ON SOME DUAL SERIES EQUATIONS AND THEIR APPLICATION TO ELECTROSTATIC PROBLEMS FOR SPHEROIDAL CAPS

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1. *Introduction.* This paper is concerned with dual series equations of the form

$$\sum_{n=0}^{\infty} (1+H_n) A_n T^{-m}_{m+n}(\cos\theta) = f(\theta) \quad (0 < \theta < \alpha), \tag{1·1}$$

$$\sum_{n=0}^{\infty} (2n+2m+1) A_n T^{-m}_{m+n}(\cos\theta) = g(\theta) \quad (\alpha < \theta < \pi), \tag{1·2}$$

where $T^{-m}_{m+n}(\cos\theta)$ is the associated Legendre function of degree $n+m$ and order $-m$ of the first kind, H_n is a known coefficient of n, $f(\theta)$ and $g(\theta)$ are given functions of the variable θ and the equations are to be solved for the unknown coefficients A_n. In hydrodynamics, elasticity and electromagnetism such equations occur in many problems involving a spherical or spheroidal boundary when different conditions are prescribed on different parts of the boundary. These dual series equations are analogous to the dual integral equations which arise in the application of Hankel transforms to problems involving plane or paraboloidal boundaries, the theory and applications of these latter equations having been discussed by several writers amongst whom are Sneddon, Noble and Lebedev (1), (2), (3). The main result of this paper is that the solution of the dual series equations (1·1) and (1·2) for m a positive integer or zero can be reduced to the solution of a Fredholm integral equation of the second kind in one independent variable. The restriction that m be a positive integer or zero covers those dual series equations which arise in three-dimensional boundary-value problems but excludes those which arise in two-dimensional ones where m is $\pm\frac{1}{2}$. These latter equations have, however, been discussed in a recent paper by Tranter (13).

In order to solve equations (1·1) and (1·2) it is first necessary to solve the auxiliary equations obtained by putting $H_n = 0$. This is done in § 3, certain results on associated Legendre polynomials, which are used in the solution, being given in § 2. Section 4 gives the reduction of the solution of the general dual series equations to the solution of a Fredholm integral equation, the method used being based on Noble's solution (2) of the corresponding dual integral equations. In § 5 further dual series equations, which arise in certain hydrodynamic problems, are considered, while in § 6 some electrostatic and hydrodynamic problems for a spherical cap are solved. The problem of a prolate spheroidal cap maintained at a given electrostatic potential, symmetrical about the axis of the cap, is considered in § 6, the Fredholm equation to which the dual

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series equations in this problem reduce being given and an approximate expression for the capacity of the cap derived. Corresponding results for an oblate spheroidal cap are stated in § 7.

2. *Some results on associated Legendre polynomials.* Ferrers' associated Legendre function of the first kind is defined by MacRobert ((4), p. 125) as

$$T^m_{m+n}(x) = (-1)^m (1-x^2)^{\frac{1}{2}m} \frac{d^m}{dx^m} P_{m+n}(x) \quad (-1 < x < 1), \tag{2·1}$$

m and n being positive integers or zero and $P_{m+n}(x)$ the Legendre polynomial of degree $m+n$. Further, we have the relation ((4), p. 328)

$$T^{-m}_{m+n}(x) = (-1)^m \frac{\Gamma(n+1)}{\Gamma(n+2m+1)} T^m_{m+n}(x). \tag{2·2}$$

We next require an inversion theorem for associated Legendre polynomials. This follows from a result of MacRobert ((4), p. 336) on the expansion of a function $f(\theta)$ in a series of associated Legendre polynomials. If we assume that the expansion

$$f(\theta) = \sum_{n=0}^{\infty} (2n+2m+1) C_n T^{-m}_{m+n}(\cos\theta) \tag{2·3}$$

is valid for $0 \leqslant \theta \leqslant \pi$ and that it can be integrated term by term, the coefficients C_n are given by

$$C_n = \tfrac{1}{2}(-1)^m \int_0^{\pi} f(\chi)\, T^m_{m+n}(\cos\chi) \sin\chi\, d\chi. \tag{2·4}$$

Finally, we require an integral representation of the associated Legendre polynomial $T^{-m}_{m+n}(\cos\theta)$, to obtain which we make use of the following result. If

$$\int_0^{\theta} \frac{p(u)\, du}{(\cos u - \cos\theta)^{\frac{1}{2}}} = q(\theta) \quad (0 < \theta < \alpha), \tag{2·5}$$

then

$$p(u) = \frac{1}{\pi} \frac{d}{du} \int_0^{u} \frac{q(\theta) \sin\theta\, d\theta}{(\cos\theta - \cos u)^{\frac{1}{2}}} \quad (0 < u < \alpha). \tag{2·6}$$

Equation (2·5) is an integral equation for $p(u)$, which can be reduced to Abel's equation, but which is readily solved by multiplying the equation by $\sin\theta(\cos\theta - \cos v)^{-\frac{1}{2}}$ and integrating with respect to θ from 0 to v $(0 < v < \alpha)$. We have

$$\int_0^{v} \frac{\sin\theta\, d\theta}{(\cos\theta - \cos v)^{\frac{1}{2}}} \int_0^{\theta} \frac{p(u)\, du}{(\cos u - \cos\theta)^{\frac{1}{2}}} = \int_0^{v} \frac{q(\theta) \sin\theta\, d\theta}{(\cos\theta - \cos v)^{\frac{1}{2}}},$$

and, on interchanging the order of integration on the left-hand side, we obtain

$$\int_0^{v} p(u)\, du = \frac{1}{\pi} \int_0^{v} \frac{q(\theta) \sin\theta\, d\theta}{(\cos\theta - \cos v)^{\frac{1}{2}}}.$$

Differentiation of this equation with respect to v gives equation (2·6).

We now represent the associated Legendre polynomial $T^{-m}_{m+n}(\cos\theta)$ as an integral of the form

$$T^{-m}_{m+n}(\cos\theta) = \cot^m \tfrac{1}{2}\theta \int_0^{\theta} \frac{R^m_{m+n}(u)\, du}{(\cos u - \cos\theta)^{\frac{1}{2}}} \quad (0 < \theta < \pi). \tag{2·7}$$

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From equations (2·5) and (2·6) we find that the function $R^m_{m+n}(u)$ is given by

$$R^m_{m+n}(u) = \frac{1}{\pi}\frac{d}{du}\int_0^u \frac{\sin\theta \tan^m \tfrac{1}{2}\theta\, T^{-m}_{m+n}(\cos\theta)\, d\theta}{(\cos\theta - \cos u)^{\frac{1}{2}}}.$$

Since ((4), p. 335)

$$\tan^m \tfrac{1}{2}\theta\, T^{-m}_{m+n}(\cos\theta) = \frac{\sin^{2m}\tfrac{1}{2}\theta}{\Gamma(m+1)} F(-n, n+2m+1;\ m+1;\ \tfrac{1}{2}(1-\cos\theta)),$$

we have

$$R^m_{m+n}(u) = \frac{1}{2^m\pi\Gamma(m+1)}\frac{d}{du}\sum_{r=0}^{n}\frac{(-n)_r\,(n+2m+1)_r}{2^r(m+1)_r\,\Gamma(r+1)}\int_0^u \frac{(1-\cos\theta)^{m+r}\sin\theta\, d\theta}{(\cos\theta-\cos u)^{\frac{1}{2}}},$$

where $(d)_r = \Gamma(d+r)/\Gamma(d)$. On making the substitution

$$\cos\theta = 1 - t(1-\cos u)$$

we find that

$$\begin{aligned}\int_0^u (\cos\theta-\cos u)^{-\frac{1}{2}}(1-\cos\theta)^{m+r}\sin\theta\, d\theta &= (1-\cos u)^{m+r+\frac{1}{2}}\int_0^1 t^{m+r}(1-t)^{-\frac{1}{2}}\, dt\\ &= (1-\cos u)^{m+r+\frac{1}{2}} B(m+r+1, \tfrac{1}{2})\\ &= \frac{\pi^{\frac{1}{2}}\Gamma(m+r+1)}{\Gamma(m+r+\frac{3}{2})}(1-\cos u)^{m+r+\frac{1}{2}};\end{aligned}$$

so

$$\begin{aligned}R^m_{m+n}(u) &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{\sin^{2m}\tfrac{1}{2}u\cos\tfrac{1}{2}u}{\Gamma(m+\frac{1}{2})}\sum_{r=0}^{n}\frac{(-n)_r\,(n+2m+1)_r}{2^r(m+\frac{1}{2})_r\,\Gamma(r+1)}(1-\cos u)^r\\ &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{\sin^{2m}\tfrac{1}{2}u\cos\tfrac{1}{2}u}{\Gamma(m+\frac{1}{2})}F(-n, n+2m+1;\ m+\tfrac{1}{2};\ \tfrac{1}{2}(1-\cos u)).\end{aligned}$$

From this we obtain

$$R^m_{m+n}(u) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{\Gamma(n+1)}{\Gamma(n+m+\frac{1}{2})}\sin^{2m}\tfrac{1}{2}u\cos\tfrac{1}{2}u\, P_n^{(m-\frac{1}{2},\, m+\frac{1}{2})}(\cos u), \tag{2·8}$$

$P_n^{(m-\frac{1}{2},\, m+\frac{1}{2})}(\cos u)$ being a Jacobi polynomial ((5), p. 170), defined by

$$P_n^{(m-\frac{1}{2},\, m+\frac{1}{2})}(\cos u) = \frac{\Gamma(n+m+\frac{1}{2})}{\Gamma(n+1)\,\Gamma(m+\frac{1}{2})}F(-n, n+2m+1;\ m+\tfrac{1}{2};\ \tfrac{1}{2}(1-\cos u)).$$

Using various results on hypergeometric functions ((6), pp. 101–2, 105), we can express this polynomial as

$$\begin{aligned}&P_n^{(m-\frac{1}{2},\, m+\frac{1}{2})}(\cos u)\\ &= \frac{2^{m+\frac{1}{2}}\Gamma(n+m+\frac{1}{2})\,(1+\cos u)^{-m-\frac{1}{2}}}{\Gamma(n+1)\,\Gamma(m+\frac{1}{2})}F(n+m+\tfrac{1}{2}, -n-m-\tfrac{1}{2};\ m+\tfrac{1}{2};\ \tfrac{1}{2}(1-\cos u))\\ &= \frac{(-1)^m\,2^m\Gamma(n+m+\frac{1}{2})}{\pi^{\frac{1}{2}}\Gamma(n+2m+1)}\left(\frac{1}{\sin u}\frac{d}{du}\right)^m[\cos^{-1}\tfrac{1}{2}u F(n+m+\tfrac{1}{2}, -n-m-\tfrac{1}{2};\ \tfrac{1}{2};\ \tfrac{1}{2}(1-\cos u))]\\ &= \frac{(-1)^m\,2^m\Gamma(n+m+\frac{1}{2})}{\pi^{\frac{1}{2}}\Gamma(n+2m+1)}\left(\frac{1}{\sin u}\frac{d}{du}\right)^m\left(\frac{\cos(n+m+\frac{1}{2})u}{\cos\frac{1}{2}u}\right).\end{aligned} \tag{2·9}$$

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Thus we find that

$$R^m_{m+n}(u) = \frac{(-1)^m 2^{m+\frac{1}{2}}\Gamma(n+1)}{\pi\Gamma(n+2m+1)} \sin^{2m}\tfrac{1}{2}u \cos\tfrac{1}{2}u \left(\frac{1}{\sin u}\frac{d}{du}\right)^m \left(\frac{\cos(n+m+\frac{1}{2})u}{\cos\frac{1}{2}u}\right). \quad (2{\cdot}10)$$

When $m = 0$, equation (2·10) gives

$$R_n(u) = \frac{2^{\frac{1}{2}}}{\pi} \cos(n+\tfrac{1}{2})u, \quad (2{\cdot}11)$$

and (2·7) is thus the Mehler–Dirichlet integral for $P_n(\cos\theta)$. When $m = 1$, (2·10) gives

$$R^1_{n+1}(u) = \frac{2^{-\frac{1}{2}}}{\pi} \sec\tfrac{1}{2}u \tan\tfrac{1}{2}u \left(\frac{\sin(n+1)u}{n+1} + \frac{\sin(n+2)u}{n+2}\right). \quad (2{\cdot}12)$$

3. *The solution of the auxiliary dual series equations for integer m.* In order to solve the general dual series equations (1·1) and (1·2) we must first consider the auxiliary equations

$$\sum_{n=0}^{\infty} A_n T^{-m}_{m+n}(\cos\theta) = f(\theta) \quad (0 < \theta < \alpha), \quad (3{\cdot}1)$$

$$\sum_{n=0}^{\infty} (2n+2m+1) A_n T^{-m}_{m+n}(\cos\theta) = g(\theta) \quad (\alpha < \theta < \pi). \quad (3{\cdot}2)$$

When $g(\theta) = 0$, these equations occur in the problem of determining the surface density induced on a thin uniform spherical cap maintained at a given potential. To solve (3·1) and (3·2) a method based on a solution to this problem given by the author (7) is used.

We assume that $\sin^{-m}\theta f(\theta)$ and $\sin^{-m}\theta g(\theta)$ are continuous in the closed intervals $(0, \alpha)$ and (α, π), respectively. The analysis in this and in the next section is formal, it being assumed that orders of integration and summation can be interchanged as necessary. If we write

$$\sum_{n=0}^{\infty} (2n+2m+1) A_n T^{-m}_{m+n}(\cos\theta) = h(\theta) \quad (0 \leqslant \theta \leqslant \alpha), \quad (3{\cdot}3)$$

then on applying the inversion theorem (2·4) to (3·2) and (3·3) we obtain

$$A_n = \tfrac{1}{2}(-1)^m \int_0^{\alpha} h(\chi) T^m_{m+n}(\cos\chi) \sin\chi d\chi + \tfrac{1}{2}(-1)^m \int_{\alpha}^{\pi} g(\chi) T^m_{m+n}(\cos\chi) \sin\chi d\chi \quad (n = 0, 1, 2, \ldots). \quad (3{\cdot}4)$$

On substituting this expression for A_n into equation (3·1) we have

$$\tfrac{1}{2}(-1)^m \sum_{n=0}^{\infty} T^{-m}_{m+n}(\cos\theta) \int_0^{\alpha} h(\chi) T^m_{m+n}(\cos\chi) \sin\chi d\chi = f(\theta) - \tfrac{1}{2}(-1)^m \sum_{n=0}^{\infty} T^{-m}_{m+n}(\cos\theta) \int_{\alpha}^{\pi} g(\chi) T^m_{m+n}(\cos\chi) \sin\chi d\chi \quad (0 < \theta < \alpha).$$

If we now interchange the order of summation and integration, this equation becomes

$$\tfrac{1}{2}(-1)^m \int_0^{\alpha} h(\chi) S_m(\theta, \chi) \sin\chi d\chi = f(\theta) - \tfrac{1}{2}(-1)^m \int_{\alpha}^{\pi} g(\chi) S_m(\theta, \chi) \sin\chi d\chi \quad (0 < \theta < \alpha), \quad (3{\cdot}5)$$

where

$$S_m(\theta, \chi) = \sum_{n=0}^{\infty} T^{-m}_{m+n}(\cos\theta) T^m_{m+n}(\cos\chi). \quad (3{\cdot}6)$$

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To sum $S_m(\theta, \chi)$ we make use of the addition theorem for Legendre polynomials, which is given by MacRobert ((4), p. 328) as

$$P_r(\cos\gamma) = P_r(\cos\theta)\,P_r(\cos\chi) + 2\sum_{m=1}^{r}(-1)^m \cos m\psi\, T_r^{-m}(\cos\theta)\,T_r^m(\cos\chi) \quad (r = 1, 2, \ldots), \tag{3·7}$$

where $\cos\gamma = \cos\theta\cos\chi + \sin\theta\sin\chi\cos\psi$. We can thus write

$$S_m(\theta,\chi) = \frac{(-1)^m}{2\pi}\sum_{n=0}^{\infty}\int_0^{2\pi} P_{m+n}(\cos\gamma)\cos m\psi\, d\psi$$

$$= \frac{(-1)^m}{2\pi}\sum_{n=0}^{\infty}\int_0^{2\pi} P_n(\cos\gamma)\cos m\psi\, d\psi,$$

since it follows from (3·7) that

$$\int_0^{2\pi} P_n(\cos\gamma)\cos m\psi\, d\psi = 0 \quad (n < m).$$

On interchanging the order of summation and integration in $S_m(\theta, \chi)$ we obtain

$$S_m(\theta,\chi) = \frac{(-1)^m}{2\pi}\int_0^{2\pi}\frac{\cos m\psi\, d\psi}{(2-2\cos\gamma)^{\frac{1}{2}}}$$

$$= \frac{(-1)^m}{2\pi}\int_0^{2\pi}\frac{\cos m\psi\, d\psi}{(s_1^2+s_2^2-2s_1s_2\cos\psi)^{\frac{1}{2}}},$$

where $s_1 = 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\chi$, $s_2 = 2\sin\frac{1}{2}\chi\cos\frac{1}{2}\theta$ and both s_1 and s_2 are positive for all θ and χ since $0 < \theta, \chi < \pi$. Using a lemma of Copson (8), we find that

$$S_m(\theta,\chi) = \frac{2(-1)^m}{\pi(s_1s_2)^m}\int_0^{\mathrm{Min}\,(s_1,s_2)}\frac{s^{2m}\,ds}{[(s_1^2-s^2)(s_2^2-s^2)]^{\frac{1}{2}}}. \tag{3·8}$$

On substituting (3·8) into equation (3·6) and noting that $s_1 < s_2$ when $\theta < \chi$ and $s_1 > s_2$ when $\theta > \chi$, we obtain

$$\frac{\sin^{-m}\theta}{\pi}\left[\int_0^{\theta} h(\chi)\sin^{1-m}\chi\int_0^{s_2}\frac{s^{2m}\,ds}{[(s_1^2-s^2)(s_2^2-s^2)]^{\frac{1}{2}}}\,d\chi + \int_{\theta}^{\alpha} h(\chi)\sin^{1-m}\chi\int_0^{s_1}\frac{s^{2m}\,ds}{[(s_1^2-s^2)(s_2^2-s^2)]^{\frac{1}{2}}}\,d\chi\right]$$

$$= f(\theta) - \frac{\sin^{-m}\theta}{\pi}\int_{\alpha}^{\pi} g(\chi)\sin^{1-m}\chi\int_0^{s_1}\frac{s^{2m}\,ds}{[(s_1^2-s^2)(s_2^2-s^2)]^{\frac{1}{2}}}\,d\chi \quad (0 < \theta < \alpha). \tag{3·9}$$

We next make the change of variable

$$s = 2\cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}\chi\tan\tfrac{1}{2}u$$

and invert the orders of integration in (3·9) to find that

$$\int_0^{\theta}\frac{H(u)\tan^{2m}\frac{1}{2}u\,du}{(\cos u-\cos\theta)^{\frac{1}{2}}} = f(\theta)\tan^m\tfrac{1}{2}\theta - \int_0^{\theta}\frac{G(u,\alpha)\tan^{2m}\frac{1}{2}u\,du}{(\cos u-\cos\theta)^{\frac{1}{2}}} \quad (0 < \theta < \alpha), \tag{3·10}$$

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where
$$H(u) = \frac{1}{2\pi}\int_u^\alpha \frac{h(\chi)\cot^m \tfrac{1}{2}\chi \sin\chi d\chi}{(\cos u - \cos\chi)^{\frac{1}{2}}} \tag{3·11}$$

and
$$G(u,\sigma) = \frac{1}{2\pi}\int_\sigma^\pi \frac{g(\chi)\cot^m \tfrac{1}{2}\chi \sin\chi d\chi}{(\cos u - \cos\chi)^{\frac{1}{2}}}. \tag{3·12}$$

The solution of the integral equation (3·10) for the unknown function $H(u)$ is found from (2·5) and (2·6) as

$$H(u) = \frac{\cot^{2m}\tfrac{1}{2}u}{\pi}\frac{d}{du}\int_0^u \frac{f(\theta)\tan^m \tfrac{1}{2}\theta \sin\theta d\theta}{(\cos\theta - \cos u)^{\frac{1}{2}}} - G(u,\alpha) \quad (0 < u < \alpha). \tag{3·13}$$

The function $h(\chi)$ is now found from the integral equation (3·10), the solution of which is

$$h(\chi) = -2\tan^m \tfrac{1}{2}\chi \operatorname{cosec}\chi \frac{d}{d\chi}\int_\chi^\alpha \frac{H(u)\sin u du}{(\cos\chi - \cos u)^{\frac{1}{2}}} \quad (0 < \chi < \alpha). \tag{3·14}$$

When $h(\chi)$ is known, the coefficients A_n can be found from (3·4) and the solution of the dual series equations completed. It is however unnecessary to determine $h(\chi)$ since the coefficients A_n can be expressed in terms of $H(u)$. From equations (2·2), (2·7) and (2·8) we find that for $n = 0, 1, 2, \ldots$,

$$\begin{aligned} A_n &= \frac{1}{2}\frac{\Gamma(n+2m+1)}{\Gamma(n+1)}\left[\int_0^\alpha h(\chi)\, T_{m+n}^{-m}(\cos\chi)\sin\chi d\chi + \int_\alpha^\pi g(\chi)\, T_{m+n}^{-m}(\cos\chi)\sin\chi d\chi\right] \\ &= \frac{1}{(2\pi)^{\frac{1}{2}}}\frac{\Gamma(n+2m+1)}{\Gamma(n+m+\frac{1}{2})}\left[\int_0^\alpha h(\chi)\cot^m \tfrac{1}{2}\chi \sin\chi \int_0^\chi \frac{\sin^{2m}\tfrac{1}{2}u \cos\tfrac{1}{2}u\, P_n^{(m-\frac{1}{2},\, m+\frac{1}{2})}(\cos u)\, du d\chi}{(\cos u - \cos\chi)^{\frac{1}{2}}}\right. \\ &\qquad \left. + \int_\alpha^\pi g(\chi)\cot^m \tfrac{1}{2}\chi \sin\chi \int_0^\chi \frac{\sin^{2m}\tfrac{1}{2}u \cos\tfrac{1}{2}u\, P_n^{(m-\frac{1}{2},\, m+\frac{1}{2})}(\cos u)\, du d\chi}{(\cos u - \cos\chi)^{\frac{1}{2}}}\right]. \end{aligned}$$

On interchanging the orders of integration and using (3·11) and (3·12) we obtain

$$\begin{aligned} A_n = \frac{(2\pi)^{\frac{1}{2}}\Gamma(n+2m+1)}{\Gamma(n+m+\frac{1}{2})}&\left[\int_0^\alpha H(u)\sin^{2m}\tfrac{1}{2}u \cos\tfrac{1}{2}u\, P_n^{(m-\frac{1}{2},\, m+\frac{1}{2})}(\cos u)\, du\right. \\ &\left. + \int_0^\pi G_0(u)\sin^{2m}\tfrac{1}{2}u \cos\tfrac{1}{2}u P_n^{(m-\frac{1}{2},\, m+\frac{1}{2})}(\cos u)\, du\right] \quad (n = 0, 1, 2, \ldots), \end{aligned} \tag{3·15}$$

where
$$\begin{aligned} G_0(u) &= G(u,\alpha) \quad (0 < u < \alpha), \\ &= G(u,u) \quad (\alpha < u < \pi), \end{aligned}$$

$G(u,\alpha)$ and $G(u,u)$ being defined by (3·12). When $m = 0$, equation (3·15) gives

$$A_n = 2^{\frac{1}{2}}\left[\int_0^\alpha H(u)\cos(n+\tfrac{1}{2})u du + \int_0^\pi G_0(u)\cos(n+\tfrac{1}{2})u du\right] \quad (n = 0, 1, 2, \ldots), \tag{3·16}$$

and, when $m = 1$,

$$\begin{aligned} A_n = 2^{-\frac{1}{2}}&\left[\int_0^\alpha H(u)\tan\tfrac{1}{2}u \sec\tfrac{1}{2}u\{(n+1)\sin(n+2)u + (n+2)\sin(n+1)u\} du\right. \\ &\left. + \int_0^\pi G_0(u)\tan\tfrac{1}{2}u \sec\tfrac{1}{2}u\{(n+1)\sin(n+2)u + (n+2)\sin(n+1)u\} du\right] \\ &(n = 0, 1, 2, \ldots). \end{aligned} \tag{3·17}$$

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4. *The solution of the general dual series equations for integer m.* In the dual series equations (1·1) and (1·2) we now suppose that H_n is a given coefficient of n and is $O(n^{-1})$ for large n and as before that $\sin^{-m}\theta f(\theta)$ and $\sin^{-m}\theta g(\theta)$ are continuous in the closed intervals $(0,\alpha)$ and (α,π), respectively. As in the solution of the auxiliary equations we write

$$\sum_{n=0}^{\infty}(2n+2m+1)A_n T^{-m}_{m+n}(\cos\theta) = h(\theta) \quad (0 \leqslant \theta \leqslant \alpha), \tag{4·1}$$

and use the inversion theorem (2·4) to obtain

$$A_n = \tfrac{1}{2}(-1)^m \int_0^{\alpha} h(\chi)\, T^m_{m+n}(\cos\chi)\sin\chi\, d\chi + \tfrac{1}{2}(-1)^m \int_{\alpha}^{\pi} g(\chi)\, T^m_{m+n}(\cos\chi)\sin\chi\, d\chi \quad (n = 0, 1, 2, \ldots). \tag{4·2}$$

On substituting for A_n in equation (1·1) we have that

$$\tfrac{1}{2}(-1)^m \sum_{n=0}^{\infty}(1+H_n)\, T^{-m}_{m+n}(\cos\theta)\int_0^{\alpha} h(\chi)\, T^m_{m+n}(\cos\chi)\sin\chi\, d\chi$$
$$= f(\theta) - \tfrac{1}{2}(-1)^m \sum_{n=0}^{\infty}(1+H_n)\, T^{-m}_{m+n}(\cos\theta)\int_{\alpha}^{\pi} g(\chi)\, T^m_{m+n}(\cos\chi)\sin\chi\, d\chi \quad (0 < \theta < \alpha). \tag{4·3}$$

From the analysis of § 3 it follows that

$$\tfrac{1}{2}(-1)^m \sum_{n=0}^{\infty} T^{-m}_{m+n}(\cos\theta)\int_0^{\alpha} h(\chi)\, T^m_{m+n}(\cos\chi)\sin\chi\, d\chi = \cot^m\tfrac{1}{2}\theta \int_0^{\theta} \frac{H(u)\tan^{2m}\tfrac{1}{2}u\, du}{(\cos u - \cos\theta)^{\frac{1}{2}}}, \tag{4·4}$$

and

$$\tfrac{1}{2}(-1)^m \sum_{n=0}^{\infty} T^{-m}_{m+n}(\cos\theta)\int_{\alpha}^{\pi} g(\chi)\, T^m_{m+n}(\cos\chi)\sin\chi\, d\chi = \cot^m\tfrac{1}{2}\theta \int_0^{\theta} \frac{G(u,\alpha)\tan^{2m}\tfrac{1}{2}u\, du}{(\cos u - \cos\theta)^{\frac{1}{2}}}, \tag{4·5}$$

where

$$H(u) = \frac{1}{2\pi}\int_u^{\alpha} \frac{h(\chi)\cot^m\tfrac{1}{2}\chi\sin\chi\, d\chi}{(\cos u - \cos\chi)^{\frac{1}{2}}} \tag{4·6}$$

and

$$G(u,\sigma) = \frac{1}{2\pi}\int_{\sigma}^{\pi} \frac{g(\chi)\cot^m\tfrac{1}{2}\chi\sin\chi\, d\chi}{(\cos u - \cos\chi)^{\frac{1}{2}}}. \tag{4·7}$$

We next consider the remaining series on the left-hand side of equation (4·3) and use (2·2) and (2·7) to obtain

$$I = \tfrac{1}{2}(-1)^m \sum_{n=0}^{\infty} H_n\, T^{-m}_{m+n}(\cos\theta)\int_0^{\alpha} h(\chi)\, T^m_{m+n}(\cos\chi)\sin\chi\, d\chi$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{H_n\,\Gamma(n+2m+1)}{\Gamma(n+1)}\cot^m\tfrac{1}{2}\theta\int_0^{\theta}\frac{R^m_{m+n}(u)\, du}{(\cos u - \cos\theta)^{\frac{1}{2}}}$$
$$\times \int_0^{\alpha} h(\chi)\sin\chi\cot^m\tfrac{1}{2}\chi\int_0^{\chi}\frac{R^m_{m+n}(v)\, dv}{(\cos v - \cos\chi)^{\frac{1}{2}}}\, d\chi.$$

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After some manipulation of the orders of integration and summation we find that

$$I = \cot^m \tfrac{1}{2}\theta \int_0^\theta \frac{\tan^m \tfrac{1}{2}u}{(\cos u - \cos\theta)^{\frac{1}{2}}} \int_0^\alpha \tan^m \tfrac{1}{2}v K_1(u,v)\, H(v)\, dv\, du, \tag{4·8}$$

where

$$\begin{aligned} K_1(u,v) &= \pi \cot^m \tfrac{1}{2}u \cot^m \tfrac{1}{2}v \sum_{n=0}^{\infty} \frac{\Gamma(n+2m+1)}{\Gamma(n+1)} H_n R^m_{m+n}(u) R^m_{m+n}(v) \\ &= 2^{1-2m} \sin^m u \sin^m v \cos \tfrac{1}{2}u \cos \tfrac{1}{2}v \\ &\quad \times \sum_{n=0}^{\infty} \frac{\Gamma(n+2m+1)\,\Gamma(n+1)}{[\Gamma(n+m+\frac{1}{2})]^2} H_n P_n^{(m-\frac{1}{2},\, m+\frac{1}{2})}(\cos u)\, P_n^{(m-\frac{1}{2},\, m+\frac{1}{2})}(\cos v). \end{aligned} \tag{4·9}$$

The interchanges of order performed in obtaining (4·8) are valid provided H_n is $O(n^{-1})$ for large n.

Similarly we find that the remaining series on the right-hand side of equation (4·3) can be put in the form

$$\begin{aligned} &\tfrac{1}{2}(-1)^m \sum_{n=0}^{\infty} H_n T^{-m}_{m+n}(\cos\theta) \int_\alpha^\pi g(\chi)\, T^m_{m+n}(\cos\chi) \sin\chi\, d\chi \\ &\qquad = \cot^m \tfrac{1}{2}\theta \int_0^\theta \frac{\tan^m \tfrac{1}{2}u}{(\cos u - \cos\theta)^{\frac{1}{2}}} \int_0^\pi \tan^m \tfrac{1}{2}v K_1(u,v)\, G_0(v)\, dv\, du, \end{aligned} \tag{4·10}$$

where

$$\begin{aligned} G_0(v) &= G(v,\alpha) \quad (0 < v < \alpha), \\ &= G(v,v) \quad (\alpha < v < \pi), \end{aligned} \tag{4·11}$$

$G(v,\alpha)$ and $G(v,v)$ being defined by (4·7).

If we now substitute (4·4), (4·5), (4·8) and (4·10) into (4·3), we obtain

$$\begin{aligned} &\cot^m \tfrac{1}{2}\theta \int_0^\theta \frac{\tan^m \tfrac{1}{2}u}{(\cos u - \cos\theta)^{\frac{1}{2}}} \left(\tan^m \tfrac{1}{2}u H(u) + \int_0^\alpha K_1(u,v) \tan^m \tfrac{1}{2}v H(v)\, dv \right) du \\ &\qquad = f(\theta) - \cot^m \tfrac{1}{2}\theta \int_0^\theta \frac{\tan^m \tfrac{1}{2}u}{(\cos u - \cos\theta)^{\frac{1}{2}}} \\ &\qquad\quad \times \left(\tan^m \tfrac{1}{2}u G(u,\alpha) + \int_0^\pi K_1(u,v) \tan^m \tfrac{1}{2}v\, G_0(v)\, dv \right) du \quad (0 < \theta < \alpha). \end{aligned}$$

If we write

$$J(u) = \tan^m \tfrac{1}{2}u (H(u) + G(u,\alpha)) \tag{4·12}$$

and use (2·5) and (2·6), we find that $J(u)$ satisfies the equation

$$\begin{aligned} J(u) + \int_0^\alpha K_1(u,v) J(v)\, dv &= \frac{1}{\pi} \cot^m \tfrac{1}{2}u \frac{d}{du} \int_0^u \frac{f(\theta) \tan^m \tfrac{1}{2}\theta \sin\theta\, d\theta}{(\cos\theta - \cos u)^{\frac{1}{2}}} \\ &\quad - \int_\alpha^\pi \tan^m \tfrac{1}{2}v K_1(u,v)\, G(v,v)\, dv \quad (0 < u < \alpha). \end{aligned} \tag{4·13}$$

Equation (4·13) is a Fredholm integral equation of the second kind for $J(u)$. The kernel $K_1(u,v)$ of the equation is given by (4·9) and is symmetric.

From the definition (2·1) of $T^{-m}_{m+n}(\cos\theta)$ and the dual series equations (1·1) and (1·2) it follows that $\sin^{-m}\theta f(\theta)$ and $\sin^{-m}\theta g(\theta)$ must be taken as even functions of θ. Equation (4·13) then shows that $J(u)$ is an even or an odd function of u according as m is

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even or odd, from which it follows that $H(u)$ is an even function for any value of m. Further, under the assumptions made on $f(\theta)$ and $g(\theta)$, $H(u)$ can be shown to be continuous in the interval $(0, \alpha)$.

Once equation (4·13) is solved for $J(u)$, $H(u)$ follows from (4·12) and $h(\chi)$ by solving the integral equation (4·6). However, in order to determine the coefficients A_n given by (4·2), this last step is unnecessary since these coefficients can be expressed in terms of $H(u)$ by an analysis exactly similar to that performed in the derivation of equation (3·15). The resulting expressions for A_n are in fact given by (3·15), $H(u)$ and $G_0(u)$ now being given by (4·6) and (4·11).

When $m = 0$, we have that

$$J(u) = H(u) + G(u, \alpha), \tag{4·14}$$

$J(u)$ being an even function of u. If we write $J(u) = J(-u)$ for $u < 0$, $J(u)$ satisfies the Fredholm equation

$$J(u) + \frac{1}{\pi}\int_{-\alpha}^{\alpha} K_0(u-v)\,J(v)\,dv = \frac{1}{\pi}\frac{d}{du}\int_0^u \frac{f(\theta)\sin\theta\,d\theta}{(\cos\theta - \cos u)^{\frac{1}{2}}}$$
$$-\frac{2}{\pi}\sum_{n=0}^{\infty} H_n \cos(n+\tfrac{1}{2})u \int_\alpha^\pi G(v,v)\cos(n+\tfrac{1}{2})v\,dv \quad (-\alpha < u < \alpha), \tag{4·15}$$

where
$$K_0(u-v) = \sum_{n=0}^{\infty} H_n \cos(n+\tfrac{1}{2})(u-v).$$

5. *Further dual series equations.* The dual series equations considered in the previous sections are not of course the only ones which arise in practice. Another pair of dual series equations of interest have the form

$$\sum_{n=0}^{\infty} (n+m)(n+m+1)(1+H_n)A_n T^{-m}_{m+n}(\cos\theta) = f(\theta) \quad (0 < \theta < \alpha), \tag{5·1}$$

$$\sum_{n=0}^{\infty} (2n+2m+1)A_n T^{-m}_{m+n}(\cos\theta) = g(\theta) \quad (\alpha < \theta < \pi), \tag{5·2}$$

where $f(\theta)$ and $g(\theta)$ are given functions such that $\sin^{-m}\theta f(\theta)$ and $\sin^{-m}\theta g(\theta)$ are continuous in the closed intervals $(0, \alpha)$ and (α, π), respectively. These equations with H_n and $g(\theta)$ zero arise in the Neumann problem for a spherical cap.

We show that equations (5·1) and (5·2) can be replaced by equations of the form (1·1) and (1·2). To do this we note that the associated Legendre function $T^{-m}_{m+n}(\cos\theta)$ satisfies the differential equation

$$(n+m)(n+m+1)\sin^2\theta T^{-m}_{m+n}(\cos\theta)$$
$$= -\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}T^{-m}_{m+n}(\cos\theta)\right) + m^2 T^{-m}_{m+n}(\cos\theta).$$

For $m > 0$ equation (5·1) can thus be replaced by the equation

$$\sum_{n=0}^{\infty} (1+H_n)A_n T^{-m}_{m+n}(\cos\theta) = e(\theta) \quad (0 < \theta < \alpha), \tag{5·3}$$

where $e(\theta)$ is a solution of the equation

$$\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{de(\theta)}{d\theta}\right) - m^2 e(\theta) = -f(\theta)\sin^2\theta. \tag{5·4}$$

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Equations (5·3) and (5·2) are now dual series equations of the type already considered. The function $e(\theta)$ is chosen so that $\sin^{-m}\theta e(\theta)$ is continuous in the closed interval $(0, \alpha)$. Further, since the series (5·1) is divergent for any value of θ in the range $0 \leqslant \theta \leqslant \alpha$ for which $h(\theta)$ defined by (4·1) is not bounded, we must also choose $e(\theta)$ so that $h(\theta)$ is bounded for $0 \leqslant \theta \leqslant \alpha$. From (4·6) we obtain

$$h(\theta) = -2\operatorname{cosec}\theta \tan^m \tfrac{1}{2}\theta \frac{d}{d\theta}\int_\theta^\alpha \frac{H(u)\sin u\,du}{(\cos\theta - \cos u)^{\frac{1}{2}}}$$

$$= 2\tan^m \tfrac{1}{2}\theta \left(\frac{H(\alpha)}{(\cos\theta - \cos\alpha)^{\frac{1}{2}}} - \int_\theta^\alpha \frac{H'(u)\,du}{(\cos\theta - \cos\alpha)^{\frac{1}{2}}}\right),$$

and hence find that $e(\theta)$ must be chosen so that $H(\alpha) = 0$. From (5·4) it follows that $e(\theta)$ consists of the particular integral of this equation together with a suitable choice of complementary function, this latter being $k_0 \tan^m \frac{1}{2}\theta$, where k_0 is an undetermined constant. Equations (5·3) and (5·2) are now solved as in § 4 and k_0 determined from the condition that $H(\alpha)$ be zero.

When $m = 0$, this procedure needs modification. On using the differential equation for $P_n(\cos\theta)$ we obtain

$$\sum_{n=0}^{\infty} (1 + H_n) A_n P_n(\cos\theta) = e(\theta) \quad (0 < \theta < \alpha), \tag{5·5}$$

$$\sum_{n=0}^{\infty} (2n+1) A_n P_n(\cos\theta) = g(\theta) \quad (\alpha < \theta < \pi), \tag{5·6}$$

where $e(\theta)$ is obtained as before, the complementary function now being k_0. Further, we add a constant $A_0(1 + H_0)$ to the left-hand side of the first equation, $A_0(1 + H_0)$ being a solution of (5·4) when $m = 0$. Equations (5·5) and (5·6) can now be solved as in § 4 and k_0 determined from the condition that $H(\alpha)$ be zero.

Since

$$(n+1)(n+2) T^{-1}_{n+1}(\cos\theta) = -\frac{d}{d\theta} P_{n+1}(\cos\theta), \tag{5·7}$$

equations (5·5) and (5·6) can be differentiated with respect to θ to obtain dual series equations in associated Legendre functions of order -1, whose solution does not involve the determination of the constant k_0. These equations, however, can be more readily obtained from (5·1) and (5·2) if we make use of the integral ((6), p. 159):

$$T^{-1}_{n+1}(\cos\theta) = \sin^{-1}\theta \int_0^\theta \sin\chi P_{n+1}(\cos\chi)\,d\chi. \tag{5·8}$$

On multiplying (5·1) for $m = 0$ by $\sin\theta$ and integrating from 0 to θ $(0 < \theta < \alpha)$ we obtain

$$\sum_{n=0}^{\infty} (n+1)(n+2)(1 + H_{n+1}) A_{n+1} T^{-1}_{n+1}(\cos\theta)$$

$$= \sin^{-1}\theta \int_0^\theta f(\chi)\sin\chi\,d\chi \quad (0 < \theta < \alpha), \tag{5·9}$$

whilst, if we differentiate (5·2) for $m = 0$ with respect to θ and use (5·7), we obtain

$$\sum_{n=0}^{\infty} (2n+3)(n+1)(n+2) A_{n+1} T^{-1}_{n+1}(\cos\theta) = -\frac{dg(\theta)}{d\theta} \quad (\alpha < \theta < \pi). \tag{5·10}$$

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If we put $$B_n = (n+1)(n+2)A_{n+1},$$

these dual series equations are of the form already considered in § 4 and can thus be solved for B_n.

In conclusion it may be noted that, if equations (5·3) and (5·2) for $m > 0$ are multiplied by $\cot^m \frac{1}{2}\theta$ and differentiated with respect to θ, the resulting dual series equations do not involve an undetermined constant but are, however, equations in Jacobi polynomials.

6. *Some electrostatic and hydrodynamic problems for a spherical cap.* In this section we consider the application of the theory of dual series equations developed in the previous sections to some electrostatic and hydrodynamic problems for a spherical cap.

We first determine the electrostatic potential V due to a thin spherical cap maintained at a constant potential V_0. We use spherical polar coordinates (r, θ, ω) referred to the centre of the sphere $r = a$ as origin and the axis of the cap as polar axis, the cap being given by $r = a$ $(0 \leqslant \theta \leqslant \alpha)$. Since the potential V is a solution of Laplace's equation, symmetrical about the axis of the cap, continuous across the sphere $r = a$ and $O(r^{-1})$ for large r, we take

$$V = \sum_{n=0}^{\infty} A_n \left(\frac{r}{a}\right)^n P_n(\cos\theta) \qquad (r < a),$$

$$= \sum_{n=0}^{\infty} A_n \left(\frac{a}{r}\right)^{n+1} P_n(\cos\theta) \quad (r > a).$$

The conditions that V be equal to V_0 on the cap and that $\partial V/\partial r$ be continuous over that part of the sphere $r = a$ not occupied by the cap lead to the dual series equations

$$\sum_{n=0}^{\infty} A_n P_n(\cos\theta) = V_0 \quad (0 < \theta < \alpha),$$

$$\sum_{n=0}^{\infty} (2n+1) A_n P_n(\cos\theta) = 0 \quad (\alpha < \theta < \pi).$$

These are of the type considered in § 3. From (3·13) we find that

$$H(u) = \frac{2^{\frac{1}{2}} V_0}{\pi} \cos\tfrac{1}{2}u, \tag{6·1}$$

so that from (3·16) we obtain

$$A_n = \frac{V_0}{\pi}\left(\frac{\sin n\alpha}{n} + \frac{\sin(n+1)\alpha}{n+1}\right).$$

The capacity C of the cap is given by

$$C = \frac{A_0 a}{V} = \frac{a}{\pi}(\alpha + \sin\alpha).$$

We next determine the velocity potential ϕ due to the motion of a spherical cap along its axis with constant velocity U in a perfect fluid at rest at large distances from the cap. We use the same coordinate system as in the electrostatic problem. The velocity potential ϕ is a solution of Laplace's equation, symmetrical about the axis of the cap

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and $O(r^{-2})$ for large r. Further, we require $r\partial\phi/\partial r$ to be continuous across the sphere $r = a$. We therefore take

$$\phi = \sum_{n=0}^{\infty} (n+1) A_n \left(\frac{r}{a}\right)^n P_n(\cos\theta) \quad (r < a),$$

$$= -\sum_{n=1}^{\infty} n A_n \left(\frac{a}{r}\right)^{n+1} P_n(\cos\theta) \quad (r > a).$$

On the cap $r = a$ $(0 \leqslant \theta \leqslant \alpha)$ we have ((9), p. 160)

$$r\frac{\partial\phi}{\partial r} = -Ua\cos\theta,$$

whilst over the remainder of the sphere $r = a$ we require ϕ to be continuous. These conditions lead to the dual series equations

$$\sum_{n=1}^{\infty} n(n+1) A_n P_n(\cos\theta) = -Ua\cos\theta \quad (0 < \theta < \alpha),$$

$$\sum_{n=0}^{\infty} (2n+1) A_n P_n(\cos\theta) = 0 \quad (\alpha < \theta < \pi).$$

Solving these equations by the second method given for the case $m = 0$ in § 5, we obtain

$$A_n = \frac{-Ua}{2\pi n(n+1)} \left[\left(\frac{n+1}{n-1}\right) \sin(n-1)\alpha + \sin n\alpha - \sin(n+1)\alpha \right.$$
$$\left. - \left(\frac{n}{n+2}\right) \sin(n+2)\alpha\right] \quad (n = 1, 2, \ldots).$$

In the region $r < a$ it is sufficient to determine V to within an arbitrary constant A_0. The virtual mass M of the cap ((9), p. 165) is found as

$$M = -\frac{4\pi\rho a^2 A_1}{U} = \rho a^3(2\alpha + \sin\alpha - \sin 2\alpha - \tfrac{1}{3}\sin 3\alpha).$$

We next suppose the cap moves through the fluid in the direction $\omega = 0$ perpendicular to its axis of symmetry with uniform velocity U. The velocity potential ϕ for the motion is a solution of Laplace's equation and is $O(r^{-2})$ for large r. Further ϕ and $\partial\phi/\partial r$ are continuous over that part of the sphere $r = a$ not occupied by the cap, while on the cap $r = a$ $(0 \leqslant \theta \leqslant \alpha)$

$$r\frac{\partial\phi}{\partial r} = -Ua\sin\theta\cos\omega.$$

The appropriate solutions of Laplace's equation are thus

$$\phi = \sum_{n=0}^{\infty} (n+2) A_n \left(\frac{r}{a}\right)^{n+1} T_{n+1}^{-n}(\cos\theta)\cos\omega \quad (r < a),$$

$$= -\sum_{n=0}^{\infty} (n+1) A_n \left(\frac{a}{r}\right)^{n+2} T_{n+1}^{-1}(\cos\theta)\cos\omega \quad (r > a),$$

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the coefficients A_n being determined from the dual series equations

$$\sum_{n=0}^{\infty}(n+1)(n+2)A_n T_{n+1}^{-1}(\cos\theta) = -Ua\sin\theta \quad (0<\theta<\alpha),$$

$$\sum_{n=0}^{\infty}(2n+3)A_n T_{n+1}^{-1}(\cos\theta) = 0 \quad (\alpha<\theta<\pi).$$

Following the method of § 5 we replace these equations by the equations

$$\sum_{n=0}^{\infty}A_n T_{n+1}^{-1}(\cos\theta) = -\tfrac{1}{2}Ua\sin\theta + k_0\tan\tfrac{1}{2}\theta \quad (0<\theta<\alpha),$$

$$\sum_{n=0}^{\infty}(2n+3)A_n T_{n+1}^{-1}(\cos\theta) = 0 \quad (\alpha<\theta<\pi),$$

and solve these equations as in § 3. From (3·13) we find that

$$H(u) = -\frac{2\sqrt{2}\,Ua}{\pi}\cos^3\tfrac{1}{2}u + \frac{k_0}{\pi\sqrt{2}}\operatorname{cosec}\tfrac{1}{2}u(u+\sin u).$$

On imposing the condition that $H(\alpha)=0$ we determine k_0 as

$$k_0 = \frac{2Ua\sin\alpha\cos^2\tfrac{1}{2}\alpha}{\alpha+\sin\alpha}.$$

The coefficients A_n can now be found from (3·17). In particular, the virtual mass M ((9), p. 165) is given by

$$M = -\frac{2\pi\rho a^2 A_0}{U}$$

$$= \rho a^3\left[\frac{(2\alpha-\sin 2\alpha)(2\alpha+4\sin\alpha+\sin 2\alpha)}{2(\alpha+\sin\alpha)} - \tfrac{8}{3}\sin^3\alpha\right].$$

7. *An electrostatic problem for a prolate spheroidal cap.* We next consider a prolate spheroidal cap maintained at a given electrostatic potential, symmetrical about the axis of the cap. If (z,ϖ,ω) are the cylindrical polar coordinates of a point referred to the centre of the spheroid of which the cap is a part as origin and the axis of the cap as polar axis, we define prolate spheroidal coordinates (ξ,η,ω) by

$$z = c\cosh\xi\cos\eta, \quad \varpi = c\sinh\xi\sin\eta \quad (0\leqslant\xi<\infty,\ 0\leqslant\eta\leqslant\pi). \tag{7·1}$$

The cap is given by $\xi=\xi_0$ $(0\leqslant\eta\leqslant\eta_0)$, the semi-axes of the spheroid of which it is a part being $c\cosh\xi_0$ and $c\sinh\xi_0$. The interior of the spheroid is given by $0\leqslant\xi<\xi_0$, $0\leqslant\eta\leqslant\pi$, and the exterior by $\xi_0<\xi<\infty$, $0\leqslant\eta\leqslant\pi$.

We require a potential function $V(\xi,\eta)$ which is $O(r^{-1})$ for large r $(r^2=z^2+\varpi^2)$, which is continuously differentiable at all points except possibly those on the rim $(\xi=\xi_0,\ \eta=\eta_0)$ of the cap and which equals a given function $V_0(\eta)$ on the cap. Using results given by Hobson ((10), p. 413), we find that the appropriate forms for V are

$$V(\xi,\eta) = \sum_{n=0}^{\infty}L_n P_n(\cosh\xi)P_n(\cos\eta) \quad (0\leqslant\xi<\xi_0)$$

$$= \sum_{n=0}^{\infty}M_n Q_n(\cosh\xi)P_n(\cos\eta) \quad (\xi_0<\xi). \tag{7·2}$$

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Since V is continuous across $\xi = \xi_0$, we require

$$L_n P_n(\cosh\xi_0) = M_n Q_n(\cosh\xi_0), \tag{7·3}$$

whilst the condition satisfied by V on the cap gives

$$\sum_{n=0}^{\infty} M_n Q_n(\cosh\xi_0) P_n(\cos\eta) = V_0(\eta) \quad (0 < \eta < \eta_0). \tag{7·4}$$

If V is to be continuously differentiable at all points except possibly those on the rim of the cap, $\partial V/\partial\xi$ must be continuous across $\xi = \xi_0$ $(\eta_0 < \eta < \pi)$. We thus obtain

$$\sum_{n=0}^{\infty} \left[L_n \frac{d}{d\xi} P_n(\cosh\xi) - M_n \frac{d}{d\xi} Q_n(\cosh\xi) \right]_{\xi=\xi_0} P_n(\cos\eta) = 0 \quad (\eta_0 < \eta < \pi).$$

Since ((6), p. 146)

$$P_n(\cosh\xi) \frac{d}{d\xi} Q_n(\cosh\xi) - Q_n(\cosh\xi) \frac{d}{d\xi} P_n(\cosh\xi) = \operatorname{cosech}\xi,$$

we find on using (7·3) that

$$\sum_{n=0}^{\infty} \frac{M_n}{\sinh\xi_0 \, P_n(\cosh\xi_0)} P_n(\cos\eta) = 0 \quad (\eta_0 < \eta < \pi). \tag{7·5}$$

If we write

$$M_n = (2n+1)\sinh\xi_0 P_n(\cosh\xi_0) A_n, \tag{7·6}$$

equations (7·4) and (7·5) become

$$\sum_{n=0}^{\infty} (2n+1)\sinh\xi_0 \, P_n(\cosh\xi_0) Q_n(\cosh\xi_0) A_n P_n(\cos\eta) = V_0(\eta) \quad (0 < \eta < \eta_0), \tag{7·7}$$

$$\sum_{n=0}^{\infty} (2n+1) A_n P_n(\cos\eta) = 0 \quad (\eta_0 < \eta < \pi). \tag{7·8}$$

Using asymptotic expansions of the Legendre functions given by Hobson ((10), pp. 305–6), we can show that for large n

$$(2n+1)\sinh\xi_0 P_n(\cosh\xi_0) Q_n(\cosh\xi_0) = 1 + O(n^{-2}).$$

Equations (7·7) and (7·8) are thus dual series equations of the type (1·1) and (1·2) with

$$H_n = (2n+1)\sinh\xi_0 P_n(\cosh\xi_0) Q_n(\cosh\xi_0) - 1. \tag{7·9}$$

From (3·16) and (4·15) the coefficients A_n can be expressed in the form

$$A_n = \sqrt{2} \int_0^{\eta_0} H(u) \cos(n+\tfrac{1}{2}) u \, du, \tag{7·10}$$

where $H(u)$ satisfies the Fredholm integral equation of the second kind

$$H(u) + \frac{1}{\pi} \int_{-\eta_0}^{\eta_0} K_0(u-v) H(v) \, dv = \frac{1}{\pi} \frac{d}{du} \int_0^u \frac{V_0(\eta)\sin\eta \, d\eta}{(\cos\eta - \cos v)^{\frac{1}{2}}} \quad (-\eta_0 < u < \eta_0), \tag{7·11}$$

the kernel $K_0(u-v)$ of this equation being given by

$$K_0(u-v) = \sum_{n=0}^{\infty} \{(2n+1)\sinh\xi_0 P_n(\cosh\xi_0) Q_n(\cosh\xi_0) - 1\} \cos(n+\tfrac{1}{2})(u-v). \tag{7·12}$$

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To sum the kernel we note that from the Mehler–Dirichlet integral ((6), p. 159)

$$P_n(\cos\chi) = \frac{\sqrt{2}}{\pi}\int_0^{\chi} \frac{\cos(n+\frac{1}{2})x\,dx}{(\cos x-\cos\chi)^{\frac{1}{2}}} \quad (0<\chi<\pi)$$

and equations (2·5) and (2·6) we can deduce that

$$\cos(n+\tfrac{1}{2})x = \frac{1}{\sqrt{2}}\frac{d}{dx}\int_0^x \frac{P_n(\cos\chi)\sin\chi\,d\chi}{(\cos\chi-\cos x)^{\frac{1}{2}}}.$$

Hence, if $x = u-v$, $K_0(u-v)$ can be written as

$$K_0(x) = \frac{1}{\sqrt{2}}\frac{d}{dx}\int_0^x \frac{\sin\chi}{(\cos\chi-\cos x)^{\frac{1}{2}}}\left(\sum_{n=0}^{\infty}\{(2n+1)\sinh\xi_0 P_n(\cosh\xi_0)Q_n(\cosh\xi_0)-1\} \times P_n(\cos\chi)\right)d\chi.$$

On summing the series which occurs in this expression ((10), p. 415) we find that

$$K_0(x) = \frac{1}{\sqrt{2}}\frac{d}{dx}\int_0^x \frac{\sin\chi}{(\cos\chi-\cos x)^{\frac{1}{2}}}\left(\frac{\sinh\xi_0}{(2\cosh^2\xi_0-2\cosh^2\xi_0\cos\chi-\sin^2\chi)^{\frac{1}{2}}} - \frac{1}{(2-2\cos\chi)^{\frac{1}{2}}}\right)d\chi.$$

We now evaluate the integrals in this expression, the first by means of a table of elliptic integrals ((11), p. 74) and find that

$$K_0(x) = \tfrac{1}{2}\left|\cot\tfrac{1}{2}x\right| k'\{E(k)-K(k)\}, \tag{7·13}$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kinds and

$$k^2 = \frac{\sin^2\frac{1}{2}x}{\sinh^2\xi_0+\sin^2\frac{1}{2}x}, \quad k'^2 = 1-k^2 \quad (k, k' \geqslant 0).$$

If the cap is maintained at a constant potential V_0, equation (7·11) becomes

$$H(u)+\frac{1}{\pi}\int_{-\eta_0}^{\eta_0} K_0(u-v)H(v)\,dv = \frac{\sqrt{2}V_0}{\pi}\cos\tfrac{1}{2}u \quad (-\eta_0<u<\eta_0). \tag{7·14}$$

The capacity C of the cap is given by

$$V_0C = \lim_{r\to\infty} rV.$$

If we note that for large ξ, $r = \frac{1}{2}ce^{\xi}$ and use an expression for $Q_n(\cosh\xi)$ given by Bateman ((6), p. 122), we find that

$$\begin{aligned} V_0C = cM_0 &= c\sinh\xi_0 A_0 \\ &= \sqrt{2}\,c\sinh\xi_0\int_0^{\eta_0} H(u)\cos\tfrac{1}{2}u\,du. \end{aligned} \tag{7·15}$$

When $\xi_0\to\infty$ and $c\to 0$ in such a way that ce^{ξ_0} tends to a finite limit $2a$, the cap becomes a spherical cap of radius a, η_0 being the semi-angle of the cap. It can then be shown that H_n, given by (7·9), tends to zero, so that $K_0(u-v)$ also tends to zero. The integral equation (7·14) thus reduces to equation (6·1) already found in the problem of a spherical cap maintained at a constant potential.

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When ξ_0 is large, the cap is almost spherical. Since the eccentricity $e = \operatorname{sech}\xi_0$ of the cap in this case is small, we can obtain a solution of equation (7·14) as a power series in e by iteration and, provided e is sufficiently small, this power series is convergent. Further, the first few terms of this series can be taken as an approximation to $H(u)$, the first term being the solution for a spherical cap. We thus obtain

$$H(u) = \frac{\sqrt{2V_0}}{\pi}\cos\tfrac{1}{2}u + \frac{\sqrt{2V_0}e^2}{8\pi}\left(\tfrac{4}{3}\cos\tfrac{1}{2}u - \cos u(\cos\tfrac{1}{2}\eta_0 + \tfrac{1}{3}\cos\tfrac{3}{2}\eta_0)\right)$$
$$+ \frac{\sqrt{2V_0}e^4}{128\pi}\{\tfrac{688}{45}\cos\tfrac{1}{2}u + \cos u(-\tfrac{121}{12}\cos\tfrac{1}{2}\eta_0 - \tfrac{113}{36}\cos\tfrac{3}{2}\eta_0 + \tfrac{1}{4}\cos\tfrac{5}{2}\eta_0 + \tfrac{1}{12}\cos\tfrac{7}{2}\eta_0)$$
$$- \tfrac{9}{2}\cos 2u(\tfrac{1}{3}\cos\tfrac{3}{2}\eta_0 + \tfrac{1}{5}\cos\tfrac{5}{2}\eta_0) - u\sin u(\cos\tfrac{1}{2}\eta_0 + \tfrac{1}{3}\cos\tfrac{3}{2}\eta_0)\} + O(e^6).$$

If $b = c\sinh\xi_0$, the capacity C of the cap is then found as

$$C = \frac{b}{\pi}(\eta_0 + \sin\eta_0) + \frac{be^2}{24\pi}(4\eta_0 + \sin\eta_0 - 2\sin 2\eta_0 - \tfrac{1}{3}\sin 3\eta_0)$$
$$+ \frac{be^4}{128\pi}(\tfrac{82}{5}\eta_0 + \tfrac{5}{3}\eta_0\cos\eta_0 + \tfrac{2}{3}\eta_0\cos 2\eta_0 + \tfrac{1}{9}\eta_0\cos 3\eta_0 + \tfrac{301}{90}\sin\eta_0$$
$$- \tfrac{23}{3}\sin 2\eta_0 - \tfrac{1057}{108}\sin 3\eta_0 - \tfrac{13}{30}\sin 4\eta_0 - \tfrac{137}{900}\sin 5\eta_0) + O(e^6).$$

8. *An electrostatic problem for an oblate spheroidal cap.* The problem of determining the electrostatic potential due to an oblate spheroidal cap maintained at a given potential symmetrical about its axis can be treated in exactly the same manner as that for a prolate spheroidal cap. Only the results are given here.

In terms of cylindrical polar coordinates (z, ϖ, ω) referred to the centre of the oblate spheroid of which the cap is a part as origin and the axis of the cap as polar axis, we define oblate spheroidal coordinates (ξ, η, ω) by

$$z = c\sinh\xi\cos\eta, \quad \varpi = c\cosh\xi\sin\eta \quad (0 \leqslant \xi < \infty,\ 0 \leqslant \eta \leqslant \pi). \tag{8·1}$$

The cap is given by $\xi = \xi_0$ $(0 \leqslant \eta \leqslant \eta_0)$, the semi-axes of the spheroid of which it is a part being $c\sinh\xi_0$ and $c\cosh\xi_0$, whilst the interior of the spheroid is given by $0 \leqslant \xi < \xi_0$, $0 \leqslant \eta \leqslant \pi$, and the exterior by $\xi_0 < \xi < \infty$, $0 \leqslant \eta \leqslant \pi$.

The potential function $V(\xi, \eta)$, which is $O(r^{-1})$ for large r $(r^2 = z^2 + \varpi^2)$, which is continuously differentiable at all points except those on the rim $(\xi = \xi_0,\ \eta = \eta_0)$ of the cap and which equals a given function $V_0(\eta)$ on the cap, is given by

$$V(\xi,\eta) = \sum_{n=0}^{\infty}(2n+1)A_n\cosh\xi_0\, q_n(\sinh\xi_0)\, p_n(\sinh\xi)\, P_n(\cos\eta) \quad (0 \leqslant \xi < \xi_0),$$
$$= \sum_{n=0}^{\infty}(2n+1)A_n\cosh\xi_0\, p_n(\sinh\xi_0)\, q_n(\sinh\xi)\, P_n(\cos\eta) \quad (\xi_0 < \xi), \tag{8·2}$$

where the functions $p_n(\sinh\xi)$ and $q_n(\sinh\xi)$ are the oblate spheroidal harmonic functions of Nicholson (12), defined by

$$p_n(\sinh\xi) = i^{-n}P_n(i\sinh\xi), \quad q_n(\sinh\xi) = i^{n+1}Q_n(i\sinh\xi),$$

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$P_n(i\sinh\xi)$ and $Q_n(i\sinh\xi)$ being Legendre functions of the first and second kinds, respectively. The coefficients A_n in the expressions (8·2) satisfy the dual series equations

$$\sum_{n=0}^{\infty}(2n+1)\cosh\xi_0 p_n(\sinh\xi_0)q_n(\sinh\xi_0)A_nP_n(\cos\eta)=V_0(\eta)\quad(0<\eta<\eta_0),$$
$$\sum_{n=0}^{\infty}(2n+1)A_nP_n(\cos\eta)=0\quad(\eta_0<\eta<\pi). \tag{8·3}$$

If these coefficients are expressed in the form

$$A_n=\sqrt{2}\int_0^{\eta_0}H(u)\cos(n+\tfrac{1}{2})u\,du, \tag{8·4}$$

the function $H(u)$ in (8·4) satisfies the Fredholm integral equation

$$H(u)+\frac{1}{\pi}\int_{-\eta_0}^{\eta_0}K_0(u-v)H(v)\,dv=\frac{1}{\pi}\frac{d}{du}\int_0^u\frac{V_0(\eta)\sin\eta\,d\eta}{(\cos\eta-\cos v)^{\frac{1}{2}}}\quad(-\eta_0<u<\eta_0), \tag{8·5}$$

where, if $x=u-v$, $\qquad K_0(x)=\tfrac{1}{2}\left|\cot\tfrac{1}{2}x\right|k'\{E(ik)-K(ik)\}$,

$K(ik)$ and $E(ik)$ being complete elliptic integrals of the first and second kinds respectively with

$$k^2=\frac{\sin^2\frac{1}{2}x}{\cosh^2\xi_0-\sin^2\frac{1}{2}x},\quad k'^2=1+k^2\quad(k,k'\geqslant 0).$$

When the cap is maintained at a constant potential V_0, equation (8·5) becomes

$$H(u)+\frac{1}{\pi}\int_{-\eta_0}^{\eta_0}K_0(u-v)H(v)\,dv=\frac{\sqrt{2}\,V_0}{\pi}\cos\tfrac{1}{2}u\quad(-\eta_0<u<\eta_0), \tag{8·6}$$

the capacity C of the cap being given by

$$CV_0=\sqrt{2}\,c\cosh\xi_0\int_0^{\eta_0}H(u)\cos\tfrac{1}{2}u\,du. \tag{8·7}$$

For an almost spherical cap the eccentricity $e=\operatorname{sech}\xi_0$ of the cap is small and we can obtain a solution for $H(u)$ as a power series in e by iteration. The first few terms of this series are

$$H(u)=\frac{\sqrt{2}\,V_0}{\pi}\cos\tfrac{1}{2}u-\frac{\sqrt{2}\,V_0e^2}{8\pi}\{\tfrac{4}{3}\cos\tfrac{1}{2}u-\cos u(\cos\tfrac{1}{2}\eta_0+\tfrac{1}{3}\cos\tfrac{3}{2}\eta_0)\}$$
$$+\frac{\sqrt{2}\,V_0e^4}{128\pi}\{-\tfrac{272}{45}\cos\tfrac{1}{2}u+\cos u(\tfrac{71}{12}\cos\tfrac{1}{2}\eta_0+\tfrac{79}{36}\cos\tfrac{3}{2}\eta_0+\tfrac{1}{4}\cos\tfrac{5}{2}\eta_0+\tfrac{1}{12}\cos\tfrac{7}{2}\eta_0)$$
$$-\tfrac{9}{2}\cos 2u(\tfrac{1}{3}\cos\tfrac{3}{2}\eta_0+\tfrac{1}{5}\cos\tfrac{5}{2}\eta_0)-u\sin u(\cos\tfrac{1}{2}\eta_0+\tfrac{1}{3}\cos\tfrac{3}{2}\eta_0)\}+O(e^6).$$

If $b=c\cosh\xi_0$, the capacity C of the cap is found as

$$C=\frac{b}{\pi}(\eta_0+\sin\eta_0)-\frac{be^2}{24\pi}(4\eta_0+\sin\eta_0-2\sin 2\eta_0-\tfrac{1}{3}\sin 3\eta_0)$$
$$+\frac{be^4}{128\pi}(-\tfrac{74}{15}\eta_0+\tfrac{5}{3}\eta_0\cos\eta_0+\tfrac{2}{3}\eta_0\cos 2\eta_0+\tfrac{1}{9}\eta_0\cos 3\eta_0$$
$$-\tfrac{179}{90}\sin\eta_0+3\sin 2\eta_0+\tfrac{35}{108}\sin 3\eta_0-\tfrac{13}{30}\sin 4\eta_0-\tfrac{137}{900}\sin 5\eta_0)+O(e^6).$$

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