap \{a_1 = 0\}$ on the quotient X_1 is the double curve $y = w = z = 0$.

We have thus recovered the description given in Construction 5.2.
3. Notice that $X_2 = X \cap (y = w = z = 0)$ is the hypersurface of weighted degree 12 defined by $(u^2 = v\tilde{g}_8 + x^2\tilde{k}_{10})$ in $\mathbb{P}(1, 1, 4, 6)$. For general $\tilde{g}$, $\tilde{k}$ this has one A_1 singularity at $(0 : 0 : 1_v : 1_u)$. The curve E is cut out by $x = 0$ and thus passes through the singular point. Thus X_2 is a singular K3 surface as described in Construction 5.3. □

5.3. Locating $\mathfrak{D}_B$ inside $\overline{\mathfrak{M}}_{1,3}$

In this section we show that every 2-Gorenstein stable I-surface with reduced canonical curve of type B is $\mathbb{Q}$-Gorenstein smoothable. Together with a parameter count and some further analysis we get the following.

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Proposition 5.8. *Consider the closure of the set of 2-Gorenstein stable I-surfaces with reduced canonical curve of type B,*

$$\mathfrak{D}_B = \overline{\left\{ [X] \in \overline{\mathfrak{M}}_{1,3} \,\middle|\, \begin{array}{l} X \text{ 2-Gorenstein with reduced} \\ \text{canonical curve of type } B \end{array} \right\}} \subset \overline{\mathfrak{M}}_{1,3}.$$

1. *The set $\mathfrak{D}_B$ is an irreducible divisor in the closure of the Gieseker moduli space.*
2. *The intersection of divisors $\mathfrak{D}_A \cap \mathfrak{D}_B$ contains the irreducible subset of codimension* 2

$$\mathfrak{R}_{AB} = \{[X] \in \mathfrak{D}_B \mid \tilde{g}_8 \text{ in (5.2) does not contain } y^4\}.$$

Proof. Let us consider again the equations (5.2) defining X. For $\tilde{g}_8$ we have 13 free parameters while for $\tilde{k}_{10}$ we have 21 free parameters. So $\mathfrak{D}_B$ is dominated by a linear space of dimension 34 and it is thus irreducible.

From the parameter count in Proposition 5.4 we see that $\dim \mathfrak{D}_B = 27$.

To prove that $\mathfrak{D}_B$ is in the closure of the Gieseker component, it is enough to prove that the general surface X in $\mathfrak{D}_B$ is $\mathbb{Q}$-Gorenstein smoothable.

We are going to show that the following family $\mathcal{X} \to \mathbb{A}^1_\lambda$ in $\mathbb{P}(1,1,2,3,4,5,6) \times \mathbb{A}^1_\lambda$ gives a smoothing of X:

$$\bigwedge^2 \begin{pmatrix} \lambda & y & w & z \\ x & w & v & u \end{pmatrix} = 0, \quad \begin{cases} z^2 = y\tilde{g}_8(x_0, y, w, v) + \lambda^2 \tilde{k}_{10}(x_0, x, v) \\ zu = w\tilde{g}_8(x_0, y, w, v) + \lambda x \tilde{k}_{10}(x_0, x, v) \\ u^2 = v\tilde{g}_8(x_0, y, w, v) + x^2 \tilde{k}_{10}(x_0, x, v) \end{cases}$$

Comparing the equations of $\mathcal{X}_0$ with (5.2) it is clear that the central fibre describes a surface in $\mathfrak{D}_B$. Now assume that λ is invertible. Then the equations allow us to eliminate w, v, u:

$$w = \frac{xy}{\lambda}, \; v = \frac{xw}{\lambda} = \frac{x^2 y}{\lambda^2}, \; u = \frac{xz}{\lambda}. \tag{5.5}$$

Hence the minors of the 2×4 matrix vanish identically. If we define

$$f = z^2 - y\tilde{g}_8 - \lambda^2 \tilde{k}_{10},$$

then after substituting using (5.5), the remaining three equations become

$$f, \; \frac{x}{\lambda} f, \; \frac{x^2}{\lambda^2} f.$$

In other words, for $\lambda \neq 0$ the fibre

$$\mathcal{X}_\lambda \cong \mathsf{Proj}(\mathbb{C}[x_0, x, y, z]/f)$$

is a smooth I-surface if λ is sufficiently general, because $\tilde{g}_8$ and $\tilde{k}_{10}$ are general by hypothesis. The family is equidimensional over a curve and hence flat. The relative canonical bundle $\omega_{\mathcal{X}/\mathbb{A}^1} = \mathscr{O}_{\mathcal{X}/\mathbb{A}^1}(1)$ is $\mathbb{Q}$-Cartier, so this is a $\mathbb{Q}$-Gorenstein smoothing by [Kol23, Definition-Theorem 3.1]. This concludes the proof of 1.

To prove 2. we assume that $\tilde{g}$ does not contain the monomial y^4 with nonzero coefficient, which is a codimension one condition on $\mathfrak{D}_B$ by our parameter count. Since $\tilde{k}_{10}$ does not contain powers of y, every general fibre is isomorphic to a hypersurface of degree 10 in $\mathbb{P}(1,1,2,5)$ passing through the point $(0:0:1:0)$, which are the surfaces parametrised by $\mathfrak{D}_A$ by Theorem 4.1. So $\mathfrak{R}_{AB} \subset \mathfrak{D}_A \cap \mathfrak{D}_B$ as claimed. □

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Remark 5.9. The fibrewise projection of the family constructed in the proof of Proposition 5.8 gives a family $\mathcal{W}/\mathbb{A}^1_\lambda$ in $\mathbb{P}(1,1,2,3,4)\times\mathbb{A}^1_\lambda$ defined by

$$\bigwedge^2\begin{pmatrix}\lambda & y & w\\ x & w & v\end{pmatrix}=0$$

of which $\mathcal{X}$ is a double cover. Thus $\mathcal{W}$ is a degeneration of $\mathbb{P}(1,1,2)$ to the reducible rational surface $W=W_1\cup W_2$ from Corollary 5.7.

Remark 5.10. The total space of $\mathcal{X}/\mathbb{A}^1_\lambda$ constructed in the proof can be expressed as the 4×4 Pfaffians of the following extrasymmetric skew 6×6 matrix, where we omit the diagonal zero-entries and thus only show the elements above the diagonal:

$$\begin{pmatrix}0 & \lambda & y & w & z & \\ & x & w & v & u & \\ & & z & u & \tilde{g} & \\ & & & 0 & \lambda\tilde{k} & \\ & & & & x\tilde{k} & \end{pmatrix}.$$

On the first component of the central fibre $\mathcal{X}_0\cap(x=0)$, the Pfaffians are equal to the 2×2 minors of the upper-right 3×3 symmetric block. These already appeared in (5.3).

Using [Ilt12] one can check that every deformation of a general X fits this format, so the versal deformation is smooth of dimension 28, so also the moduli space should be smooth at the general point of $\mathfrak{D}_B$.

Maybe this format can also be used to study the interaction with other known strata in the moduli space.

Remark 5.11. In [CFPR22] we explained how to compute the Deligne mixed Hodge structure of a stable I-surface. If X has a nonreduced canonical curve of type B, then the components of the normalisation satisfy $p_g(X_1)+p_g(X_2)=1$ and it follows that the mixed Hodge structure on $H^2(X,\mathbb{Z})$ is not pure. In the notation of Robles [Rob17], the general such surface has Hodge type $\diamond_{0,1}$.

From her results one can also infer that no stable I-surface with Hodge type $\diamond_{0,0}$, that is, pure Hodge structure in the second cohomology, is in $\mathfrak{D}_B$. This applies in particular to the stratum $\mathfrak{R}_{DD}$ described in Theorem 7.1.

6. Components of stable I-surfaces with only nonreduced canonical curves

We now classify 2-Gorenstein stable I-surfaces with only nonreduced canonical curves. The method employed in Section 4 and 5, lifting the canonical ring from the canonical curve, did not lead us to success in this case because we were unable to classify the relevant curves and ggs structures. Therefore, we take a more geometric approach, classifying the component surfaces directly.

It will turn out that each component is a singular K3 surface of a particular type and we will remain with two cases, type DD and type DE. These will be treated in Section 7 and Section 8 respectively.

6.1. Numerical restrictions

Let X be a 2-Gorenstein stable I-surface and assume from now on that every canonical curve $C\in|K_X|$ is nonreduced. By Proposition 3.3, X is reducible with normalisation $(\bar{X},\bar{D})=(\bar{X}_1,\bar{D}_1)\sqcup(\bar{X}_2,\bar{D}_2)$, $\bar{D}_i=\bar{\Gamma}_i+\bar{Z}_i$, where $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ are identified in X and $K_{\bar{X}_i}+\bar{Z}_i=0$. Moreover $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ are irreducible curves since $1=2(K_{\bar{X}_i}+\bar{D}_i)(K_{\bar{X}_i}+\bar{D}_i)=2(K_{\bar{X}_i}+\bar{D}_i)\bar{\Gamma}_i$.

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We write $X = X_1 \cup_\Gamma X_2$. For both components consider the refinement of the standard pushout glueing diagram arising from Kollár's glueing theorem [Kol13, Thm. 5.13]:

$$\begin{array}{ccccc}
\bar{X}_i & \hookleftarrow & \bar{D}_i = \bar{\Gamma}_i + \bar{Z}_i & \hookleftarrow & \bar{D}_i^\nu \\
\downarrow{\scriptstyle \text{normalisation}} & & \downarrow & & \\
\tilde{X}_i & \hookleftarrow & \tilde{D}_i = \tilde{\Gamma}_i + \tilde{Z}_i & & \\
\downarrow{\scriptstyle S_2\text{-fication}} & & \downarrow & & \\
X_i & \hookleftarrow & D_i = \Gamma + Z_i & &
\end{array}.$$

The idea is to consider the properties of the components individually, which is problematic because they are in general not S_2, so not stable log surfaces. The purpose of the S_2-fication is to remedy this, so $(\tilde{X}_i, \tilde{\Gamma}_i)$ is a stable log surface with non-normal locus $\tilde{Z}_i$ (possibly empty). One could alternatively describe this by glueing only part of the boundary of $(\bar{X}_i, \bar{D}_i)$.

Lemma 6.1. *In the above situation we have $K_{\tilde{X}_i} = 0$, in particular $\tilde{X}_i$ is Gorenstein with slc singularities. The holomorphic Euler characteristics satisfy*

$$\chi(X) + \chi(\tilde{\Gamma}_1) + \chi(\tilde{\Gamma}_2) = \chi(\tilde{X}_1) + \chi(\tilde{X}_2) + \chi(\Gamma). \tag{6.1}$$

The curves $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are irreducible, and their normalisations are isomorphic.

Proof. Outside a subset of codimension two we have that the pullback of $K_{\tilde{X}_i}$ is $K_{\bar{X}_i} + \bar{Z}_i$, which is trivial by Proposition 3.3. Since $\tilde{X}_i$ is S_2, hence Cohen–Macaulay, also the canonical sheaf is S_2, thus trivial.

To prove the second part consider the map $\tilde{\pi}\colon (\tilde{X}, \tilde{\Gamma}) = (\tilde{X}_1, \tilde{\Gamma}_1) \sqcup (\tilde{X}_2, \tilde{\Gamma}_2) \to X$ inducing the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{I}_\Gamma & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_\Gamma & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{\pi}_*\mathcal{O}_{\tilde{X}}(-\tilde{\Gamma}) & \longrightarrow & \tilde{\pi}_*\mathcal{O}_{\tilde{X}} & \longrightarrow & \tilde{\pi}_*\mathcal{O}_{\tilde{\Gamma}} & \longrightarrow & 0.
\end{array}$$

The component Γ of the non-normal locus of X which connects the two components is a reduced subscheme of pure codimension one, so $\mathcal{O}_\Gamma$ is S_1. By the depth Lemma $\mathcal{I}_\Gamma$ is S_2 and coincides in codimension one with $\tilde{\pi}_*\mathcal{O}_{\tilde{X}}(-\tilde{\Gamma})$ (as in the usual case of the normalisation), so they are isomorphic. Hence from the additivity of the Euler characteristic we get

$$\chi(\mathcal{O}_X) - \chi(\mathcal{O}_\Gamma) = \chi(\mathcal{I}_\Gamma) = \chi(\tilde{\pi}_*\mathcal{O}_{\tilde{X}}(-\tilde{\Gamma})) = \chi(\tilde{\pi}_*\mathcal{O}_{\tilde{X}}(-\tilde{\Gamma})) - \chi(\tilde{\pi}_*\mathcal{O}_{\tilde{\Gamma}}),$$

which is equivalent to Formula (6.1).

The curve $\tilde{\Gamma}_i$ is irreducible because $\bar{\Gamma}_i$ is. Since $\tilde{\Gamma}_1$ is glued to $\tilde{\Gamma}_2$, their normalisations are exchanged by the glueing involution and hence are isomorphic. □

We need the following general lemma.

Lemma 6.2. *Let S be an irreducible projective slc surface such that $K_S = 0$. If $\chi(S) > 0$, then $\chi(S) = 2$.*

Proof. Since $p_g(S) = 1$ because $K_S = 0$, we have $\chi(S) \leq 2$ and it is enough to show that $\chi(S) = 1$ cannot occur. So assume by contradiction that $\chi(S) = 1$, i.e., $q(S) = 1$. Pick a very ample line bundle H on S and let $C \in |H|$ be a general curve. By the generalised Kodaira vanishing theorem ([LR14, Prop. 3.1]), $h^1(\mathcal{O}_S(-C)) = 0$ and therefore the restriction sequence for C induces an injection $H^1(\mathcal{O}_S) \to H^1(\mathcal{O}_C)$. It follows that the restriction map $\mathsf{Pic}^0(S) \to \mathsf{Pic}^0(C)$ has finite kernel. The curve C, being general, has nodal singularities, hence $\mathsf{Pic}^0(C)$ is a quasi-abelian variety, namely there is an exact sequence $0 \to (\mathbb{C}^*)^m \to \mathsf{Pic}^0(C) \to A \to 0$, where A is an abelian variety. So the image of

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$\mathrm{Pic}^0(S)$ in $\mathrm{Pic}^0(C)$ is isomorphic either to $\mathbb{C}^*$ or to an elliptic curve, and therefore $\mathrm{Pic}^0(S)$ contains a torsion element ζ of order $m \geq 2$. Let $Y \to S$ be the connected étale cover given by ζ. Then K_Y is trivial and $p_g(Y) = \chi(Y) + q(Y) - 1 \geq m + 1 - 1 \geq 2$, a contradiction. □

Lemma 6.3. *In the set-up of Lemma 6.1 there are the following possibilities for the invariants, up to renumbering the components:*

type	$\chi(\tilde{\Gamma}_1)$	$\chi(\tilde{\Gamma}_2)$	$\chi(\tilde{X}_1)$	$\chi(\tilde{X}_2)$	$\chi(\Gamma)$
DD	1	1	2	2	1
DE	1	0	2	2	0

Proof. Recall that Γ and the $\tilde{\Gamma}_i$ are irreducible with $\chi = 0$ or 1. Since the maps $\tilde{\Gamma}_i \to \Gamma$ are birational, we also have $\chi(\Gamma) \leq \chi(\tilde{\Gamma}_i)$, $i = 1, 2$. So if $\chi(\Gamma) = 1$ we get $\chi(X_1) + \chi(X_2) = 4$ by (6.1) and therefore $\chi(\tilde{X}_1) = \chi(\tilde{X}_2) = 2$ by Lemma 6.2. If $\chi(\Gamma) = 0$ then (6.1) gives $\chi(\tilde{X}_1) + \chi(\tilde{X}_2) \geq 3$ and so again $\chi(\tilde{X}_1) = \chi(\tilde{X}_2) = 2$ by Lemma 6.2. □

6.2. Identifying components

Next we classify the possible components. The first one is a double plane branched over a sextic.

Proposition 6.4 (type D). *Let (S, Δ) be an irreducible stable log surface pair with the following properties:*

$$K_S = 0,\ \Delta^2 = \frac{1}{2},\ 2\Delta \text{ Cartier},\ p_a(\Delta) = 0, \chi(S) = 2.$$

Then $|2\Delta|$ induces a double cover $f\colon S \to \mathbb{P}^2$ branched over a sextic $L + B_5$, where L is a line such that $f^*L = 2\Delta$ and $(\mathbb{P}^2, \frac{1}{2}(L + B_5))$ is lc.

For a general choice of B_5, the surface S is a singular K3 surface with five A_1 singularities.

Moreover, we have

$$R(S, \Delta) \cong \mathbb{C}[x, y_1, y_2, z]/\left(z^2 - f_{10}(x, y_1, y_2)\right)$$

with $\deg(x, y_1, y_2, z) = (1, 2, 2, 5)$.

Proof. Riemann–Roch and generalised Kodaira vanishing give $h^0(2\Delta) = \chi(2\Delta) = 3$. The restriction sequence together with Kodaira vanishing gives an exact sequence

$$0 \to H^0(S, \Delta) \to H^0(S, 2\Delta) \to H^0(\Delta, \mathscr{O}_\Delta(2\Delta)) \to 0.$$

Since $\Delta \cong \mathbb{P}^1$ we see that $|2\Delta|$ has no base points, because it has no base points on Δ.

Since Δ is ample and $(2\Delta)^2 = 2$, the system $|2\Delta|$ induces a finite map of degree two $S \to \mathbb{P}^2$ that maps Δ isomorphically to a line L, which is necessarily a component of the branch locus. The total degree of the branch locus and the fact that $(\mathbb{P}^2, \frac{1}{2}(L + B_5))$ is lc follow from the Hurwitz formula (cf. [AP12, Prop. 2.5]).

Concerning the ring $R(S, \Delta)$, we can write $H^0(S, \Delta) = \langle x \rangle$ and $H^0(S, 2\Delta) = \langle x^2, y_1, y_2 \rangle$. We see that $h^0(S, m\Delta) = 2 + \frac{1}{4}m^2$ if m is even and restricting to Δ (which has genus 0), we get $h^0(S, m\Delta) = 1 + \frac{1}{4}(m^2 - 1)$ if m is odd.

So we need a further generator z in degree 5, and we see that the first relations are in degree 10 and we can assume that it is of the form $z^2 = f_{10}(x, y_1, y_2)$, then considering the map $\mathbb{C}[x, y_1, y_2, z]/\left(z^2 - f_{10}(x, y_1, y_2)\right) \to R(S, \Delta)$ and looking at the Hilbert series we can conclude that they are isomorphic.

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The double cover $S \to \mathbb{P}^2 \cong \mathbb{P}(1,2,2)$ corresponds to the truncation in degree 2 and its branch locus consists of the curve B_5 defined by $\{f_{10}(x, y_1, y_2) = 0\}$ together with the distinguished line L defined by $\{x = 0\}$. □

Proposition 6.5 (type E). *Let (S, Δ) be an irreducible stable log surface with Δ irreducible and the following properties:*

$$K_S = 0,\ \Delta^2 = \frac{1}{2},\ 2\Delta \text{ Cartier},\ p_a(\Delta) = 1,\ \chi(S) = 2.$$

Then $|4\Delta|$ induces a double cover $f : S \to \mathcal{C}_4 \subset \mathbb{P}^5$, where $\mathcal{C}_4$ is the cone over the rational normal curve of degree four. The morphism f is branched over the vertex and a cubic section B such that $(\mathcal{C}_4, \frac{1}{2}B)$ is lc.

For general choice of B, we get a singular elliptic K3 surface with one A_1 singularity over the vertex of the cone, which is also a base point for the elliptic pencil $|\Delta|$ (compare Construction 5.3).

Moreover, we have

$$R(S, \Delta) \cong \mathbb{C}[u_0, u_1, v, w] / \left(w^2 - g_{12}(u_0, u_1, v)\right)$$

with $\deg(u_0, u_1, v, w) = (1, 1, 4, 6)$.

Proof. Riemann–Roch and generalised Kodaira vanishing give $h^0(2\Delta) = \chi(2\Delta) = 3$. The restriction sequence together with Kodaira vanishing gives an exact sequence

$$0 \to H^0(S, \Delta) \to H^0(S, 2\Delta) \to H^0(\Delta, \mathcal{O}_\Delta(2\Delta)) \to 0.$$

Since $h^0(\Delta, \mathcal{O}_\Delta(2\Delta)) = 1$, we get $h^0(\Delta) = 2$. All the curves of $|\Delta|$ are irreducible and contain the point p defined by $\mathcal{O}_\Delta(2\Delta) = \mathcal{O}_\Delta(p)$.

Similarly, one has $h^0(4\Delta) = 6$ and an exact sequence

$$0 \to H^0(S, 3\Delta) \to H^0(S, 4\Delta) \to H^0(\Delta, \mathcal{O}_\Delta(4\Delta)) \to 0.$$

Since $p_a(\Delta) = 1$ and $4\Delta^2 = 2$, we see that $|4\Delta|$ has no base points, because it has no base points on Δ. Its restriction to the curves of $|\Delta|$ defines a double cover $\Delta \to \mathbb{P}^1$, so the image Σ of the morphism φ given by $|4\Delta|$ is ruled by lines and $\deg \varphi \geq 2$. More precisely, Σ is a cone, since the images of the curves of Δ all go through the point $\varphi(p)$. Since $8 = (4\Delta)^2 = \deg \varphi \deg \Sigma \geq 2 \cdot 4$, we have $\deg \varphi = 2$ and Σ is the cone $\mathcal{C}_4$ over the rational normal curve of degree 4. For every $\Delta' \in |\Delta|$ we see that $|4\Delta|_{|\Delta'}$ contains $2p$, so φ is branched over p. Finally the Hurwitz formula shows that φ is branched on a divisor of $|-2K_{\mathcal{C}_4}| = |\mathcal{O}_{\mathcal{C}_4}(3)|$.

Concerning the ring $R(S, \Delta)$, we can write $H^0(S, \Delta) = \langle u_0, u_1 \rangle$ and restricting to Δ we see that we need further generators v, w in degree 4, resp. 6. The double cover $S \to \mathcal{C}_4 \cong \mathbb{P}(1,1,4)$ corresponds to the truncation in degree 4 and its branch locus consists of the vertex of the cone and the curve $B = \{g_{12}(u_0, u_1, v) = 0\}$. This corresponds to the only relation we need. □

7. I-surfaces of type DD

The surfaces of type DD were already described in [FPR17, Example 4.7], so we can be brief here. Let $\mathfrak{R}^{DD}$ be the closure of the locus of such surfaces in the moduli space.

Theorem 7.1. *Let X be a 2-Gorenstein stable I-surface such that every canonical curve is nonreduced and the S_2-fication of both components is of type D (compare Lemma 6.3).*

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Then X is a complete intersection of degree $(2, 10)$ *in* $\mathbb{P}(1, 1, 2, 2, 5)$ *and is smoothable. More precisely*

$$R(X, K_X) \cong \mathbb{C}[x_1, x_2, y_1, y_2, z]/(x_1x_2, z^2 - f_{10}(x_1, x_2, y_1, y_2)).$$

The set $\mathfrak{R}_{DD}$ *is irreducible of dimension 26 and satisfies*

$$\mathfrak{R}_{DD} \subset \mathfrak{D}_A \subset \overline{\mathfrak{M}}_{1,3}^{class}.$$

Proof. It is easy to check that the linear system $|2K_X|$ defines a double cover of the union of two planes in $\mathbb{P}^3$ branched, as in the smooth case, over a quintic section.

These surfaces give a 26-dimensional locus inside the moduli space: the linear system of quintics in the two planes that match on the intersection line is of dimension 35 and the automorphism group of the union of two planes in $\mathbb{P}^3$ has dimension 9. (cf. [FPR17, Example 4.7]).

To show that $\mathfrak{R}_{DD} \subset \mathfrak{D}_A$, we write in the equation

$$f_{10} \equiv g(y_1, y_2) \mod (x_1, x_2).$$

We may assume after a coordinate change that y_1 is a factor of g. Considering the family $\mathcal{X}/\mathbb{A}^1_\lambda$ given by equations

$$x_1x_2 - \lambda y_1 = z^2 - f_{10} = 0$$

we see that for $\lambda \neq 0$ we can eliminate y_1. The resulting equation $z^2 - f_{10}(x_1, x_2, x_1x_2/\lambda, y_2) = 0$ in $\mathbb{P}(1, 1, 2, 5)$ does not contain the monomial y_2^5 and thus the general fibre contains the point $(0 : 0 : 1 : 0)$. Hence the general fibre is in $\mathfrak{D}_A$ by Theorem 4.1. □

8. I-surfaces of type DE

We now treat the case of 2-Gorenstein stable I-surfaces with only nonreduced canonical curves of type DE defined in Lemma 6.3. Let us consider the set of these surfaces

$$\mathfrak{U}^{DE} = \{[X] \in \overline{\mathfrak{M}}_{1,3} \mid X \text{ of type DE}\}.$$

in the moduli space of stable surfaces.

The results of this section are summed up in the following result, where we relegate the precise geometric and algebraic descriptions to the subsequent subsections.

Theorem 8.1. *Every surface $X = X_1 \cup X_2$ of type DE is glued from two particular singular K3 surfaces along a nodal rational curve. It is canonically embedded in* $\mathbb{P}(1, 1, 2, 2, 3, 4, 5, 6, 7)$ *and its ideal is as in (8.5).*

The subset $\mathfrak{U}^{DE}$ *is irreducible of dimension* 30 *and does not intersect the closure of the Gieseker component. Its closure*

$$\overline{\mathfrak{M}}^{DE} = \overline{\mathfrak{U}^{DE}} \subset \overline{\mathfrak{M}}_{1,3}$$

is an irreducible component of the moduli space.

The proof of Theorem 8.1 will occupy the rest of the section and follow from Propositions 8.2, 8.5 and 8.11.

8.1. Geometric description and moduli

Let X be a 2-Gorenstein stable I-surface with only nonreduced canonical curves of type DE. Then $X = X_1 \cup_\Gamma X_2$ and Propositions 6.4 and 6.5 describe the S_2-fication of the components as stable pairs

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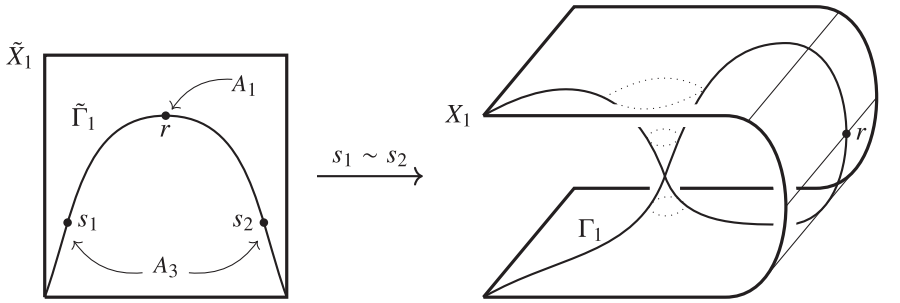


Figure 2. *S_2-fication of the (generic) component X_1.*

$(\tilde{X}_i, \tilde{\Gamma}_i)$. Incorporating the information $\chi(\tilde{\Gamma}_1) = 1 > \chi(\Gamma) = 0 = \chi(\tilde{\Gamma}_2)$ we can sum up the information in the following diagram:

$$\begin{array}{ccccccc}
\mathbb{P}^2 & \xleftarrow{\text{2:1, br. over } L+B_5} & \tilde{X}_1 & & \tilde{X}_2 & \xrightarrow{\text{2:1, br. over } B_3+\text{vertex}} & \mathbb{P}(1,1,4) \\
\uparrow & & \uparrow & \searrow \quad \swarrow & \uparrow & & \uparrow \\
L & \xleftarrow{\cong} & \tilde{\Gamma}_1 & & \tilde{\Gamma}_2 & \xrightarrow{\text{2:1, br. over 4 points}} & \mathbb{P}(1,4) \\
 & & & \searrow^{\text{bir.}} \quad \swarrow_{\cong} & & & \\
 & & & \Gamma & & &
\end{array} \qquad (8.1)$$

Proposition 8.2. *Let $(\tilde{X}_1, \tilde{\Gamma}_1)$ be of type D (Proposition 6.4) and $(\tilde{X}_2, \tilde{\Gamma}_2)$ of type E (Proposition 6.5). In particular, there is a double cover $f\colon \tilde{X}_1 \to \mathbb{P}^2$ ramified over $B_5 + L$ and $2\tilde{\Gamma}_1 = f^*L$, for a line L.*

1. *The pairs $(\tilde{X}_1, \tilde{\Gamma}_1)$ and $(\tilde{X}_2, \tilde{\Gamma}_2)$ occur as the S_2-fications of the components of a 2-Gorenstein stable I-surface X if and only if*
   (a) *$\tilde{\Gamma}_2$ is a nodal rational curve (of arithmetic genus 1),*
   (b) *the line L is bitangent to B_5, that is, $B_5|_L = r + 2(s_1 + s_2)$, where r, s_1 and s_2 are distinct points.*
2. *The general $\tilde{X}_1$ as in 1. is a singular K3 surface with exactly one A_1 singularity at r and A_3 singularities at s_1 and s_2.*
3. *The general $\tilde{X}_2$ as in 1. is a singular K3 surface with a unique A_1 singularity at a point p mapping to the vertex in $\mathcal{C}_4 = \mathbb{P}(1,1,4)$.*
4. *The map $\tilde{\Gamma}_1 \to \Gamma \cong \tilde{\Gamma}_2$ and therefore also the S_2-fication map $\tilde{X}_1 \to X_1$, identifies the two points s_1 and s_2 as shown in Figure 2.*

Proof. We use Kollár's glueing principle [Kol13, Thm. 5.13]. In order to be able to glue the two surfaces, the normalisations of the boundaries have to be isomorphic, so $\tilde{\Gamma}_2^\nu \cong \tilde{\Gamma}_1^\nu = \tilde{\Gamma}_1 \cong \mathbb{P}^1$. Since the pair is slc we have that $\tilde{\Gamma}_2$ is a nodal curve of arithmetic genus 1, proving *(a)*.

The glueing involution in Kollár's glueing principle also has to preserve the different. On $\tilde{\Gamma}_2^\nu$, we have $\mathrm{Diff}_{\tilde{\Gamma}_2^\nu}(0) = \frac{1}{2}p + q_1 + q_2$, where p is the basepoint of $|\tilde{\Gamma}_2|$, an A_1 singularity in $\tilde{X}_2$, and the q_i are the preimages of the node of $\tilde{\Gamma}_2$.

Let us compute the different on $\tilde{\Gamma}_1$. Let $\iota\colon \tilde{\Gamma}_1 \to \tilde{X}_1$ be the inclusion and recall that $\tilde{\Gamma}_1$ is a smooth rational curve and that

$$\omega_{\tilde{X}_1}(\tilde{\Gamma}_1)^{[2]} = \omega_{\tilde{X}_1}^{[2]}(2\tilde{\Gamma}_1) = \omega_{\tilde{X}_1}^{[2]}(f^*L) = f^*\mathscr{O}_{\mathbb{P}^2}(2K_{\mathbb{P}^2} + B_5 + L)(f^*L) = f^*\mathscr{O}_{\mathbb{P}^2}(2K_{\mathbb{P}^2} + B_5 + 2L)$$

is locally free. Let $U \subset \tilde{X}_1$ be the smooth locus and note that the generic point of $\tilde{\Gamma}_1$ lies in U.

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Then by definition [Kol13, (4.3.7), p.154] the divisor $2\mathsf{Diff}_{\tilde{\Gamma}_1}(0)$ is uniquely defined by the property that the square of the Poincaré residue morphism $\omega_U(\tilde{\Gamma}_1 \cap U)|_{\tilde{\Gamma}_1 \cap U} \cong \omega_{\tilde{\Gamma}_1 \cap U}$ extends to an isomorphism

$$\omega_{\tilde{\Gamma}_1}^{\otimes 2}\Big(2\mathsf{Diff}_{\tilde{\Gamma}_1}(0)\Big) \cong \iota^*\omega_{\tilde{X}_1}(\tilde{\Gamma}_1)^{[2]} \cong \iota^* f^* \mathscr{O}_{\mathbb{P}^2}(2K_{\mathbb{P}^2} + B_5 + 2L).$$

In total, we get $\mathsf{Diff}_{\tilde{\Gamma}_1}(0) = \frac{1}{2}B_5|_L$ under the isomorphism $\tilde{\Gamma}_1 \cong L$, so compatibility for glueing forces *(b)*.

Conversely, $\mathsf{Aut}(\mathbb{P}^1)$ acts 3-transitively, so any pair with the above properties can be glued, thus we have proved the first item.

With this extra information about the branch locus, the second and the third item follow from Proposition 6.4 and Proposition 6.5. The last point follows from the definition of X_1 as a component of X and the description of the glueing. □

Remark 8.3. In order to glue surfaces of type D and E as in Proposition 8.2, we need to pick an isomorphism $\tilde{\Gamma}_1 \to \tilde{\Gamma}_2^{\nu}$ that preserves the different. This boils down to mapping s to p and the points s_1, s_2 to q_1, q_2, so we have two choices. However, acting on $\tilde{X}_1$ with the covering involution and on $\tilde{X}_2$ with the identity gives an isomorphism between the surfaces corresponding to the two choices.

Lemma 8.4. *Let $X = X_1 \cup X_2$ be a 2-Gorenstein stable I-surface of type DE. If σ is an involution of X that acts as the identity on $|mK_X|$ for $m \leq 4$, then $\chi(X/\sigma) \geq 2$.*

Proof. Let σ be an involution as in the statement and consider the induced involution of $\tilde{X} = \tilde{X}_1 \sqcup \tilde{X}_2$: since $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are not isomorphic by Proposition 8.2, they cannot be exchanged by the involution and so σ induces involutions σ_i of $\tilde{X}_i$ that preserve $\tilde{\Gamma}_i$, $i = 1, 2$.

Let $\alpha \in H^0(\tilde{X}_1, \tilde{\Gamma}_1)$ be a section vanishing on $\tilde{\Gamma}_1$. There is an injective pull back map $r = (r_1, r_2)\colon H^0(4K_X) \to H^0(\tilde{X}_1, 4\tilde{\Gamma}_1) \oplus H^0(\tilde{X}_2, 4\tilde{\Gamma}_2)$, so we may identify $H^0(4K_X)$ with the image of r. So $H^0(4K_X)$ contains all the sections of the form $(\alpha^2\gamma, 0)$, where γ varies in $H^0(2\tilde{\Gamma}_1) = \pi_1^* H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1))$, where π_1 is the double cover given by $|2\tilde{\Gamma}_1|$ (cf. Proposition 6.4). So σ_1 is either the identity or the involution associated with π_1 and, in either case it restricts to the identity on $\tilde{\Gamma}_1$. As a consequence σ restricts to the identity on Γ, and therefore also σ_2 restricts to the identity on $\tilde{\Gamma}_2$. An argument similar to the previous one shows that σ_2 preserves the curves of $|\tilde{\Gamma}_2|$ and acts on $\mathcal{C}_4 \subset \mathbb{P}^5$, the image of the 2-to-1 map π_2 defined by $|4\tilde{\Gamma}_2|$ (cf. 6.5), mapping each ruling to itself. Let τ be the automorphism on $\mathcal{C}_4 \cong \mathbb{P}(1,1,4)$ induced by σ. If τ is the identity then σ_2 is either the identity or the covering involution. Since $\tilde{\Gamma}_2$ is not in the branch locus of $\tilde{X}_2 \to \mathcal{C}_4$, the covering involution is not the identity on $\tilde{\Gamma}_2$, so in this case σ_2 must be the identity. Now assume that τ is a nontrivial involution of $\mathcal{C}_4$: since τ maps each ruling to itself, there are weighted homogeneous coordinates x_0, x_1, y on $\mathbb{P}(1,1,4)$ such that τ is given by $(x_0, x_1, y) \mapsto (x_0, x_1, -y)$. So τ is nontrivial on every ruling of $\mathcal{C}_4$ and, a fortiori, σ_2 is not trivial on $\tilde{\Gamma}_2$. So the only possibility is that σ_2 is the identity, and therefore if σ is nontrivial then σ_1 is the covering involution of $\tilde{X}_1 \to \mathbb{P}^2$.

If σ is the identity, then $\chi(X/\sigma) = \chi(X) = 3$. If σ is not the identity, then it is easy to see that $Y := X/\sigma$ is defined by the push-out diagram:

$$\begin{array}{ccccc}
\tilde{X}/\sigma & \hookleftarrow & \tilde{\Gamma}/\sigma & \xleftarrow{\bar{\nu}} & \tilde{\Gamma}^{\nu}/\sigma \\
\downarrow & & \downarrow & & \downarrow \\
Y & \hookleftarrow & \Gamma & \xleftarrow{\nu} & \Gamma^{\nu}.
\end{array}$$

Now Lemma 6.1 gives $1 = \chi(Y) = \chi(\Gamma) + \chi(\tilde{X}/\sigma) - \chi(\tilde{\Gamma}) = 0 + (1+2) - (1+0) = 2$. □

Proposition 8.5. *The subset $\mathfrak{U}^{DE}$ is irreducible of dimension 30 and does not intersect the closure of the Gieseker component. Its closure*

$$\overline{\mathfrak{M}}^{DE} = \overline{\mathfrak{U}^{DE}} \subset \overline{\mathfrak{M}}_{1,3}$$

is an irreducible component of the moduli space.

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Proof. First we count the moduli of the construction. Since the elliptic fibration of a surface of type E has finitely many singular fibres, which are all nodal if the surface is general, by Proposition 8.2 it is enough to count parameters for surfaces of type E and for surfaces of type D such that the double cover $\tilde{X}_1 \to \mathbb{P}^2$ is branched over a sextic $L + B_5$ with the line L bitangent to the quintic B_5. Fixing the line $L = \{x_0 = 0\}$ and points $p, q_1, q_2 \in L$, we see that system of plane quintics tangent to L at q_1 and q_2 and passing through p has dimension 15. Subtracting the dimension of the subgroup of $\mathsf{Aut}(\mathbb{P}^2)$ that fixes L pointwise we obtain $15 - 3 = 12$ moduli.

Surfaces of type E are double covers of $\mathcal{C}_4 \cong \mathbb{P}(1,1,4)$, branched over a cubic section B and the vertex. The curve B moves in a system of dimension 27, so subtracting the dimension of $\mathsf{Aut}(\mathcal{C}_4)$ we get $27 - 9 = 18$ moduli. Summing up, surfaces of type DE depend on $12 + 18 = 30$ moduli. We can consider the surfaces of type E such that the branch locus of the map to $\mathcal{C}_4$ is simply tangent to a fixed ruling R_0 and define $\tilde{\Gamma}_2$ to be the preimage of R_0; since $\mathsf{Aut}(\mathcal{C}_4)$ acts transitively on the set of rulings, in this way we obtain an irreducible family of pairs $(\tilde{X}_2, \tilde{\Gamma}_2)$ that contains every isomorphism class of such pairs. We can take the double cover of this family obtained by labelling the preimages of the node of $\tilde{\Gamma}_2$ in the normalisation map $\tilde{\Gamma}_2^\nu \to \tilde{\Gamma}_2$. By Remark 8.3, an irreducible component of this double cover contains all the isomorphism classes of pairs of type E + labelling of the preimages of the node, and therefore surfaces of type DE give an irreducible locus $\mathfrak{U}^{DE} \subset \overline{\mathfrak{M}}_{1,3}$ of dimension 30.

Now let X be a surface of type DE and assume that there is a one parameter deformation $\mathcal{X} \to C$ over a smooth curve such that the general fiber is not of type DE. Since the Cartier index is lower semicontinuous, because $\omega^{[m]}_{\mathcal{X}/C}$ being locally free is an open condition, up to shrinking C we may assume that all fibres of $\mathcal{X}$ are 2-Gorenstein and not of type DE except the central fibres. Then by our classification results (Theorem 1.1) the fibres of X are smoothable, so by picking another family we may assume that every fibre except X is smooth.

The general fibre X_t thus carries an involution σ_t, induced by the bi-canonical map, which acts trivially on m-canonical system for $m \leq 4$ and such that $\chi(X_t/\sigma_t) = 1$. By [FPRR22, Prop. 2.6], the involutions on the general fibres extend to give a global fibrewise involution on the family $\mathcal{X}/C$, so there is an involution σ on X contradicting Lemma 8.4.

Therefore any small deformation of a surface of type DE is again a surface of type DE, the locus $\mathfrak{U}^{DE}$ does not intersect the closure of the Gieseker component and its closure $\overline{\mathfrak{M}}^{DE}$ is an irreducible component of the moduli space. □

8.2. *Algebraic description*

Here we reverse the approach we used in the nonreduced case, and deduce the equations of the surfaces of type DE from their geometric description via Kollár's glueing. More precisely, by the glueing principle for pluricanonical sections [Kol13, Prop. 5.8] the glueing diagram (8.1) lets us compute the canonical ring as a pullback ring in the diagram

$$\begin{array}{ccc} R(X, K_X) & \longrightarrow & R(\Gamma^\nu, K_X|_{\Gamma^\nu}) \\ \downarrow{\scriptstyle \tilde{\pi}^*} & & \downarrow \\ R(\tilde{X}_1, \tilde{\Gamma}_1) \times R(\tilde{X}_2, \tilde{\Gamma}_2) & \longrightarrow & R(\tilde{\Gamma}_1^\nu, \tilde{\Gamma}_1|_{\tilde{\Gamma}_1^\nu}) \times R(\tilde{\Gamma}_2^\nu, \tilde{\Gamma}_2|_{\tilde{\Gamma}_2^\nu}), \end{array}$$

where we use the superscript $(-)^\nu$ to denote normalisation and have incorporated the fact that $K_{\tilde{X}_i} = 0$.

Since the glueing involution τ identifies $\tau(\tilde{\Gamma}_1) \cong \tilde{\Gamma}_2^\nu \cong \Gamma^\nu$ we can rewrite the pullback diagram as

$$\begin{array}{ccc} R(X, K_X) & \longrightarrow & R(\tilde{X}_1, \tilde{\Gamma}_1) \\ \downarrow & & \downarrow{\scriptstyle \alpha^*} \\ R(\tilde{X}_2, \tilde{\Gamma}_2) & \xrightarrow{\beta^*} & R(\Gamma^\nu, K_X|_{\Gamma^\nu}). \end{array} \tag{8.2}$$

To make the above explicit, we fix algebraic descriptions of the S_2-fication of the two components.

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Lemma 8.6. *Let $(\tilde{X}_1, \tilde{\Gamma}_1)$ be a surface of type D with B_5 bitangent to L as in Proposition 8.2. Then we can choose generators for the section ring $R(\tilde{X}_1, \tilde{\Gamma}_1)$ such that*

$$\begin{aligned} \tilde{X}_1 \colon &\quad \{z^2 = y_0(y_1+y_0)^2(y_1-y_0)^2 + x^2 f_8(x, y_0, y_1)\} \subset \mathbb{P}(1_x, 2_{y_0}, 2_{y_1}, 5_z) \\ \tilde{\Gamma}_1 \colon &\quad \tilde{X}_1 \cap \{x = 0\} \\ &\quad r = (0:0:1:0),\ s_1 = (0:1:1:0),\ s_2 = (0:1:-1:0) \end{aligned} \tag{8.3}$$

where $f_8(x, y_0, y_1)$ is homogeneous of weighted degree 8.

Conversely, every sufficiently general f_8 will define a pair $(\tilde{X}_1, \tilde{\Gamma}_1)$ as in Proposition 8.2.

Proof. As shown in Proposition 6.4, $\tilde{X}_1$ is a hypersurface $z^2 = f_{10}(x, y_0, y_1)$ in $\mathbb{P}(1, 2, 2, 5)$. The branch curve of the double cover $\tilde{X}_1 \to \mathbb{P}^2 \cong \mathbb{P}(1, 2, 2)$ consists of the curve B_5 defined by $\{f_{10} = 0\}$ together with the distinguished line L defined by $\{x = 0\}$. The bitangency condition means that the quintic $f_{10}|_{x=0}$ has two double roots, and after a coordinates change we may write $f_{10}|_{x=0} = y_0(y_1+y_0)^2(y_1-y_0)^2$. □

Lemma 8.7. *Let $(\tilde{X}_2, \tilde{\Gamma}_2)$ be a surface of elliptic type E with $\tilde{\Gamma}_2$ a nodal rational curve as in Proposition 8.2. Then we can choose generators for the section ring $R(\tilde{X}_2, \tilde{\Gamma}_2)$ such that*

$$\begin{aligned} \tilde{X}_2 \colon &\quad \{w^2 = v(v - u_1^4)^2 + u_0 g_{11}(u_0, u_1, v)\} \subset \mathbb{P}(1_{u_0}, 1_{y_1}, 4_v, 6_w) \\ \tilde{\Gamma}_2 \colon &\quad \tilde{X}_1 \cap \{u_0 = 0\} = \{w^2 = v(v - u_1^4)^2\} \subset \mathbb{P}(1_{y_1}, 4_v, 6_w) \\ &\quad q = (0:1:1:0),\ p = (0:0:1:1), \end{aligned} \tag{8.4}$$

where $g_{11}(u_0, u_1, v)$ is homogeneous of weighted degree 11, p is the base point of the elliptic pencil $|\tilde{\Gamma}_2|$, and $\tilde{\Gamma}_2$ has a node at q.

Conversely, every sufficiently general g_{11} will define a pair $(\tilde{X}_2, \tilde{\Gamma}_2)$ as in Proposition 8.2.

Proof. As shown in Proposition 6.5, $\tilde{X}_2$ is a hypersurface $w^2 = g_{12}(u_0, u_1, v)$ in $\mathbb{P}(1, 1, 4, 6)$. The elliptic pencil is spanned by $\langle u_0, u_1 \rangle$ and we suppose that the nodal rational curve $\tilde{\Gamma}_2$ is defined by $\{u_0 = 0\}$. Since $\tilde{\Gamma}_2$ is nodal, it follows that $g_{12}|_{u_0=0}$ has a double root and we choose coordinates so that $g_{12}|_{u_0=0} = v(v - u_1^4)^2$. The node q of $\tilde{\Gamma}_2$ corresponds to this double root. □

Having identified the components, we now turn to the glueing.

Lemma 8.8. *Identify $\Gamma^\nu \cong \mathbb{P}^1 \cong \mathbb{P}(1_t, 2_s)$. Then we can choose the glueing involution τ such that the maps α^* and β^* in (8.2) are, with respect to the coordinates chosen above, induced by*

$$\begin{aligned} \alpha \colon \mathbb{P}(1,2) &\longrightarrow \mathbb{P}(1,2,2,5) & \beta \colon \mathbb{P}(1,2) &\longrightarrow \mathbb{P}(1,1,4,6) \\ (t,s) &\longmapsto (0, t^2, s, t(s^2 - t^4)) & (t,s) &\longmapsto (0, t, s^2, s(s^2 - t^4)) \end{aligned}.$$

Proof. By definition, the image of α is $\tilde{\Gamma}_1 \subset \tilde{X}_1$ and the image of β is $\tilde{\Gamma}_2 \subset \tilde{X}_2$.

By Remark 8.3 the choice of involution does not matter as long as it identifies the preimages of the node $q \in \tilde{\Gamma}_2$ with s_1 and s_2 and the point p with the point r. This is satisfied here, because

$$\begin{aligned} \alpha(1:\pm 1) &= (0:1:\pm 1:0) = s_i \quad \text{and} \quad & \beta(1, \pm 1) &= (0:1:1:0) = q, \\ \alpha(0:1) &= (0:0:1:0) = r \quad \text{and} \quad & \beta(0:1) &= (0:0:1:1) = p. \end{aligned}$$

Note that our choice of grading respects the gradings of the rings on the components, which concludes the proof. □

These descriptions and (8.2) now allow us to compute the canonical ring,

$$R(X, K_X) = \{(\xi, \eta) \in R(\tilde{X}_1, \tilde{\Gamma}_1) \times R(\tilde{X}_2, \tilde{\Gamma}_2) \mid \alpha^*(\xi) = \beta^*(\eta)\},$$

of X explicitly, but it turns out that it is convenient to treat the components $X_i \subset X$ first individually.

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Table 2. *Generators of the canonical ring of type DE and their restriction to the components.*

degree	name	$R(X, K_X)$	on $\tilde{X}_1$	on $\tilde{X}_2$
1	a_0	$(x, 0)$	x	
1	a_1	$(0, u_0)$		u_0
2	b_0	(y_0, u_1^2)	y_0	u_1^2
2	b_1	$(0, u_0u_1)$		u_0u_1
3	c	$(xy_1, 0)$	xy_1	
4	d	(y_1^2, v)	y_1^2	v
5	e	$(z, u_1(v - u_1^4))$	z	$u_1(v - u_1^4)$
6	f	$(y_1(y_1^2 - y_0^2), w)$	$y_1(y_1^2 - y_0^2)$	w
7	g	(y_1z, u_1w)	y_1z	u_1w

It is elementary but tedious to check that a minimal set of generators for $R(X, K_X)$ is listed in Table 2. For example, suppose that ξ is a generator of $R(\tilde{X}_1, \tilde{\Gamma}_1)$. If there is a complementary η in $R(\tilde{X}_2, \tilde{\Gamma}_2)$ whose preimage $\beta^*(\eta)$ is equal to $\alpha^*(\xi)$, then (ξ, η) is necessarily a generator of the canonical ring. This approach yields a_0, b_0, e in Table 2. Vice versa, starting from a generator η of $R(\tilde{X}_2, \tilde{\Gamma}_2)$, we get a_1, d, f. The remaining three generators were found by an ad hoc method. It follows from Proposition 8.11 that these are all the generators. Consequently, we have $X_1 = X \cap \{a_1 = b_1 = 0\}$ and $X_2 = X \cap \{a_0 = c = 0\}$.

Lemma 8.9. *The generators in Table 2 induce an immersion*

$$\tilde{X}_1 \to X_1 \subset \mathbb{P}(1_{a_0}, 2_{b_0}, 3_c, 4_d, 5_e, 6_f, 7_g)$$

which maps both points s_i to $(0 : 1 : 0 : 0 : 0 : 0 : 0)$ and is an embedding elsewhere (compare Figure 2). The ideal of X_1 is generated by

$$\operatorname{rk}\begin{pmatrix} a_0 & d - b_0^2 & e & c & f & g \\ c & f & g & a_0d & (d - b_0^2)d & ed \end{pmatrix} \le 1,$$

$$e^2 = b_0(d - b_0^2)^2 + a_0^2A_8 + a_0cB_6,$$
$$eg = b_0f(d - b_0^2) + a_0cA_8 + a_0^2dB_6,$$
$$g^2 = b_0d(d - b_0^2)^2 + a_0^2dA_8 + a_0cdB_6 \quad (= de^2),$$

where A_8, B_6 are certain polynomials in a_0, b_0, c, d, f.

Proof. This can be checked with the computer, but also by hand: the determinantal relations encode the image of

$$\mathbb{P}(1, 2, 2, 5) \xrightarrow{(x:y_0:xy_1:y_1^2:z:y_1(y_1^2-y_0^2):y_1z)} \mathbb{P}(1, 2, 3, 4, 5, 6, 7)$$

and the polynomials A_8 and B_6 are determined from (8.3) by

$$e^2 = z^2 = y_0(y_1 + y_0)^2(y_1 - y_0)^2 + x^2f_8(x, y_0, y_1) = b_0(d - b_0^2)^2 + a_0^2A_8 + a_0cB_6.$$

The remaining two equations are induced by $g^2 = de^2$ and $eg = y_1z^2$ using the determinantal relations. □

Lemma 8.10. *The generators in Table 2 induce an embedding*

$$\tilde{X}_2 \to \mathbb{P}(1, 1, 4, 6) \to \mathbb{P}(1_{a_1}, 2_{b_0}, 2_{b_1}, 4_d, 5_e, 6_f, 7_g)$$

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identifying $X_2 \cong \tilde{X}_2$. The ideal of X_2 is generated by

$$\operatorname{rk}\begin{pmatrix} a_1 & d-b_0^2 & f & b_1 & e & g \\ b_1 & e & g & a_1b_0 & (d-b_0^2)b_0 & fb_0 \end{pmatrix} \le 1$$

$$\begin{aligned} f^2 &= d(d-b_0^2)^2 + a_1C_{11} + b_1D_{10} \\ fg &= de(d-b_0^2) + b_1C_{11} + a_1b_0D_{10} \\ g^2 &= b_0d(d-b_0^2)^2 + a_1b_0C_{11} + b_0b_1D_{10} \quad (= b_0f^2) \end{aligned}$$

where C_{11}, D_{10} are general polynomials in a_1, b_0, b_1, d, e.

Proof. Note that a_1^2, b_0, b_1, d, f generate the even subring of $\mathbb{C}[u_0, u_1, v, w]$, so the map is an embedding.

As in the proof of Lemma 8.9, the determinantal equations cut out the image of $\mathbb{P}(1,1,4,6)$ and the remaining equations are induced from (8.4) by the equation defining $\tilde{X}_2$. □

The canonical model of the glued surface X in $\mathbb{P}(1,1,2,2,3,4,5,6,7)$ can be computed as the intersection of the two former ideals:

Proposition 8.11. *Let X be a 2-Gorenstein stable I-surface with every canonical curve nonreduced and such that the S_2-fication of one component is of type D and the other component is of elliptic type E. Then the canonical model of X is defined by the following 20 equations in $\mathbb{P}(1_{a_0}, 1_{a_1}, 2_{b_0}, 2_{b_1}, 3_c, 4_d, 5_e, 6_f, 7_g)$:*

$$\operatorname{rk}\begin{pmatrix} 0 & 0 & a_0 & c \\ 0 & 0 & c & a_0d \\ a_1 & b_1 & d-b_0^2 & f \\ b_1 & a_1b_0 & e & g \end{pmatrix} \le 1, \qquad \begin{cases} e^2 &= b_0(d-b_0^2)^2 + a_0^2A_8 + a_0cB_6 \\ eg &= b_0f(d-b_0^2) + a_0cA_8 + a_0^2dB_6 \\ f^2 &= d(d-b_0^2)^2 + a_1C_{11} + b_1D_{10} \\ fg &= de(d-b_0^2) + b_1C_{11} + a_1b_0D_{10} \\ g^2 &= de^2 + a_1b_0C_{11} + b_1^2D_{10} \\ (g^2 = b_0f^2 + a_0^2dA_8 + a_0cdB_6) \end{cases} . \tag{8.5}$$

If A, B, C, D are general, then X is a stable I-surface, with $\omega_X = \mathscr{O}_X(1)$ and $\omega_X^{[2]} = \mathscr{O}_X(2)$ is invertible.

Proof. The intersection of ideals of the components described in Lemma 8.9 and Lemma 8.10 can be calculated by computer or by hand giving (8.5). The rank condition is expressed by the vanishing of the 36 minors of size 2×2 of the matrix, but we claim that only 15 of these are ideal generators. The nine minors involving at least two entries from the upper left 2×2 block give the equation $0 = 0$. Moreover, a further seven minors are repeated or redundant. For example, the equation $a_1c = 0$ appears twice while $b_1c = 0$ appears four times as different minors. Then six minors are not needed to generate the ideal. For example, $a_1b_0c = 0$ appears twice but is not needed because we already know that $a_1c = 0$. Finally, of the six equations listed on the right hand side, the last one follows from the other ones. In total, the ideal is generated by 20 equations.

The two components of X are $X_1 = X \cap \{a_1 = b_1 = 0\}$ and $X_2 = X \cap \{a_0 = c = 0\}$ and one can confirm that this recovers the ideals of the components.

Now consider a surface X defined by these equations with A, B, C, D general. The Hilbert series of X can be determined by computer or follows from Proposition 3.3 as soon as we have established that X is 2-Gorenstein stable.

The canonical sheaf $\omega_X = \mathscr{O}_X(1)$ can be computed as in the proof of Proposition 5.6.

The sheaf $\mathscr{O}_{\mathbb{P}}(2)$ is invertible on $\mathbb{P}(1,1,2,2,3,4,5,6,7)$ away from the loci $\mathbb{P}(3_c, 6_f)$, $\mathbb{P}(4_d)$, $\mathbb{P}(5_e)$ and $\mathbb{P}(7_g)$. Clearly, X does not meet the last three of these loci, because monomials d^3, e^2, g^2 appear with nonzero coefficient in some equations defining X. Moreover, $X \cap \mathbb{P}(3,6)$ is also empty, because c^2 and f^2 also appear in separate equations defining X. Hence the restriction $\omega_X^{[2]} = \mathscr{O}_{\mathbb{P}}(2)|_X$ is invertible.

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Using the description of the components, one can see that for general equations the resulting surface has slc singularities. □

Remark 8.12. The nonexistence of an involution as in Lemma 8.4 can also be checked in this algebraic model.

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