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Note on a hyperbolic equation having p-Laplacian and Kirchhoff damping

Liao Zhang and Fengjie Li

Abstract. In this article, we investigate a high-order quasilinear hyperbolic equation that involves Kirchhoff damping and logarithmic source:

$$u_{tt} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \sigma(\|\nabla u\|_2^2)u_t + \Delta^2 u = |u|^{q-2}u\log|u|,$$

in $\Omega \times (0, T_{\max})$, subject to null Neumann boundary value conditions, where $\Omega \subset \mathbb{R}^n$ is an open bounded domain with smooth boundary and $p, q > 2$ and $\sigma(\|\nabla u\|_2^2)$ is the Kirchhoff-type coefficient of the damping u_t. Through the utilization of the Faedo–Galerkin approximation, we gain the well-posedness of local weak solutions. When $q \geq p$, we construct algebraic and exponential decay estimates for the energy of global weak solutions. At the same time, relying on the contradiction argument and the auxiliary function method, we prove that the weak solutions blow-up with negative initial energy for $q > p$. When $q < p$, we achieve that the weak solutions are globally bounded.

1 Introduction

In this article, we carry out a study on the following fourth-order p-Laplace hyperbolic problem having Kirchhoff-type weak damping and logarithmic source:

(1.1)
$$\begin{cases} u_{tt} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \sigma(\|\nabla u\|_2^2)u_t + \Delta^2 u \\ \quad = |u|^{q-2}u\log|u|, & (x,t) \in \Omega \times (0, T_{\max}), \\ u = 0, \quad \frac{\partial u}{\partial \eta} = 0, & (x,t) \in \partial\Omega \times (0, T_{\max}), \\ u(x,0) := u_0(x) \in H_0^2(\Omega) \cap W_0^{1,p}(\Omega), \quad u_t(x,0) \\ \quad := u_1(x) \in L^2(\Omega), & x \in \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded domain with smooth boundary $\partial\Omega$; $T_{\max}(\leq +\infty)$ denotes the maximal existence time of (1.1); and $p, q > 2$ are constants. The Kirchhoff coefficient $\sigma(\|\nabla u\|_2^2) := a + b\|\nabla u\|_2^2$, where $\|\cdot\|_s := (\int_\Omega |\cdot|^s dx)^{\frac{1}{s}}$ denotes the $L^s(\Omega)$ norm for $1 \leq s \leq \infty$, and a, b are positive constants. High-order hyperbolic equations find extensive application in numerous fields, for instance, physics, ecology, architecture, fluid mechanics, and so on. During the motion process, the phenomenon

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where the energy of the system gradually dissipates as a result of the interaction between the system itself and the external environment is termed damping. Generally, damping can be classified into strong damping, weak damping, and the structural damping falling between the two. For the Kirchhoff damping $\sigma(\|\nabla u\|_2^2)u_t$, it has been firstly proposed in [12] by Lange and Menzala in the elastic beam equation

$$u_{tt} + \Delta^2 u + \sigma(\|\nabla u\|_2^2)u_t = 0, \quad (x,t) \in \mathbb{R}^n \times (0, T_{\max}).$$

This type of damping mimics the u-dependent mean friction mechanism that operates on an object. After that, Cavalcanti et al. added the memory term $-(g * \Delta u)(t)$ to the above equation in [4]. The global existence and energy decay estimate of the solutions were obtained for any unbounded set and bounded set through the Galerkin estimate and multiplier method. In the Kirchhoff damping where $\sigma(\|\nabla u\|_2^2) \neq 1$, the majority of the literature is centered around the long-time dynamic behavior of the solutions to quasilinear wave equations and fourth-order wave equations (see, e.g., [9]).

The impact of the logarithmic source on the characteristics of solutions for the semilinear wave equations has garnered extensive attention. Yang and Fang in [19] studied the Dirichlet initial-boundary value problem for a class of semilinear wave equations

$$u_{tt} - \Delta u + \sigma(\|\nabla u\|_2^2)|u_t|^{\gamma-2}u_t = |u|^{p-2}u\log|u|, \quad (x,t) \in \Omega \times (0, T_{\max}).$$

They achieved the well-posedness of weak solutions. The global solution of the equation was acquired for $p < \gamma$, and the finite time blow-up solution was obtained for $p > \gamma$. They also investigated a semilinear wave equation featuring Kirchhoff damping and a logarithmic source (see [20])

$$u_{tt} - \Delta u + \sigma(\|\nabla u\|_2^2)u_t = |u|^{q-2}u\log|u|, \quad q > 2, \ (x,t) \in \Omega \times (0, T_{\max}).$$

By enhancing the Faedo–Galerkin method in conjunction with the potential well theory, they established the local existence and global existence of solutions, along with decay estimates of the energy. Simultaneously, relying on the contradiction argument, they demonstrated that there are blow-up solutions for negative initial energy. When the source term is superlinear, Zhang and Li in [21] studied the asymptotic stability and blow-up of the wave equation

$$u_{tt} - \Delta u + \sigma(\|\nabla u\|_2^2)|u_t|^{m-2}u_t = |u|^{p-2}u, \quad (x,t) \in \Omega \times (0, T_{\max}),$$

where $\sigma(\|\nabla u\|_2^2) := \|\nabla u\|_2^{2l}$. They discovered that when the initial energy is small, the asymptotic stability of energy is achieved for polynomial degenerate damping. Moreover, they provided some sufficient conditions for the solution to blow-up in finite time when $p > m + lp$ by employing the potential well theory. For other relevant equations, the interested readers can refer to [3, 8, 13, 15, 17].

Ma and Fang in [16] studied the Dirichlet initial-boundary value problem for semilinear equation

$$u_{tt} - \Delta u - \Delta u_t = u\log|u|^2, \quad (x,t) \in \Omega \times (0, T_{\max}).$$

By employing the potential well theory and the logarithmic Sobolev inequality, the global solvability, energy decay, and infinite blow-up were obtained. For other relevant

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research about potential well method or equations involving logarithmic sources, the interested readers can refer to [1, 2, 5–7, 10, 11, 13, 18, 22, 23].

It can be verified through the above references that the logarithmic sources not only can influence the stability of the solutions of the constant coefficient wave equations but also have an effect on the singularity of the solutions. The interaction among different dampings and the viscoelastic terms has a profound impact on the property of the weak solutions. To the best of our knowledge, the quasilinear hyperbolic equation (1.1) with Kirchhoff nonlocal damping has seldom been taken into consideration previously. Inspired by the works [16, 19–21], we aim to investigate the global existence, decay estimate, and blow-up of weak solutions to (1.1). The p-Laplacian can contribute to the existence of global solutions, whereas the logarithmic source can result in the blow-up of solutions, which can be reflected in the relationship between p and q. When addressing these issues, the Kirchhoff nonlocal damping term, p-Laplacian, and logarithmic source pose significant challenges in applying the potential well method. In the proof of the well-posedness of weak solutions, the Kirchhoff damping term requires stronger regularity of the approximation solutions in the Galerkin procedure. When the initial energy is less than or equal to the depth of the potential well and Nehari's initial energy is nonnegative, the traditional Aubin–Lions lemma is ineffective in the approximation. Hence, it is necessary to modify the Faedo–Galerkin approximation technique, utilize the density theory and the compactness theory to obtain the existence of global weak solutions and their energy decay estimates. The traditional convexity method is also unsuccessful in handling the blow-up of solutions even if the initial energy is negative, and the difficulty stems from the introduced Kirchhoff damping. Consequently, it is necessary to construct new auxiliary functions and prove the differential inequalities to demonstrate the existence of the blow-up solution. The logarithmic source renders some embedding inequalities inapplicable in certain exponential regions. For example, the Sobolev space $W_0^{1,p}(\Omega)$ cannot be embedded into $L^{q+\mu}(\Omega)$ directly when $q+\mu < p$. Therefore, it is necessary to add other conditions so that $W_0^{1,p}(\Omega) \hookrightarrow L^{p+\mu}(\Omega) \subset L^{q+\mu}(\Omega)$.

In the next section, we provide the definition of the weak solution and some crucial lemmas. In Section 3, we prove the well-posedness of the weak solution of (1.1). Section 4 is concerned with the global solvability, decay estimates, and blow-up of the solutions to (1.1) when $q > p$. Section 5 investigates the global existence and decay estimate of the solutions to (1.1) when $q = p$. Then, in Section 6, we prove that the weak solution of (1.1) is globally bounded when $q < p$.

2 Preliminary lemmas

Firstly, we introduce some notations. The norms in $H_0^2(\Omega)$ and $W_0^{1,p}(\Omega)$ are defined by $\|u\|_{H_0^2} := \|\Delta u\|_2$ and $\|u\|_{W_0^{1,p}} := \|\nabla u\|_p$, respectively. The inner product in $L^2(\Omega)$ is denoted by $(f,g) := \int_\Omega f(x)g(x)\mathrm{d}x$. We define $H := (H_0^2(\Omega) \cap W_0^{1,p}(\Omega)) \times L^2(\Omega)$. We have inequalities $\lambda_1\|u\|_2^2 \le \|\Delta u\|_2^2$ and $\lambda_1^{\frac{1}{2}}\|\nabla u\|_2^2 \le \|\Delta u\|_2^2$, where λ_1 denotes the first eigenvalue of the operator Δ^2 subject to null Dirichlet condition. For any $u \in H_0^2(\Omega) \cap W_0^{1,p}(\Omega)$, we define

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$$\text{the potential well functional } J(u) := \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{q}\int_\Omega |u|^q \log|u| \mathrm{d}x + \frac{1}{q^2}\|u\|_q^q,$$

$$\text{the Nehari functional } I(u) := \|\Delta u\|_2^2 + \|\nabla u\|_p^p - \int_\Omega |u|^q \log|u| \mathrm{d}x,$$

$$\text{the energy functional } E(t) := E(u(t), u_t(t)) = \frac{1}{2}\|u_t\|_2^2 + J(u).$$

The initial energy is $E(0) = \frac{1}{2}\|u_1\|_2^2 + J(u_0)$. Because $E'(t) = -\sigma(\|\nabla u\|_2^2)\|u_t\|_2^2 \le 0$, $E(t)$ is decreasing with respect to t. By the definitions of $J(u), I(u)$, and $E(t)$, we get

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + J(u) = \frac{1}{2}\|u_t\|_2^2 + \frac{q-2}{2q}\|\Delta u\|_2^2 + \frac{q-p}{pq}\|\nabla u\|_p^p + \frac{1}{q^2}\|u\|_q^q + \frac{1}{q}I(u).$$

Next, we define the potential well $\mathcal{W} := \mathcal{W}_1 \cup \mathcal{W}_2$, where

$$\mathcal{W}_1 := \left\{(u, u_t) \in (H_0^2(\Omega) \cap W_0^{1,p}(\Omega))\backslash\{0\} \times L^2(\Omega)\backslash\{0\} | I(u) > 0, E(t) < d\right\},$$
$$\mathcal{W}_2 := \left\{(u, u_t) \in (H_0^2(\Omega) \cap W_0^{1,p}(\Omega))\backslash\{0\} \times L^2(\Omega)\backslash\{0\} | I(u) > 0, E(t) = d\right\},$$

and the outside of potential well $\mathcal{V}_0 := \left\{(u, u_t) \in H | I(u) < 0, \|\Delta u\|_2^2 \neq 0, \|u_t\|_2^2 \neq 0\right\}$, and the Nehari manifold $\mathcal{N} := \left\{u \in H_0^2(\Omega) \cap W_0^{1,p}(\Omega) | I(u) = 0, \|\Delta u\|_2^2 \neq 0\right\}$. The potential well depth is defined by $d := \inf_{u\in\mathcal{N}} J(u)$. Define two specific constants

$$r(\alpha) := \left(\mathrm{e}\alpha B_\alpha^{-q-\alpha}\right)^{\frac{1}{q+\alpha-2}}, \quad M := \frac{q-2}{2q}\left(\mathrm{e}\alpha B_\alpha^{-q-\alpha}\right)^{\frac{2}{q+\alpha-2}}, \tag{2.1}$$

where α is a positive constant and B_α is the embedding constant of $H_0^2(\Omega) \hookrightarrow L^{q+\alpha}(\Omega)$.

There are the embedding results, which will play an important role in the proof of the main results:

$$H_0^2(\Omega) \hookrightarrow L^{q+\alpha}(\Omega), \quad H_0^2(\Omega) \hookrightarrow L^{2(q+\alpha-1)}(\Omega), \quad H_0^2(\Omega) \hookrightarrow L^{2(q+\alpha-2)\frac{q-1}{q-2}}(\Omega),$$

provided

$$(\mathcal{H}) \qquad \begin{cases} 2 < q < \dfrac{2n-4}{n-4},\ 0 < \alpha < \left(\dfrac{2n-4}{n-4} - q\right)\dfrac{q-2}{q-1}, & \text{for } n > 4, \\ 2 < q < +\infty,\ 0 < \alpha < +\infty, & \text{for } n \le 4. \end{cases}$$

Secondly, we give the definition of the weak solution for (1.1), which will be proved in Theorem 3.1 of the present article.

Definition 2.1 A function $u(x,t)$ is called a weak solution of problem (1.1) for $(x,t) \in \Omega \times [0,T] (0 < T < +\infty)$ in the sense of $u \in C([0,T]; H_0^2(\Omega) \cap W_0^{1,p}(\Omega))$,

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$u_t \in L^\infty([0,T];H_0^1(\Omega)\cap L^2(\Omega))$, such that $u(x,0)=u_0(x)$, $u_t(x,0)=u_1(x)$ for $x\in\Omega$, and

$$\int_\Omega u_t\phi \mathrm{d}x + \int_0^t\int_\Omega (\Delta u\Delta\phi + |\nabla u|^{p-2}\nabla u\cdot\nabla\phi)\mathrm{d}x\mathrm{d}s + \int_0^t \sigma(\|\nabla u\|_2^2)\int_\Omega u_s\phi \mathrm{d}x\mathrm{d}s = \int_0^t\int_\Omega |u|^{q-2}u\phi\log|u|\mathrm{d}x\mathrm{d}s + \int_\Omega u_1\phi\mathrm{d}x \tag{2.2}$$

for any test function $\phi\in H_0^2(\Omega)\cap W_0^{1,p}(\Omega)$ and a.e. $t\in[0,T]$.

Thirdly, we present several lemmas, which will be utilized in the proof of the main results.

Lemma 2.1 *Let $u\in H_0^2(\Omega)\cap W_0^{1,p}(\Omega)$, $\|\Delta u\|_2^2\neq 0$, $q\geq p$. There exists a unique $\lambda_*>0$ such that*

$$I(\lambda u)>0 \text{ for } 0<\lambda<\lambda_*;\quad I(\lambda u)=0 \text{ for } \lambda=\lambda_*;\quad I(\lambda u)<0 \text{ for } \lambda>\lambda_*. \tag{2.3}$$

Proof By the definition of $I(u)$, we have

$$I(\lambda u)=\lambda^q\left(\lambda^{2-q}\|\Delta u\|_2^2+\lambda^{p-q}\|\nabla u\|_p^p-\int_\Omega |u|^q\log|\lambda u|\mathrm{d}x\right).$$

Let $K(\lambda):=\lambda^{2-q}\|\Delta u\|_2^2+\lambda^{p-q}\|\nabla u\|_p^p-\int_\Omega|u|^q\log|\lambda u|\mathrm{d}x$. It is easy to get that $\lim_{\lambda\to 0^+}K(\lambda)=+\infty$, $\lim_{\lambda\to+\infty}K(\lambda)=-\infty$, and $K(\lambda)$ is strictly decreasing with respect to λ in $(0,+\infty)$. Therefore, there exists a unique $\lambda_*\in(0,+\infty)$ such that $K(\lambda_* u)=0$; $K(\lambda u)>0$ for $0<\lambda<\lambda_*$; and $K(\lambda u)<0$ for $\lambda>\lambda_*$. From $I(\lambda u)=\lambda^q K(\lambda)$, we get (2.3). ∎

From definitions of $J(u)$ and $I(u)$, we have $I(\lambda u)=\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda u)$. Then $J(\lambda u)$ is strictly increasing in $(0,\lambda_*)$ and strictly decreasing in $(\lambda_*,+\infty)$, respectively. It takes its maximum at $\lambda=\lambda_*$.

Here is the logarithmic inequality:

$$s^q\log s\leq\frac{1}{\mathrm{e}\alpha}s^{q+\alpha},\ \text{for any } s>0,\ \alpha>0. \tag{2.4}$$

Lemma 2.2 *Let $u\in H_0^2(\Omega)\cap W_0^{1,p}(\Omega)$ and $\|\Delta u\|_2\neq 0$. For any α defined in (2.1) satisfying $(\mathcal{H})$, we have the results: (i) if $0<\|\Delta u\|_2\leq r(\alpha)$, then $I(u)\geq 0$ and (ii) if $I(u)<0$, then $\|\Delta u\|_2>r(\alpha)$.*

Proof Using (2.4) and the embedding $H_0^2(\Omega)\hookrightarrow L^{q+\alpha}(\Omega)$, we get

$$\begin{aligned} I(u)&\geq\|\Delta u\|_2^2-\int_\Omega|u|^q\log|u|\mathrm{d}x\geq\|\Delta u\|_2^2-\frac{1}{\mathrm{e}\alpha}\|u\|_{q+\alpha}^{q+\alpha}\\ &\geq\|\Delta u\|_2^2\left(1-\frac{1}{\mathrm{e}\alpha}B_\alpha^{q+\alpha}\|\Delta u\|_2^{q+\alpha-2}\right). \end{aligned} \tag{2.5}$$

Consequently, if $0<\|\Delta u\|_2\leq r(\alpha)$, then $I(u)\geq 0$. Moreover, if $I(u)<0$, it is evident that $\|\Delta u\|_2>r(\alpha)$. ∎

Lemma 2.3 *Let $(\mathcal{H})$ hold and $q\geq p$. The potential well depth $d>M$, where M is defined in (2.1).*

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Proof It follows from Lemma 2.2 and definitions of $I(u)$ and $J(u)$ that

$$J(\lambda_* u) = \frac{(q-2)\lambda_*^2}{2q}\|\Delta u\|_2^2 + \frac{(q-p)\lambda_*^p}{pq}\|\nabla u\|_p^p + \frac{\lambda_*^q}{q^2}\|u\|_q^q. \tag{2.6}$$

Just like the proof for Lemma 2.2, we get

$$0 = I(\lambda_* u) \geq \lambda_*^2\|\Delta u\|_2^2\left(1 - \lambda_*^{q+\alpha-2}\frac{1}{e\alpha}B_\alpha^{q+\alpha}\|\Delta u\|_2^{q+\alpha-2}\right).$$

It follows from (2.5) that $\left(e\alpha B_\alpha^{-q-\alpha}\right)^{\frac{1}{q+\alpha-2}} \leq \lambda_*\|\Delta u\|_2$. Hence, we get $J(\lambda_* u) > \frac{q-2}{2q}\left(e\alpha B_\alpha^{-q-\alpha}\right)^{\frac{2}{q+\alpha-2}}$. Therefore, by the definitions of M and d, we have $d > M$. ∎

Lemma 2.4 *For any $u \in H_0^2(\Omega) \cap W_0^{1,p}(\Omega)$, if $\int_\Omega |u|^q \log|u|\mathrm{d}x \geq 0$ and there exists a constant $D > 0$ such that $\|\Delta u\|_2 \leq D$, then*

$$\|u\|_2^2 \leq K_1\left[\left(\int_\Omega |u|^q \log|u|\mathrm{d}x\right)^{\frac{2}{q}} + \|\Delta u\|_2^{\frac{4}{q}}\right], \quad \textit{where } K_1 := \lambda_1^{-1}D^{2-\frac{4}{q}}.$$

Proof Using the embedding inequality $\lambda_1\|u\|_2^2 \leq \|\Delta u\|_2^2$, we have

$$\|u\|_2^2 \leq \lambda_1^{-1}\|\Delta u\|_2^2 \leq \lambda_1^{-1}D^{2-\frac{4}{q}}\|\Delta u\|_2^{\frac{4}{q}} \leq K_1\left[\left(\int_\Omega |u|^q \log|u|\mathrm{d}x\right)^{\frac{2}{q}} + \|\Delta u\|_2^{\frac{4}{q}}\right].$$

∎

Lemma 2.5 *Let $(\mathcal{H})$ hold. For any $u \in H_0^2(\Omega) \cap W_0^{1,p}(\Omega)$ and $2 \leq s \leq q$, if $\int_\Omega |u|^q \log|u|\mathrm{d}x \geq 0$, then*

$$\left(\int_\Omega |u|^q \log|u|\mathrm{d}x\right)^{\frac{s}{q}} \leq K_2\left(\int_\Omega |u|^q \log|u|\mathrm{d}x + \|\Delta u\|_2^2\right), \tag{2.7}$$

where $K_2 := \max\left\{1, B_\alpha^2(e\alpha)^{-\frac{2}{q+\alpha}}\right\}$ and B_α is defined in (2.1).

Proof If $\int_\Omega |u|^q \log|u|\mathrm{d}x \geq 1$, then

$$\left(\int_\Omega |u|^q \log|u|\mathrm{d}x\right)^{\frac{s}{q}} \leq \int_\Omega |u|^q \log|u|\mathrm{d}x \leq \int_\Omega |u|^q \log|u|\mathrm{d}x + \|\Delta u\|_2^2.$$

If $\int_\Omega |u|^q \log|u|\mathrm{d}x < 1$, then we denote $\Omega_1 := \{x \in \Omega \| |u| > 1\}$ and, for any $\beta \in (0,2)$,

$$\begin{aligned}\left(\int_\Omega |u|^q \log|u|\mathrm{d}x\right)^{\frac{s}{q}} &\leq \left(\int_{\Omega_1} |u|^q \log|u|\mathrm{d}x\right)^{\frac{\beta}{q}} \\ &\leq \left(\frac{1}{e\alpha}\int_\Omega |u|^{q+\alpha}\mathrm{d}x\right)^{\frac{\beta}{q}} \\ &= (e\alpha)^{-\frac{\beta}{q}}\|u\|_{q+\alpha}^{\frac{(q+\alpha)\beta}{q}}.\end{aligned}$$

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Choosing $\beta := \frac{2q}{q+\alpha} \in (0,2)$ and using the embedding $H_0^2(\Omega) \hookrightarrow L^{q+\alpha}(\Omega)$, we get

$$\left(\int_\Omega |u|^q \log|u| dx\right)^{\frac{s}{q}} \le (e\alpha)^{-\frac{2}{q+\alpha}} \|u\|_{q+\alpha}^2 \le B_\alpha^2 (e\alpha)^{-\frac{2}{q+\alpha}} \|\Delta u\|_2^2$$
$$\le \int_\Omega |u|^q \log|u| dx + B_\alpha^2 (e\alpha)^{-\frac{2}{q+\alpha}} \|\Delta u\|_2^2.$$

Summarizing the above two cases, we get (2.7). ∎

3 Well-posedness of weak solution

In this section, our focus is on the well-posedness of the solution as defined in Definition 2.1.

Theorem 3.1 *Let initial data $(u_0, u_1) \in H$ and $(\mathcal{H})$ hold. There exists a positive time T_0 such that problem (1.1) admits a unique weak solution $u(x,t)$ in $\Omega \times [0, T_0]$ in the sense of $u \in C([0,T_0]; H_0^2(\Omega) \cap W_0^{1,p}(\Omega))$, $u_t \in L^\infty([0,T_0]; H_0^1(\Omega)) \cap L^2(\Omega)$, $u(x,0) = u_0(x)$, $u_t(x,0) = u_1(x)$ in Ω, and for any test function $\phi \in H_0^2(\Omega) \cap W_0^{1,p}(\Omega)$ and a.e. $t \in [0,T_0]$,*

$$\int_\Omega u_t \phi dx + \int_0^t \int_\Omega (\Delta u \Delta\phi + |\nabla u|^{p-2} \nabla u \cdot \nabla \phi) dx ds$$
$$+ \int_0^t \sigma(\|\nabla u\|_2^2) \int_\Omega u_s \phi dx ds = \int_0^t \int_\Omega |u|^{q-2} u \phi \log|u| dx ds + \int_\Omega u_1 \phi dx.$$

The following cases only have one that holds:
(i) $T_{\max} = +\infty$, *which means the weak solution is globally existent.*
(ii) $T_{\max} < +\infty$, *which indicates the weak solution undergoes blow-up in a finite time in this regard*

$$\lim_{t \to T_{\max}^-} \left(\|u_t\|_2^2 + \|\Delta u\|_2^2 + \|\nabla u\|_p^p\right) = +\infty.$$

Proof We divided the proof into five steps.
Step 1. *Approximation.* Let $\{\omega_j\}_{j=1}^\infty$ be an orthonormal basis in $H_0^2(\Omega) \cap W_0^{1,p}(\Omega)$, which are the eigenfunctions of Δ^2 with the boundary value conditions in (1.1). We define the finite-dimensional space $V_m := \text{span}\{\omega_1, \omega_2, \dots, \omega_m\}$, $m \in \mathbb{Z}^+$, and construct the approximate solution $u_m(x,t) := \sum_{j=1}^m b_{jm}(t)\omega_j(x)$, where $u_m(x,t)$ satisfies the following Cauchy problem:

$$\int_\Omega (u_{mtt}\omega_j + \Delta u_m \Delta \omega_j + |\nabla u_m|^{p-2} \nabla u_m \cdot \nabla \omega_j) dx$$
$$+ \sigma(\|\nabla u_m\|_2^2) \int_\Omega u_{mt} \omega_j dx = \int_\Omega |u_m|^{q-2} u_m \omega_j \log|u_m| dx, \tag{3.1}$$

$$u_m(x,0) := u_{0m} = \sum_{j=1}^m b_{jm}(0)\omega_j(x) \to u_0 \quad \text{in } H_0^2(\Omega) \cap W_0^{1,p}(\Omega), \tag{3.2}$$

$$u_{mt}(x,0) := u_{1m} = \sum_{j=1}^m b'_{jm}(0)\omega_j(x) \to u_1 \quad \text{in } L^2(\Omega). \tag{3.3}$$

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By standard methods in ODEs, we prove the existence and uniqueness of solution to approximate problem (3.1)–(3.3) on some of intervals $[0, T_m)$ for $0 < T_m < +\infty$.

Step 2. *Priori estimate I.* Define a function

$$\overline{E}_m(t) := \frac{1}{2}\|u_{mt}\|_2^2 + \frac{1}{2}\|\Delta u_m\|_2^2 + \frac{1}{p}\|\nabla u_m\|_p^p. \tag{3.4}$$

Multiplying (3.1) by $b'_{jm}(t)$, summing on $j = 1, 2, \ldots, m$, and integrating with respect to t from 0 to $t(\leq T_m)$, we obtain

$$\begin{aligned} &\overline{E}_m(t) + \int_0^t \sigma(\|\nabla u_m(s)\|_2^2)\|u_{ms}(s)\|_2^2 \mathrm{d}s \\ &= \overline{E}_m(0) + \int_0^t \int_\Omega |u_m(s)|^{q-2}u_m(s)u_{ms}(s)\log|u_m(s)|\mathrm{d}x\mathrm{d}s. \end{aligned} \tag{3.5}$$

From (3.2)–(3.4), we have $\overline{E}_m(0) < +\infty$. On the other hand, by (3.5), Young's inequality, and the embedding $H_0^2(\Omega) \hookrightarrow L^{2(q+\alpha-1)}(\Omega)$, we arrive at

$$\begin{aligned} \int_\Omega |u_m|^{q-2}u_m u_{mt}\log|u_m|\mathrm{d}x &\leq \frac{1}{\mathrm{e}\alpha}\int_\Omega |u_m|^{q+\alpha-1}|u_{mt}|\mathrm{d}x \\ &\leq \frac{1}{2\varepsilon_1\alpha^2\mathrm{e}^2}\int_\Omega |u_m|^{2(q+\alpha-1)}\mathrm{d}x + \frac{\varepsilon_1}{2}\|u_{mt}\|_2^2 \\ &\leq \frac{B_1^{2(q+\alpha-1)}}{2\varepsilon_1\alpha^2\mathrm{e}^2}\|\Delta u_m\|_2^2 + \frac{\varepsilon_1}{2}\|u_{mt}\|_2^2, \end{aligned} \tag{3.6}$$

where α is defined in $(\mathcal{H})$ and B_1 is the embedding constant of $H_0^2(\Omega) \hookrightarrow L^{2(q+\alpha-1)}(\Omega)$. Replacing (3.6) in (3.5) and using $\sigma(s) \geq a$, we have

$$\begin{aligned} &\overline{E}_m(t) + \left(a - \frac{\varepsilon_1}{2}\right)\int_0^t \|u_{ms}(s)\|_2^2\mathrm{d}s \\ &\leq \overline{E}_m(0) + \frac{B_1^{2(q+\alpha-1)}}{2\varepsilon_1\alpha^2\mathrm{e}^2}\int_0^t \|\Delta u_m\|_2^{2(q+\alpha-1)}\mathrm{d}s \\ &\leq \overline{E}_m(0) + \frac{B_1^{2(q+\alpha-1)}}{2\varepsilon_1\alpha^2\mathrm{e}^2}\int_0^t \left(\|u_{ms}(s)\|_2^2 + \|\Delta u_m\|_2^2 + \|\nabla u_m\|_p^p\right)^{q+\alpha-1}\mathrm{d}s \\ &\leq \overline{E}_m(0) + \frac{B_1^{2(q+\alpha-1)}}{2\varepsilon_1\alpha^2\mathrm{e}^2}p^{q+\alpha-1}\int_0^t \overline{E}_m^{q+\alpha-1}(s)\mathrm{d}s, \end{aligned}$$

where $q + \alpha - 1 > 1$. We take ε_1 small enough that $\varepsilon_1 < 2a$. Then by Gronwall's inequality, there exists a constant $C_{T_0} > 0$ such that

$$\|u_{mt}\|_2^2 + \|\Delta u_m\|_2^2 + \|\nabla u_m\|_p^p + \int_0^t \|u_{ms}(s)\|_2^2\mathrm{d}s \leq C_{T_0}, \quad \text{for all } m \in N_+. \tag{3.7}$$

Hence,

$$\{u_m\} \text{ is uniformly bounded in } L^\infty([0, T_0]; H_0^2(\Omega)), \tag{3.8}$$

$$\{u_m\} \text{ is uniformly bounded in } L^\infty([0, T_0]; W_0^{1,p}(\Omega)), \tag{3.9}$$

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$$\{u_{mt}\} \text{ is uniformly bounded in } L^\infty([0,T_0];L^2(\Omega)), \tag{3.10}$$

$$\{u_{mt}\} \text{ is uniformly bounded in } L^2([0,T_0]\times\Omega). \tag{3.11}$$

Step 3. *Priori estimate II.* Define $z_m := u_{m_1} - u_{m_2}$ for any $m_1, m_2 \in \mathbb{Z}^+$. By (2.2), we get

$$\begin{aligned}
&\int_\Omega z_{mtt}\omega_j \mathrm{d}x + \int_\Omega \Delta z_m \Delta\omega_j \mathrm{d}x + \int_\Omega |\nabla u_{m_1}|^{p-2}\nabla u_{m_1}\cdot\nabla\omega_j \mathrm{d}x\\
&- \int_\Omega |\nabla u_{m_2}|^{p-2}\nabla u_{m_2}\cdot\nabla\omega_j \mathrm{d}x + \sigma(\|\nabla u_{m_1}\|_2^2)\int_\Omega u_{m_1t}\omega_j \mathrm{d}x\\
&\quad - \sigma(\|\nabla u_{m_2}\|_2^2)\int_\Omega u_{m_2t}\omega_j \mathrm{d}x\\
&= \int_\Omega |u_{m_1}|^{q-2}u_{m_1}\omega_j \log|u_{m_1}|\mathrm{d}x - \int_\Omega |u_{m_2}|^{q-2}u_{m_2}\omega_j\log|u_{m_2}|\mathrm{d}x,
\end{aligned}$$

where the initial data $(z_{0m}, z_{1m}) := (u_{0m_1} - u_{0m_2}, u_{1m_1} - u_{1m_2}) \in H$. Replacing ω_j by z_{mt} in the above equation, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|z_{mt}\|_2^2 + \|\Delta z_m\|_2^2) + \sigma(\|\nabla u_{m_1}\|_2^2)\|z_{mt}\|_2^2\\
&+ \int_\Omega \left(|\nabla u_{m_1}|^{p-2}\nabla u_{m_1} - |\nabla u_{m_2}|^{p-2}\nabla u_{m_2}\right)\cdot\nabla z_{mt}\mathrm{d}x\\
&= \left[\sigma(\|\nabla u_{m_2}\|_2^2) - \sigma(\|\nabla u_{m_1}\|_2^2)\right]\int_\Omega u_{m_2t}z_{mt}\mathrm{d}x + \int_\Omega |u_{m_1}|^{q-2}u_{m_1}z_{mt}\log|u_{m_1}|\mathrm{d}x\\
&- \int_\Omega |u_{m_2}|^{q-2}u_{m_2}z_{mt}\log|u_{m_2}|\mathrm{d}x.
\end{aligned}$$

For all $x, y \in \mathbb{R}^n$ and $p > 2$, using $2^{2-p}|x-y|^p \le \left(|x|^{p-2}x - |y|^{p-2}y\right)\cdot(x-y)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|z_{mt}\|_2^2 + \frac{1}{2}\|\Delta z_m\|_2^2 + \frac{1}{2^{p-2}p}\|\nabla z_m\|_p^p\right) \le I_1 + I_2 + I_3, \tag{3.12}$$

where the notations are defined as

$$I_1 := -\sigma(\|\nabla u_{m_1}\|_2^2)\|z_{mt}\|_2^2, \quad I_2 := \left(\sigma(\|\nabla u_{m_2}\|_2^2) - \sigma(\|\nabla u_{m_1}\|_2^2)\right)\int_\Omega u_{m_2t}z_{mt}\mathrm{d}x,$$

$$I_3 := \int_\Omega \left(|u_{m_1}|^{q-2}u_{m_1}\log|u_{m_1}| - |u_{m_2}|^{q-2}u_{m_2}\log|u_{m_2}|\right)z_{mt}\mathrm{d}x.$$

Now, let us estimate I_1, I_2, and I_3. First, from $\sigma(s) \ge a$, we have

$$I_1 \le -a\|z_{mt}\|_2^2. \tag{3.13}$$

Using $\sigma(s) = a + bs \in C^1([0,+\infty))$, and Hölder's and Young's inequalities, we have

$$\begin{aligned}
I_2 &\le b|\|\nabla u_{m_1}\|_2 - \|\nabla u_{m_2}\|_2|(\|\nabla u_{m_1}\|_2 + \|\nabla u_{m_2}\|_2)\int_\Omega |u_{m_2t}||z_{mt}|\mathrm{d}x\\
&\le b\|\nabla z_m\|_2(\|\nabla u_{m_1}\|_2 + \|\nabla u_{m_2}\|_2)\int_\Omega |u_{m_2t}||z_{mt}|\mathrm{d}x\\
&\le 2b\lambda_1^{-\frac{1}{4}}C_{T_0}^{\frac{1}{2}}\|\nabla z_m\|_2\|u_{m_2t}\|_2\|z_{mt}\|_2\\
&\le \frac{b^2\lambda_1^{-\frac{1}{2}}C_{T_0}}{\varepsilon_2}\|\nabla z_m\|_2^2\|u_{m_2t}\|_2^2 + \varepsilon_2\|z_{mt}\|_2^2.
\end{aligned} \tag{3.14}$$

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Using Hölder's inequality again, we obtain

$$I_3 \le \|G(u_{m_1}) - G(u_{m_2})\|_2 \|z_{mt}\|_2, \tag{3.15}$$

where $G(s) := |s|^{q-2}s\log|s|$. From (2.4) and the Mean Value theorem, there exists a constant $\xi_1 \in (0,1)$ such that

$$\begin{aligned}
&\|G(u_{m_1}) - G(u_{m_2})\|_2 \le \|G'(\xi_1 u_{m_1} + (1-\xi_1)u_{m_2})z_m\|_2 \\
&\le \left\| [1 + (q-1)\log|\xi_1 u_{m_1} + (1-\xi_1)u_{m_2}|]\,[\xi_1 u_{m_1} + (1-\xi_1)u_{m_2}]^{q-2} z_m \right\|_2 \\
&\le \left\| \left[|\xi_1 u_{m_1} + (1-\xi_1)u_{m_2}|^{q-2} + \frac{q-1}{e\alpha}|\xi_1 u_{m_1} + (1-\xi_1)u_{m_2}|^{q+\alpha-2} \right] z_m \right\|_2 \\
&\le \left\| (|u_{m_1}| + |u_{m_2}|)^{q-2} |z_m| \right\|_2 + \frac{q-1}{e\alpha} \left\| (|u_{m_1}| + |u_{m_2}|)^{q+\alpha-2} z_m \right\|_2 .
\end{aligned} \tag{3.16}$$

Applying Hölder's inequality, the embedding $H_0^2(\Omega) \hookrightarrow L^{2(q-1)}(\Omega)$, and the inequality $|a+b|^r \le 2^{r-1}(|a|^r + |b|^r)$, $r \ge 1$, to the right side of (3.16), we get

$$\begin{aligned}
\left\| (|u_{m_1}| + |u_{m_2}|)^{q-2} z_m \right\|_2^2 &\le \| |u_{m_1}| + |u_{m_2}| \|_{2(q-1)}^{2(q-2)} \|z_m\|_{2(q-1)}^2 \\
&\le 2^{\frac{(2q-3)(q-2)}{q-1}} \left(\|u_{m_1}\|_{2(q-1)}^{2(q-1)} + \|u_{m_2}\|_{2(q-1)}^{2(q-1)} \right)^{\frac{q-2}{q-1}} \|z_m\|_{2(q-1)}^2 \\
&\le C_1 \left(\|\Delta u_{m_1}\|_2^{2(q-1)} + \|\Delta u_{m_2}\|_2^{2(q-1)} \right)^{\frac{q-2}{q-1}} \|\Delta z_m\|_2^2,
\end{aligned} \tag{3.17}$$

where $C_1 := 2^{\frac{(2q-3)(q-2)}{q-1}} B_2^{2(q-1)}$ and B_2 is the embedding constant of $H_0^2(\Omega) \hookrightarrow L^{2(q-1)}(\Omega)$. Similarly, for $q^* := 2(q+\alpha-2)\frac{q-1}{q-2}$, we arrive at

$$\begin{aligned}
&\left\| (|u_{m_1}| + |u_{m_2}|)^{q+\alpha-2} z_m \right\|_2^2 \\
&\le 2^{\frac{(q^*-1)(q-2)}{q-1}} B_3^{2(q+\alpha)-2} |\Omega|^{\frac{\alpha}{(q+\alpha-2)(q-1)}} \left(\|\Delta u_{m_1}\|_2^{q^*} + \|\Delta u_{m_2}\|_2^{q^*} \right)^{\frac{q-2}{q-1}} \|\Delta z_m\|_2^2.
\end{aligned} \tag{3.18}$$

Denoting $C_2 := 2^{\frac{(q^*-1)(q-2)}{q-1}} B_3^{2(q+\alpha)-2} |\Omega|^{\frac{\alpha}{(q+\alpha-2)(q-1)}}$ and substituting (3.16)–(3.18) into (3.15) and using Young's inequality, we get

$$\begin{aligned}
I_3 \le{}& \frac{1}{2} C_1^{\frac{1}{2}} \left(\|\Delta u_{m_1}\|_2^{2(q-1)} + \|\Delta u_{m_2}\|_2^{2(q-1)} \right)^{\frac{q-2}{2(q-1)}} \left(\|\Delta z_m\|_2^2 + \|z_{mt}\|_2^2 \right) \\
&+ \frac{q-1}{2e\alpha} C_2^{\frac{1}{2}} \left(\|\Delta u_{m_1}\|_2^{q^*} + \|\Delta u_{m_2}\|_2^{q^*} \right)^{\frac{q-2}{2(q-1)}} \left(\|\Delta z_m\|_2^2 + \|z_{mt}\|_2^2 \right).
\end{aligned} \tag{3.19}$$

Now replacing (3.13), (3.14), and (3.19) into (3.12), choosing $\varepsilon_2 < a$, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2}\|z_{mt}\|_2^2 + \frac{1}{2}\|\Delta z_m\|_2^2 + \frac{1}{2^{p-2}p}\|\nabla z_m\|_p^p \right) \\
&\le (\varepsilon_2 - a)\|z_{mt}\|_2^2 + \frac{1}{2}\max\left\{ C_1^{\frac{1}{2}}, \frac{q-1}{2e\alpha} C_2^{\frac{1}{2}} \right\} \left\{ \left[\|\Delta u_{m_1}\|_2^{2(q-1)} + \|\Delta u_{m_2}\|_2^{2(q-1)} \right]^{\frac{q-2}{2(q-1)}} \right. \\
&\left. + \frac{b^2 \lambda_1^{-\frac{1}{2}} C_{T_0}}{\varepsilon_2} \|\nabla z_m\|_2^2 \|u_{m_2 t}\|_2^2 + \left(\|\Delta u_{m_1}\|_2^{q^*} + \|\Delta u_{m_2}\|_2^{q^*} \right)^{\frac{q-2}{2(q-1)}} \right\} \left(\|\Delta z_m\|_2^2 + \|z_{mt}\|_2^2 \right)
\end{aligned}$$

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$$\leq \frac{b^2\lambda_1^{-\frac{5}{2}}C_{T_0}^2}{\varepsilon_2}\|\Delta z_m\|_2^2 + C_3\left(\|\Delta z_m\|_2^2 + \|z_{mt}\|_2^2\right)$$
$$\leq C_4\left(\frac{1}{2}\|z_{mt}\|_2^2 + \frac{1}{2}\|\Delta z_m\|_2^2 + \frac{1}{2^{p-2}p}\|\nabla z_m\|_p^p\right),$$

where the positive constants are defined as

$$C_3 := \frac{1}{2}\max\left\{C_1^{\frac{1}{2}}, \frac{q-1}{2e\alpha}C_2^{\frac{1}{2}}\right\}\left[2^{\frac{q-2}{2(q-1)}}C_{T_0}^{\frac{q-2}{2}} + 2^{\frac{q^*(q-2)}{4(q-1)}}C_{T_0}^{\frac{q^*(q-2)}{4(q-1)}}\right],$$

$$C_4 := \max\left\{2, p2^{p-2}\right\}\max\left\{1, \frac{b^2\lambda_1^{-\frac{5}{2}}C_{T_0}^2}{\varepsilon_2} + C_3\right\}.$$

By Gronwall's inequality, there exists a constant $\overline{C}_{T_0} > 0$ such that

(3.20)
$$\|z_{mt}\|_2^2 + \|\Delta z_m\|_2^2 + \|\nabla z_m\|_p^p \leq \overline{C}_{T_0}\left(\|z_{1m}\|_2^2 + \|\Delta z_{0m}\|_2^2 + \|\nabla z_{0m}\|_p^p\right), \quad \text{for all } m \in N_+.$$

Step 4. *Passing to the limit.* It follows from (3.8)–(3.11) that there exists a function u and a subsequence of $\{u_m\}_{m=1}^{\infty}$ (still denoted by it) such that $u_m \overset{*}{\rightharpoonup} u$ in $L^\infty([0,T_0];H_0^2(\Omega))$, $u_m \overset{*}{\rightharpoonup} u$ in $L^\infty([0,T_0];W_0^{1,p}(\Omega))$, $u_{mt} \overset{*}{\rightharpoonup} u_t$ in $L^\infty([0,T_0];L^2(\Omega))$, and $u_{mt} \rightharpoonup u_t$ in $L^2([0,T_0];L^2(\Omega))$. The estimate (3.20) guarantees the limit for the Kirchhoff damping. From (3.20) and taking (3.2) and (3.3) into account, we deduce $z_m(0) := u_{m_1}(0) - u_{m_2}(0) \to 0$ in $H_0^2(\Omega) \cap W_0^{1,p}(\Omega)$, $z_{mt}(0) := u_{m_1t}(0) - u_{m_2t}(0) \to 0$ in $L^2(\Omega)$, and $\|z_{mt}\|_2^2 + \|\Delta z_m\|_2^2 + \|\nabla z_m\|_p^p \to 0$. Hence,

(3.21)
$$u_m \to u \text{ in } C^0([0,T_0];H_0^2(\Omega)\cap W_0^{1,p}(\Omega)), \quad u_{mt} \to u_t \text{ in } C^0([0,T_0];L^2(\Omega)).$$

Considering the convergence in (3.21), we have $u_m \to u$, a.e. in $\Omega\times(0,T_0)$. Hence, $\sigma(\|\nabla u_m\|_2^2) \to \sigma(\|\nabla u\|_2^2)$ for $t \in (0,T_0)$. On the other hand, using (2.4), (3.8), and (3.9), the embedding $H_0^2(\Omega) \hookrightarrow L^{2(q+\alpha-1)}(\Omega)$, and the inequality $|x^{q-1}\log x| \leq [e(q-1)]^{-1}$ for $x \in (0,1)$, we get

$$\begin{aligned}\int_\Omega (|u_m|^{q-2}u_m\log|u_m|)^2\mathrm{d}x &\leq [e(q-1)]^{-2}|\Omega| + \int_{\{x\in\Omega||u_m|>1\}}(|u_m|^{q-2}u_m\log|u_m|)^2\mathrm{d}x\\ &\leq [e(q-1)]^{-2}|\Omega| + \left(\frac{1}{e\alpha}\right)^2\|u_m\|_{2(q+\alpha-1)}^{2(q+\alpha-1)}\\ &\leq [e(q-1)]^{-2}|\Omega| + \frac{B_1^{2(q+\alpha-1)}}{(e\alpha)^2}\|\Delta u_m\|_2^{2(q+\alpha-1)}\\ &\leq [e(q-1)]^{-2}|\Omega| + \frac{B_1^{2(q+\alpha-1)}}{(e\alpha)^2}C_{T_0}^{q+\alpha-1}. \qquad (3.22)\end{aligned}$$

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Then $|u_m|^{q-2}u_m \log|u_m| \overset{*}{\rightharpoonup} |u|^{q-2}u\log|u|$ in $L^\infty([0,T_0];L^2(\Omega))$. Integrating (3.1) over $(0,t)$, we obtain

$$\int_\Omega u_{mt}\omega_j \mathrm{d}x + \int_0^t \int_\Omega (\Delta u_m \Delta\omega_j + |\nabla u_m|^{p-2}\nabla u_m \cdot \nabla\omega_j)\mathrm{d}x\mathrm{d}s + \int_0^t \sigma(\|\nabla u_m\|_2^2)\int_\Omega u_{ms}\omega_j \mathrm{d}x\mathrm{d}s = \int_0^t \int_\Omega |u_m|^{q-2}u_m\omega_j \log|u_m|\mathrm{d}x\mathrm{d}s + \int_\Omega u_{1m}\omega_j \mathrm{d}x.$$

Therefore, from (3.1) and the conditions (3.2) and (3.3) about the initial data, we pass the limit to the above equation and get the local weak solution satisfying $u(x,0) := u_0(x)$, $u_t(x,0) := u_1(x)$ in Ω.

The uniqueness of the weak solution can be acquired through the method in the work [14].

Step 5. *Extension of local weak solutions.* By using contradiction argument, we suppose that $T_{\max} < +\infty$ and $\lim_{t\to T_{\max}^-}(\|u_t\|_2^2 + \|\Delta u\|_2^2 + \|\nabla u\|_p^p) < +\infty$. Then there would be a sequence $\{t_j\}_{j=1}^\infty$ and a constant $K > 0$ such that $t_j \to T_{\max}$ as $j \to \infty$ and $\|u_t\|_2^2 + \|\Delta u\|_2^2 + \|\nabla u\|_p^p < K$. For each $j \in N_+$, there exists a unique solution of (1.1) on $[t_j, t_j + T^*]$, where $u(x,t_j)$ is the initial datum and $T^* > 0$ only depends on K. It can be found out that there is a contradiction, that is, for large $j \in N_+$, $T_{\max} < t_j + T^*$. ■

4 Case $q > p$

The diffusion $\mathrm{div}(|\nabla u|^{p-2}\nabla u)$ and the reaction $|u|^{q-2}u\log|u|$ play crucial roles in determining the existence of global and blow-up solutions. When $q > p$, we state that the reaction term is prevailing.

Theorem 4.1 *Let $q > p$. If $(u_0,u_1) \in \mathcal{W}_1$ and $(\mathcal{H})$ hold, then problem (1.1) possesses a unique global solution u as defined in Definition 2.1. The energy function $E(t)$ satisfies*

$$E(t) + \int_0^t \sigma(\|\nabla u\|_2^2)\|u_s(s)\|_2^2 \mathrm{d}s \le E(0). \tag{4.1}$$

Moreover, there is the decay estimate of energy function

$$E(t) \le \frac{C_5}{L+t}, \quad \text{for all } 0 \le t < +\infty, \tag{4.2}$$

where L is a positive constant and

$$C_4 := \frac{(1+\sigma_1)qd}{(q-2)\lambda_1} + d + \frac{d}{a} + \frac{1}{2}\|u_0\|_2^2 + \frac{1}{2}\|u_1\|_2^2,$$

$$C_5 := \frac{1}{q}C_4 + \frac{(q-2)\lambda_*^q}{2q(\lambda_*^q - \lambda_*^2)}C_4 + \frac{(q-p)\lambda_*^q}{pq(\lambda_*^q - \lambda_*^p)}C_4 + \frac{C_4}{q^2\log\lambda_*} + \frac{d}{2a} + Ld > 0.$$

In particular, if $E(0) < M$ (M is defined in (2.1)), then there are positive constants C_8 and C_9 such that $E(t)$ satisfies the exponential decay estimate

$$E(t) \le C_8 \mathrm{e}^{-C_9 t}, \quad \text{for all } 0 \le t < +\infty, \tag{4.3}$$

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where the constants are defined as

$$C_6 := 1 - \varepsilon_3 - \frac{\varepsilon_3(1+\sigma_1)q}{\lambda_1(q-2)}, \quad C_8 := \left(E(0) + \varepsilon_3 \int_\Omega u_0 u_1 dx\right)/C_6,$$
$$C_7 := 1 + \varepsilon_3 + \frac{\varepsilon_3(1+\sigma_1)q}{\lambda_1(q-2)}, \quad C_9 := \beta\varepsilon_3/C_7,$$

and β is a small constant satisfying

$$\beta < p, \quad \frac{\beta}{2} - 1 + \frac{\beta B_0^q}{q^2}\left(\frac{2qE(0)}{q-2}\right)^{\frac{q-2}{2}} + \frac{B_\alpha^{q+\alpha}}{e\alpha}\left(\frac{2qE(0)}{q-2}\right)^{\frac{q+\alpha-2}{2}} < 0, \tag{4.4}$$

and ε_3 is a small constant satisfying $C_6, C_8 > 0$ and

$$\varepsilon_3 + \frac{\beta\varepsilon_3}{2} - a < 0, \quad E(0) + \varepsilon_3 \int_\Omega u_0 u_1 \mathrm{d}x \geq 0. \tag{4.5}$$

Proof We divide the proof into five steps.

Step 1. *Global existence for the case of* $(u_0, u_1) \in \mathcal{W}_1$. Let the sequences $\{\omega_j\}_{j=1}^{+\infty}$, $\{u_{0m}\}_{m=1}^{+\infty}$, $\{u_{1m}\}_{m=1}^{+\infty}$, $\{u_m\}_{m=1}^{+\infty}$ be the same as those stated in the proof of Theorem 3.1. By the definitions of $I(u)$, $E(t)$, and $J(u)$, we obtain

(4.6)

$$J(u_m) = \frac{q-2}{2q}\|\Delta u_m\|_2^2 + \frac{q-p}{pq}\|\nabla u_m\|_p^p + \frac{1}{q^2}\|u_m\|_q^q + \frac{1}{q}I(u_m), \quad \text{for all } 0 \leq t < +\infty.$$
$$E_m(t) = \frac{1}{2}\|u_{mt}\|_2^2 + J(u_m), \quad \text{for all } 0 \leq t < +\infty.$$

Multiplying both sides of (3.1) by $b'_{jm}(t)$, summing over $j = 1, 2, \ldots, m$, and integrating it from 0 to t, for sufficiently large m, we have

$$E_m(t) + \int_0^t \sigma(\|\nabla u_m\|_2^2)\|u_{ms}(s)\|_2^2 \mathrm{d}s \leq E_m(0) < d, \quad \text{for all } 0 \leq t < +\infty, \tag{4.7}$$

$$E_m(t) = \frac{1}{2}\|u_{mt}\|_2^2 + J(u_m) \leq E_m(0) < d, \quad \text{for all } 0 \leq t < +\infty. \tag{4.8}$$

We claim that $(u_m, u_{mt}) \in \mathcal{W}_1$ for large m. By contradiction argument, if it was false, then there would be a constant t_0 such that $(u_m(t_0), u_{mt}(t_0)) \in \partial\mathcal{W}_1$, and hence $I(u_m(t_0)) = 0$ or $E(u_m(t_0), u_{mt}(t_0)) = d$. Obviously, (4.7) means that the second equation is impossible. On the other hand, if the first equation holds, then we have $u_m(t_0) \in \mathcal{N}$. By the definition of potential depth d, we have $J(u_m(t_0)) \geq d$, which contradicts (4.8). Thus, $(u_m, u_{mt}) \in \mathcal{W}_1$ for sufficiently large m and $0 \leq t < +\infty$. Using (4.6) and (4.8), we get

$$\frac{q-2}{2q}\|\Delta u_m\|_2^2 + \frac{q-p}{pq}\|\nabla u_m\|_p^p + \frac{1}{q^2}\|u_m\|_q^q < d, \quad \text{for all } 0 \leq t < +\infty. \tag{4.9}$$

By using $2 < p < q$, (4.7), and (4.9), we get

$$u_m \text{ is uniformly bounded in } L^\infty((0,\infty); H_0^2(\Omega)), \tag{4.10}$$

$$u_m \text{ is uniformly bounded in } L^\infty((0,\infty); W_0^{1,p}(\Omega)), \tag{4.11}$$

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$$u_{mt} \text{ is uniformly bounded in } L^\infty((0,\infty);L^2(\Omega)), \tag{4.12}$$

$$u_{mt} \text{ is uniformly bounded in } L^2((0,\infty);L^2(\Omega)). \tag{4.13}$$

From (4.10)–(4.13), there exists a function u and a subsequence still denoted by $\{u_m\}_{m=1}^{+\infty}$ such that

$$u_m \overset{*}{\rightharpoonup} u \text{ in } L^\infty((0,\infty);H_0^2(\Omega)), \tag{4.14}$$

$$u_m \overset{*}{\rightharpoonup} u \text{ in } L^\infty((0,\infty);W_0^{1,p}(\Omega)), \tag{4.15}$$

$$u_{mt} \overset{*}{\rightharpoonup} u_t \text{ in } L^\infty((0,\infty);L^2(\Omega)), \tag{4.16}$$

$$u_{mt} \rightharpoonup u_t \text{ in } L^2((0,\infty);L^2(\Omega)). \tag{4.17}$$

By using (4.10)–(4.13) and repeating the arguments in Priori estimate II of Theorem 3.1, we obtain that there exists a positive constant $\overline{C}$ independent on T_0 such that

$$\begin{aligned}&\|z_{mt}\|_2^2 + \|\Delta z_m\|_2^2 + \|\nabla z_m\|_p^p\\ &\quad\le \overline{C}\left(\|z_{1m}\|_2^2 + \|\Delta z_{0m}\|_2^2 + \|\nabla z_{0m}\|_p^p\right), \text{ for all } m\in\mathbb{Z}^+,\ 0\le t<+\infty.\end{aligned}$$

Hence, we obtain $u_m \to u$ in $C^0((0,\infty);H_0^2(\Omega)\cap W_0^{1,p}(\Omega))$ and $u_m \to u$ in $C^0((0,\infty);L^2(\Omega))$. Therefore, we have $\sigma(\|\nabla u_m\|_2^2)\to\sigma(\|\nabla u\|_2^2)$ for all $0\le t<+\infty$. In addition, similar to (3.22), we get

$$|u_m|^{q-2}u_m\log|u_m| \overset{*}{\rightharpoonup} |u|^{q-2}u\log|u| \text{ in } L^\infty((0,\infty);L^2(\Omega)). \tag{4.18}$$

Therefore, together with (3.1) and initial data condition (3.2), (3.3), (4.14)–(4.18), we obtain

$$\begin{aligned}&u_{tt}+\Delta^2 u-\operatorname{div}(|\nabla u|^{p-2}\nabla u)+\sigma(\|\nabla u\|_2^2)u_t=|u|^{q-2}u\log|u|,\quad (x,t)\in\Omega\times(0,+\infty),\\ &\qquad u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x),\quad x\in\Omega.\end{aligned} \tag{4.19}$$

Finally, from (4.19), it can be seen that u is a global weak solution of problem (1.1).

Step 2. *Global existence for the case of* $(u_0,u_1)\in\mathcal{W}_2$. Let $\gamma_m := 1-\frac{1}{m}$, $u_{m0}:=\gamma_m u_0$, $u_{m1}:=\gamma_m u_1$, $\gamma_m\in(0,1)$ for $m\ge 2$ and $m\in\mathbb{Z}^+$. Now, we consider the following problem:

$$\begin{cases} u_{tt}+\Delta^2 u-\operatorname{div}(|\nabla u|^{p-2}\nabla u)+\sigma(\|\nabla u\|_2^2)u_t & \\ \quad =|u|^{q-2}u\log|u|, & (x,t)\in\Omega\times(0,T_{\max}),\\ u=0,\quad \frac{\partial u}{\partial\eta}=0, & (x,t)\in\partial\Omega\times(0,T_{\max}),\\ u(x,0)=u_{m0}(x),\quad u_t(x,0)=u_{m1}(x), & x\in\Omega. \end{cases} \tag{4.20}$$

We claim that $I(u_{m0})>0$ and $E(u_{m0},u_{m1})<d$. Actually, using $I(u_0)\ge 0$ and Lemma 2.2, we observe $\lambda_*=\lambda_*(u_0)\ge 1$. Since $\gamma_m\in(0,1)$, we obtain $I(u_{m0})=I(\gamma_m u_0)>0$. In addition,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\gamma_m}J(\gamma_m u_0)&=\frac{1}{\gamma_m}\left(\|\Delta\gamma_m u_0\|_2^2+\|\nabla\gamma_m u_0\|_p^p-\int_\Omega|\gamma_m u_0|^q\log|\gamma_m u_0|\,\mathrm{d}x\right)\\ &=\frac{1}{\gamma_m}I(\gamma_m u_0)>0.\end{aligned}$$

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Hence, $J(\gamma_m u_0)$ is increasing with respect to γ_m and

$$E(u_{m0}, u_{m1}) = \frac{1}{2}\|u_{m1}\|_2^2 + J(u_{m0}) = \frac{1}{2}\|\gamma_m u_1\|_2^2 + J(\gamma_m u_0) < E(0) < d.$$

Similar to Step 1, we obtain that problem (1.1) admits a global solution.

Step 3. *Energy inequality (4.1).* Now, we prove that

$$\lim_{m\to\infty}\int_\Omega |u_m|^q \log|u_m| \mathrm{d}x = \int_\Omega |u|^q \log|u| \mathrm{d}x, \tag{4.21}$$

$$\lim_{m\to\infty}\int_\Omega |u_m|^q \mathrm{d}x = \int_\Omega |u|^q \mathrm{d}x. \tag{4.22}$$

In fact, from Steps 1 and 2, we have $u \in L^\infty((0,\infty); H_0^2(\Omega)\cap W_0^{1,p}(\Omega))$ and $u_t \in L^\infty((0,\infty); L^2(\Omega))$. Using similar discussion to (3.16)–(3.18) and letting $m \to +\infty$, we have

$$\left|\int_\Omega |u_m|^q \log|u_m| \mathrm{d}x - \int_\Omega |u|^q \log|u| \mathrm{d}x\right| \le C\|\Delta u_m - \Delta u\|_2 \to 0, \quad \text{for all } 0 \le t < +\infty.$$

For some of a constant $\theta \in (0,1)$, using the Mean Value theorem, Hölder's inequality, the embedding $H_0^2(\Omega) \hookrightarrow L^{2(q-1)}(\Omega)$, and taking $m \to +\infty$, we have

$$\begin{aligned}\int_\Omega \left(|u_m|^q - |u|^q\right) \mathrm{d}x &\le q\int_\Omega \left(\theta|u_m| + (1-\theta)|u|\right)^{q-1}|u_m - u| \mathrm{d}x\\ &\le q\|\theta|u_m| + (1-\theta)|u|\|_{2(q-1)}^{q-1}\|u_m - u\|_2\\ &\le q\lambda_1^{-1}B^{q-1}\||\Delta u| + |\Delta u_m|\|_2^{q-1}\|\Delta u - \Delta u_m\|_2\\ &\le C\left(\|\Delta u\|_2 + \|\Delta u_m\|_2\right)^{q-1}\|\Delta u_m - \Delta u\|_2 \to 0.\end{aligned}$$

Hence, (4.21) and (4.22) hold. By (3.2) and (3.3), we have $\lim\limits_{m\to+\infty} E(u_{0m}, u_{1m}) = E(u_0, u_1) = E(0)$. Making use of Fatou's Lemma, (4.7), (4.21), and (4.22), we deduce

$$\begin{aligned}&\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{p}\|\nabla u\|_p^p + \int_0^t \sigma(\|\nabla u\|_2^2)\|u_s\|_2^2 \mathrm{d}s\\ &\quad\le \frac{1}{2}\liminf_{m\to+\infty}\|u_{mt}\|_2^2 + \frac{1}{2}\liminf_{m\to+\infty}\|\Delta u_m\|_2^2 + \frac{1}{p}\liminf_{m\to+\infty}\|\nabla u_m\|_p^p\\ &\qquad+ \liminf_{m\to+\infty}\int_0^t \sigma(\|\nabla u_m\|_2^2)\|u_{ms}\|_2^2 \mathrm{d}s\\ &\quad\le \liminf_{m\to+\infty}\left[E_m(0) + \frac{1}{q}\int_\Omega |u_m|^q \log|u_m| \mathrm{d}x + \frac{1}{q^2}\|u_m\|_q^q\right]\\ &\quad= E(0) + \frac{1}{q}\int_\Omega |u|^q \log|u| \mathrm{d}x + \frac{1}{q^2}\|u\|_q^q.\end{aligned}$$

Then the energy inequality (4.1) holds.

Step 4. *Decay estimate (4.2) of energy.* From Steps 1 and 2, we know $(u, u_t) \in \mathcal{W}$ for $0 \le t < +\infty$ and $I(u) \ge 0$ provided $(u_0, u_1) \in \mathcal{W}$. By the definitions of $E(t)$, $\sigma(s)$, and

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$I(u) \geq 0$, we get

$$E(0) \geq E(t) + \int_0^t \sigma(\|\nabla u\|_2^2)\|u_s(s)\|_2^2 \mathrm{d}s \geq \frac{1}{2}\|u_t\|_2^2 + J(u) + a\int_0^t \|u_s\|_2^2 \mathrm{d}s$$

$$\geq \frac{1}{2}\|u_t\|_2^2 + \frac{q-2}{2q}\|\Delta u\|_2^2 + \frac{q-p}{pq}\|\nabla u\|_p^p + \frac{1}{q^2}\|u\|_q^q + a\int_0^t \|u_s\|_2^2 \mathrm{d}s. \tag{4.23}$$

Equation (4.23) and $E(0) \leq d$ yield

$$\|u_t\|_2^2 \leq 2d, \quad \|\Delta u\|_2^2 \leq \frac{2qd}{q-2}, \quad \|\nabla u\|_p^p \leq \frac{pqd}{q-p}, \quad \int_0^t \|u_s\|_2^2 \mathrm{d}s \leq \frac{d}{a}. \tag{4.24}$$

Using Hölder's inequality, we have $\|\nabla u\|_2^2 \leq |\Omega|^{\frac{p-2}{p}}\|\nabla u\|_p^2 \leq |\Omega|^{\frac{p-2}{p}}\left(\frac{pqd}{q-p}\right)^{\frac{2}{p}}$. Due to the continuity of $\sigma(s)$ in $[0, +\infty)$, there exists a constant $\sigma_1 > 0$ such that

$$\sigma(\|\nabla u\|_2^2) \leq \max_{0 \leq \|\nabla u\|_2^2 \leq C'} \sigma(\|\nabla u\|_2^2) := \sigma_1. \tag{4.25}$$

Now, taking $\phi := u$ in (2.2) and applying (4.24) and (4.25), Young's and Hölder's inequalities, and $\lambda_1\|u\|_2^2 \leq \|\Delta u\|_2^2$, we get

$$\begin{aligned}\int_0^t I(u)\mathrm{d}s &= -\int_0^t \int_\Omega u_{ss}u\mathrm{d}x\mathrm{d}s - \int_0^t \sigma(\|\nabla u\|_2^2)\int_\Omega u_s u\mathrm{d}x\mathrm{d}s \\ &\leq -\int_\Omega u_t u\mathrm{d}x + \int_\Omega u_0 u_1 \mathrm{d}x + \int_0^t \|u_s\|_2^2\mathrm{d}s + \sigma_1\int_\Omega \left|\int_0^t u_s u\mathrm{d}s\right|\mathrm{d}x \\ &\leq \frac{1+\sigma_1}{2\lambda_1}\|\Delta u\|_2^2 + \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|u_0\|_2^2 + \frac{1}{2}\|u_1\|_2^2 + \int_0^t \|u_s\|_2^2\mathrm{d}s. \end{aligned} \tag{4.26}$$

Furthermore, using (4.24) and (4.26), we arrive at

$$\int_0^t I(u)\mathrm{d}s \leq \frac{(1+\sigma_1)qd}{(q-2)\lambda_1} + d + \frac{d}{a} + \frac{1}{2}\|u_0\|_2^2 + \frac{1}{2}\|u_1\|_2^2 =: C_4, \quad \text{for all } 0 \leq t < +\infty. \tag{4.27}$$

By the definitions of $J(u)$ and $I(u)$, we have

$$J(u) = \frac{q-2}{2q}\|\Delta u\|_2^2 + \frac{q-p}{pq}\|\nabla u\|_p^p + \frac{1}{q^2}\|u\|_q^q + \frac{1}{q}I(u). \tag{4.28}$$

Since $I(u) \geq 0$ and by Lemma 2.2, there exists a constant $\lambda_* \geq 1$ such that $I(\lambda_* u) = 0$. By the definition of d and (4.28), we have

$$\begin{aligned} d \leq J(\lambda_* u) &= \lambda_*^q\left(\frac{q-2}{2q}\lambda_*^{2-q}\|\Delta u\|_2^2 + \frac{q-p}{pq}\lambda_*^{p-q}\|\nabla u\|_p^p + \frac{1}{q^2}\|u\|_q^q\right) \\ &\leq \lambda_*^q\left(\frac{q-2}{2q}\|\Delta u\|_2^2 + \frac{q-p}{pq}\|\nabla u\|_p^p + \frac{1}{q^2}\|u\|_q^q + \frac{1}{q}I(u)\right) = \lambda_*^q J(u). \end{aligned} \tag{4.29}$$

By the definition of $E(t)$, $I(u) \geq 0$, and (4.28), we have

$$\frac{q-2}{2q}\|\Delta u\|_2^2 + \frac{q-p}{pq}\|\nabla u\|_p^p + \frac{1}{q^2}\|u\|_q^q \leq J(u) < E(t) \leq E(0) \leq d. \tag{4.30}$$

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It follows from (4.29) and (4.30) that

$$1 \le \left(\frac{d}{E(0)}\right)^{\frac{1}{q}} \le \left(\frac{d}{E(t)}\right)^{\frac{1}{q}} < \left(\frac{d}{J(u)}\right)^{\frac{1}{q}} \le \lambda_*. \tag{4.31}$$

On the other hand, together with (4.31), we have $0 = I(\lambda_* u) = \lambda_*^q I(u) + (\lambda_*^2 - \lambda_*^q)\|\Delta u\|_2^2 + (\lambda_*^p - \lambda_*^q)\|\nabla u\|_p^p - \lambda_*^q \|u\|_q^q \log \lambda_*$, which implies

$$I(u) = \frac{\lambda_*^q - \lambda_*^2}{\lambda_*^q}\|\Delta u\|_2^2 + \frac{\lambda_*^q - \lambda_*^p}{\lambda_*^q}\|\nabla u\|_p^p + \|u\|_q^q \log \lambda_*. \tag{4.32}$$

Using (4.27) and (4.32), we have

$$\int_0^t \|\Delta u\|_2^2 \mathrm{d}s \le \frac{\lambda_*^q}{\lambda_*^q - \lambda_*^2}\int_0^t I(u)\mathrm{d}s \le \frac{\lambda_*^q}{\lambda_*^q - \lambda_*^2} C_4, \tag{4.33}$$

$$\int_0^t \|\nabla u\|_p^p \mathrm{d}s \le \frac{\lambda_*^q}{\lambda_*^q - \lambda_*^p}\int_0^t I(u)\mathrm{d}s \le \frac{\lambda_*^q}{\lambda_*^q - \lambda_*^p} C_4, \tag{4.34}$$

$$\int_0^t \|u\|_q^q \mathrm{d}s \le \frac{1}{\log \lambda_*}\int_0^t I(u)\mathrm{d}s \le \frac{C_4}{\log \lambda_*}. \tag{4.35}$$

Due to the monotonicity of $E(t)$, we conclude that $\frac{\mathrm{d}}{\mathrm{d}t}[(L+t)E(t)] \le E(t)$. By integration, we infer

$$\begin{aligned} E(t) \le \frac{1}{L+t}\Bigg(&\frac{1}{q}\int_0^t I(u)\mathrm{d}s + \frac{q-2}{2q}\int_0^t \|\Delta u\|_2^2 \mathrm{d}s + \frac{q-p}{pq}\int_0^t \|\nabla u\|_p^p \mathrm{d}s \\ &+ \frac{1}{q^2}\int_0^t \|u\|_q^q \mathrm{d}s + \frac{1}{2}\int_0^t \|u_s\|_2^2 \mathrm{d}s + LE(0)\Bigg). \end{aligned}$$

Hence, using $E(0) \le d$, (4.24), (4.26), and (4.33)–(4.35), we get decay estimate (4.2).

Step 5. *Exponential decay estimate (4.3) of energy.* Define a function

$$F(t) := E(t) + \varepsilon_3 \int_\Omega u u_t \mathrm{d}x + \varepsilon_3 \int_0^t \sigma(\|\nabla u\|_2^2)\int_\Omega u u_s \mathrm{d}x\mathrm{d}s,$$

where ε_3 is a positive constant to be specified later. We claim that there exist two positive constants C_6, C_7 such that $C_6 E(t) \le F(t) \le C_7 E(t)$ for all $0 \le t < +\infty$. In fact, by the definition of $\sigma(s)$, (4.23), (4.25), Young's inequality, and $\lambda_1\|u\|_2^2 \le \|\Delta u\|_2^2$, we compute

$$\begin{aligned} F(t) &\ge E(t) - \frac{\varepsilon_3}{2}\|u\|_2^2 - \frac{\varepsilon_3}{2}\|u_t\|_2^2 - \varepsilon_3\sigma_1 \int_\Omega \left|\int_0^t u u_s \mathrm{d}s\right| \mathrm{d}x \\ &\ge E(t) - \frac{\varepsilon_3}{2}\|u\|_2^2 - \frac{\varepsilon_3}{2}\|u_t\|_2^2 - \frac{\varepsilon_3\sigma_1}{2}\|u\|_2^2 \\ &\ge E(t) - \frac{\varepsilon_3(1+\sigma_1)}{2\lambda_1}\|\Delta u\|_2^2 - \frac{\varepsilon_3}{2}\|u_t\|_2^2 \ge C_6 E(t). \end{aligned}$$

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Now, we choose a small positive constant ε_3 such that (4.5) holds, where β will be specified later. Similar to the above procedure, we have

$$\begin{aligned} F(t) &\le E(t) + \frac{\varepsilon_3}{2}\|u\|_2^2 + \frac{\varepsilon_3}{2}\|u_t\|_2^2 + \varepsilon_3\sigma_1 \int_\Omega \left| \int_0^t u u_s \mathrm{d}s \right| \mathrm{d}x \\ &\le E(t) + \frac{\varepsilon_3}{2}\|u_t\|_2^2 + \frac{\varepsilon_3(1+\sigma_1)}{2\lambda_1}\|\Delta u\|_2^2 \le C_7 E(t). \end{aligned}$$

Differentiating $F(t)$ with respect to t, using (2.2), and definitions of $E(t)$ and $\sigma(s)$, we get

$$\begin{aligned} F'(t) &= -E'(t) + \varepsilon_3\|u_t\|_2^2 + \varepsilon_3 \int_\Omega u_{tt} u \mathrm{d}x + \varepsilon_3 \sigma(\|\nabla u\|_2^2) \int_\Omega u_t u \mathrm{d}x \\ &= -\sigma(\|\nabla u\|_2^2)\|u_t\|_2^2 + \varepsilon_3\|u_t\|_2^2 + \varepsilon_3 \int_\Omega |u|^q \log|u| \mathrm{d}x - \varepsilon_3\|\Delta u\|_2^2 - \varepsilon_3\|\nabla u\|_p^p \\ &\le -\beta\varepsilon_3 E(t) + \left(\frac{\beta\varepsilon_3}{2} + \varepsilon_3 - a\right)\|u_t\|_2^2 + \left(\frac{\beta\varepsilon_3}{2} - \varepsilon_3\right)\|\Delta u\|_2^2 + \left(\frac{\beta\varepsilon_3}{p} - \varepsilon_3\right)\|\nabla u\|_p^p \\ &\quad + \left(\varepsilon_3 - \frac{\beta\varepsilon_3}{q}\right)\int_\Omega |u|^q \log|u| \mathrm{d}x + \frac{\beta\varepsilon_3}{q^2}\|u\|_q^q. \end{aligned} \tag{4.36}$$

Now, we consider the right side of (4.36). Using (2.4), (4.24), and the embedding $H_0^2(\Omega) \hookrightarrow L^{q+\alpha}(\Omega)$, we arrive at

$$\int_\Omega |u|^q \log|u| \mathrm{d}x \le \frac{1}{\mathrm{e}\alpha}\|u\|_{q+\alpha}^{q+\alpha} \le \frac{B_\alpha^{q+\alpha}}{\mathrm{e}\alpha}\|\Delta u\|_2^{q+\alpha} \le \frac{B_\alpha^{q+\alpha}}{\mathrm{e}\alpha}\left(\frac{2qE(0)}{q-2}\right)^{\frac{q+\alpha-2}{2}}\|\Delta u\|_2^2. \tag{4.37}$$

Similarly, we have

$$\|u\|_q^q \le B_0^q\|\Delta u\|_2^q = B_0^q\|\Delta u\|_2^{q-2}\|\Delta u\|_2^2 \le B_0^q\left(\frac{2qE(0)}{q-2}\right)^{\frac{q-2}{2}}\|\Delta u\|_2^2. \tag{4.38}$$

Since (4.37) and (4.38), equation (4.36) can be estimated by

$$\begin{aligned} F'(t) &\le \left(\frac{\beta\varepsilon_3}{2} + \varepsilon_3 - a\right)\|u_t\|_2^2 - \frac{\beta\varepsilon_3}{\mathrm{e}q\alpha}B_\alpha^{q+\alpha}\left(\frac{2qE(0)}{q-2}\right)^{\frac{q+\alpha-2}{2}}\|\Delta u\|_2^2 + \varepsilon_3\left(\frac{\beta}{p} - 1\right)\|\nabla u\|_p^p \\ &\quad + \varepsilon_3\left[\frac{\beta}{2} + \frac{\beta B_0^q}{q^2}\left(\frac{2qE(0)}{q-2}\right)^{\frac{q-2}{2}} + \frac{B_\alpha^{q+\alpha}}{\mathrm{e}\alpha}\left(\frac{2qE(0)}{q-2}\right)^{\frac{q+\alpha-2}{2}} - 1\right]\|\Delta u\|_2^2 - \beta\varepsilon_3 E(t). \end{aligned} \tag{4.39}$$

By assumption $E(0) < M$, we get $\frac{B_\alpha^{q+\alpha}}{\mathrm{e}\alpha}\left(\frac{2qE(0)}{q-2}\right)^{\frac{q+\alpha-2}{2}} < 1$. Meanwhile, we select small β such that (4.4) holds. Choosing ε_3 such that (4.5) holds and substituting (4.4) into

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(4.39), we have

$$F'(t) \le -\beta\varepsilon_3 E(t) \le \frac{-\beta\varepsilon_3}{C_7} F(t), \quad \text{for all } 0 \le t < +\infty. \tag{4.40}$$

Integrating (4.40) from 0 to t, we have the decay estimate (4.3). ∎

The next theorem shows the existence of blow-up solution of problem (1.1). We note that the positive constant $r(\alpha)$ is defined in (2.1).

Theorem 4.2 *Let $q > p$ and the initial data $(u_0, u_1) \in \mathcal{V}_0$ and $(\mathcal{H})$ hold. Denote u the weak solution as defined in Definition 2.1. If $E(0) < 0$, then the solution u blows up in finite time or infinite time. Moreover, if $T_{\max} < +\infty$, then there are the results: (i) If $-(r(\alpha))^2 \le E(0) < 0$, then u blows up in finite time in the sense of $\lim_{t \to T_{\max}^-} \|\Delta u\|_2 = +\infty$. (ii) If $-(r(\alpha))^2 \le E(0) < 0$ and p, q satisfy*

$$2 < q + \alpha < \begin{cases} \dfrac{np}{n-p}, & \text{for } n > p, \\ +\infty, & \text{for } n \le p, \end{cases} \tag{$\mathcal{H}'$}$$

then u blows up in finite time in the sense of $\lim_{t \to T_{\max}^-} \|\nabla u\|_p = +\infty$.

Proof The proof is divided into two steps.

Step 1. To prove that the solution blows up in finite or infinite time, we only need to rule out the case where the solution is globally bounded, and we use the proof by contradiction for this purpose. If u was globally bounded, there exists a positive constant C independent of t such that $\|u_t\|_2^2 + \|\Delta u\|_2^2 + \|\nabla u\|_p^p \le C$. We define $K(t) := \|u\|_2^2$ and $H(t) := -E(t)$ for all $0 \le t \le T_1 < +\infty$, where T_1 denotes any of a positive constant. Using the monotonicity of $E(t)$, we have $H'(t) = -E'(t) \ge 0$,

$$H(t) \ge H(0) = -E(0) > 0, \quad \text{for all } 0 \le t \le T_1. \tag{4.41}$$

We define $G(t) := \int_\Omega \sigma(\|\nabla u\|_2^2) u u_t \mathrm{d}x$ for $0 \le t \le T_1$. By the assumption of the global existence of the solution, there are three positive constants D_1, D_2, and D_3, which are independent of T_1, satisfying

$$\|\Delta u\|_2^2 \le D_1, \quad \|\nabla u\|_p^p \le D_2, \quad \|u_t\|_2^2 \le D_3, \quad \text{for all } 0 \le t \le T_1. \tag{4.42}$$

Using Hölder's inequality, we have

$$\|\nabla u\|_2^2 \le |\Omega|^{\frac{p-2}{p}} \|\nabla u\|_p^2 \le |\Omega|^{\frac{p-2}{p}} D_2^{\frac{2}{p}} =: D_4. \tag{4.43}$$

Since $\sigma(s)$ is continuous, we have

$$\sigma(\|\nabla u\|_2^2) \le \max_{0 \le \|\nabla u\|_2^2 \le D_4} \sigma(\|\nabla u\|_2^2) =: \sigma_2. \tag{4.44}$$

Using (4.44), and Young's and Hölder's inequalities, we get

$$\begin{aligned} |G(t)| &= \left| \int_\Omega \sigma(\|\nabla u\|_2^2) u u_t \mathrm{d}x \right| \le \sigma(\|\nabla u\|_2^2) \|u\|_2 \|u_t\|_2 \\ &\le \frac{1}{2} \sigma(\|\nabla u\|_2^2) \left(\frac{1}{\eta} \|u_t\|_2^2 + \eta \|u\|_2^2 \right) \le \frac{1}{2\eta} H'(t) + \frac{\sigma_2 \eta}{2} \|u\|_2^2, \end{aligned} \tag{4.45}$$

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where η is a positive constant which will be specified later. Differentiating $K(t)$ directly, we obtain $K'(t) = 2\int_\Omega uu_t \mathrm{d}x$, $K''(t) = 2\|u_t\|_2^2 + 2\int_\Omega uu_{tt}\mathrm{d}x$. Define a function $y(t) := H^{1-\zeta}(t) + \varepsilon_4 \int_\Omega uu_t \mathrm{d}x$, where ζ satisfies

$$\frac{2(q-2)}{q^2} < \zeta < \frac{q-2}{2q} < \frac{1}{2}, \tag{4.46}$$

and ε_4 will be specified later.

Next, we will estimate $y'(t)$. By using (2.2) and Young's inequality, we have

$$\begin{aligned} y'(t) &= (1-\zeta)H^{-\zeta}(t)H'(t) + \varepsilon_4\|u_t\|_2^2 + \varepsilon_4\int_\Omega |u|^q \log|u|\mathrm{d}x - \varepsilon_4\|\Delta u\|_2^2 \\ &\quad - \varepsilon_4\|\nabla u\|_p^p - \varepsilon_4\sigma(\|\nabla u\|_2^2)\int_\Omega u_t u\mathrm{d}x \\ &\geq \left[(1-\zeta)H^{-\zeta}(t) - \frac{\varepsilon_4}{2\eta}\right]H'(t) + \varepsilon_4\|u_t\|_2^2 + \varepsilon_4\int_\Omega |u|^q\log|u|\mathrm{d}x \\ &\quad - \varepsilon_4\|\Delta u\|_2^2 - \varepsilon_4\|\nabla u\|_p^p - \frac{\varepsilon_4\eta\sigma_2}{2}\|u\|_2^2. \end{aligned}$$

Furthermore, we introduce a constant $c \in (0,1)$ and the above inequality can be written by

$$\begin{aligned} y'(t) &\geq \left(1-\zeta-\frac{\varepsilon_4}{2\eta}H^\zeta(t)\right)H^{-\zeta}(t)H'(t) + \varepsilon_4 q(1-c)H(t) + \varepsilon_4\frac{q(1-c)+2}{2}\|u_t\|_2^2 \\ &\quad + \varepsilon_4\frac{q(1-c)-2}{2}\|\Delta u\|_2^2 + \varepsilon_4\frac{q(1-c)-p}{p}\|\nabla u\|_p^p + \varepsilon_4 c\int_\Omega |u|^q\log|u|\mathrm{d}x \\ &\quad + \frac{\varepsilon_4(1-c)}{q}\|u\|_q^q - \frac{\varepsilon_4\eta\sigma_2}{2}H^{-\zeta}(t)H^\zeta(t)\|u\|_2^2. \end{aligned} \tag{4.47}$$

Now, we prove $\int_\Omega |u|^q\log|u|dx \geq 0$. In fact, by the monotonicity of $E(t)$, we have

$$\frac{1}{q}\int_\Omega |u|^q\log|u|\mathrm{d}x \geq \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q^2}\|u\|_q^q - E(0) \geq 0. \tag{4.48}$$

Due to (4.48), the condition of Lemma 2.4 holds. Now, let us estimate the last term on the right side of (4.47). Applying Lemma 2.4, we have $\|u\|_2^2 \leq K_1\left[\left(\int_\Omega |u|^q\log|u|\mathrm{d}x\right)^{\frac{2}{q}} + \|\Delta u\|_2^{\frac{4}{q}}\right]$. Combining (4.41) with Young's inequality, we obtain

$$\begin{aligned} H^\zeta\|u\|_2^2 &\leq \left(\frac{1}{q}\int_\Omega |u|^q\log|u|\mathrm{d}x\right)^\zeta \|u\|_2^2 \\ &\leq K_1 q^{-\zeta}\left[\left(\int_\Omega |u|^q\log|u|\mathrm{d}x\right)^{\frac{2}{q}+\zeta} + \left(\int_\Omega |u|^q\log|u|\mathrm{d}x\right)^{\frac{q\zeta}{q-2}} + \|\Delta u\|_2^2\right]. \end{aligned} \tag{4.49}$$

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From (4.46), it is easy to know that $\frac{2}{q} < \frac{2+q\zeta}{q} \le 1$ and $\frac{2}{q} < \frac{q\zeta}{q-2} \le 1$. Making use of (4.49) and Lemma 2.5, we obtain

$$\begin{aligned} H^{\zeta}(t)\|u\|_2^2 &\le K_1 q^{-\zeta}\left[2K_2\int_\Omega |u|^q \log|u|\mathrm{d}x + (2K_2+1)\|\Delta u\|_2^2\right] \\ &\le 2K_1(K_2+1)q^{-\zeta}\left(\int_\Omega |u|^q\log|u|\mathrm{d}x + \|\Delta u\|_2^2\right). \end{aligned}$$

Substituting the above inequality into (4.47), we have

$$\begin{aligned} y'(t) \ge{}& \left(1-\zeta-\frac{\varepsilon_4}{2\eta}H^{\zeta}(t)\right)H^{-\zeta}(t)H'(t) + \varepsilon_4 q(1-c)H(t) + \varepsilon_4\frac{q(1-c)+2}{2}\|u_t\|_2^2 \\ &+ \varepsilon_4\left[\frac{q(1-c)-2}{2} - H^{-\zeta}(t)\frac{\eta\sigma_2 K_1(K_2+1)}{q^{\zeta}}\right]\|\Delta u\|_2^2 + \varepsilon_4\frac{q(1-c)-p}{p}\|\nabla u\|_p^p \\ &+ \varepsilon_4\left[c - H^{-\zeta}(t)\frac{\eta\sigma_2 K_1(K_2+1)}{q^{\zeta}}\right]\int_\Omega |u|^q\log|u|\mathrm{d}x + \frac{\varepsilon_4(1-c)}{q}\|u\|_q^q. \end{aligned} \tag{4.50}$$

Combining (2.4), (4.42), (4.43), the definition of $E(t)$, and the embedding $H_0^2(\Omega) \hookrightarrow L^{q+\alpha}(\Omega)$, we get

$$H(t) \le \frac{1}{q}\int_\Omega |u|^q\log|u|\mathrm{d}x \le \frac{1}{eq\alpha}\|u\|_{q+\alpha}^{q+\alpha} \le \frac{B_\alpha^{q+\alpha}}{eq\alpha}\|\Delta u\|_2^{q+\alpha} \le \frac{B_\alpha^{q+\alpha}}{eq\alpha}D_1^{\frac{q+\alpha}{2}}.$$

Then by (4.41) and (4.50), we have

$$\begin{aligned} y'(t) \ge{}& \left[1-\zeta-\frac{\varepsilon_4}{2\eta}\left(\frac{B_\alpha^{q+\alpha}}{eq\alpha}D_1^{\frac{q+\alpha}{2}}\right)^{\zeta}\right]H^{-\zeta}(t)H'(t) + \varepsilon_4 q(1-c)H(t) \\ &+ \varepsilon_4\frac{q(1-c)+2}{2}\|u_t\|_2^2 + \varepsilon_4\left[\frac{q(1-c)-2}{2} - H^{-\zeta}(0)\frac{\eta\sigma_2 K_1(K_2+1)}{q^{\zeta}}\right]\|\Delta u\|_2^2 \\ &+ \varepsilon_4\frac{q(1-c)-p}{p}\|\nabla u\|_p^p + \varepsilon_4\left[c - H^{-\zeta}(0)\frac{\eta\sigma_2 K_1(K_2+1)}{q^{\zeta}}\right]\int_\Omega |u|^q\log|u|\mathrm{d}x \\ &+ \frac{\varepsilon_4(1-c)}{q}\|u\|_q^q. \end{aligned} \tag{4.51}$$

We choose the constants c and η so small that $q(1-c)-p>0$ and

$$\begin{aligned} \frac{q(1-c)-2}{2} - H^{-\zeta}(0)\eta\sigma_2 K_1(K_2+1)q^{-\zeta} &> 0, \\ c - H^{-\zeta}(0)\eta\sigma_2 K_1(K_2+1)q^{-\zeta} &> 0. \end{aligned}$$

Once c and η are fixed, we pick some of a constant ε_4 small enough that

$$\begin{cases} 1-\zeta-\dfrac{\varepsilon_4}{2\eta}\left(\dfrac{B_\alpha^{q+\alpha}}{eq\alpha}D_1^{\frac{q+\alpha}{2}}\right)^{\zeta} > 0, \\ H^{1-\zeta}(0) + \varepsilon_4\displaystyle\int_\Omega u_0u_1\mathrm{d}x > 0. \end{cases} \tag{4.52}$$

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By (4.52), the estimate (4.51) becomes

$$y'(t) \geq C_{10}\left(H(t) + \|u_t\|_2^2 + \|\Delta u\|_2^2 + \|\nabla u\|_p^p + \|u\|_q^q\right) \geq 0, \tag{4.53}$$

where the positive constant

$$C_{10} := \min\left\{\varepsilon_4 \frac{q(1-c)-p}{p},\ \varepsilon_4\left[\frac{q(1-c)-2}{2} - H^{-\zeta}(0)\frac{\eta\sigma_2 K_1(K_2+1)}{q^\zeta}\right],\ \varepsilon_4\frac{1-c}{q}\right\} > 0.$$

Hence, from (4.53), $y(t)$ is increasing in $(0, T_{\max})$, and hence $y(t) \geq H^{1-\zeta}(0) + \varepsilon_4 \int_\Omega u_0 u_1 \mathrm{d}x > 0$. By the choice of ζ, we have $\zeta \in (0, 1/2)$. Let $r := \frac{1}{1-\zeta}$. According to the inequality $|a+b|^r \leq 2^{r-1}\left(|a|^r + |b|^r\right)$ for $r \geq 1$, using Hölder's and Young's inequalities, we have

$$\begin{aligned} y^r(t) &\leq 2^{r-1}\left(H(t) + \varepsilon_4^r \Big|\int_\Omega u u_t \mathrm{d}x\Big|^r\right) \\ &\leq 2^{r-1}\left(H(t) + \varepsilon_4^r \|u\|_2^r \|u_t\|_2^r\right) \\ &\leq 2^{r-1}\left[H(t) + \frac{2-r}{2}\left(\varepsilon_4\|u\|_2\right)^{\frac{2}{1-2\zeta}} + \frac{r}{2}\|u_t\|_2^2\right] \\ &\leq C_{11}\left(H(t) + \|u\|_2^{\frac{2}{1-2\zeta}} + \|u_t\|_2^2\right), \end{aligned} \tag{4.54}$$

where $C_{11} := 2^{r-1}\max\left\{1, \frac{2-r}{2}(\varepsilon_4)^{\frac{2}{1-2\zeta}}, \frac{r}{2}\right\} > 0$. Next, we estimate the second term on the right side of (4.54). It follows from (4.46) that $\frac{1}{2} - \zeta > \frac{1}{q}$. Applying the inequality $x^\tau \leq (1+\frac{1}{m})(m+x)$, $x \geq 0$, $0 \leq \tau \leq 1$, $m > 0$, and taking $x := \|u\|_2^q$, $\tau := \frac{2}{(1-2\zeta)q} < 1$, $m := H(0)$, we obtain

$$\|u\|_2^{\frac{2}{1-2\zeta}} \leq \left(1 + \frac{1}{H(0)}\right)\left(H(t) + |\Omega|^{\frac{q-2}{2}}\|u\|_q^q\right) \leq C_{12}\left(H(t) + \|u\|_q^q\right), \tag{4.55}$$

where $C_{12} := \left(1 + \frac{1}{H(0)}\right)\max\left\{1, |\Omega|^{\frac{q-2}{2}}\right\} > 0$. Substituting (4.55) into (4.54), we have

$$y^r(t) \leq C_{11}(1+C_{12})\left(H(t) + \|u\|_q^q + \|u_t\|_2^2\right) \leq \frac{C_{11}(1+C_{12})}{C_{10}} y'(t),$$

which means

$$y'(t) \geq \frac{C_{10}}{C_{11}(1+C_{12})} y^r(t). \tag{4.56}$$

A simple integration of (4.56) over $(0,t)$ yields $y(t) \geq \left[y(0)^{1-r} - \frac{C_{10}(r-1)t}{C_{11}(1+C_{12})}\right]^{\frac{1}{1-r}}$ for all $0 \leq t < T_1$. We choose $T_1 \geq T_* := \frac{C_{11}(1+C_{12})y(0)^{1-r}}{C_{10}(r-1)}$. Combining $r = \frac{1}{1-\zeta} > 1$ and $y(0) > 0$, we have

$$\lim_{t \to T_*^-} y(t) = +\infty. \tag{4.57}$$

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However, we claim that there exists a positive constant C_{13} such that

$$y(t) \le C_{13}, \quad \text{for all } 0 \le t \le T_1. \tag{4.58}$$

Actually, from (4.42), Young's inequality, and $\lambda_1\|u\|_2^2 \le \|\Delta u\|_2^2$, we get

$$\begin{aligned} y(t) &= H^{1-\zeta}(t) + \varepsilon_4 \int_\Omega u_t u \mathrm{d}x \\ &= \left(\int_0^t \sigma(\|\nabla u\|_2^2)\|u_s\|_2^2 \mathrm{d}s - E(0)\right)^{1-\zeta} + \varepsilon_4 \int_\Omega u_t u \mathrm{d}x \\ &\le \left(\sigma_2 \int_0^t \|u_s\|_2^2 \mathrm{d}s - E(0)\right)^{1-\zeta} + \varepsilon_4\|u\|_2^2 + \varepsilon_4\|u_t\|_2^2 \\ &\le (\sigma_2 D_3 T_1 - E(0))^{1-\zeta} + \frac{\varepsilon_4}{\lambda_1} D_1 + \varepsilon_4 D_3 =: C_{13}. \end{aligned}$$

Obviously, (4.58) is contrary to (4.57). We conclude the solution cannot remain globally bounded. Then $\limsup_{t\to T_{\max}^-} \left(\|u_t\|_2^2 + \|\Delta u\|_2^2 + \|\nabla u\|_p^p\right) = +\infty$. Especially, if $T_{\max} < +\infty$, we obtain by Theorem 3.1 that $\lim_{t\to T_{\max}^-} \left(\|u_t\|_2^2 + \|\Delta u\|_2^2 + \|\nabla u\|_p^p\right) = +\infty$.

Step 2. If $T_{\max} < +\infty$, from Theorem 3.1, we have $\lim_{t\to T_{\max}^-} \left(\|u_t\|_2^2 + \|\Delta u\|_2^2 + \|\nabla u\|_p^p\right) = +\infty$. Using the definition of $E(t)$, we have $\|u_t\|_2^2 + \|\Delta u\|_2^2 + \|\nabla u\|_p^p \le pE(0) + \frac{p}{q}\int_\Omega |u|^q \log|u| \mathrm{d}x$. By the definitions of $E(t)$ and $I(u)$, we get

$$\frac{1}{q} I(u) \le \frac{1}{2}\|u_t\|_2^2 + \frac{q-2}{2q}\|\Delta u\|_2^2 + \frac{q-p}{pq}\|\nabla u\|_p^p + \frac{1}{q^2}\|u\|_q^q + \frac{1}{q} I(u) = E(t) \le E(0) < 0.$$

Hence, $I(u) < 0$. By Lemma 2.2, we have $\|\Delta u\|_2 > r(\alpha) > 0$. Together with the definition of $I(u)$, we have $\frac{1}{q}\int_\Omega |u|^q \log|u| \mathrm{d}x > \|\Delta u\|_2^2 > 0$. By $-(r(\alpha))^2 \le E(0) < 0$, we have $E(0) + \frac{1}{q}\int_\Omega |u|^q \log|u| \mathrm{d}x > 0$.

(i) Using (2.4) and the embedding $H_0^2(\Omega) \hookrightarrow L^{q+\alpha}(\Omega)$, we obtain

$$\begin{aligned} \|u_t\|_2^2 + \|\Delta u\|_2^2 + \|\nabla u\|_p^p &\le p\left(E(0) + \frac{1}{q}\int_\Omega |u|^q \log|u| \mathrm{d}x\right) \\ &\le p\left(E(0) + \frac{1}{eq}\|u\|_{q+\alpha}^{q+\alpha}\right) \le p\left(E(0) + \frac{B_\alpha^{q+\alpha}}{eq}\|\Delta u\|_2^{q+\alpha}\right). \end{aligned}$$

Hence, $\lim_{t\to T_{\max}^-} \|\Delta u\|_2 = +\infty$.

(ii) Similarly, using (2.4) and the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{q+\alpha}(\Omega)$, we obtain

$$\|u_t\|_2^2 + \|\Delta u\|_2^2 + \|\nabla u\|_p^p \le pE(0) + \frac{pB_3^{q+\alpha}}{eq}\|\nabla u\|_p^{q+\alpha},$$

where B_3 is the embedding constant in $W_0^{1,p}(\Omega) \hookrightarrow L^{q+\alpha}(\Omega)$. Hence, $\lim_{t\to T_{\max}^-} \|\nabla u\|_p = +\infty$. ∎

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5 Case $q = p$

In this section, we will consider the global solvability of problem (1.1) under the state of the equal strength of the diffusion and the reaction (i.e., $q = p$).

Theorem 5.1 *Let $(\mathcal{H})$ hold. If $q = p$ and the initial data $(u_0, u_1) \in \mathcal{W}$, then problem (1.1) possesses a unique global solution u as defined in Definition 2.1. Moreover, the energy function $E(t)$ satisfies the following decay estimate:*

$$E(t) \le \frac{C_{15}}{L + t}, \quad \text{for all } 0 \le t < +\infty, \tag{5.1}$$

where L is a positive constant and

$$C_{14} := \frac{(1+\sigma_3)pd}{(p-2)\lambda_1} + d + \frac{d}{a} + \frac{1}{2}\|u_0\|_2^2 + \frac{1}{2}\|u_1\|_2^2,$$

$$C_{15} := \frac{1}{p}C_{14} + \frac{(p-2)\lambda_*^p}{2p(\lambda_*^p - \lambda_*^2)}C_{14} + \frac{C_{14}}{p^2 \log \lambda *} + \frac{d}{2a} + Ld.$$

In particular, if $E(0) < M$, then the energy functional $E(t)$ satisfies the exponential decay estimate

$$E(t) \le C_{16} e^{-C_{17} t}, \quad \text{for all } 0 \le t < +\infty, \tag{5.2}$$

where the positive constants

$$C_{16} := \frac{E(0) + \varepsilon_5 \int_\Omega u_0 u_1 \mathrm{d}x}{1 - \varepsilon_5 - \frac{\varepsilon_5(1+\sigma_1)p}{\lambda_1(p-2)}}, \quad C_{17} := \frac{\beta\varepsilon_5}{1 + \varepsilon_5 + \frac{\varepsilon_5(1+\sigma_1)p}{\lambda_1(p-2)}},$$

and β_1 is a small constant such that

$$\beta_1 < p, \quad \frac{\beta_1}{2} + \frac{\beta_1 B_0^p}{p^2}\left(\frac{2pE(0)}{p-2}\right)^{\frac{p-2}{2}} + \frac{B_\alpha^{p+\alpha}}{e\alpha}\left(\frac{2pE(0)}{p-2}\right)^{\frac{p+\alpha-2}{2}} - 1 < 0,$$

and ε_5 is a small constant such that

$$1 - \varepsilon_5 - \frac{\varepsilon_5(1+\sigma_1)p}{\lambda_1(p-2)} > 0, \quad \varepsilon_5 + \frac{\beta\varepsilon_5}{2} - a < 0, \quad E(0) + \varepsilon_5 \int_\Omega u_0 u_1 \mathrm{d}x \ge 0.$$

Proof The proof of Theorem 5.1 is divided into four steps.

Step 1. *Global existence for the case of $(u_0, u_1) \in \mathcal{W}_1$.* Let the sequences $\{\omega_j\}_{j=1}^{+\infty}$, $\{u_{0m}\}_{m=1}^{+\infty}$, $\{u_{1m}\}_{m=1}^{+\infty}$, and $\{u_m\}_{m=1}^{+\infty}$ be the same ones as those stated in the proof of Theorem 3.1. Multiplying both sides of (3.1) by $b'_{jm}(t)$, summing over $j = 1, 2, \ldots, m$, and integrating it from 0 to t, for sufficiently large m, we obtain

$$E_m(t) + \int_0^t \sigma(\|\nabla u_m\|_2^2)\|u_{ms}(s)\|_2^2 \mathrm{d}s \le E_m(0) < d, \quad \text{for all } 0 \le t < +\infty. \tag{5.3}$$

$$E_m(t) = \frac{1}{2}\|u_{mt}\|_2^2 + J(u_m) \le E_m(0) < d, \quad \text{for all } 0 \le t < +\infty. \tag{5.4}$$

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Next, we prove that $(u_m, u_{mt}) \in \mathcal{W}_1$ for sufficiently large m. Similar to the proof in Theorem 4.1, we get $(u_m, u_{mt}) \in \mathcal{W}_1$ and $I(u_m) > 0$ for sufficiently large m and $0 \le t < +\infty$.

Next, by the definitions of $I(u)$, $E(t)$, $J(u)$, and $I(u) \ge 0$, we obtain

$$J(u_m) = \frac{p-2}{2p}\|\Delta u_m\|_2^2 + \frac{1}{p^2}\|u_m\|_p^p + \frac{1}{p}I(u_m), \quad \text{for all } 0 \le t < +\infty, \tag{5.5}$$
$$E_m(t) := \frac{1}{2}\|u_{mt}\|_2^2 + J(u_m) > 0, \quad \text{for all } 0 \le t < +\infty.$$

Using (5.4) and (5.5) and $I(u) \ge 0$, we get $\frac{p-2}{2p}\|\Delta u_m\|_2^2 + \frac{1}{p^2}\|u_m\|_p^p < d$ for all $0 \le t < +\infty$. From the embedding $H_0^2(\Omega) \hookrightarrow W_0^{1,p}(\Omega)$, we get $\|\nabla u\|_p \le \overline{S}\|\Delta u\|_2 \le \overline{S}\sqrt{\frac{2pd}{p-2}}$ for all $0 \le t < +\infty$. We observe that for $q = p > 2$, with (5.3)–(5.5), we obtain that $\{u_m\}$ is uniformly bounded in $L^\infty((0,\infty); H_0^2(\Omega))$, $\{u_m\}$ is uniformly bounded in $L^\infty((0,\infty); W_0^{1,p}(\Omega))$, $\{u_{mt}\}$ is uniformly bounded in $L^\infty((0,\infty); L^2(\Omega))$, and $\{u_{mt}\}$ is uniformly bounded in $L^2((0,\infty); L^2(\Omega))$. The rest of the proof can be obtained similarly to that of Theorem 4.1.

Step 2. *Global existence for the case of* $(u_0, u_1) \in \mathcal{W}_2$. The proof can be obtained similarly to that of Theorem 4.1.

Step 3. *Decay estimate (5.1) of energy.* From Steps 1 and 2, we know that if $(u_0, u_1) \in \mathcal{W}$, then $(u, u_t) \in \mathcal{W}$ for $0 \le t < +\infty$, and $I(u) \ge 0$. With the definitions of $E(t)$, $\sigma(s)$, and $I(u) \ge 0$, we have

$$\begin{aligned} E(0) &\ge E(t) + \int_0^t \sigma(\|\nabla u\|_2^2)\|u_s(s)\|_2^2 \mathrm{d}s \ge \frac{1}{2}\|u_t\|_2^2 + J(u) + a\int_0^t \|u_s\|_2^2 \mathrm{d}s \\ &= \frac{1}{2}\|u_t\|_2^2 + \frac{p-2}{2p}\|\Delta u\|_2^2 + \frac{1}{p^2}\|u\|_p^p + \frac{1}{p}I(u) + a\int_0^t \|u_s\|_2^2 \mathrm{d}s \\ &\ge \frac{1}{2}\|u_t\|_2^2 + \frac{p-2}{2p}\|\Delta u\|_2^2 + \frac{1}{p^2}\|u\|_p^p + a\int_0^t \|u_s\|_2^2 \mathrm{d}s. \end{aligned} \tag{5.6}$$

A combination of (5.6) with $E(0) \le d$ yields that $\|u_t\|_2^2 \le 2d$, $\|\Delta u\|_2^2 \le \frac{2pd}{p-2}$, $\|\nabla u\|_p^p \le \overline{S}^p\|\Delta u\|_2^p \le \overline{S}^p\left(\frac{2pd}{p-2}\right)^{\frac{p}{2}}$, and $\int_0^t \|u_s\|_2^2 \le \frac{d}{a}$. Using the inequality $\lambda_1^{\frac{1}{2}}\|\nabla u\|_2^2 \le \|\Delta u\|_2^2$, we have

$$\|\nabla u\|_2^2 \le \lambda_1^{-\frac{1}{2}}\|\Delta u\|_2^2 \le \lambda_1^{-\frac{1}{2}}\frac{2pd}{p-2} =: C''.$$

Because $\sigma(s)$ is a positive continuous function on $[0, +\infty)$, there is a constant $\sigma_3 > 0$ such that

$$\sigma(\|\nabla u\|_2^2) \le \max_{0 \le \|\nabla u\|_2^2 \le C''} \sigma(\|\nabla u\|_2^2) =: \sigma_3. \tag{5.7}$$

The rest of the proof can be obtained similarly to that of Theorem 4.1.

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Step 4. *Exponential decay estimate (5.2) of energy.* Let

$$F_1(t) := E(t) + \varepsilon_5 \int_\Omega u u_t \mathrm{d}x + \varepsilon_5 \int_0^t \sigma(\|\nabla u\|_2^2) \int_\Omega u u_s \mathrm{d}x\mathrm{d}s.$$

Similar to the proof of Theorem 4.1, we get the exponential decay estimate (5.2). ■

Unfortunately, we failed to obtain the existence of blow-up solutions in the critical case $p = q$. However, we are certain that blow-up might occur when the initial data lie outside the potential well.

6 Case $q < p$

This section is concerned with the case where the reaction is dominant (i.e., $q < p$).

Theorem 6.1 *Let $q < p$, $(u_0, u_1) \in H$, and u be the weak solution as given in Definition 2.1. If $E(0) + \widetilde{C} \geq 0$ and μ is a positive constant which satisfies $q + \mu < p$ and $2 < p + \mu < \begin{cases} \frac{np}{n-p}, & \text{for } n > p, \\ +\infty, & \text{for } n \leq p, \end{cases}$ then u remains globally bounded for any $t \in [0, +\infty)$, which satisfies*

$$\|u_t\|_2 \leq \sqrt{2}\left(E(0) + \widetilde{C}\right)^{\frac{1}{2}}, \quad \|\Delta u\|_2 \leq \sqrt{2}\left(E(0) + \widetilde{C}\right)^{\frac{1}{2}}, \tag{6.1}$$

$$\|\nabla u\|_p \leq \left[\frac{m_2 p\left(E(0) + \widetilde{C}\right)}{m_2 - 1}\right]^{\frac{1}{p}}, \quad \|u\|_q \leq \left[q^2\left(E(0) + \widetilde{C}\right)\right]^{\frac{1}{q}}, \tag{6.2}$$

where the positive constants are defined as $m_1 := \frac{p}{p-q-\mu}$, $m_2 := \frac{p}{q+\mu}$, $\widetilde{C} := \frac{\widetilde{S}^{m_1(q+\mu)}}{m_1(qe\mu)^{m_1}} |\Omega|^{\frac{(p-q)m_1}{p+\mu}} p^{\frac{m_1}{m_2}}$, and $\widetilde{S}$ is the embedding constant in $W_0^{1,p}(\Omega) \hookrightarrow L^{p+\mu}(\Omega)$.

Proof Taking $\phi := u_t$ in (2.2), we get

$$\begin{aligned}
&\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q^2}\|u\|_q^q \\
=&\frac{1}{2}\|u_1\|_2^2 + \frac{1}{2}\|\Delta u_0\|_2^2 + \frac{1}{p}\|\nabla u_0\|_p^p + \frac{1}{q^2}\|u_0\|_q^q + \frac{1}{q}\int_\Omega |u|^q \log|u| \mathrm{d}x \\
&- \frac{1}{q}\int_\Omega |u_0|^q \log|u_0| \mathrm{d}x - \int_0^t \sigma(\|\nabla u\|_2^2) \int_\Omega \|u_t\|_2^2 \mathrm{d}x\mathrm{d}s \\
\leq& E(0) + \frac{1}{q}\int_\Omega |u|^q \log|u| \mathrm{d}x.
\end{aligned} \tag{6.3}$$

By using the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p+\mu}(\Omega)$ and Hölder's inequality, we obtain

$$\|u\|_{q+\mu} \leq |\Omega|^{\frac{p-q}{(p+\mu)(q+\mu)}} \|u\|_{p+\mu} \leq \widetilde{S}|\Omega|^{\frac{p-q}{(p+\mu)(q+\mu)}} \|\nabla u\|_p. \tag{6.4}$$

As for the logarithmic term, using (2.4), (6.4), Hölder's inequality, and the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p+\mu}(\Omega)$, we get

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$$\frac{1}{q}\int_\Omega |u|^q \log|u|\mathrm{d}x = \frac{1}{q}\int_{\{x\in\Omega||u|\le 1\}} |u|^q \log|u|\mathrm{d}x + \frac{1}{q}\int_{\{x\in\Omega||u|>1\}} |u|^q \log|u|\mathrm{d}x$$
$$\le \frac{1}{qe\mu}\|u\|_{q+\mu}^{q+\mu} \le \frac{\widetilde{S}^{q+\mu}}{qe\mu}|\Omega|^{\frac{p-q}{p+\mu}}\|\nabla u\|_p^{q+\mu}.$$

Finally, using Young's inequality, we get

$$\frac{\widetilde{S}^{q+\mu}}{qe\mu}|\Omega|^{\frac{p-q}{p+\mu}}\|\nabla u\|_p^{q+\mu} \le \frac{\widetilde{S}^{m_1(q+\mu)}}{m_1(qe\mu)^{m_1}}|\Omega|^{\frac{(p-q)m_1}{p+\mu}} p^{\frac{m_1}{m_2}} + \frac{1}{m_2 p}\|\nabla u\|_p^p. \tag{6.5}$$

Substituting (6.5) into (6.3), we get $\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\Delta u\|_2^2 + \left(\frac{1}{p} - \frac{1}{m_2 p}\right)\|\nabla u\|_p^p + \frac{1}{q^2}\|u\|_q^q \le E(0) + \widetilde{C}$. Hence, from (6.1) and (6.2), the solution u of (1.1) remains globally bounded. ∎

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