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Vertex-Bipartition Method for Colouring Minor-Closed Classes of Graphs

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Thomas conjectured that there is an absolute constant c such that for every proper minor-closed class $\mathcal{C}$ of graphs, there is a polynomial-time algorithm that can colour every $G \in \mathcal{C}$ with at most $\chi(G) + c$ colours. We introduce a parameter $\rho(\mathcal{C})$, called the degenerate value of $\mathcal{C}$, which is defined to be the smallest r such that every $G \in \mathcal{C}$ can be vertex-bipartitioned into a part of bounded tree-width (the bound depending only on $\mathcal{C}$), and a part that is r-degenerate. Although the existence of one global bound for the degenerate values of all proper minor-closed classes would imply Thomas's conjecture, we prove that the values $\rho(\mathcal{C})$ can be made arbitrarily large. The problem lies in the clique sum operation. As our main result, we show that excluding a planar graph with a fixed number of apex vertices gives rise to a minor-closed class with small degenerate value. As corollaries, we obtain that (i) the degenerate value of every class of graphs of bounded local tree-width is at most 6, and (ii) the degenerate value of the class of K_n-minor-free graphs is at most $n + 1$. These results give rise to P-time approximation algorithms for colouring any graph in these classes within an error of at most 7 and $n + 2$ of its chromatic number, respectively.

1. Introduction

In this paper we consider graph colouring, one of the most difficult problems in combinatorial optimization. It is well known [7] that, for general graphs, chromatic number cannot be approximated even within a factor of $|V|^{1-\varepsilon}$ for any $\varepsilon > 0$, unless co-RP = NP. In contrast, there are better algorithms for minor-closed classes of graphs. Probably the best-known result in this area is the following theorem of Arnborg and Proskurowski [1] on graphs of bounded tree-width.

Theorem 1.1 ([1]). *For each positive integer w there exists a linear-time algorithm that can optimally colour every graph of tree-width at most w.*

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Our work was motivated by the following conjecture made by Robin Thomas at the Banff workshop on topological graph theory and crossing numbers (Banff, Canada, October 2006). A minor-closed class of graphs is *proper* if it is not the class of all graphs.

Conjecture 1.2. *There exists a constant c with the following property: for each proper minor-closed class $\mathcal{C}$ of graphs, there exists a polynomial-time algorithm (the complexity depends on $\mathcal{C}$) that can colour every graph $G \in \mathcal{C}$ with at most $\chi(G) + c$ colours.*

Note that $\chi(G) + c$ cannot be replaced by $\chi(G)$ in the conjecture, as the problem of optimally colouring planar graphs is NP-hard [8]. In this sense we can say that the conjecture cannot be further improved (other than finding the best value for c).

It is an immediate consequence of the paper by DeVos and co-workers [5] that this conjecture is true if we are allowed to approximate $\chi(G)$ within a constant factor.

Theorem 1.3. *For each proper minor-closed class $\mathcal{C}$ of graphs, there exists a polynomial-time algorithm that can colour every graph $G \in \mathcal{C}$ with at most $2\chi(G)$ colours.*

Proof. The authors of [5] proved that every $G \in \mathcal{C}$ has a vertex partition (V_1, V_2) such that both $G[V_1]$ and $G[V_2]$ have bounded tree-width (the bound depending only on $\mathcal{C}$). Hence, by Theorem 1.1, for each $i \in \{1, 2\}$, colouring $G[V_i]$ with $\chi(G[V_i])$ colours takes polynomial time. Therefore, using distinct sets of colours for $G[V_1]$ and $G[V_2]$, we can colour G with at most $\chi(G[V_1]) + \chi(G[V_2]) \leqslant 2\chi(G)$ colours in polynomial time. □

Hence, the above conjecture, if true, would yield a stronger result than the above theorem. Before discussing our results we make another remark on this conjecture. A classical theorem of Mader [10] states: *For each positive integer n there exists a number d_n such that every graph of minimum degree at least d_n contains K_n as a minor*. Let us call a graph *r-degenerate* if it can be reduced to K_1 by repeatedly deleting vertices of degree at most r. Note that r-degenerate graphs can be $(r + 1)$-coloured in polynomial time, and, by Mader's theorem, K_n-minor-free graphs are $(d_n - 1)$-degenerate. Hence K_n-minor-free graphs can be d_n-coloured in polynomial time. This observation implies that the above conjecture is true if the constant c is allowed to depend on the class $\mathcal{C}$, because we may choose n with $K_n \notin \mathcal{C}$ and let $c = d_n$.

In this paper we develop a vertex-bipartition method with which we attempt to solve Thomas's conjecture. We will follow the terminology used in [6] and [12].

For any proper minor-closed class $\mathcal{C}$ of graphs, let $\rho(\mathcal{C})$ be the smallest non-negative integer r for which there exists an integer w, depending only on $\mathcal{C}$, such that every graph $G \in \mathcal{C}$ has a vertex partition (V', V'') such that $G[V']$ is r-degenerate and $G[V'']$ has tree-width at most w. We call $\rho(\mathcal{C})$ the *degenerate value* of $\mathcal{C}$. It follows immediately from the aforementioned theorem of Mader that for every proper minor-closed class $\mathcal{C}$, $\rho(\mathcal{C})$ is a finite number, and thus is well-defined.

Our approach in trying to prove Conjecture 1.2 was to find one global upper bound for the degenerate values of all proper minor-closed classes. If there were such an upper bound ρ_0, then from Theorem 1.1 and the fact that r-degenerate graphs can be $(r + 1)$-coloured

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in polynomial time, it would follow that Conjecture 1.2 holds with $c = \rho_0 + 1$. However, in Theorem 4.1, we show that such an upper bound does not exist, by constructing for every integer $d \geqslant 0$ a proper minor-closed class of graphs $\mathcal{F}(d)$, such that $\rho(\mathcal{F}(d)) \geqslant d$. The problem lies in the operation of a clique sum.

For $r \geqslant 1$, define $\mathcal{A}_r := \{H : H$ has a set X of at most r vertices such that $H - X$ is planar$\}$. Alternatively, a graph in $\mathcal{A}_r$ is formed by taking a planar graph and joining it by any number of edges to some graph on at most r vertices. Note that members of $\mathcal{A}_1$ are precisely *apex graphs*. The following is our main result in this paper.

Theorem 1.4. *For any $r \geqslant 1$ and any minor-closed class $\mathcal{C} \not\supseteq \mathcal{A}_r$, $\rho(\mathcal{C}) \leqslant r + 5$.*

Our proof is algorithmic, which, for every minor-closed class $\mathcal{C} \not\supseteq \mathcal{A}_r$, yields a polynomial-time algorithm that can colour every $G \in \mathcal{A}_r$ with at most $\chi(G) + r + 6$ colours.

Theorem 1.4 has two interesting corollaries. According to [3], a minor-closed class $\mathcal{C}$ has bounded *local tree-width* if and only if $\mathcal{C} \not\supseteq \mathcal{A}_1$.

Corollary 1.5. *If $\mathcal{C}$ has bounded local tree-width, then $\rho(\mathcal{C}) \leqslant 6$.*

The next corollary follows obviously from the fact that for $n \geqslant 5$, $K_n \in \mathcal{A}_{n-4}$.

Corollary 1.6. *If $\mathcal{C}$ is the class of K_n-minor-free graphs ($n \geqslant 5$), then $\rho(\mathcal{C}) \leqslant n + 1$.*

The algorithmic implication of the last two corollaries is that graphs in these classes can be coloured in polynomial time using at most $\chi(G) + 7$ and $\chi(G) + n + 2$ colours, respectively. Results similar to the two corollaries are also obtained in [4].

One of the currently most well-known and challenging problems in graph theory, Hadwiger's conjecture, states that K_n-minor-free graphs can be $(n-1)$-coloured. While the statement has been shown to be true for $n \leqslant 6$, and remains open for $n \geqslant 7$, nothing is known about polynomial-time algorithms that colour graphs in such classes, even approximately. Our algorithm colours K_n-minor-free graphs (for $n \geqslant 5$) in polynomial time within an error of at most $n + 2$ of the chromatic number.

We close this section with an outline of the rest of the paper. In Section 2, we explain the structure theorem of Robertson and Seymour [15]. Then, in Section 3, we discuss results on tree-width that we will use in our proof. Next, by constructing certain proper minor-closed classes in Section 4, we show that $\rho(\mathcal{C})$ can be made arbitrarily large. Finally, in Section 5, we complete the proof of our main result, Theorem 1.4.

2. The structure theorem of Robertson and Seymour

Our proof will use the structure theorem of Robertson and Seymour [15] for graphs in a general proper minor-closed class. We first introduce the necessary terminology for the statement of the theorem.

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Surfaces. A *surface* is a compact connected 2-manifold with (possibly empty) boundary; a surface is *closed* if its boundary is empty. The unique surface obtained from a closed surface Σ by removing the interiors of k disjoint closed disks will be denoted by $\Sigma - k$. The components of the boundary of $\Sigma - k$ are the *cuffs*. Thus, each cuff of a surface is homeomorphic to the unit circle.

Vortices. This notion was first defined by Robertson and Seymour in [15]. Roughly, a *vortex* is a graph with some of its vertices arranged in a circular order, so that however these special vertices are partitioned into two intervals, there are only a bounded number of disjoint paths from one interval to the other. The authors of [15] give a more precise definition of an 'r-ring with perimeter $t_1, \ldots, t_n$'. The definition we will use is the one reformulated by Thomas in [18] as follows. Let G be a graph, and let U be a cyclically ordered subset of $V(G)$. We say that $(X_u)_{u \in U}$ is a *vortex-decomposition* of the pair (G, U) if:

(V1) $u \in X_u$ for every $u \in U$,
(V2) $\bigcup_{u \in U} X_u = V(G)$, and every edge of G has both ends in some X_u, and
(V3) for every $v \in V(G)$, the set of $u \in U$ with $v \in X_u$ is a contiguous interval (which could be the whole U).

The vortex-decomposition $(X_u)_{u \in U}$ is said to have *width less than* k if $|X_u| \leqslant k$ for every $u \in U$.

Nearly embedding a graph in surface $\Sigma - k$. Let G be a graph, and let $\Sigma - k$ be a surface with cuffs $C_1, \ldots C_k$. We say that G *can be nearly embedded in* $\Sigma - k$ if G has a set X (of so-called *apex vertices*) of at most k vertices such that $G - X$ can be written as $G_0 \cup G_1 \cup \cdots \cup G_k$, where:

(N1) G_0 is embedded in $\Sigma - k$,
(N2) the graphs G_i $(i = 1, \ldots, k)$ are pairwise disjoint, and $U_i := V(G_0) \cap V(G_i) = V(G_0) \cap C_i$ for each $i = 1, \ldots, k$,
(N3) for each $i = 1, \ldots, k$, the pair (G_i, U_i) has a vortex-decomposition $(X_u)_{u \in U_i}$ of width less than k, where the ordering of U_i is determined by the cyclic ordering of points on C_i.

We will call the vertices of $\bigcup_{i=1}^{k} V(G_i)$ the *vortex vertices* of G.

Clique sums. Let G_1 and G_2 be graphs with disjoint vertex sets, let $k \geqslant 0$ be an integer, and for $i = 1, 2$ let $X_i \subseteq V(G_i)$ be a set of k pairwise adjacent vertices. For $i = 1, 2$ let G_i' be obtained from G_i by deleting a (possibly empty) set of edges with both ends in X_i. Let $f : X_1 \longrightarrow X_2$ be a bijection, and let G be the graph obtained from the union of G_1' and G_2' by identifying x with $f(x)$ for all $x \in X_1$. In those circumstances we say that G is a *k-sum* of G_1 and G_2. Also, we say that G is a *clique sum* of G_1 and G_2 if G is a j-sum of G_1 and G_2 for some integer $j \geqslant 0$.

We are finally ready to state the general structure theorem of Robertson and Seymour [15] (as stated in [18]) for graphs with no minor isomorphic to a given graph H.

Theorem 2.1. *For every graph H there exists an integer $k \geqslant 0$, such that every graph with no H-minor can be obtained by means of clique sums from graphs that can be nearly embedded in $\Sigma - k$ for some closed surface Σ in which H cannot be embedded.*

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Remark 2.2. It is an unpublished consequence of the work by Robertson and Seymour [16] that the above decomposition of G into nearly embeddable graphs can be found in polynomial time. More precisely, for each proper minor-closed class $\mathcal{C}$ of graphs, there exists a polynomial-time algorithm (the complexity depending only on $\mathcal{C}$) that, given $G \in \mathcal{C}$ as input, outputs nearly embeddable graphs $G_1, G_2, \ldots, G_l$ such that G is a clique sum of $G_1, G_2, \ldots, G_l$. Furthermore, for each G_i, the set of its apex vertices X_i and its vortex vertices Y_i can also be identified in polynomial time.

3. Results on tree-width

A key concept that we need is tree-width, introduced by Robertson and Seymour in [13].

Definition. Let G be a graph, let T be a tree, and let $\mathcal{V} = \{V_t\}_{t \in V(T)}$ be a family of vertex sets $V_t \subseteq V(G)$. The pair $(T, \mathcal{V})$ is called a *tree-decomposition* of G if it satisfies the following two conditions:

(T1) $V(G) = \bigcup_{t \in V(T)} V_t$, and every edge of G has both ends in some V_t,
(T2) for every $v \in V(G)$, the subgraph of T induced by those t for which $v \in V_t$ is connected.

The *width* of a tree-decomposition $(T, \mathcal{V})$ is $\max_{t \in V(T)}\{|V_t| - 1\}$. The *tree-width* of G, denoted by $\mathbf{tw}(G)$, is the minimum width over all possible tree-decompositions of G. The following standard notion and property can be found in [6].

Definition. An $(n \times n)$-*grid*, or more simply an *n-grid*, is the graph with vertices v_{ij} $(1 \leqslant i, j \leqslant n)$, in which v_{ij} and $v_{i'j'}$ are adjacent if $|i - i'| + |j - j'| = 1$.

Lemma 3.1. *The tree-width of an n-grid is n.*

A deeper connection between grids and tree-width is shown by the following theorem of Robertson and Seymour [14], which can also be found in [6] as Theorem 12.4.4.

Theorem 3.2. *For every integer $n > 1$ there is an integer $f_{3.2}(n)$ such that every graph G with $\mathbf{tw}(G) \geqslant f_{3.2}(n)$ has an n-grid minor.*

Remark 3.3. In fact, Robertson, Seymour and Thomas showed in [17] that $f_{3.2}(n) = 20^{2n^5}$ suffices.

The next simple lemma follows easily from the definition of tree-decomposition.

Lemma 3.4.

(i) *If G' is a minor of G, then $\mathbf{tw}(G') \leqslant \mathbf{tw}(G)$,*
(ii) *if $x \in V(G)$, then $\mathbf{tw}(G) \leqslant \mathbf{tw}(G - x) + 1$,*
(iii) *if G is a clique sum of G_1 and G_2, then $\mathbf{tw}(G) \leqslant \max\{\mathbf{tw}(G_1), \mathbf{tw}(G_2)\}$.*

We also need the following technical lemma, which explains how two tree-decompositions can be combined to get a new one.

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Lemma 3.5. *Let $G = G_1 \cup G_2$ with $U = V(G_1) \cap V(G_2)$. Suppose that for every $u \in U$ there exists $X_u \subseteq V(G_2)$ such that:*

(i) *$u \in X_u$ for every $u \in U$,*
(ii) *$\bigcup_{u \in U} X_u = V(G_2)$, and every edge of G_2 has both ends in some X_u, and*
(iii) *for each $v \in V(G_2)$, the set of $u \in U$ with $v \in X_u$ induces a connected subgraph on G_1.*

Then $\mathbf{tw}(G) \leqslant (\mathbf{tw}(G_1) + 1) \cdot \max\{|X_u| : u \in U\} - 1$.

Proof. Let $(T, \mathcal{V})$ be an optimal tree-decomposition of G_1, where $\mathcal{V} = \{V_t\}_{t \in V(T)}$. Construct a new family $\mathcal{W} = \{W_t\}_{t \in V(T)}$ as follows. For each $t \in V(T)$, let $W_t := V_t \cup (\bigcup_{u \in V_t} X_u)$. We claim that the resulting pair $(T, \mathcal{W})$ is a tree-decomposition of G.

We first check condition (T1). Clearly,

$$\bigcup_{t \in V(T)} W_t = \left(\bigcup_{t \in V(T)} V_t \right) \cup \left(\bigcup_{u \in U} X_u \right) = V(G_1) \cup V(G_2) = V(G).$$

Moreover, since every edge e of G is an edge of either G_1 or G_2 (or both), both ends of e belong to either some V_t or some X_u, which implies that they belong to some W_t.

To verify condition (T2), we first introduce the following notation. For each $v \in V(G)$, let T_v be the subgraph of T induced by those vertices t for which $v \in W_t$; for each $v \in V(G_1)$, let T'_v be the subgraph of T induced by those vertices t for which $v \in V_t$. We need to show that every T_v is connected. Observe that, for each $v \in V(G_1) - V(G_2)$, $T_v = T'_v$, which is connected, as $(T, \mathcal{V})$ is a tree-decomposition of G_1. Hence we only need to consider vertices $v \in V(G_2)$. For $v \in V(G_2)$, let U_v be the set of vertices $u \in U$ for which $v \in X_u$. It follows by assumption (iii) that $G_1[U_v]$ is connected. Also, by construction of $(T, \mathcal{W})$, it follows that W_t contains v if and only if V_t contains at least one member of U_v. Consequently, $T_v = \bigcup_{u \in U_v} T'_u$. Note that, for every edge xy of $G_1[U_v]$, $V(T'_x) \cap V(T'_y) \neq \varnothing$, since there exists t with $V_t \supseteq \{x, y\}$. Hence, it follows that T_v is connected, as required. This completes the verification of (T2), which proves that $(T, \mathcal{W})$ is a tree-decomposition of G.

To compute the width of $(T, \mathcal{W})$, note that

$$|W_t| = \left| V_t \cup \left(\bigcup_{u \in V_t} X_u \right) \right| \leqslant |V_t| \cdot \max\{|X_u| : u \in V_t\} \leqslant (\mathbf{tw}(G_1) + 1) \cdot \max\{|X_u| : u \in U\},$$

which concludes the proof of the lemma. □

To prove our last result on tree-width we need two theorems from [11] and [19] (see also Propositions 7.3.1 and 7.3.2 in [12]). We combine these two results as follows. Let G be a graph embedded in a closed surface Σ. A subgraph G' of G is called *flat* (with respect to the embedding) if there is a disc of Σ that contains all vertices and edges of G'. Note that every flat subgraph is planar but there could be planar subgraphs that are not flat (with respect to the given embedding).

Theorem 3.6. *For every closed surface Σ and every integer $n > 1$, there exists an integer $f_{3.6}(\Sigma, n)$ such that, if G is a connected graph embedded in Σ with $\mathbf{tw}(G) \geqslant f_{3.6}(\Sigma, n)$, then G has a flat subgraph that contains an n-grid minor.*

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Remark 3.7. In fact, it can be deduced from Remark 3.3 and Propositions 7.3.1 and 7.3.2 in [12] that $f_{3.6}(\Sigma, n) = 20^{2(400\sqrt{g}(n+4g+4))^5}$ suffices, where g is the genus of G.

The following is our last lemma in this section.

Lemma 3.8. *Let G be a connected graph embedded in a closed surface Σ. Let G' be obtained from G by deleting all vertices of k of its faces. Then* $\mathbf{tw}(G) < f_{3.6}(\Sigma, \mathbf{tw}(G') + 1 + 2k)$.

Proof. Suppose, on the contrary, that $\mathbf{tw}(G) \geqslant f_{3.6}(\Sigma, \mathbf{tw}(G') + 1 + 2k)$. By Theorem 3.6, G has a flat subgraph H that contains an n-grid minor, where $n = \mathbf{tw}(G') + 1 + 2k$. Since H is flat, every face of G is part of a face of H. Moreover, since H is planar, deleting any number of vertices of one face of H results in a graph that contains an $(n-2)$-grid minor. It follows that G' contains an $(n-2k)$-grid minor. Therefore, by Lemmas 3.4(i) and 3.1, $\mathbf{tw}(G') \geqslant n - 2k = \mathbf{tw}(G') + 1$, a contradiction. □

4. Classes with large degenerate values

Our approach in trying to prove Thomas's conjecture (Conjecture 1.2) was to try to find one global bound for the degenerate values of all proper minor-closed classes. However, such a global constant does not exist since the values $\rho(\mathcal{C})$ can be made arbitrarily large, as the following theorem shows.

Theorem 4.1. *For every integer $d \geqslant 0$ there is a proper minor-closed class of graphs $\mathcal{F}(d)$, such that $\rho(\mathcal{F}(d)) \geqslant d$.*

We first define $\mathcal{F}(d)$, and then we prove that it has the required property.

Construction of $G(d, n)$ and $\mathcal{F}(d)$. Let $n \geqslant 0$ be a fixed integer, and let R_n be an n-grid. Let $A_{0,n} := R_n$, and for each $i = 1, 2, \ldots, d$, let $A_{i,n} := R_n + K_i$, where K_i is the clique on i vertices, and the '+' binary operation takes a disjoint union of two graphs and adds all possible edges between vertices of one graph and vertices of the other. Hence each $A_{i,n}$ has a natural vertex partition into $V(R_n)$ and $V(K_i)$, whose vertices we will call the *grid vertices* and the *apex vertices* of $A_{i,n}$, respectively.

To construct graph $G(d, n)$ we will need one copy of $A_{0,n}$, n^2 copies of $A_{1,n}$, $(n^2)^2$ copies of $A_{2,n}, \ldots, (n^2)^d$ copies of $A_{d,n}$. For the sake of clarity, in the following construction we will denote each $A_{i,n}$ simply by A_i. The construction of $G(d, n)$ takes d steps as follows.

Step 1. To every (grid) vertex of the grid A_0 adjoin a copy of A_1, by taking a 1-sum identifying each (grid) vertex of A_0 with the apex vertex of A_1. Note that at this point, every grid vertex of every copy of A_1 forms a 2-clique with a grid vertex of A_0, and conversely, every grid vertex of A_0 forms a 2-clique with every grid vertex of the copy of A_1 to which it is adjoined.

Step 2. At this step, every grid vertex of every copy of A_1 forms a 2-clique with a grid vertex of A_0. To every such 2-clique adjoin a copy of A_2, by taking a 2-sum identifying the

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2-clique with the apex vertices of A_2. Note that at this point, every grid vertex of every copy of A_2 forms a 3-clique with a grid vertex of A_1 and a grid vertex of A_0.

So, the general ith step for $i = 2, 3, \ldots, d$ is as follows.

Step i. At this step, every grid vertex of every copy of A_{i-1} forms an i-clique with a grid vertex of A_{i-2}, a grid vertex of $A_{i-3}, \ldots$, and a grid vertex of A_0. To every such i-clique adjoin a copy of A_i, by taking an i-sum identifying the i-clique with the apex vertices of A_i. Note that at this point, every grid vertex of every copy of A_i forms an $(i+1)$-clique with a grid vertex of A_{i-1}, a grid vertex of $A_{i-2}, \ldots$, and a grid vertex of A_0.

Hence, at the end of the dth step, we will have that in the resulting graph $G(d,n)$, every grid vertex of every copy of A_d forms a $(d+1)$-clique with a grid vertex of A_{d-1}, a grid vertex of $A_{d-2}, \ldots$, and a grid vertex of A_0.

Now we define $\mathcal{F}(d) = \{G : G \text{ is a minor of } G(d,n) \text{ for some } n \geqslant 1\}$. Clearly, the class $\mathcal{F}(d)$ is proper since, for instance, K_{d+5} does not belong to it.

Proof of Theorem 4.1. We must show that $\rho(\mathcal{F}(d)) \geqslant d$. Assume, on the contrary, that $\rho(\mathcal{F}(d)) < d$. This implies that there exists an integer w such that every graph $G \in \mathcal{F}(d)$ has a vertex partition (V', V'') such that $G[V']$ is $(d-1)$-degenerate and $\mathbf{tw}(G[V'']) \leqslant w$. Let $n > w$ and consider the graph $G(d,n) \in \mathcal{F}(d)$. For clarity of notation let $H := G(d,n)$. Consider the vertices of A_0 in H. Since A_0 is an n-grid, we cannot have that $V(A_0) \subseteq V''$, because by Lemma 3.1, $\mathbf{tw}(A_0) = \mathbf{tw}(R_n) = n > w$. Hence, there is a (grid) vertex $v_0 \in V(A_0)$ such that $v_0 \in V'$. Similarly, in the copy of A_1 that was adjoined to v_0 in Step 1 (of the construction of $H = G(d,n)$), there is a grid vertex v_1 such that $v_1 \in V'$. Note that, by construction, $H[v_0, v_1] = K_2$. Similarly, in the copy of A_2 that was adjoined to $H[v_0, v_1]$ in Step 2, there is a grid vertex v_2 such that $v_2 \in V'$. Note that, by construction, $H[v_0, v_1, v_2] = K_3$. Continuing in this way, we obtain grid vertices $v_i \in V(A_i)$ $(i = 0, 1, \ldots, d)$ such that $v_i \in V'$, and $H[v_0, v_1, \ldots, v_d] = K_{d+1}$. This is a contradiction, because it implies that K_{d+1} is an induced subgraph of $H[V']$, which is $(d-1)$-degenerate. □

5. Bounding the degenerate value

In this section we prove our main result, Theorem 1.4. To prove it, we need a key technical lemma concerning nearly embeddable graphs. For the proof of the key lemma, we need, in turn, the following Ramsey theorem for bipartite graphs (Theorem 5.1.1 in [9]), which we state using matrix language.

Lemma 5.1. *For any positive integers m and p, there exists a positive integer $q = f_{5.1}(m,p)$ such that if A is a $q \times q$ matrix whose entries take at most m different values, then A has a $p \times p$ submatrix whose entries take only one value.*

Remark 5.2. In fact, it can be shown directly that $f_{5.1}(m,p) = pm^{pm}$ suffices.

We will also need the following definition. A set of vertices U of a graph H is said to have *forward degree* $\leqslant h$ in H, and is denoted by $\mathbf{fd}_H(U) \leqslant h$, if H can be reduced to

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$H - U$ by repeatedly deleting vertices of degree $\leqslant h$. Also, recall the following definition from Section 1. For $r \geqslant 1$, $\mathcal{A}_r := \{H : H$ has a set X of at most r vertices such that $H - X$ is planar$\}$. Also, for any integer $n \geqslant 1$, we let $A_{r,n}$ denote the graph in $\mathcal{A}_r$ obtained from K_r and a disjoint n-grid by adding all possible edges between them.

The following is the lemma crucial to the proof of our main theorem.

Lemma 5.3. *Let $k \geqslant 0$, $r, n \geqslant 1$ be integers, and let G be a graph nearly embedded in $\Sigma - k$ with X as its set of apex vertices. Let C be a clique subgraph of $G[X]$, and let (R, S) be a partition of $V(C)$. Then, there exists a positive integer $f_{5.3}(\Sigma, k, r, n)$ such that G has a vertex partition (V', V'') satisfying:*

(i) $\mathbf{fd}_{G[V']}(V' - R) \leqslant r + 5$, *and* $V' \cap C = R$,

(ii) $\mathbf{tw}(G[V'']) < f_{5.3}(\Sigma, k, r, n)$, *unless $G \backslash E(C)$ has an $A_{r,n}$-minor.*

Furthermore, the partition (V', V'') can be found in polynomial time.

Proof. Let χ_Σ be the Euler characteristic of Σ. Let Y be the set of vortex vertices of G, and $Z = V(G) - X - Y$. We first prove the following two intermediate steps (a) and (b).

(a) There exists $T \subseteq Z$ with $|T| \leqslant 12 - 6\chi_\Sigma$ such that $G[Z - T]$ is 6-degenerate.

Starting from $G[Z]$, repeatedly delete vertices of degree at most six until this is no longer possible. Let T be the (possibly empty) set of remaining vertices. Note that computing T takes at most $O(|Z|^2)$-time, and so $O(|V(G)|^2)$-time, since each removal of a vertex of degree at most six in $G[Z]$ may require a traversal of the whole list of remaining vertices, which is a subset of Z. Clearly $G[Z - T]$ is 6-degenerate. It remains to show that $|T| \leqslant 12 - 6\chi_\Sigma$.

If $T = \varnothing$ then the inequality holds since $\chi_\Sigma \leqslant 2$. Hence we assume $T \neq \varnothing$. Note that $G[T]$ is embedded in Σ. By Euler's formula (formula (3.7) in [12]), $\chi_\Sigma = |T| - e + f$, where e and f are the number of edges and faces of $G[T]$, respectively. Since each face is bounded by at least three edges and each edge is on the boundary of at most two faces, it follows that $3f \leqslant 2e$ and thus $\chi_\Sigma \leqslant |T| - e + \frac{2}{3}e = |T| - \frac{1}{3}e$. Note that $7|T| \leqslant 2e$ since the minimum degree of $G[T]$ is at least seven. Therefore, $\chi_\Sigma \leqslant |T| - \frac{7}{6}|T| = -\frac{1}{6}|T|$, which proves (a).

Next, we prove the following step.

(b) Let (Z', Z'') be the partition of $Z - T$ such that Z' consists of vertices that are adjacent to fewer than r vertices of R. Then, there exists a positive number $g(k, r, n)$ such that $\mathbf{tw}(G[Z'']) < g(k, r, n)$, unless $G \backslash E(C)$ has an $A_{r,n}$-minor.

First, note that computing Z' requires one traversal of the list of vertices of $Z - T$, and for each vertex, a check whether it has fewer than r neighbours in R. Hence, computing Z' takes $O(|E(G[Z])|)$-time, and so $O(|E(G)|)$-time.

We now prove that $g(k, r, n) := f_{3.2}(f_{5.1}(\binom{k}{r}, n + r))$ has the required property. Suppose, on the contrary, that $G \backslash E(C)$ has no $A_{r,n}$-minor, yet $\mathbf{tw}(G[Z'']) \geqslant f_{3.2}(f_{5.1}(\binom{k}{r}, n + r))$. By Theorem 3.2, $G[Z'']$ contains a q-grid minor Q, where $q = f_{5.1}(\binom{k}{r}, n + r)$. Since each vertex of Z'' has at least r neighbours in R, each vertex of Q also has at least r neighbours in R. Let M be a $q \times q$ matrix whose entry in row s and column t is an arbitrarily chosen

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r-element subset of neighbours in R of the vertex in row s and column t of the grid Q. Then the entries of M take on at most $\binom{|R|}{r} \leqslant \binom{k}{r}$ different values. By Lemma 5.1, M has an $(n+r) \times (n+r)$ submatrix whose entries take on only one value, which means that Q can be contracted to an $(n+r)$-grid Q^*, all of whose vertices are adjacent to all vertices of some $R^* \subseteq R$ with $|R^*| = r$. Let F be a matching of size r that is between R^* and one of the four sides of Q^*. Then $G \backslash E(C)/F$ contains $A_{r,n+r-1}$ as a minor, a contradiction. Hence, we have shown (b).

We are now ready to prove the conclusion of the lemma.

Let $f_{5.3}(\Sigma, k, r, n) := k \cdot f_{3.6}(\Sigma, g(k,r,n) + 1 + 2k) + (12 - 6\chi_\Sigma) + k$. Note that $f_{5.3}$ depends only on Σ, k, r, n. Now, let $V' = Z' \cup R$ and $V'' = V(G) - V'$. Note that computing the partition (V', V'') only requires computing the sets T and Z', and thus takes $O(|V(G)|^2 + |E(G)|)$-time.

We now prove that the partition (V', V'') of $V(G)$ satisfies conditions (i) and (ii).

(i) Clearly, by definition of V' and Z', we have $V' \cap C = R$. Also, since $Z' \subseteq Z - T$, it follows from (a) that $G[Z']$ is 6-degenerate. Hence, from the definition of Z', we have that $\mathbf{fd}_{G[V']}(Z') \leqslant 6 + (r-1) = r + 5$. Since $Z' = V' - R$, (i) is proved.

(ii) Assume that $G \backslash E(C)$ does not contain $A_{r,n}$ as a minor. Let U be the set of vertices of G on the cuffs of $\Sigma - k$. Note that $U \subseteq Y$. Let G_Σ be the graph obtained from $G[Z'' \cup U]$ by deleting all the vortex edges. Note that $V(G_\Sigma)$ is embedded in $\Sigma - k$. Assume, without loss of generality, that G_Σ has edges connecting consecutive vertices on each cuff, so that in G_Σ, the vertices of U are covered by k faces (when viewing G_Σ as being embedded in Σ). Since $G \backslash E(C)$ has no $A_{r,n}$-minor, it follows from (b) and Lemma 3.8 that $\mathbf{tw}(G_\Sigma) < f_{3.6}(\Sigma, g(k,r,n) + 1 + 2k)$.

Let H be the graph formed by the disjoint union of the k vortices $H_1, \ldots, H_k$ of G. We will verify that G_Σ and H satisfy the conditions of Lemma 3.5. Note that $G[Z'' \cup Y] = G_\Sigma \cup H$ with $U = V(G_\Sigma) \cap V(H)$. Let $C_1, \ldots, C_k$ be the cuffs of $\Sigma - k$, and for $i = 1, \ldots, k$, let U_i be the set of vertices of G on C_i, so that $U = \bigcup_{i=1}^k U_i$. By (N3) in the definition of a nearly embeddable graph, for each $i = 1, \ldots, k$, the pair (H_i, U_i) has a vortex-decomposition $(X_u)_{u \in U_i}$ of width less than k, where the ordering of U_i is determined by the cyclic ordering of points on C_i. By (V1) and (V2) in the definition of vortex-decomposition, and the fact that $U = \bigcup_{i=1}^k U_i$, it follows that conditions (i) and (ii) of Lemma 3.5 are satisfied. Condition (iii) follows from (V3), and the fact that G_Σ has edges connecting consecutive vertices on each C_i in its cyclic ordering. Hence it follows that $\mathbf{tw}(G[Z'' \cup Y]) < (\mathbf{tw}(G_\Sigma) + 1) \cdot \max\{|X_u| : u \in U\} \leqslant k \cdot f_{3.6}(\Sigma, g(k,r,n) + 1 + 2k)$.

Since $V'' = (Z'' \cup Y) \cup T \cup (X - R)$, $|T| \leqslant 12 - 6\chi_\Sigma$ (by (a)), and $|X - R| \leqslant |X| \leqslant k$, it follows from Lemma 3.4(ii) that $\mathbf{tw}(G[V'']) \leqslant \mathbf{tw}(G[Z'' \cup Y]) + |T| + |X - R| < k \cdot f_{3.6}(\Sigma, g(k,r,n) + 1 + 2k) + (12 - 6\chi_\Sigma) + k = f_{5.3}(\Sigma, k, r, n)$, which proves (ii) and the lemma. □

With the previous key lemma now in place, we are finally ready to prove our main result.

Proof of Theorem 1.4. Let r be a positive integer and let $\mathcal{C}$ be a minor-closed class of graphs with $\mathcal{C} \not\supseteq \mathcal{A}_r$. By definition of $\mathcal{A}_r$, there must exist a positive integer n, which

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depends only on $\mathcal{C}$, such that $A_{r,n} \notin \mathcal{C}$, where $A_{r,n}$ is the graph obtained from K_r and a disjoint n-grid by adding all possible edges between them. Let $\mathcal{C}^*$ be the class of graphs that do not contain $A_{r,n}$ as a minor. Since $\mathcal{C} \subseteq \mathcal{C}^*$, it is enough for us to prove that $\rho(\mathcal{C}^*) \leqslant r+5$. Therefore, without loss of generality, we may assume $\mathcal{C} = \mathcal{C}^*$.

By Theorem 2.1, there exists a closed surface Σ and an integer $k \geqslant 0$ such that every graph $G \in \mathcal{C}$ is a clique sum of $G_1, G_2, \ldots, G_\ell$, where each G_i is a graph nearly embedded in $\Sigma - k$. To be more specific, let us assume that G is obtained by clique-summing G_1 with G_2 over clique C_2; then the result is clique-summed with G_3 over clique C_3, and then the result is clique-summed with G_4 over C_4, and so on. Let C_1 be any clique of G_1. In each G_i $(i = 1, 2, \ldots, \ell)$, let X_i be the set of its apex vertices. As discussed in [5], we may assume that $V(C_i) \subseteq X_i$.

Note that by Remark 2.2, the graphs G_i $(i = 1, 2, \ldots, \ell)$ as well as their corresponding sets of apex vertices X_i and vortex vertices Y_i can be identified in polynomial time (the complexity depending only on $\mathcal{C}$).

Let $w := f_{5.3}(\Sigma, k, r, n)$. Note that w depends only on $\mathcal{C}$, since the parameters Σ, k, r, n depend only on $\mathcal{C}$. We prove that $V(G)$ admits a partition (V', V'') such that $G[V']$ is $(r+5)$-degenerate and $\mathbf{tw}(G[V'']) \leqslant w$.

Our proof is by induction on ℓ. First, we consider the base case $\ell = 1$. Recall that C_1 can be chosen as any clique of $G = G_1$. Thus we may choose $C_1 = K_1$. Under this choice, it is easy to see that the partition given in Lemma 5.3 satisfies the requirements.

Let H be a clique sum of $G_1, G_2, \ldots, G_{\ell-1}$, and let G be a clique sum of H and G_ℓ. By induction, we may assume that H admits a vertex partition (U', U'') that satisfies the requirements. Since $V(C_\ell)$ is a subset of $V(H)$, (U', U'') defines a partition (R_ℓ, S_ℓ) of $V(C_\ell)$ with $R_\ell = U' \cap V(C_\ell)$ and $S_\ell = U'' \cap V(C_\ell)$. Note that $G_\ell \backslash E(C_\ell)$ is a subgraph of G, hence it does not contain $A_{r,n}$ as a minor.

It follows that $V(G_\ell)$ has a partition (V'_ℓ, V''_ℓ) as defined in Lemma 5.3. Let $V' = U' \cup V'_\ell$ and $V'' = U'' \cup V''_\ell$. We verify that (V', V'') satisfies the requirements. Since $V(C_\ell)$ is the intersection of $V(H)$ and $V(G_\ell)$, and it is partitioned by (U', U'') and (V'_ℓ, V''_ℓ) in the same way, it follows that (V', V'') is a partition of $V(G)$. By Lemma 3.4(iii), $\mathbf{tw}(G[V'']) \leqslant w$ since $G[V'']$ is a clique sum of $H[U'']$ and $G_\ell[V''_\ell]$, both of which have tree-width at most w. Finally, by Lemma 5.3(i), $G_\ell[V'_\ell]$ can be reduced to $G_\ell[R_\ell]$ by repeatedly deleting vertices of degree at most $r+5$. Hence, it follows that $G[V']$ can be reduced to $G[U']$ by repeatedly deleting vertices of degree at most $r+5$. Therefore, the induction hypothesis on $G[U']$ implies that $G[V']$ is $(r+5)$-degenerate. This completes the induction as well as the proof of the theorem. □

6. Colouring algorithm

In conclusion, if $\mathcal{C} \not\supseteq \mathcal{A}_r$, for some r, then there exists an integer w (depending only on $\mathcal{C}$), such that every graph G in $\mathcal{C}$ admits a partition (V', V'') of $V(G)$ for which $G[V']$ is $(r+5)$-degenerate and $\mathbf{tw}(G[V'']) \leqslant w$. Note that computing the partition (V', V'') of $V(G)$ takes polynomial time. This is because, by Remark 2.2, the graphs G_i $(i = 1, 2, \ldots, \ell)$ in the decomposition of G, as well as their corresponding sets of apex vertices X_i and vortex vertices Y_i can be identified in polynomial time (the complexity depending only on

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$\mathcal{C}$). Then, for each G_i ($i = 1, 2, \ldots, \ell$), the partitions (V_i', V_i'') of G_i satisfying the conditions of Lemma 5.3 can be found in polynomial time (by the same Lemma 5.3). With each successive clique sum of the G_i, a new partition of the current graph is defined as in the proof of Theorem 1.4 (each repartition takes $O(1)$-time). Since there are $\ell - 1$ clique sums, computing the final partition (V', V'') takes an extra $O(\ell)$-time, and so an extra $O(|V(G)|)$-time.

The above vertex-bipartition method immediately gives rise to the following polynomial-time colouring algorithm. For any graph $G \in \mathcal{C}$, we act as follows.

- We find the desired partition (V', V'') of $V(G)$ in polynomial time (as above).
- We apply the algorithm of Bodlaender [2], which outputs a tree-decomposition of $G[V'']$ of width at most w in time $O(|V''|)$ (note that the constant hidden in the big-O notation is exponential in w).
- We input the tree-decomposition of $G[V'']$ of width at most w into the algorithm from Theorem 1.1, which will colour $G[V'']$ with exactly $\chi(G[V''])$ colours in $O(|V''|)$-time (again, the constant hidden in the big-O notation is exponential in w).
- Using the fact that $G[V']$ is $(r + 5)$-degenerate, we colour $G[V']$ in $O(|V'|)$-time with at most $(r + 6)$ colours distinct from those used to colour $G[V'']$.

Thus, the above sequence of steps colours G with at most $\chi(G) + (r + 6)$ colours in polynomial time.

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