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Output Feedback Admissible Control for Singular Systems: Delta Operator (Discretised) Approach

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Abstract. Singular systems simultaneously capture the dynamics and algebraic constraints in many practical applications. Output feedback admissible control for singular systems through a delta operator method is considered in this article. Two novel admissibility conditions, derived for the singular delta operator system (SDOS) from a singular continuous system through sampling, can not only produce unified admissibility for both continuous and discrete singular systems but also practical procedures. To solve the problem of output feedback admissible control for the SDOS, an existence condition and design procedure is given for the determination of a physically realisable observer for the state estimation, and then a suitable state-feedback-like admissible controller design based on the observer is developed. All of the conditions presented are necessary and sufficient, involving strict linear matrix inequalities (LMI) with feasible solutions obtained with low computational costs. Numerical examples illustrate our approach.

AMS subject classifications: 93B07, 93B51, 93B52

Key words: Singular delta operator system (SDOS), admissibility, output feedback, state observer, linear matrix inequalities (LMI).

1. Introduction

Singular systems simultaneously capturing both dynamics and algebraic constraints arise in many practical applications such as economic systems [1], electrical networks [2], chemical processes [3], or highly interconnected large-scale systems [4]. Extensive study of singular systems during recent decades has been due to their important state variable characteristics, and many fundamental issues related to performance analysis and synthesis have been investigated — cf. [4–9] and references therein. Moreover,

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singular systems have also been adopted to study some state-space systems in order to incorporate state constraints [10, 11]. The fundamental requirement to implement a singular system in practice is its admissibility (its regularity, stability and impulse elimination for the continuous case or causality for the discrete case), whereas for state-space systems the main concern is usually stability.

In most digital approaches, a standard shift operator is used for discrete systems resulting from continuous dynamical systems. However, the response of the discrete system may not converge smoothly to its continuous counterpart when the sampling period tends to zero [12], mainly due to numerical ill-conditioning. A delta operator introduced to avoid this deficiency appears to be more effective than the shift operator in certain applications [13], and in particular required smaller word length when it was implemented in fixed-point digital control processors [14]. Moreover, the delta operator method has been shown to be significantly less sensitive than the shift operator setting at high sampling rates [15]. Furthermore, the delta operator model can provide a theoretically unified formulation of continuous and discrete systems [16]. Many results on state-space systems using the delta operator have been presented in the literature, including stability, sliding mode control and positive real control [16–18]. Systematic results for delta operator systems can be found in Refs. [16, 19]; and recently the delta operator method has been applied to singular systems, with progress under the framework of SDOS — e.g. see Refs. [20–26]. Specifically, the delta operator model has been established for a singular continuous system through two different approaches [22, 23], directly obtained from a suitable singular discrete model and convergent to the corresponding continuous system as the sampling period tends to zero. The delta operator model thus provides not only a practical approach both a unified description of a singular continuous system and its discrete model, including in high sample rate cases. The analysis of controllability and observability were provided in Refs. [20] and [21] respectively, and various necessary and sufficient admissibility conditions have been obtained [22]. Results on admissible control found in Refs. [23–25] are given in terms of matrix inequalities and linear matrix inequalities (LMI), respectively. Different D-admissibility conditions and design methods for a suitable D-admissible controller are derived in Ref. [26]. It is notable that the controllers obtained are all state feedback forms, with sufficient LMI-based conditions for the existence of admissible and D-admissible controllers [23–26].

State feedback control is fundamental in the controller design for dynamical systems in control system theory. If all states are available for the controller design, then the output feedback control is a full state feedback. However, in many applications not all states of the system are available for measurement due to the system limitation or inhibited high cost. Instead, the information available for controller design is usually the output measurement that has much smaller dimension than the state space, so the study of output feedback control for singular systems has both theoretical and practical significance. Some interesting results have been reported in the observer design for singular systems, in order to estimate the states [27–30]. In Ref. [24] the problem of observer-based admissible control for SDOSs was considered, where via LMI

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an asymptotic and physically realisable state observer is first determined and then a state-feedback-like admissible controller designed based on this observer. However, the existence conditions obtained for both the observer and the admissible controller were only sufficient and also depend on certain pre-determined parameter matrices, which leads to some restrictions in application.

Motivated by the results in Refs. [22–24], in this article we establish a practical and less conservative approach for the design of an output feedback admissible control for SDOSs. Due to the nature of singular systems, the output feedback controller design is much more challenging than that for state-space systems, and the design procedure requires some subtle analysis that is quite different from state feedback design. Thus we first design an observer to estimate the state of the SDOS resulting from a singular continuous system, and then construct a state-feedback-like admissible controller based on the state estimation. Two novel admissibility conditions are presented for the SDOSs. Based on these conditions, necessary and sufficient LMI-based conditions are provided for the existence of an asymptotic and physically realisable observer and a suitable admissible controller, respectively. The design methods of the observer and the controller are also presented through the solutions of the corresponding LMI. It is notable that the forms of the observer and the admissible controller are similar in Ref. [24]. However, our results are all necessary and sufficient conditions and involve no parameter matrices, whereas the results in Ref. [24] are only sufficient and involve some pre-determined parameter matrices, so are less restricted. Section 2 provides some preliminaries and the problem formulation, and our two novel admissibility conditions are given in Section 3. The design procedure of the asymptotic and physically realisable observer is discussed in Section 4, and the construction of an output feedback admissible controller is presented in Section 5. Some numerical examples to illustrate our theoretical results and design procedure are provided in Section 6, and our concluding remarks are in Section 7.

Throughout, the n-dimensional real vector and $m \times n$ real matrix spaces are denoted by R^n and $R^{m\times n}$, respectively. The notation $P > 0$ ($P < 0$) means that P is symmetric positive definite (symmetric negative definite), the superscript T denotes the vector or matrix transpose, $det(X)$ is the determinant of a square matrix X, X^{-1} the inverse of a square invertible matrix X, and $diag(M_1, M_2, \cdots, M_q)$ a block diagonal matrix with diagonal block matrices $M_1, M_2, \cdots, M_q$. The notation $deg(f)$ represents the degree of a polynomial f, $D_{int}(a, r)$ is the interior of the region in the complex plane with centre $(a, 0)$ and radius r, $\lambda(A, B) = \{z \mid \det(zA - B) = 0\}$, C^- denotes the left open complex plane, and I_r is the identity matrix with dimension $r \times r$. If h denotes a sampling period, the delta operator is defined by

$$\delta x(t) = \begin{cases} \dot{x}(t)\,, & h = 0\,, \\ \dfrac{x(t+h) - x(t)}{h}\,, & h \neq 0\,. \end{cases}$$

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2. Preliminaries and Problem Formulation

A singular continuous system may be described by

$$\begin{aligned} E\dot{x}(t) &= A_s x(t) + B_s u(t) \,, \\ y(t) &= Cx(t) \,, \end{aligned} \tag{2.1}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ the control input, $y(t) \in R^q$ the measure output, and $E, A_s \in R^{n\times n}$, $B_s \in R^{n\times m}$ and $C \in R^{q\times n}$ are known constant matrices where $0 < rank(E) = r < n$. Then for a prescribed sampling period $h > 0$, one can set up the delta operator model for the system (2.1) as follows — cf. [22] or [23]:

$$\begin{aligned} E\delta x(t_k) &= Ax(t_k) + Bu(t_k) \,, \\ y(t_k) &= Cx(t_k) \,, \end{aligned} \tag{2.2}$$

where $x(t_k) \in R^n$ is the state, $u(t_k) \in R^m$ the control input, $y(t_k) \in R^q$ the measure output, $t_k = kh$ the time intervals, E and C are as in the system (2.1), and A and B are known constant matrices depending on E, A_s, and B_s respectively, as well as the sampling period h.

We assume throughout that the full state $x(t_k)$ of the system (2.2) is unavailable, so a state feedback controller cannot be used. Our purpose is to design an output feedback controller for the system (2.2) such that the resulting closed-loop system is admissible, as is crucial for singular systems. To do this, we first estimate the state $x(t_k)$ of the system (2.2), and then consider the design of a suitable admissible controller based on the error dynamics.

Corresponding to (2.1), we consider the unforced system

$$E\dot{x}(t) = A_s x(t) \,. \tag{2.3}$$

Definition 2.1 (cf. Ref. [5]). The system (2.3) is said to be regular if $det(sE–A_s)$ is not identically zero; impulse-free if $\deg(\det(sE-A_s)) = rank(E)$; stable if $\lambda(E, A_s) \subset C^-$; and admissible if it is regular, impulse-free and stable.

Lemma 2.1 (cf. Ref. [5]). *The system* (2.3) *is admissible if and only if there exist matrices* $Z > 0$ *and* Y *satisfying*

$$A_s(ZE^T + LY) + (ZE^T + LY)^T A_s^T < 0 \,, \tag{2.4}$$

where L *is any matrix of full column rank that satisfies* $EL = 0$.

Lemma 2.2 (cf. Ref. [5]). *The system* (2.3) *is admissible if and only if there exist matrices* $X > 0$ *and* Q *satisfying*

$$A_s^T(XE + SQ^T) + (XE + SQ^T)^T A_s < 0 \,, \tag{2.5}$$

where S *is any matrix of full column rank that satisfies* $E^T S = 0$.

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Corresponding to (2.2), we consider the unforced system

$$E\delta x(t_k) = Ax(t_k)\ . \tag{2.6}$$

Definition 2.2 (cf. Ref. [22])**.** The system (2.6) is said to be regular if $det(\eta E - A)$ is not identically zero; causal if $\deg(\det(\eta E - A)) = rank(E)$; stable if $\lambda(E, A) \subset D_{int}(-h^{-1}, h^{-1})$; and admissible if it is regular, causal and stable.

Remark 2.1. When the sampling period h tends to zero, the domain $D_{int}(-h^{-1}, h^{-1})$ tends to C^- and the matrix A tends to A_s [22, 23], such that from Definitions 2.1 and 2.2 the admissibility definition for the SDOS (2.6) is consistent with the definition for the continuous system (2.3). .

Lemma 2.3 (cf. Ref. [22])**.** *The system* (2.6) *is admissible if and only if there exist matrices* $X > 0$ *and* Q *satisfying*

$$A^T(XE + SQ^T) + (XE + SQ^T)^T A + hA^T XA < 0\ , \tag{2.7}$$

where S *is any matrix of full column rank that satisfies* $E^T S = 0$.

Remark 2.2. When the sampling period h tends to zero, the inequality (2.7) tends to the inequality (2.5) because the matrix A tends to A_s. Thus Lemma 2.3 provides a unified admissibility condition for both singular continuous and singular discrete systems.

3. Novel Admissibility Conditions

In this section, we present some admissibility conditions that will be used later. Since $0 < rank(E) = r < n$, the matrix E permits a decomposition such that

$$MEN = diag(I_r, 0) = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\ , \tag{3.1}$$

where the matrices M and N are invertible. Denoting

$$U_1 = M^{-1} \begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix}\ , \tag{3.2}$$

we have the following result.

Theorem 3.1. *The system* (2.6) *is admissible if and only if there exist matrices* $Z > 0$, $V_1 > 0$ *and* Y *satisfying*

$$\begin{bmatrix} AP_1 + P_1^T A^T & P_1^T A^T \\ AP_1 & -h^{-1}(EZE^T + U_1 V_1 U_1^T) \end{bmatrix} < 0\ , \tag{3.3}$$

where $P_1 = ZE^T + LY$, L *is any matrix with a full column rank that satisfies* $EL = 0$.

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Proof. (Sufficiency) If we assume that the inequality (3.3) holds, then it follows from (3.3) that

$$AP_1 + P_1^T A^T < 0.$$

This inequality means that the matrices P_1 and A are both invertible, which can be proved as follows. If the matrix P_1 were not invertible, then a nonzero vector ξ would exist such that $P_1\xi = 0$ and hence $\xi^T(AP_1 + P_1^T A^T)\xi = 0$, which obviously contradicts $AP_1 + P_1^T A^T < 0$. Thus the matrix P_1 is invertible, and similarly the matrix A is also invertible. Now let

$$L = N \begin{bmatrix} 0 \\ \Upsilon \end{bmatrix}, \qquad Y = \Upsilon^{-1} \begin{bmatrix} Y_1 & Y_2 \end{bmatrix} M^{-T}, \qquad Z = N \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^T & Z_{22} \end{bmatrix} N^T, \quad (3.4)$$

where $\Upsilon \in R^{(n-r)\times(n-r)}$ is any invertible matrix and the partitions are compatible with the structure in Eqs. (3.1) and (3.2). Then from the invertibility of N, Υ and in Eqs. (3.4), we have that the matrix L is of full column rank and also

$$EL = M^{-1} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} N^{-1} N \begin{bmatrix} 0 \\ \Upsilon \end{bmatrix} = 0 .$$

From Eqs. (3.1) and (3.4), the matrix $P_1 = ZE^T + LY$ can be computed as

$$\begin{aligned} ZE^T + LY &= N \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^T & Z_{22} \end{bmatrix} N^T N^{-T} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} M^{-T} + N \begin{bmatrix} 0 \\ \Upsilon \end{bmatrix} \Upsilon^{-1} \begin{bmatrix} Y_1 & Y_2 \end{bmatrix} M^{-T} \\ &= N \begin{bmatrix} Z_{11} & 0 \\ Z_{12}^T + Y_1 & Y_2 \end{bmatrix} M^{-T} . \end{aligned}$$

Since the matrices P_1, M and N are all invertible, the matrix Y_2 must also be invertible from the above equality. Now let

$$S = M^T \begin{bmatrix} 0 \\ \Gamma \end{bmatrix}, \; X = M^T \begin{bmatrix} Z_{11}^{-1} & 0 \\ 0 & V_1^{-1} \end{bmatrix} M , \; Q = N^{-T} \begin{bmatrix} -Z_{11}^{-1}(Z_{12} + Y_1^T)Y_2^{-T} \\ Y_2^{-T} \end{bmatrix} \Gamma^{-T}, \quad (3.5)$$

where $\Gamma \in R^{(n-r)\times(n-r)}$ is any invertible matrix. From $V_1 > 0$ we have $V_1^{-1} > 0$, and from $Z > 0$ and the invertibility of N we have $Z_{11} > 0$ so that $Z_{11}^{-1} > 0$. Then we can obtain $X > 0$ from $Z_{11}^{-1} > 0$, $V_1^{-1} > 0$ and the invertibility of M. Moreover, from the invertibility of M and Γ we know that the matrix S is of full column rank and also satisfies

$$E^T S = N^{-T} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} M^{-T} M^T \begin{bmatrix} 0 \\ \Gamma \end{bmatrix} = 0 .$$

Thus from Eqs. (3.1) and (3.5) it follows that

$$\begin{aligned} XE + SQ^T =& M^T \begin{bmatrix} Z_{11}^{-1} & 0 \\ 0 & V_1^{-1} \end{bmatrix} M M^{-1} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} N^{-1} \\ &+ M^T \begin{bmatrix} 0 \\ \Gamma \end{bmatrix} \Gamma^{-1} \begin{bmatrix} -Y_2^{-1}(Z_{12}^T + Y_1)Z_{11}^{-1} & Y_2^{-1} \end{bmatrix} N^{-1} \\ =& M^T \begin{bmatrix} Z_{11}^{-1} & 0 \\ -Y_2^{-1}(Z_{12}^T + Y_1)Z_{11}^{-1} & Y_2^{-1} \end{bmatrix} N^{-1} , \end{aligned}$$

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whence

$$\begin{aligned}&(ZE^T+LY)(XE+SQ^T)\\ =&N\begin{bmatrix} Z_{11} & 0\\ Z_{12}^T+Y_1 & Y_2\end{bmatrix}M^{-T}M^T\begin{bmatrix} Z_{11}^{-1} & 0\\ -Y_2^{-1}(Z_{12}^T+Y_1)Z_{11}^{-1} & Y_2^{-1}\end{bmatrix}N^{-1}=I_n\end{aligned}$$

such that

$$(ZE^T+LY)^{-1}=XE+SQ^T\,. \tag{3.6}$$

From Eqs. (3.1), (3.2) and (3.4) we have

$$\begin{aligned}EZE^T+U_1V_1U_1^T=&M^{-1}\begin{bmatrix} I_r & 0\\ 0 & 0\end{bmatrix}N^{-1}N\begin{bmatrix} Z_{11} & Z_{12}\\ Z_{12}^T & Z_{22}\end{bmatrix}N^TN^{-T}\begin{bmatrix} I_r & 0\\ 0 & 0\end{bmatrix}M^{-T}\\ &+M^{-1}\begin{bmatrix} 0\\ I_{n-r}\end{bmatrix}V_1\begin{bmatrix} 0 & I_{n-r}\end{bmatrix}M^{-T}\\ =&M^{-1}\begin{bmatrix} Z_{11} & 0\\ 0 & V_1\end{bmatrix}M^{-T}\,,\end{aligned}$$

such that for the matrix X in Eq. (3.5) we obtain

$$X^{-1}=EZE^T+U_1V_1U_1^T\,. \tag{3.7}$$

From Eqs. (3.6), (3.7) and pre-multiplying and post-multiplying the inequality (3.3) with $diag\ (E^TX+QS^T,I_n)$ and $diag(XE+SQ^T,I_n)$ respectively, we then have that Eq. (3.3) is equivalent to the following inequality:

$$\begin{bmatrix}(E^TX+QS^T)A+A^T(XE+SQ^T) & A^T\\ A & -h^{-1}X^{-1}\end{bmatrix}<0\,. \tag{3.8}$$

From the Schur complement, the inequality (3.8) is equivalent to $X>0$ and

$$(E^TX+QS^T)A+A^T(XE+SQ^T)+hA^TXA<0\,,$$

so by Lemma 2.3 we have that the system (2.6) is admissible.

(Necessity) Assume that the system (2.6) is admissible, and let

$$MAN=\begin{bmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix} \tag{3.9}$$

where the partition is compatible with the structure in (3.1). Then the matrix A_{22} is invertible so that the system (2.6) is regular and causal, and $\lambda(I_r,\hat{A})\subset D_{int}(-h^{-1},h^{-1})$ so it is stable, where $\hat{A}=A_{11}-A_{12}A_{22}^{-1}A_{21}$. From Ref. [16], $\lambda(I_r,\hat{A})\subset D_{int}(-h^{-1},h^{-1})$ if and only if there exists a matrix $X_1>0$ such that

$$\hat{A}^TX_1+X_1\hat{A}+h\hat{A}^TX_1\hat{A}<0\,. \tag{3.10}$$

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Thus a sufficiently small scalar $\alpha > 0$ can always be found such that

$$\hat{A}^T X_1 + X_1 \hat{A} + h\hat{A}^T X_1 \hat{A} + \alpha h A_{21}^T A_{22}^{-T} A_{22}^{-1} A_{21} < 0 \,. \tag{3.11}$$

Since $\hat{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}$, the inequality (3.11) is also the same as

$$\begin{aligned} J =& A_{11}^T X_1 + X_1 A_{11} + h A_{11}^T X_1 A_{11} - A_{21}^T A_{22}^{-T} A_{12}^T X_1 - X_1 A_{12} A_{22}^{-1} A_{21} \\ & - h A_{11}^T X_1 A_{12} A_{22}^{-1} A_{21} - h A_{21}^T A_{22}^{-T} A_{12}^T X_1 A_{11} + h A_{21}^T A_{22}^{-T} \Omega A_{22}^{-1} A_{21} < 0 \end{aligned}$$

where $\Omega = \alpha I_{n-r} + A_{12}^T X_1 A_{12}$, so from $\alpha > 0$ and $X_1 > 0$ we obtain $\Omega > 0$ such that $\Omega^{-1} > 0$. Let

$$S = M^T \begin{bmatrix} 0 \\ \Gamma \end{bmatrix}, \qquad X = M^T \begin{bmatrix} X_1 & 0 \\ 0 & V_1^{-1} \end{bmatrix} M \,, \tag{3.12}$$

$$Q = N^{-T} \begin{bmatrix} -X_1 A_{12} A_{22}^{-1} - h A_{11}^T X_1 A_{12} A_{22}^{-1} \\ -h A_{22}^T V_1^{-1} \end{bmatrix} \Gamma^{-T} \,, \tag{3.13}$$

where $\Gamma \in R^{(n-r)\times(n-r)}$ is any invertible matrix and $V_1 = A_{22}\Omega^{-1}A_{22}^T$. Then from $\Omega^{-1} > 0$ and the invertibility of A_{22} we have $V_1 > 0$ such that $V_1^{-1} > 0$, so that $X > 0$ from $X_1 > 0$, $V_1^{-1} > 0$ and the invertibility of M. Further, from the invertibility of M and Γ the matrix S is of full column rank, and it can therefore be shown that S also satisfies $E^T S = 0$. From Eqs. (3.1), (3.12) and (3.13) we have

$$\begin{aligned} llXE + SQ^T =& M^T \begin{bmatrix} X_1 & 0 \\ 0 & V_1^{-1} \end{bmatrix} M M^{-1} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} N^{-1} \\ & + M^T \begin{bmatrix} 0 \\ \Gamma \end{bmatrix} \Gamma^{-1} \begin{bmatrix} -A_{22}^{-T} A_{12}^T X_1 - h A_{22}^{-T} A_{12}^T X_1 A_{11} & -h V_1^{-1} A_{22} \end{bmatrix} N^{-1} \\ =& M^T \begin{bmatrix} X_1 & 0 \\ -A_{22}^{-T} A_{12}^T X_1 - h A_{22}^{-T} A_{12}^T X_1 A_{11} & -h V_1^{-1} A_{22} \end{bmatrix} N^{-1} \,, \end{aligned}$$

hence from Eq. (3.9) we obtain

$$\begin{aligned} & A^T (XE + SQ^T) \\ =& N^{-T} \begin{bmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{bmatrix} \begin{bmatrix} X_1 & 0 \\ -A_{22}^{-T} A_{12}^T X_1 - h A_{22}^{-T} A_{12}^T X_1 A_{11} & -h V_1^{-1} A_{22} \end{bmatrix} N^{-1} \\ =& N^{-T} \begin{bmatrix} A_{11}^T X_1 - A_{21}^T A_{22}^{-T} A_{12}^T X_1 - h A_{21}^T A_{22}^{-T} A_{12}^T X_1 A_{11} & -h A_{21}^T V_1^{-1} A_{22} \\ -h A_{12}^T X_1 A_{11} & -h A_{22}^T V_1^{-1} A_{22} \end{bmatrix} N^{-1} \,. \end{aligned}$$

From Eqs. (3.9) and (3.12) we obtain

$$\begin{aligned} h A^T X A &= h N^{-T} \begin{bmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{bmatrix} M^{-T} M^T \begin{bmatrix} X_1 & 0 \\ 0 & V_1^{-1} \end{bmatrix} M M^{-1} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} N^{-1} \\ &= h N^{-T} \begin{bmatrix} A_{11}^T X_1 A_{11} + A_{21}^T V_1^{-1} A_{21} & A_{11}^T X_1 A_{12} + A_{21}^T V_1^{-1} A_{22} \\ A_{12}^T X_1 A_{11} + A_{22}^T V_1^{-1} A_{21} & A_{12}^T X_1 A_{12} + A_{22}^T V_1^{-1} A_{22} \end{bmatrix} N^{-1} \,. \end{aligned}$$

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Thus from $\alpha > 0$, $J < 0$, the invertibility of N the expressions for Ω, V_1 and the above equalities we have

$$A^T(XE+SQ^T)+(E^TX+QS^T)A+hA^TXA = N^{-T}\begin{bmatrix} J & 0 \\ 0 & -\alpha h I_{n-r} \end{bmatrix} N^{-1} < 0 \quad (3.14)$$

such that the matrices $X > 0$ and Q in (3.12) and (3.13) satisfy the inequality (3.8). Letting

$$L = N\begin{bmatrix} 0 \\ \Upsilon \end{bmatrix}, \qquad Z = N\begin{bmatrix} X_1^{-1} & 0 \\ 0 & I_{n-r} \end{bmatrix} N^T, \quad (3.15)$$

$$Y = \Upsilon^{-1}\begin{bmatrix} -\Omega^{-1}A_{12}^T X_1 A_{11} X_1^{-1} - h^{-1}\Omega^{-1}A_{12}^T & -h^{-1}A_{22}^{-1}V_1 \end{bmatrix} M^{-T} \quad (3.16)$$

where $\Upsilon \in R^{(n-r)\times(n-r)}$ is any invertible matrix, from $X_1 > 0$ such that $X_1^{-1} > 0$ and the invertibility of N we have $Z > 0$. Then from the invertibility of N and Υ the matrix L is of full column rank, and thus L also satisfies $EL = 0$. From Eq. (3.1), (3.15), (3.16) and $V_1 = A_{22}\Omega^{-1}A_{22}^T$ we have

$$\begin{aligned} ZE^T + LY =& N\begin{bmatrix} X_1^{-1} & 0 \\ 0 & I_{n-r} \end{bmatrix} N^T N^{-T}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} M^{-T} \\ &+ N\begin{bmatrix} 0 \\ \Upsilon \end{bmatrix}\Upsilon^{-1}\begin{bmatrix} -\Omega^{-1}A_{12}^T X_1 A_{11} X_1^{-1} - h^{-1}\Omega^{-1}A_{12}^T & -h^{-1}A_{22}^{-1}V_1 \end{bmatrix} M^{-T} \\ =& N\begin{bmatrix} X_1^{-1} & 0 \\ -A_{22}^{-1}V_1 A_{22}^{-T}A_{12}^T X_1 A_{11} X_1^{-1} - h^{-1}A_{22}^{-1}V_1 A_{22}^{-T}A_{12}^T & -h^{-1}A_{22}^{-1}V_1 \end{bmatrix} M^{-T} \end{aligned}$$

such that

$$(XE + SQ^T)(ZE^T + LY) = I_n\,,$$

so

$$(XE + SQ^T)^{-1} = ZE^T + LY\,. \quad (3.17)$$

From Eqs. (3.1), (3.2) and (3.15) we also have

$$\begin{aligned} llEZE^T + U_1V_1U_1^T =& M^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} N^{-1}N\begin{bmatrix} X_1^{-1} & 0 \\ 0 & I_{n-r} \end{bmatrix} N^T N^{-T}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} M^{-T} \\ &+ M^{-1}\begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix} V_1 \begin{bmatrix} 0 & I_{n-r} \end{bmatrix} M^{-T} \\ =& M^{-1}\begin{bmatrix} X_1^{-1} & 0 \\ 0 & V_1 \end{bmatrix} M^{-T}\,, \end{aligned}$$

so comparing this equality with the matrix X in (3.12) we obtain

$$X^{-1} = EZE^T + U_1V_1U_1^T\,. \quad (3.18)$$

From Eqs. (3.17), (3.18) and pre-multiplying and post-multiplying the inequality (3.8) with $diag\ (EZ + Y^TL^T, I_n)$ and $diag(ZE^T + LY, I_n)$ respectively, we then obtain that (3.8) is equivalent to the inequality (3.3). This completes the proof. □

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From $det(\eta E - A) = det(\eta E^T - A^T)$ and Definition 2.2, the admissibility of the system (2.6) is equivalent to the admissability of the system $E^T\delta z(t_k) = A^T z(t_k)$. From $MEN = diag(I_r, 0)$ we have $N^T E^T M^T = diag(I_r, 0)$, so denoting

$$U_2 = N^{-T} \begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix} \tag{3.19}$$

we obtain the following result from Theorem 3.1.

Theorem 3.2. *The system* (2.6) *is admissible if and only if there exist matrices* $X > 0$, $V_2 > 0$ *and* Q *satisfying*

$$\begin{bmatrix} A^T P_2 + P_2^T A & P_2^T A \\ A^T P_2 & -h^{-1}(E^T X E + U_2 V_2 U_2^T) \end{bmatrix} < 0 \,, \tag{3.20}$$

where $P_2 = XE + SQ^T$, S *is any matrix of full column rank and satisfies* $E^T S = 0$.

Remark 3.1. From $h > 0$ and the Schur complement, the inequality (3.3) is equivalent to $EZE^T + U_1 V_1 U_1^T > 0$ and we have

$$AP_1 + P_1^T A^T + hP_1^T A^T (EZE^T + U_1 V_1 U_1^T)^{-1} A P_1 < 0 \,. \tag{3.21}$$

When the sampling period h tends to zero, inequality (3.21) tends to the inequality (2.4), and similarly the inequality (3.20) tends to (2.5). Thus Theorems 3.1 and 3.2 provide unified admissibility conditions for both singular continuous systems and singular discrete systems.

Remark 3.2. The necessary and sufficient admissibility conditions for SDOSs in Ref. [22] have some drawbacks — e.g. only some sufficient LMI-based conditions could be obtained for the existence of a suitable state feedback admissible controller [23, 25]. In comparison, the novel and equivalent admissibility conditions presented in Theorems 3.1 and 3.2 are very effective for further observer design and admissible control, as we show in the following sections.

4. Error Dynamics

In this section, our aim is to design an observer of the form

$$E\delta\hat{x}(t_k) = A\hat{x}(t_k) + G\big(C\hat{x}(t_k) - y(t_k)\big) + Bu(t_k) \tag{4.1}$$

for the system (2.2), where $\hat{x}(t_k)$ is the state of the observer (4.1) and G is a gain matrix to be determined such that $\hat{x}(t_k)$ converges asymptotically to $x(t_k)$. In this case, the observer (4.1) is called an *asymptotic* observer for the system (2.2), and the corresponding existence condition is given in the following theorem.

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Theorem 4.1. *There exists an asymptotic and physically realisable observer* (4.1) *for the system* (2.2) *if and only if there exist matrices* $X > 0, V_2 > 0, Q$ *and* Ψ *satisfying*

$$\begin{bmatrix} A^T P_2 + C^T \Psi^T + P_2^T A + \Psi C & P_2^T A + \Psi C \\ A^T P_2 + C^T \Psi^T & -h^{-1}(E^T X E + U_2 V_2 U_2^T) \end{bmatrix} < 0 \tag{4.2}$$

where $P_2 = XE + SQ^T$ *and* S *is any matrix of full column rank that satisfies* $E^T S = 0$. *Then with matrix* P_2 *invertible, the gain matrix* G *of the observer* (4.1) *can be designed as* $G = P_2^{-T}\Psi$. *(If* P_2 *were not invertible, we could still proceed by choosing* $P_2 = XE + SQ^T + \epsilon P$, *where* $\epsilon > 0$ *is a scalar variable and* P *is an invertible matrix satisfying* $E^T P = P^T E \geq 0$.*)*

Proof. Defining the estimation error $e(t_k) = x(t_k) - \hat{x}(t_k)$, from the systems (2.2) and (4.1) we have the error dynamic equation

$$E\delta e(t_k) = (A + GC)e(t_k) \, . \tag{4.3}$$

To ensure the existence and uniqueness of a solution to the system (4.3), we require the system (4.3) to be regular; and to ensure that the error $e(t_k)$ tends to zero asymptotically, we need the system (4.3) to be asymptotically stable. Both requirements depend on the appropriate choice of the gain matrix G. To determine a desirable G, we write (4.1) as

$$E\delta\hat{x}(t_k) = (A + GC)\hat{x}(t_k) - Ly(t_k) + Bu(t_k) \, , \tag{4.4}$$

where $y(t_k)$ and $u(t_k)$ can be viewed as the inputs of the system. To guarantee that the observer (4.1) is physically realisable, the system (4.4) must be causal, which is equivalent to requiring that the system (4.3) is causal. Thus according to our discussion above, the observer (4.1) is an asymptotic and physically realisable observer if and only if the error system (4.3) is admissible. Then from Theorem 3.2, the system (4.3) is admissible if and only if there exist matrices $X > 0, V_2 > 0$ and Q satisfying

$$\begin{bmatrix} (A + GC)^T P_2 + P_2^T (A + GC) & P_2^T (A + GC) \\ (A + GC)^T P_2 & -h^{-1}(E^T X E + U_2 V_2 U_2^T) \end{bmatrix} < 0 \, , \tag{4.5}$$

where $P_2 = XE + SQ^T$ and S is any matrix of full column rank that satisfies $E^T S = 0$. When (4.5) holds, one has $(A + GC)^T P_2 + P_2^T (A + GC) < 0$ such that the matrix P_2 is invertible. Then denoting $\Psi = P_2^T G$ we obtain the inequality (4.2) and hence $G = P_2^{-T}\Psi$, to complete the proof. □

Remark 4.1. On the existence of an asymptotic and physically realisable observer (4.1) for system (2.2), Theorem 4.1 provides a necessary and sufficient LMI-based condition that contains just four matrix variables, whereas Ref. [24] gives two sufficient LMI-based conditions — one involving five matrix variables and three other pre-determined parameter matrices, and the second four matrix variables and two other pre-determined parameter matrices. Thus Theorem 4.1 provides a much less conservative condition than the two in Ref. [24].

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Remark 4.2. If we consider the special case where $E = I_n$, then (2.2) becomes a state-space delta operator system, then (4.1) becomes

$$\delta\hat{x}(t_k) = A\hat{x}(t_k) + Bu(t_k) + G\big(y(t_k) - C\hat{x}(t_k)\big) \ ,$$

known as the Luenberger observer [16].

5. Admissible Control

Based on the above proposed observer (4.1), let the controller be of the form

$$u(t_k) = K\hat{x}(t_k) = Kx(t_k) - Ke(t_k) \ , \tag{5.1}$$

where $K \in R^{m\times n}$ is the controller gain matrix to be determined so that the closed-loop system is admissible.

The closed-loop system of the system (2.2) under the observer (4.1) and controller (5.1) can be written as

$$\begin{aligned} E\delta x(t_k) &= (A + BK)x(t_k) - BKe(t_k) \ , \\ E\delta e(t_k) &= (A + GC)e(t_k) \ , \end{aligned} \tag{5.2}$$

or in the form of an extended SDOS

$$\hat{E}\delta z(t_k) = \hat{A}z(t_k) \tag{5.3}$$

where

$$z(t_k) = \begin{bmatrix} x(t_k) \\ e(t_k) \end{bmatrix}, \qquad \hat{E} = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}, \qquad \hat{A} = \begin{bmatrix} A + BK & -BK \\ 0 & A + GC \end{bmatrix} . \tag{5.4}$$

From (5.4) we obtain $det(\eta\hat{E} - \hat{A}) = det(\eta E - (A + BK))det(\eta E - (A + GC))$. Since the matrix G has been designed such that the error system (4.3) is admissible, from Definition 2.2 the system (5.3) is admissible if and only if the system

$$E\delta x(t_k) = (A + BK)x(t_k) \tag{5.5}$$

is admissible.

Consequently, in order to design the gain matrix K, we can use a similar approach to construct a state feedback admissible controller

$$u(t_k) = Kx(t_k) \tag{5.6}$$

for the system (2.2), for which we present the following admissible control result.

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Theorem 5.1. *Based on the observer* (4.1)*, there exists an output feedback controller* (5.1) *for the system* (2.2) *such that the closed-loop system* (5.2) *is admissible if and only if there exist matrices* $Z > 0$*,* $V_1 > 0$*,* Φ *and* Y *satisfying*

$$\begin{bmatrix} AP_1 + B\Phi + P_1^T A^T + \Phi^T B^T & P_1^T A^T + \Phi^T B^T \\ AP_1 + B\Phi & -h^{-1}(EZE^T + U_1 V_1 U_1^T) \end{bmatrix} < 0 \,, \tag{5.7}$$

where $P_1 = ZE^T + LY$*,* L *is any matrix of full column rank that satisfies* $EL = 0$*. Then with matrix* P_1 *invertible, the gain matrix* K *can be designed as* $K = \Phi P_1^{-1}$*. (If* P_1 *were not invertible, we could still proceed by choosing* $P_1 = ZE^T + LY + \epsilon P$*, where* $\epsilon > 0$ *is a scalar variable and* P *is an invertible matrix satisfying* $EP = P^T E^T \geq 0$*.)*

Proof. From Theorem 3.1, the system (5.5) is admissible such that the system (5.2) is admissible if and only if there exist matrices $Z > 0, V_1 > 0$ and Y satisfying

$$\begin{bmatrix} (A + BK)P_1 + P_1^T (A + BK)^T & P_1^T (A + BK)^T \\ (A + BK)P_1 & -h^{-1}(EZE^T + U_1 V_1 U_1^T) \end{bmatrix} < 0 \,, \tag{5.8}$$

where $P_1 = ZE^T + LY$ and L is any matrix of full column rank that satisfies $EL = 0$. From (5.8), we have $(A + BK)P_1 + P_1^T (A + BK)^T < 0$, indicating that the matrix P_1 is invertible. Denoting $\Phi = KP_1$, we have that the inequality (5.8) becomes (5.7) and obtain the matrix $K = \Phi P_1^{-1}$, to complete the proof. □

Remark 5.1. On the existence of a suitable output feedback admissible controller (5.1) for the system (2.2), Theorem 5.1 presents a necessary and sufficient LMI-based condition that contains four matrix variables, whereas Ref. [24] gives two sufficient LMI-based conditions — one involving five matrix variables and three other pre-determined parameter matrices and the second four matrix variables and two other pre-determined parameter matrices). Thus Theorem 5.1 provides a much less conservative condition than the two in Ref. [24].

From the above discussion, we can readily get the following corollary.

Corollary 5.1. *There exists a state feedback controller* (5.6) *for the system* (2.2) *such that the closed-loop system* (5.5) *is admissible if and only if there exist matrices* $Z > 0$*,* $V_1 > 0, \Phi$ *and* Y *satisfying the inequality* (5.7)*, where* $P_1 = ZE^T + LY$ *and* L *is any matrix of full column rank that satisfies* $EL = 0$*. In this Then with the matrix* P_1 *invertible the gain matrix* K *can be designed as* $K = \Phi P_1^{-1}$*. (If* P_1 *were not invertible, we could still proceed by choosing* $P_1 = ZE^T + LY + \epsilon P$*, where* $\epsilon > 0$ *is a scalar variable and* P *is an invertible matrix satisfying* $EP = P^T E^T \geq 0$*).)*

Remark 5.2. On the existence of a state feedback admissible controller (5.6) for the system (2.2), Corollary 5.1 gives a necessary and sufficient LMI-based condition with four matrix variables, whereas Ref. [23] presented two sufficient LMI-based conditions through limiting the structure of some matrix variables, with one condition involving five matrix variables and three other pre-determined parameter matrices and the second four matrix variables and two other pre-determined parameter matrices. Thus the result on state feedback admissible control for SDOSs in Corollary 5.1 is much less conservative.

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6. Illustrative Numerical Examples

We now present some numerical examples to demonstrate our theoretical results.

Example 6.1. Consider a singular discrete system $Ex(t_{k+1}) = A_z x(t_k) + B_z u(t_k)$ with the following coefficient matrices:

$$E = \begin{bmatrix} 1 & 3 & 0 \\ 2 & -1 & 0 \\ 1 & 5 & 0 \end{bmatrix}, \qquad A_z = \begin{bmatrix} 4.6 & -2.8 & 0 \\ 2.2 & 2.8 & 0 \\ 6.6 & -5.2 & 0 \end{bmatrix}, \qquad B_z = \begin{bmatrix} 1.9 & -4 & -1.2 \\ 1.7 & 1 & -1.4 \\ 2.5 & -2 & 0.8 \end{bmatrix}.$$

It was shown in Ref. [5] that the above system is not admissible (it is irregular, unstable and non-causal), and while a state feedback admissible controller can be designed through matrix inequalities it is challenging to solve matrix inequalities. As follows, Corollary 5.1 appears to be more suitable to solve the corresponding admissible control problem effectively and efficiently.

First we choose the invertible matrices

$$M = \begin{bmatrix} -0.7718 & -1.4069 & 0 \\ -1.2520 & -1.0303 & 0 \\ -1.5714 & 0.2857 & 1 \end{bmatrix}, \qquad N = \begin{bmatrix} -0.1523 & -0.2080 & 0 \\ 0.1851 & -0.1141 & 0 \\ -0.2923 & -0.2943 & 1 \end{bmatrix}$$

such that $MEN = diag(I_2, 0)$, and then obtain the matrices

$$U_1 = L = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad P = \begin{bmatrix} -3.4683 & 1.1236 & 0 \\ -1.3348 & 4.4682 & 0 \\ -0.6332 & 0.8647 & 1 \end{bmatrix}$$

where P is an invertible matrix that satisfies $EP = P^T E^T \geq 0$. On adopting $h = 0.1$, $A = h^{-1}(A_z - E)$ and $B = h^{-1}B_z$ the above system can be written as the SDOS (2.2). One can verify that there is no feasible solution for the inequality (3.3), so from Theorem 3.1 the system (2.2) is not admissible. (This is of course true since the system (2.2) is irregular, non-causal and unstable with $det(\eta E - A) = 0$.)

Leting $P_1 = ZE^T + LY + \epsilon P$ and solving the inequality (5.7), we obtain a feasible solution

$$Z = \begin{bmatrix} 0.0476 & -0.0083 & 0 \\ -0.0083 & 0.0075 & 0 \\ 0 & 0 & 1.5706 \end{bmatrix}, \qquad Y = 0\,, \qquad V_1 = 0.3018\,,$$

$$\Phi = \begin{bmatrix} 4.2960 & 8.5833 & 0.0214 \\ 0.4504 & -10.9499 & -0.0472 \\ -0.8836 & 21.8968 & 0.0720 \end{bmatrix}, \quad \epsilon = 1.5706\,.$$

Thus from Corollary 5.1 there is a state feedback admissible controller (5.6) for the system (2.2) and the gain matrix K of the controller (5.6) can be designed as

$$K = \begin{bmatrix} -1.4060 & 1.6050 & -0.0108 \\ 0.5771 & -1.7198 & -0.0002 \\ -1.1558 & 3.4424 & -0.0139 \end{bmatrix}.$$

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As $deg(det(zE - A_z - B_zK)) = 2 = rank(E)$ and $\lambda(E, A_z + B_zK) = \{0.9928 + 0.0023i, 0.9928 - 0.0023i\}$ in $D_{int}(0, 1)$, it can be checked that the closed-loop singular discrete system is indeed admissible

Example 6.2. Consider the SDOS (2.2) with the following coefficient matrices:

$$E = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, \qquad A = \begin{bmatrix} -1.9287 & -0.7503 \\ 2.0713 & 4.2497 \end{bmatrix}, \qquad B = \begin{bmatrix} 1.2497 \\ 3.2497 \end{bmatrix}, \qquad h = 0.2\,.$$

It was shown in Ref. [23] that the above system is not admissible, and to design a state feedback admissible controller it was proposed that some LMI-based sufficient conditions that contain more matrix variables and additional matrices be determined in advance to ensure the feasibility of the LMIs. Corollary 5.1 presents a necessary and sufficient LMI-based condition that involves less matrix variables, and does not need any matrices to be given in advance.

We first select the invertible matrices

$$M = \begin{bmatrix} -0.4714 & -0.9428 \\ -0.7454 & 0.7454 \end{bmatrix}, \qquad N = \begin{bmatrix} -0.2357 & -0.8944 \\ -0.2357 & 0.4472 \end{bmatrix}$$

such that $MEN = diag(1, 0)$, and then obtain the matrices

$$U_1 = \begin{bmatrix} -0.8944 \\ 0.4472 \end{bmatrix}, \qquad L = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

One can show that there is no feasible solution for the inequality (3.3), so from Theorem 3.1 the system (2.2) is not admissible. (This is true because the system (2.2) is unstable, since $\lambda(E, A) = \{0.5307\}$ is not in $D_{int}(-5, 5)$.) The inequality (5.7) has a feasible solution

$$Z = \begin{bmatrix} 0.8398 & -0.4070 \\ -0.4070 & 0.2292 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0.0801 & -0.1064 \end{bmatrix},$$
$$\Phi = \begin{bmatrix} -0.0749 & -0.2055 \end{bmatrix}, \qquad V_1 = 0.3500\,,$$

so from Corollary 5.1 there is a state feedback admissible controller (5.6) for the system (2.2) and the gain matrix K can be designed as

$$K = \begin{bmatrix} -0.7398 & -2.1802 \end{bmatrix}.$$

Let $x(0) = \begin{bmatrix} 1 & -3.9406 \end{bmatrix}^T$.

Fig. 1 displays the time response of the state $x(t_k)$ of the system (2.2) under the state feedback controller (5.6), showing that the closed-loop system is indeed admissible.

Example 6.3. Consider the following 3-DOF (degree-of-freedom) helicopter produced by the Canada Quanser Company. The three degrees of freedom are the elevation ϵ,

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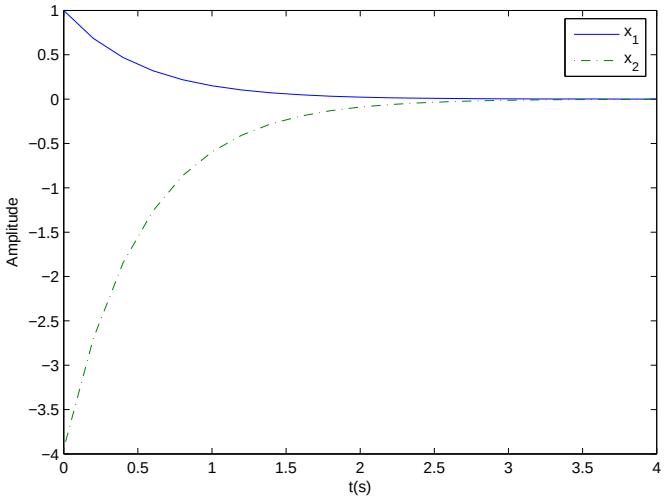


Figure 1: The time response of the state $x(t_k)$ of the system (2.2) under the controller (5.6).

pitch angle θ and travel angle velocity ω, and the linearised dynamic model of the helicopter can be represented as follows [6]:

$$\begin{aligned} E_1\dot{x}(t) &= A_1x(t) + B_1u(t) \\ y(t) &= Cx(t) \end{aligned} \tag{6.1}$$

where

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & J_e & 0 & 0 \\ 0 & 0 & 0 & J_\theta & 0 \\ 0 & 0 & 0 & 0 & J_t \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -K_{\omega\Delta} & 0 & 0 & 0 \end{bmatrix}, \quad x(t) = \begin{bmatrix} \epsilon \\ \theta \\ \dot{\epsilon} \\ \dot{\theta} \\ \omega \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ K_{f\Delta}l_a & K_{f\Delta}l_a \\ K_{f\Delta}l_h & -K_{f\Delta}l_h \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad u(t) = \begin{bmatrix} V_f \\ V_b \end{bmatrix}.$$

Here V_f and V_b are voltages applied to the front motor and the rear motor respectively, l_a is the distance between the body and the elevation axis, l_h is the distance between each motor and the pitch axis, J_e, J_θ, J_t are moments of inertia about the elevation, pitch and travel axes respectively, $K_{f\Delta}$ is a motor parameter and $K_{\omega\Delta}$ is the gravity torque. The helicopter has the parameter values as shown in Table 1. We only consider the case that the pitch moment J_θ is zero, when E_1 becomes a singular matrix denoted by E, and the system (6.1) has the same form as the system (2.1) with the above parameter values.

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Table 1: Parameters of the helicopter.

Symbol	l_a	l_h	J_e	J_t	J_θ	$K_{f\Delta}$	$K_{\omega\Delta}$
Units	m	m	$Kg*m^2$	$Kg*m^2$	$Kg*m^2$	$N/Volt$	$N*m$
Value	0.66	0.177	0.91	0.91	0	0.5	0.4528

Letting $h = 0.1s$, we set up the delta operator model (2.2) of the system (6.1) where the matrices obtained are

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -0.4528 & 0 & -0.0226 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.0181 & 0.0181 \\ 0 & 0 \\ 0.33 & 0.33 \\ 0.0885 & -0.0885 \\ 0 & 0 \end{bmatrix}.$$

By examining the inequality (3.3), it is not difficult to see that there is no feasible solution, so from Theorem 3.1 the system (2.2) is not admissible. (This is obvious because the system (2.2) is irregular, noncausal and unstable, since the entries in the fourth rows of E and A are all zeroes. It us also notable that there are in fact infinite many solutions to the open-loop system (2.2) with $u(t_k) = 0$.)

Assuming that the state of the system (2.2) is not available, we first design an asymptotic and physically realisable observer (4.1) and then construct an admissible controller (5.1). From the form of the matrix E, we can select

$$U_1 = U_2 = S = L = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T.$$

Letting $P_2 = XE + SQ^T + \epsilon I_5$ and solving the inequality (4.2), we obtain a feasible solution

$$X = \begin{bmatrix} 0.7430 & -0.2261 & -0.2691 & 0 & 0.4996 \\ -0.2261 & 0.8842 & -0.0523 & 0 & 0.2955 \\ -0.2691 & -0.0523 & 1.1885 & 0 & -0.0610 \\ 0 & 0 & 0 & 4.8751 & 0 \\ 0.4996 & 0.2955 & -0.0610 & 0 & 0.8832 \end{bmatrix},$$

$$\Psi = \begin{bmatrix} -0.3894 & -1.0713 & 0.5707 \\ -0.8728 & 1.1583 & -0.7595 \\ 0.9456 & -1.0802 & -0.3099 \\ 0.1694 & 0.6064 & -1.0537 \\ 0.1512 & -0.6754 & -0.3720 \end{bmatrix},$$

$$Q = 0\,, \qquad \epsilon = 0.0073\,, \qquad V_2 = 0.9178\,.$$

According to Theorem 4.1, there exists an asymptotic and physically realisable observer

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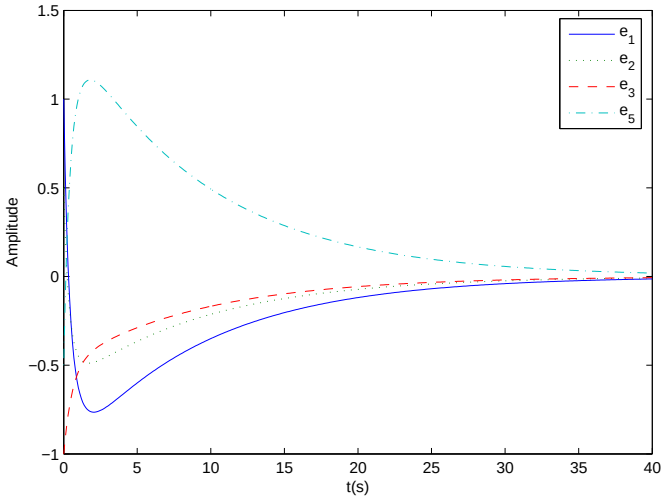


Figure 2: The time response of the state $e(t_k)$ of the system (4.3).

(4.1) for the system (2.2), where the gain matrix of (4.1) can be

$$G = \begin{bmatrix} -3.2482 & -0.9393 & 1.8260 \\ -2.7666 & 1.2580 & 0.1175 \\ 0.1666 & -1.1893 & 0.0539 \\ 23.0543 & 82.5091 & -143.3707 \\ 2.9363 & -0.8047 & -1.5177 \end{bmatrix}.$$

From the form of the matrix E, the fourth element of $e(t_k)$ in the error system (4.3) can be expressed as a linear combination of the other elements of $e(t_k)$, and therefore we just display the time response of the other elements in $e(t_k)$. Letting $e_1(0) = 1$, $e_2(0) = -0.5$, $e_3(0) = -1$ and $e_5(0) = 0.5$, we obtain the state trajectory of the system (4.3) shown in Fig. 2. Based on the observer (4.1), we now consider the design of the controller (5.1). Letting $P_1 = ZE^T + LY + \epsilon I_5$ and solving the inequality (5.7), we derive the feasible solution

$$Z = \begin{bmatrix} 0.1203 & 0 & -0.0103 & 0 & 0 \\ 0 & 0.1198 & 0 & 0 & 0.0052 \\ -0.0103 & 0 & 0.1437 & 0 & 0 \\ 0 & 0 & 0 & 1.1048 & 0 \\ 0 & 0.0052 & 0 & 0 & 0.1474 \end{bmatrix},$$

$$\Phi = \begin{bmatrix} -1.9691 & -6.8971 & -0.5619 & -2.0806 & 0.1559 \\ -1.9691 & 6.8971 & -0.5619 & 2.0806 & -0.1559 \end{bmatrix},$$

$$Y = \begin{bmatrix} 0 & -0.3687 & 0 & 0 & 0.5877 \end{bmatrix}, \quad \epsilon = 1.2208, \quad V_1 = 0.1203.$$

Then according to Theorem 5.1 there exists an admissible controller (5.1) for the system

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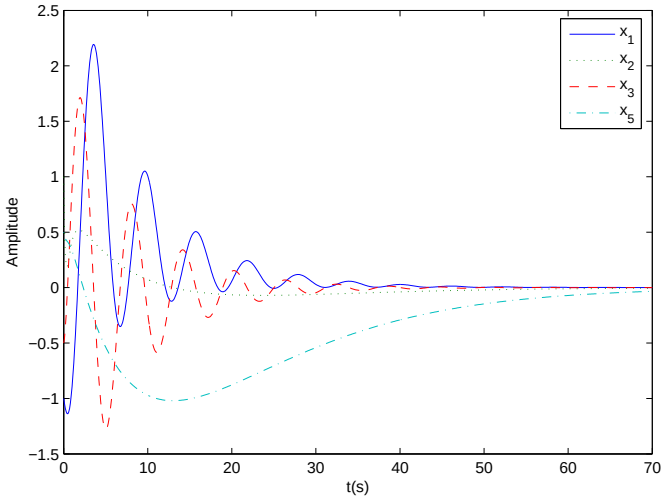


Figure 3: The time response of the state $x(t_k)$ of the system (5.2).

(2.2), and the gain matrix of (5.1) can be

$$\begin{bmatrix} -1.4715 & -5.6169 & -0.4259 & -1.7043 & 0.8739 \\ -1.4715 & 5.6169 & -0.4259 & 1.7043 & -0.8739 \end{bmatrix}.$$

Similarly, from the form of E we can obtain that the fourth element of $x(t_k)$ in the closed-loop system (5.2) can be written as a linear combination of the other elements in $x(t_k)$ and $e(t_k)$. Let $x_1(0) = -1$, $x_2(0) = 1$, $x_3(0) = -0.5$ and $x_5(0) = 0.5$. The state trajectory of the closed-loop system (5.2) is given in Fig. 3, and it is seen from Figs. 2 and 3 that both the state observer and the admissible controller we have designed are effective.

7. Concluding Remarks

The problem of output feedback admissible controller design for singular systems is a challenging task, due to the nature of singular systems and the admissible requirement that has to be met in order to obtain practical solutions. Under the delta operator approach, we have provided a detailed controller design procedure for the output feedback admissible controller construction. There are two advantages in using delta operator setting: (i) the discrete-time system converges to its corresponding continuous system when the sampling rate becomes high; and (ii) the unified procedure for both continuous and discrete systems. The necessary and sufficient conditions for the observer and controller design characterised by strict LMIs allow us to search for feasible solutions effectively. Moreover, our approach can be adapted for the local output feedback control of nonlinear singular systems, as will be reported elsewhere.

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