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OPTIMAL INVENTORY CONTROL WITH STATE-DEPENDENT JUMPS

LIJUN BO,* *Xidian University*
YIJIE HUANG,** *University of Science and Technology of China*

Abstract

We study an optimal inventory control problem under a reflected jump–diffusion netflow process with state-dependent jumps, in which the intensity of the jump process can depend on the inventory level. We examine the well-posedness of the associated integro-differential Hamilton–Jacobi–Bellman (ID-HJB) equation with Neumann boundary condition in the classical sense. To achieve this, we first establish the existence of viscosity solutions to the ID-HJB equation of an auxiliary control problem with a compact policy space, which is proved to be equivalent to the primal problem. We reformulate the ID-HJB equation as a Neumann HJB equation with the (non-local) integral term expressed in terms of the value function of the auxiliary problem and prove the existence of a unique classical solution to the Neumann HJB equation. Then, the well-posedness of the primal ID-HJB equation follows from the unique classical solution of the Neumann HJB equation and the existence of viscosity solutions to the auxiliary ID-HJB equation. Based on this classical solution, we characterize the optimal (admissible) inventory control strategy and show the verification result for the primal control problem.

Keywords: Inventory control; state-dependent jumps; integro-differential Hamilton–Jacobi–Bellman equation; Neumann boundary; viscosity solution; classical solution.

2010 Mathematics Subject Classification: Primary 93E20; 60H30
Secondary 93E03; 34A34

1. Introduction

With advances in information technology, firms can collect large amounts of data on consumer behavior relating to recent sales and decide the supply of product according to forecasted customer demand. However, due to the forecast error of demand [10], sometimes a firm may face problems of overstock, and the inventory netflow is usually used to describe the inventory backlog at the time. In this context, the firm needs to adopt some strategies in response to the overstock, for example providing customer sales discounts for the product [2, 18], and also improve storage capacity if necessary [24]. Therefore, the total storage cost of the firm consists of two components: (i) the loss of profits due to the exercise of control to reduce the backlog; and (ii) the capacity expansion cost, which is proportional to the necessary storage expansion.

Inventory management problems have attracted significant attention from researchers in economics, marketing and operations management. Cheng and Sethi [11] proposed a class of

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* Postal address: School of Mathematics and Statistics, Xidian University, Xi'an, 710126, China. Email: lijunbo@ustc.edu.cn
** Postal address: School of Mathematical Sciences, University of Science and Technology of China, Hefei, 230026, China. Email: huang1@mail.ustc.edu.cn

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Markovian demand models with backlogging, analyzed the lost-sales problem and established the optimality of an (*s*, *S*)-type policy under fairly general conditions. Interested readers may refer to [19, 31, 34] for some earlier work on this aspect. More recently, numerous papers have appeared in the literature on Brownian inventory models (see e.g. [7, 13, 14, 22]). The Brownian inventory model implies that the resulting inventory levels can be changed instantaneously over time, and this may be referred to as impulse control in the context of stochastic control [37]. In particular, Weerasinghe and Zhu [36] explored the Brownian inventory control problem subject to capacity expansion, in which the inventory netflow process is modeled by a Brownian motion with drift. In this setting, the firm aims to determine a control policy that minimizes the loss of profits due to the exercise of control and capacity expansion costs.

In this paper, we study a stochastic optimal inventory control problem with inventory space netflow following a reflected jump–diffusion process with state-dependent jumps. In fact, the jump model is widely used in the literature on inventory management; for example, see [5, 22, 30, 38]. The majority of existing studies focus on the pure jump process (not state-dependent) or a mixture of Brownian motion and a compound Poisson process (CPP) describing the inventory-level process. Specifically, Presman and Sethi [28] considered a CPP inventory model with constant rate and developed an adjusted (discounted) cycle cost formula to explore an inventory control problem under both the average and the discounted cost criteria. Liu et al. [25] studied an inventory cost minimization problem under a combination of diffusion and general compound Poisson demands and established the corresponding (*s*,*S*) policy (which is proved to be unique). The jumps of the inventory process may be caused by a surge in demand, which may occur due to rare events. In many settings, the occurrence of unexpected rare events may result in surging demand. This has been highlighted by the COVID-19 outbreak, which generated an unexpected demand surge for a range of consumer goods, and this led to a jump in the corresponding inventory level for each good. However, the intensity of jump processes considered in the work reviewed above is exogenous, which is characterized by a given jump measure. It is known that such jump processes have independent increments (which means that future jumps are independent of what has occurred in the past). In practice, one does not usually observe independent increments in real-world data (e.g. in finance, seismology and criminology). Here, we incorporate a state-dependent jump process into the controlled inventory process, such that the intensity of the jump process can depend on the inventory space level, and hence the jumps are endogenous. This class of state-dependent process models has the advantage of being very versatile in various applications (see e.g. [1, 15, 20, 29], to name just a few).

We formulate the inventory space netflow process representing the available inventory storage space as a controlled reflected jump–diffusion process with state-dependent jumps. The local time term of the netflow process is used to describe the maximum storage capacity. This means that once the netflow process hits zero, there is no spare warehouse space for product and the firm needs to improve its storage capacity. As mentioned before, to avoid expensive capacity expansion costs, the firm can adopt some strategies, such as providing customer sales discounts, to lower the product stock level. Furthermore, in our jump–diffusion setting, both positive and negative jumps may occur, and the jump arrival intensity depends on the current level of surplus inventory. In particular, the negative jumps of the controlled inventory process can make the paths of the local time process discontinuous. In other words, sometimes there may be a surge in the level of overstock product inventory, such that the firm also needs to improve the storage capacity within a short time.

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Under the above model settings, the firm seeks an optimal (admissible) inventory policy that minimizes its expected cost. Due to the occurrence of state-dependent jumps, we arrive at a class of integro-differential Hamilton–Jacobi–Bellman (ID-HJB) equations with Neumann boundary condition by using the dynamic program [27]. Cai [8] and Cai and Kou [9] proved that a certain type of linear ordinary integro-differential equations can be transformed into a homogeneous linear high-order ordinary differential equation (ODE), which would therefore have an explicit solution. In our setting, however, the ID-HJB equation is nonlinear and so no analytical solution can usually be expected. We study the well-posedness of our ID-HJB equation in the classical sense, and the optimal (admissible) inventory strategy can be characterized with the help of the classical solution to the equation.

We introduce an auxiliary stochastic control problem with a restricted admissible set in which the policy space becomes compact. The equivalence between the auxiliary problem and the primal problem is established via the verification theorem. The main motivation for proposing such an auxiliary problem is to overcome the unboundedness of the policy space in the primal control problem, which makes establishing the uniqueness of the viscosity solution to the related equation challenging. Owing to the equivalence between the auxiliary problem and the primal problem, we mainly focus on the auxiliary control problem.

The roadmap for addressing the classical solution of the ID-HJB equation satisfied by the value function of the auxiliary problem is as follows. We first establish the existence of the (continuous) viscosity solutions to the auxiliary ID-HJB equation with Neumann boundary condition, i.e. we show that the value function of the auxiliary control problem is a (continuous) viscosity solution. The proof strategy bears some resemblance to that used by Seyde [32], who considered the quasi-variational inequality. Motivated by Davis and Lleo [16], we reformulate the auxiliary ID-HJB equation as a non-homogeneous second-order ODE in which the non-homogeneous term is expressed in terms of the value function treated as a viscosity solution of the auxiliary HJB equation. We prove the uniqueness of the viscosity solution to the non-homogeneous second-order ODE. Under the compact policy space, it is worth noting that the proof strategy for the uniqueness of the viscosity solution rests on the construction of the penalty function, which is challenging due to the Neumann boundary condition and the unboundedness of the state space of the netflow process. We design a desired polynomial-like penalty function in lieu of the Neumann boundary condition and the linear growth condition satisfied by the value function of the auxiliary problem. Toward this end, we extend and develop the arguments used by Weerasinghe and Zhu [36], who studied the homogeneous case to establish existence of global (classical) solutions to the non-homogeneous second-order ODE by overcoming two main difficulties: (i) the unknown regularity of the non-homogeneous term; and (ii) the unknown monotonicity of the value function of the auxiliary problem (we will explain this point in detail before the proof of Theorem 1). The special homogeneous structure of the ODE in [36] helps us to obtain the properties of a strict local minimum (maximum) point, which is key to proving the existence of classical solutions. In this paper, we first demonstrate the Lipschitz continuity of the non-homogeneous term in the associated ODE. Subsequently, leveraging this Lipschitz property, we derive some prior estimates and conduct an analysis on the monotonicity, local extrema, boundedness and asymptotic behavior of the solution. This is necessary because the methods employed to establish well-posedness of the HJB equation in [36] are not directly applicable to our case. We finally conclude that the value function of the primal inventory control problem is in fact the unique classical solution of our primal ID-HJB equation with Neumann boundary condition. With the help of this classical solution, we characterize the optimal (admissible) inventory control strategy in the verification theorem.

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The rest of the paper is organized as follows. Section 2 formulates the optimal inventory control problem in continuous time and proves the structural property of the value function. Section 3 derives the associated ID-HJB equation with Neumann boundary condition and introduces an auxiliary stochastic control problem, which is proved to be equivalent to the primal problem. Section 4 deals with the well-posedness of the auxiliary ID-HJB equation with Neumann boundary condition in the viscosity solution sense. Section 5 shows the existence and uniqueness of classical solutions to the primal ID-HJB equation, and the optimal (admissible) inventory strategy is characterized. The proofs of some auxiliary results are provided in Appendix A.

2. The optimal inventory control problem

This section describes the infinite-horizon optimal inventory control problem with an inventory netflow fluctuating as a state-dependent jump–diffusion process, which is in essence an extension to the inventory control problem studied in [36], where the netflow process is modeled as a Brownian motion with drift.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. Under this filtered probability space, $B = (B_t)_{t\geq 0}$ is a scalar Brownian motion, and $\pi(\mathrm{d}t, \mathrm{d}z, \mathrm{d}\xi)$ defined on $\mathcal{B}_{\mathbb{R}_+^2\times\mathbb{R}}$ (Borel σ-algebra generated by $\mathbb{R}_+^2 \times \mathbb{R}$, where $\mathbb{R}_+ := [0, \infty)$) is a Poisson random measure with compensator $\mathrm{d}t\,\mathrm{d}z\,\nu(\mathrm{d}\xi)$, which is independent of the Brownian motion B. Here, $\nu(\mathrm{d}\xi)$ is referred to as the jump measure, which is assumed to be finite and satisfies $\int_{\mathbb{R}} \xi^2 \nu(\mathrm{d}\xi) < +\infty$. The reflected version of the netflow process with a capacity control $\theta = (\theta_t)_{t\geq 0}$ as an $\mathbb{F}$-adapted (non-negative) process is given by, for all $t \geq 0$,

$$\begin{cases} Y_t = Y_0 + \displaystyle\int_0^t (\alpha - \beta Y_s)\,\mathrm{d}s + \int_0^t \theta_s\,\mathrm{d}s + \sigma B_t + I_t(Y) + L_t, \\ I_t(Y) := \displaystyle\int_0^t \int_{\mathbb{R}_+\times\mathbb{R}} \gamma\xi \mathbf{1}_{z\leq\lambda(Y_{s-})}\pi(\mathrm{d}s, \mathrm{d}z, \mathrm{d}\xi), \end{cases} \tag{1}$$

where $Y_0 \geq 0, \alpha \in \mathbb{R}, \beta \geq 0$ and $\gamma \geq 0$. Here, $Y = (Y_t)_{t\geq 0}$ represents the inventory space netflow process, which describes the available inventory storage space. The local time process $L = (L_t)_{t\geq 0}$ is used to describe the maximum storage capacity. Note that $L = (L_t)_{t\geq 0}$ increases if and only if the netflow process $Y = (Y_t)_{t\geq 0}$ hits the boundary zero; that is, once there is no spare surplus inventory space, the firm needs to improve its storage capacity. In particular, if $\beta = \gamma = 0$, the inventory model (1) reduces to the one described in (2.12) of [36].

The function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is called the intensity function of the jump process $I(Y) = (I_t(Y))_{t\geq 0}$ [12, 26]. We impose the following assumption on the intensity function λ:

($\mathbf{A}_p$) The intensity function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is bounded and Lipschitz continuous, i.e. $\lambda(y) \leq M_\lambda$ for all $y \in \mathbb{R}_+$ and $|\lambda(y_1) - \lambda(y_2)| \leq K_\lambda |y_1 - y_2|$ for all $y_1, y_2 \in \mathbb{R}_+$, where $M_\lambda > 0$ is the upper bound and $K_\lambda > 0$ is the Lipschitz coefficient. Moreover, $y_2|\lambda(y_1) - \lambda(y_2)| \leq K_\lambda |y_1 - y_2|$ for all $y_1 \geq y_2 \geq 0$.

Remark 1. As in [12, 26], the jump intensity function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is also assumed to be bounded and Lipschitz continuous. Let us provide some examples of the intensity function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ that satisfy assumption ($\mathbf{A}_p$). Let $f(y) \in C^1(\mathbb{R}_+)$ satisfy $f'(y) \leq \frac{C}{(1+y)^{1+\epsilon}}$ with some constants $C > 0$ and $\epsilon \geq 0$ for all $y \in \mathbb{R}_+$. Then, for any $M > 0$, the intensity function given by $\lambda(y) := f(y) \wedge M$ for $y \in \mathbb{R}_+$ satisfies assumption ($\mathbf{A}_p$). In particular, one can take $f(y) = \ln(1 + y)$ for $y \in \mathbb{R}_+$.

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We next address the well-posedness of the stochastic differential equation (SDE) (1) with state-dependent jumps in the strong solution sense. For this, we introduce the following spaces for stochastic processes. For any $t \geq 0$, let $\mathbb{S}_t$ be the set of $\mathbb{F}$-adapted processes $Y = (Y_s)_{s\in[0,t]}$ satisfying $\mathbb{E}[\sup_{s\in[0,t]}|Y_s|] < +\infty$. Denote by $\mathbb{S}_\infty$ the set of all $\mathbb{F}$-adapted processes $Y = (Y_s)_{s\geq 0}$ such that $(Y_s)_{s\in[0,t]} \in \mathbb{S}_t$ for all $t \geq 0$. Then we have the following result.

Lemma 1. *Suppose that assumption* $(\mathbf{A}_p)$ *holds. Then the SDE (1) with initial data* $Y_0 = y \in \mathbb{R}_+$ *admits a unique strong solution* $(Y, L) \in \mathbb{S}^2_\infty$.

The proof of Lemma 1 is given in Appendix A. In (1), the non-negative $L = (L_t)_{t\geq 0}$ is a process that is non-decreasing and right-continuous with left limits (RCLL) and which satisfies $\int_0^t Y_s \, \mathrm{d}L_s = 0$ (with $L_0 = L_{0-} = 0$) almost surely (a.s.) for all $t \geq 0$. The process L is referred as to the regulator process or local time process for Y. However, in contrast to the case with Brownian netflow process in [36], the path of $t \to L_t$ is not continuous any more since the pure jump part $I(Y) = (I_t(Y))_{t\geq 0}$ may have negative jumps. Let $\Delta L_t := L_t - L_{t-}$ be the jump's size at time $t \geq 0$. Then the jump process for L can be identified as (see [6])

$$\Delta L_t = (\Delta I_t(Y) + Y_{t-})^- \quad \forall t \geq 0. \tag{2}$$

Let $\mathcal{A}$ be the admissible control space, which is the set of $\mathbb{F}$-adapted processes $\theta = (\theta_t)_{t\geq 0}$ taking values on $\mathbb{R}_+$ such that $\mathbb{E}\big[\int_0^t \theta_s \, \mathrm{d}s\big] < \infty$ for all $t > 0$. Clearly, $\mathcal{A}$ is non-empty since any non-negative constant is in $\mathcal{A}$. For a running-cost function $c : \mathbb{R}_+ \to \mathbb{R}_+$ and a capacity parameter $p > 0$, the value function of the infinite-horizon optimal inventory control problem is defined by, for all $y \in \mathbb{R}_+$,

$$V(y) := \inf_{\theta\in\mathcal{A}} J(y;\theta) := \inf_{\theta\in\mathcal{A}} \mathbb{E}_y \left[\int_0^\infty \mathrm{e}^{-\rho s} \big(c(\theta_s) \, \mathrm{d}s + p \, \mathrm{d}L_s \big) \right], \tag{3}$$

where the conditional expectation $\mathbb{E}_y[\cdot] := \mathbb{E}[\,\cdot\,|Y_0 = y]$ and $\rho > 0$ is the discount rate. We impose the following assumptions on the running-cost function and discount rate:

- $(\mathbf{A}_c)$ The running-cost function $c : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing, strictly convex and continuously differentiable. Moreover, it satisfies $c(0) = 0$ and
$$\lim_{x\to\infty} \frac{c(x)}{x} = \infty.$$
- $(\mathbf{A}_p)$ The discount rate ρ satisfies $\rho > \rho_0 := 2\gamma K_\lambda \int_{\mathbb{R}} |\xi| \nu(\mathrm{d}\xi)$.

Given the running-cost function $c : \mathbb{R}_+ \to \mathbb{R}_+$ in assumption $(\mathbf{A}_c)$, let us introduce

$$\Phi(y) := \sup_{\theta\geq 0}\{\theta y - c(\theta)\} \quad \forall y \in \mathbb{R}. \tag{4}$$

in [36] that Φ is finite on $\mathbb{R}$, non-negative and non-decreasing with $\Phi(y) = 0$ for all $y \leq 0$. Moreover, $y \to \Phi(y)$ is differentiable with $\Phi'(y) = \Psi(y) := (c')^{-1}(y)\mathbf{1}_{y>c'(0)}$ and $\Phi(y) = y\Psi(y) - c(\Psi(y))$.

We next show the Lipschitz-continuous property of the value function $y \to V(y)$ defined by (3).

Lemma 2. *Suppose that assumption* $(\mathbf{A}_p)$ *holds. Consider the value function* $V(y)$ *with* $y \in \mathbb{R}_+$ *defined by (3) with the discount factor* $\rho > 0$ *given in assumption* $(\mathbf{A}_\rho)$. *Then* $y \to V(y)$ *is Lipschitz continuous with Lipschitz coefficient* $2p$, *i.e. for all* $y_1, y_2 \in \mathbb{R}_+$,

$$|V(y_1) - V(y_2)| \leq 2p|y_1 - y_2|. \tag{5}$$

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Proof. We first prove (i). Let $Y^y = (Y_t^y)_{t\geq 0}$ be the (controlled) reflected process (1) with $Y_0^y = y \in \mathbb{R}_+$. Denote by $L^y = (L_t^y)_{t\geq 0}$ the regulator process of the reflected process Y^y with $L_0^y = 0$. Then, for any $y_1, y_2 \in \mathbb{R}_+$, we have from (3) that

$$|V(y_1) - V(y_2)| \leq \sup_{\theta\in\mathcal{A}} \mathbb{E}\left[\left|\int_0^\infty \mathrm{e}^{-\rho s} p\, \mathrm{d}(L_s^{y_1} - L_s^{y_2})\right|\right]. \tag{6}$$

Integration by parts yields that, for any $T > 0$,

$$\int_0^T \mathrm{e}^{-\rho s}\, \mathrm{d}L_s^y = \mathrm{e}^{-\rho T} L_T^y - L_0^y + \rho \int_0^T L_s^y \mathrm{e}^{-\rho s}\, \mathrm{d}s. \tag{7}$$

Define $\hat{Y}_s^y := \mathrm{e}^{\beta s} Y_s^y$ for all $s \geq 0$. Then, for all $s \geq 0$,

$$\begin{aligned} \hat{Y}_s^y &= Y_0^y + \int_0^s \mathrm{e}^{\beta v} \mathrm{d}Y_v^y + \beta \int_0^s \mathrm{e}^{\beta v} Y_v^y\, \mathrm{d}v \\ &= \underbrace{y + \int_0^s \mathrm{e}^{\beta v}\{(\alpha + \theta_v)\, \mathrm{d}v + \sigma\, \mathrm{d}B_v\} + \hat{I}_s(Y^y)}_{=:X_s^y} + \int_0^s \mathrm{d}\hat{L}_v^y. \end{aligned} \tag{8}$$

Here, $\hat{I}_s(Y^y) := \int_0^s \mathrm{e}^{\beta v}\, \mathrm{d}I_v(Y^y)$ and $\hat{L}_s^y := \int_0^s \mathrm{e}^{\beta v}\, \mathrm{d}L_v^y$ for all $s \geq 0$. Hence, $s \to \hat{L}_s^y$ is a non-decreasing RCLL process that satisfies

$$\int_0^s \mathbf{1}_{\hat{Y}_v^y = 0}\, \mathrm{d}\hat{L}_v^y = \int_0^s \mathrm{e}^{\beta v} \mathbf{1}_{Y_v^y = 0}\, \mathrm{d}L_v^y = \int_0^s \mathrm{e}^{\beta v}\, \mathrm{d}L_v^y = \int_0^s \mathrm{d}\hat{L}_v^y.$$

That is, $\hat{Y}_s^y = X_s^y + \hat{L}_s^y$ with $s \geq 0$ is a Skorokhod problem (SP). Thus, for all $s \geq 0$,

$$\hat{L}_s^y = \sup_{0\leq u\leq s}(X_u^y)^- = \sup_{0\leq u\leq s}\left(y + \int_0^u \mathrm{e}^{\beta v}\{(\alpha + \theta_v)\, \mathrm{d}v + \sigma\, \mathrm{d}B_v\} + \hat{I}_u(Y^y)\right)^-. \tag{9}$$

By this, we have that, for all $y_1, y_2 \in \mathbb{R}_+$,

$$\begin{aligned} &\left|\hat{L}_s^{y_1} - \hat{L}_s^{y_2}\right| \\ &\leq \sup_{0\leq u\leq s}\left|(X_u^{y_1})^- - (X_u^{y_2})^-\right| \leq |y_1 - y_2| + \sup_{0\leq u\leq s}\left|\hat{I}_u(Y^{y_1}) - \hat{I}_u(Y^{y_2})\right| \\ &= |y_1 - y_2| + \gamma \sup_{0\leq u\leq s}\left|\int_0^u \mathrm{e}^{\beta r} \int_{\mathbb{R}_+\times\mathbb{R}} \xi(\mathbf{1}_{z\leq\lambda(Y_{r-}^{y_1})} - \mathbf{1}_{z\leq\lambda(Y_{r-}^{y_2})})\pi(\mathrm{d}r, \mathrm{d}z, \mathrm{d}\xi)\right| \\ &\leq |y_1 - y_2| + \gamma \sup_{0\leq u\leq s}\int_0^u \mathrm{e}^{\beta r} \int_{\mathbb{R}_+\times\mathbb{R}} |\xi|\left|\mathbf{1}_{z\leq\lambda(Y_{r-}^{y_1})} - \mathbf{1}_{z\leq\lambda(Y_{r-}^{y_2})}\right| \pi(\mathrm{d}r, \mathrm{d}z, \mathrm{d}\xi) \\ &\leq |y_1 - y_2| + \gamma \int_0^s \mathrm{e}^{\beta r} \int_{\mathbb{R}_+\times\mathbb{R}} |\xi|\left|\mathbf{1}_{z\leq\lambda(Y_{r-}^{y_1})} - \mathbf{1}_{z\leq\lambda(Y_{r-}^{y_2})}\right| \pi(\mathrm{d}r, \mathrm{d}z, \mathrm{d}\xi). \end{aligned} \tag{10}$$

Taking the expectation on both sides of the inequality (10) and using assumption $(\mathbf{A}_p)$, we arrive at

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$$
\begin{aligned}
&\mathbb{E}\left[\left|\hat{L}_s^{y_1}-\hat{L}_s^{y_2}\right|\right]\\
&\le |y_1-y_2|+\gamma\mathbb{E}\left[\int_0^s \mathrm{e}^{\beta r}\int_{\mathbb{R}_+}\int_{\mathbb{R}}|\xi|\left|\mathbf{1}_{z\le\lambda(Y_{r-}^{y_1})}-\mathbf{1}_{z\le\lambda(Y_{r-}^{y_2})}\right|\nu(\mathrm{d}\xi)\,\mathrm{d}z\,\mathrm{d}r\right]\\
&= |y_1-y_2|+\gamma\int_{\mathbb{R}}|\xi|\nu(\mathrm{d}\xi)\int_0^s \mathrm{e}^{\beta r}\mathbb{E}\left[\int_{\mathbb{R}_+}\left|\mathbf{1}_{z\le\lambda(Y_{r-}^{y_1})}-\mathbf{1}_{z\le\lambda(Y_{r-}^{y_2})}\right|\,\mathrm{d}z\right]\mathrm{d}r\\
&= |y_1-y_2|+\gamma\int_{\mathbb{R}}|\xi|\nu(\mathrm{d}\xi)\int_0^s \mathrm{e}^{\beta r}\mathbb{E}\left[\left|\lambda(Y_{r-}^{y_1})-\lambda(Y_{r-}^{y_2})\right|\right]\mathrm{d}r\\
&\le |y_1-y_2|+\frac{\rho_0}{2}\int_0^s \mathrm{e}^{\beta r}\mathbb{E}\left[\left|Y_{r-}^{y_1}-Y_{r-}^{y_2}\right|\right]\mathrm{d}r,
\end{aligned}
\tag{11}
$$

where the constant $\rho_0>0$ is given in assumption $(\mathbf{A}_\rho)$. Here, we have used the fact that, for all $y_1, y_2\in\mathbb{R}_+$,

$$
\begin{aligned}
\int_{\mathbb{R}_+}\left|\mathbf{1}_{z\le\lambda(Y_{r-}^{y_1})}-\mathbf{1}_{z\le\lambda(Y_{r-}^{y_2})}\right|\mathrm{d}z &= \int_{\mathbb{R}_+}\mathbf{1}_{\lambda(Y_{r-}^{y_1})\wedge\lambda(Y_{r-}^{y_2})\le z\le\lambda(Y_{r-}^{y_1})\vee\lambda(Y_{r-}^{y_2})}\,\mathrm{d}z\\
&= \lambda(Y_{r-}^{y_1})\vee\lambda(Y_{r-}^{y_2})-\lambda(Y_{r-}^{y_1})\wedge\lambda(Y_{r-}^{y_2})=\left|\lambda(Y_{r-}^{y_1})-\lambda(Y_{r-}^{y_2})\right|.
\end{aligned}
$$

By applying a similar argument to derive (10) and (11), we also have that, for all $s\ge 0$,

$$
\mathbb{E}\left[\left|\hat{I}_s(Y^{y_1})-\hat{I}_s(Y^{y_2})\right|\right]\le\frac{\rho_0}{2}\int_0^s \mathrm{e}^{\beta r}\mathbb{E}\left[\left|Y_{r-}^{y_1}-Y_{r-}^{y_2}\right|\right]\mathrm{d}r. \tag{12}
$$

It follows from (8), (11) and (12) that

$$
\begin{aligned}
\mathbb{E}\left[\left|\hat{Y}_s^{y_1}-\hat{Y}_s^{y_2}\right|\right] &\le |y_1-y_2|+\mathbb{E}\left[\left|\hat{I}_s(Y^{y_1})-\hat{I}_s(Y^{y_2})\right|\right]+\mathbb{E}\left[\left|\hat{L}_s^{y_1}-\hat{L}_s^{y_2}\right|\right]\\
&\le 2|y_1-y_2|+\rho_0\int_0^s \mathrm{e}^{\beta r}\mathbb{E}\left[\left|Y_{r-}^{y_1}-Y_{r-}^{y_2}\right|\right]\mathrm{d}r\\
&= 2|y_1-y_2|+\rho_0\int_0^s \mathbb{E}\left[\left|\hat{Y}_{r-}^{y_1}-\hat{Y}_{r-}^{y_2}\right|\right]\mathrm{d}r.
\end{aligned}
$$

We conclude from Gronwall's inequality that, for all $s\ge 0$,

$$
\max\left\{\mathbb{E}\left[\left|\hat{Y}_s^{y_1}-\hat{Y}_s^{y_2}\right|\right],\ \mathbb{E}\left[\left|\hat{L}_s^{y_1}-\hat{L}_s^{y_2}\right|\right]\right\}\le \mathrm{e}^{\frac{\rho_0}{2}s}|y_1-y_2|. \tag{13}
$$

Since $L_0^y=0$, we obtain that, for any $T>0$,

$$
\int_0^T \mathrm{e}^{-(\rho+\beta)s}\,\mathrm{d}\hat{L}_s^y=\mathrm{e}^{-(\rho+\beta)T}\hat{L}_T^y+(\rho+\beta)\int_0^T \hat{L}_s^y\mathrm{e}^{-(\rho+\beta)s}\,\mathrm{d}s.
$$

This yields that, for all $\theta\in\mathcal{A}$ and $\rho>\rho_0$,

$$
\begin{aligned}
\mathbb{E}\left[\left|\int_0^\infty \mathrm{e}^{-\rho s}p\,\mathrm{d}(L_s^{y_2}-L_s^{y_1})\right|\right] &= \lim_{T\to\infty}\mathbb{E}\left[\left|\int_0^T \mathrm{e}^{-\rho s}p\,\mathrm{d}(L_s^{y_2}-L_s^{y_1})\right|\right]\\
&\le p\lim_{T\to\infty}\left\{\mathbb{E}\left[\mathrm{e}^{-(\rho+\beta)T}\left|\hat{L}_T^{y_2}-\hat{L}_T^{y_1}\right|\right]+(\rho+\beta)\,\mathbb{E}\left[\int_0^T\left|\hat{L}_s^{y_2}-\hat{L}_s^{y_1}\right|\mathrm{e}^{-(\rho+\beta)s}\,\mathrm{d}s\right]\right\}\\
&\le p\lim_{T\to\infty}\left\{\mathrm{e}^{-(\rho+\beta)T}\mathrm{e}^{\frac{\rho_0}{2}T}|y_1-y_2|+(\rho+\beta)|y_1-y_2|\int_0^T \mathrm{e}^{-(\rho+\beta)s}\mathrm{e}^{\frac{\rho_0}{2}s}\,\mathrm{d}s\right\}
\end{aligned}
$$

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$$= p \lim_{T\to\infty} \left\{ \mathrm{e}^{-(\rho+\beta-\frac{\rho_0}{2})T} + (\rho+\beta) \int_0^T \mathrm{e}^{-(\rho+\beta-\frac{\rho_0}{2})s}\, \mathrm{d}s \right\} |y_1 - y_2|$$
$$\leq 2p|y_1 - y_2|.$$

By the arbitrariness of the control strategy $\theta \in \mathcal{A}$, we deduce the estimate (5) in view of (6).

3. Auxiliary control problem and the ID-HJB equation

In this section, we formulate the (primal) ID-HJB equation with Neumann boundary condition satisfied by the value function $y \to V(y)$ defined by (3), and then introduce an auxiliary stochastic control problem with compact policy space.

The following lemma provides an alternative representation for the value function $y \to V(y)$, which is useful for deriving the associated HJB equation satisfied by the value function.

Lemma 3. *We have that $\int_0^t Y_{s-}\, \mathrm{d}L_s^c = 0$ for all $t \geq 0$. Moreover, the value function defined by (3) can be rewritten as, for any $y \in \mathbb{R}_+$,*

$$V(y) = \inf_{\theta\in\mathcal{A}} \mathbb{E}_y \left[\int_0^\infty \mathrm{e}^{-\rho s} \left(\left(c(\theta_s) + \lambda(Y_s) \int_{\mathbb{R}} (\gamma\xi + Y_s)^- \nu(\mathrm{d}\xi) \right) \mathrm{d}s + p\, \mathrm{d}L_s^c \right) \right], \tag{14}$$

where $L_t^c := L_t - \sum_{s\leq t} \Delta L_s$ with $t \geq 0$ is the continuous part of the local time process $L = (L_t)_{t\geq 0}$.

Proof. Note that $Y_t = 0$ whenever $\Delta L_t > 0$. Then

$$0 = \int_0^t Y_s\, \mathrm{d}L_s = \int_0^t Y_s\, \mathrm{d}L_s^c + \sum_{s\leq t} Y_s \Delta L_s = \int_0^t Y_s\, \mathrm{d}L_s^c = \int_0^t Y_{s-}\, \mathrm{d}L_s^c. \tag{15}$$

On the other hand, it follows from (2) that the jump process of $Y = (Y_t)_{t\geq 0}$ is

$$\begin{aligned} \Delta Y_t &= \Delta I_t(Y) + \Delta L_t = \Delta I_t(Y) + (\Delta I_t(Y) + Y_{t-})^- \\ &= \begin{cases} \Delta I_t(Y), & -Y_{t-} \leq \Delta I_t(Y), \\ -Y_{t-}, & -Y_{t-} > \Delta I_t(Y) \end{cases} \\ &= \max\{\Delta I_t(Y), -Y_{t-}\}. \end{aligned} \tag{16}$$

As a result, the objective functional given in (3) can be rewritten as, for all admissible control strategies $\theta \in \mathcal{A}$,

$$\begin{aligned} J(y;\theta) &= \mathbb{E}_y \left[\int_0^\infty \mathrm{e}^{-\rho s} (c(\theta_s)\, \mathrm{d}s + p\, \mathrm{d}L_s) \right] \\ &= \mathbb{E}_y \left[\int_0^\infty \mathrm{e}^{-\rho s} (c(\theta_s)\, \mathrm{d}s + p\, \mathrm{d}L_s^c) \right] + p\mathbb{E}_y \left[\sum_{s>0} \mathrm{e}^{-\rho s} \Delta L_s \right] \\ &= \mathbb{E}_y \left[\int_0^\infty \mathrm{e}^{-\rho s} (c(\theta_s)\, \mathrm{d}s + p\, \mathrm{d}L_s^c) \right] + p\mathbb{E}_y \left[\sum_{s>0} \mathrm{e}^{-\rho s} (\Delta I_s(Y) + Y_{s-})^- \right] \end{aligned}$$

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$$
\begin{aligned}
&= \mathbb{E}_y\left[\int_0^\infty \mathrm{e}^{-\rho s}(c(\theta_s)\,\mathrm{d}s + p\,\mathrm{d}L_s^c)\right] \\
&\quad + p\mathbb{E}_y\left[\int_0^\infty \int_{\mathbb{R}_+\times\mathbb{R}} \mathrm{e}^{-\rho s}(\gamma\xi \mathbf{1}_{z\le\lambda(Y_{s-})} + Y_{s-})^-\pi(\mathrm{d}s,\mathrm{d}z,\mathrm{d}\xi)\right] \\
&= \mathbb{E}_y\left[\int_0^\infty \mathrm{e}^{-\rho s}(c(\theta_s)\,\mathrm{d}s + p\,\mathrm{d}L_s^c)\right] \\
&\quad + p\mathbb{E}_y\left[\int_0^\infty \mathrm{e}^{-\rho s}\lambda(Y_s)\left(\int_{\mathbb{R}}(\gamma\xi + Y_s)^-\nu(\mathrm{d}\xi)\right)\mathrm{d}s\right] \\
&= \mathbb{E}_y\left[\int_0^\infty \mathrm{e}^{-\rho s}\left\{\left(c(\theta_s) + \lambda(Y_s)\int_{\mathbb{R}}(\gamma\xi + Y_s)^-\nu(\mathrm{d}\xi)\right)\mathrm{d}s + p\,\mathrm{d}L_s^c\right\}\right].
\end{aligned}
$$

Thus we complete the proof of the lemma.

Using the dynamic program together with Lemma 3 and equation (1), the value function $y \to V(y)$ formally satisfies the following (primal) ID-HJB equation with Neumann boundary condition: for all $y > 0$,

$$
\begin{aligned}
\rho V(y) &= (\alpha - \beta y)V'(y) + \frac{\sigma^2}{2}V''(y) + \inf_{\theta\ge 0}\left\{\theta V'(y) + c(\theta)\right\} + \lambda(y)\int_{\mathbb{R}}(y+\gamma\xi)^-\nu(\mathrm{d}\xi) \\
&\quad + \lambda(y)\int_{\mathbb{R}}\left\{V\big((y+\gamma\xi)^+\big) - V(y)\right\}\nu(\mathrm{d}\xi)
\end{aligned} \tag{17}
$$

with $V'(0) = -p$. Assume that the ID-HJB equation (17) has a smooth solution. With assumption $(\mathbf{A}_c)$, the candidate optimal control strategy θ^* satisfies

$$
c'(\theta^*) = -V'(y) \quad \forall y \in \mathbb{R}_+. \tag{18}
$$

For the function $y \to \Phi(y)$ introduced in (4), we can write

$$
\inf_{\theta\ge 0}\{\theta V'(y) + c(\theta)\} = -\sup_{\theta\ge 0}\{\theta(-V'(y)) - c(\theta)\} = -\Phi(-V'(y)).
$$

Thus, (17) can be rewritten as, for $y > 0$,

$$
\begin{aligned}
\rho V(y) &= (\alpha - \beta y)V'(y) + \frac{\sigma^2}{2}V''(y) - \Phi(-V'(y)) + \lambda(y)\int_{\mathbb{R}}(y+\gamma\xi)^-\nu(\mathrm{d}\xi) \\
&\quad + \lambda(y)\int_{\mathbb{R}}\left\{V\big((y+\gamma\xi)^+\big) - V(y)\right\}\nu(\mathrm{d}\xi).
\end{aligned} \tag{19}
$$

Considering the unboundedness of the policy space in (19), we introduce the following (restricted) admissible set with a compact policy space:

$$
\mathcal{A}_p := \{\theta = (\theta_t)_{t\ge 0} \in \mathcal{A}: \theta_t \in [0, M_p]\ \forall t \ge 0 \text{ a.s.}\}. \tag{20}
$$

Here, $M_p := (c')^{-1}(2p)$, which is finite by assumption $(\mathbf{A}_c)$. Note that for any $\theta_o \in [0, M_p]$, the constant strategy $\theta = (\theta_t)_{t\ge 0}$ with $\theta_t = \theta_o$ for all $t \ge 0$ belongs to the policy space $\mathcal{A}_p$, i.e. $\theta \in \mathcal{A}_p$. Now, the policy space in (20) becomes compact, which plays an important role in our future analysis of the uniqueness of viscosity solutions to the auxiliary HJB equation with

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Neumann boundary. We then consider the following auxiliary stochastic control problem: for all $y \in \mathbb{R}_+$,

$$u(y) := \sup_{\theta \in \mathcal{A}_p} \tilde{J}(y;\theta) := \sup_{\theta \in \mathcal{A}_p} \mathbb{E}_y \left[- \int_0^\infty \mathrm{e}^{-\rho s} \big(c(\theta_s)\,\mathrm{d}s + p\,\mathrm{d}L_s \big) \right]. \tag{21}$$

Since $\mathcal{A}_p \subset \mathcal{A}$ and $u(y) = -\inf_{\theta \in \mathcal{A}_p} \mathbb{E}_y[\int_0^\infty \mathrm{e}^{-\rho s}(c(\theta_s)\,\mathrm{d}s + p\,\mathrm{d}L_s)]$, we can see that $u(y) \geq -V(y)$ for all $y \in \mathbb{R}_+$. On the other hand, if V is smooth (which will be proved in Theorem 2 later), we have from Lemma 2 that $|V'(y)| \leq 2p$ for all $y \in \mathbb{R}_+$. It follows from (18) that $c'(\theta^*) \leq 2p$. By the strict convexity of $\theta \to c(\theta)$ in assumption $(\mathbf{A}_c)$, $\theta \to c'(\theta)$ is strictly increasing, which yields $\theta^* \leq (c')^{-1}(2p) = M_p$. Hence, we obtain

$$u(y) = -V(y) \quad \forall y \in \mathbb{R}_+. \tag{22}$$

Using a similar argument to that for proving Lemma 2, we can also conclude that

$$|u(y_1) - u(y_2)| \leq 2p|y_1 - y_2| \quad \forall y_1, y_2 \in \mathbb{R}_+. \tag{23}$$

Moreover, by using a similar proof to that of Lemma 3, we arrive at the following ID-HJB equation with Neumann boundary condition satisfied by the value function $y \to u(y)$: for all $y > 0$,

$$\begin{aligned} \rho u(y) = (\alpha - \beta y)u'(y) + \frac{\sigma^2}{2}u''(y) + \sup_{\theta \in [0, M_p]} \big\{ \theta u'(y) - c(\theta) \big\} - \lambda(y) \int_{\mathbb{R}} (y + \gamma\xi)^- \nu(\mathrm{d}\xi) \\ + \lambda(y) \int_{\mathbb{R}} \big\{ u\big((y + \gamma\xi)^+\big) - u(y) \big\} \nu(\mathrm{d}\xi) \end{aligned} \tag{24}$$

with $u'(0) = p$.

In the next section, we will consider the well-posedness of the ID-HJB equation (24) in the viscosity solution sense.

4. Viscosity solutions of the ID-HJB equation

This section focuses on the existence and uniqueness of viscosity solutions to the ID-HJB equation (24) with Neumann boundary condition. To this end, let us introduce the following differential operator and non-local integral operator:

$$\begin{aligned} \mathcal{L}\phi(y) &:= (\alpha - \beta y)\phi'(y) + \frac{\sigma^2}{2}\phi''(y) + \sup_{\theta \in [0, M_p]} \big\{ \theta\phi'(y) - c(\theta) \big\} \\ &\quad - \lambda(y) \int_{\mathbb{R}} (y + \gamma\xi)^- \nu(\mathrm{d}\xi) \quad \forall \phi \in C^2(\mathbb{R}_+), \\ \mathcal{I}\psi(y) &:= \lambda(y) \int_{\mathbb{R}} \big\{ \psi\big((y + \gamma\xi)^+\big) - \psi(y) \big\} \nu(\mathrm{d}\xi) \quad \forall \psi \in C(\mathbb{R}_+). \end{aligned}$$

The definition of viscosity solutions to the HJB equation (24) is as follows.

Definition 1. *(Viscosity solutions of equation (24).)* Let $v : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function. Then we have the following:

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(i) $v:\mathbb{R}_+\to\mathbb{R}$ is called a viscosity supersolution to (24) if for all $\phi\in C^2(\mathbb{R}_+)$ and $y\geq 0$ such that y is a local minimum point of $v-\phi$, one has that

$$\begin{aligned}\rho v(y)-(\mathcal{L}+\mathcal{I})\phi(y)&\geq 0 \quad \text{when } y>0,\\ \max\left\{\rho v(y)-(\mathcal{L}+\mathcal{I})\phi(y),\ p-\phi'(y)\right\}&\geq 0 \quad \text{when } y=0;\end{aligned}$$

(ii) $v:\mathbb{R}_+\to\mathbb{R}$ is called a viscosity subsolution to (24) if for all $\phi\in C^2(\mathbb{R}_+)$ and $y\geq 0$ such that y is a local maximum point of $v-\phi$, one has that

$$\begin{aligned}\rho v(y)-(\mathcal{L}+\mathcal{I})\phi(y)&\leq 0 \quad \text{when } y>0,\\ \min\left\{\rho v(y)-(\mathcal{L}+\mathcal{I})\phi(y),\ p-\phi'(y)\right\}&\leq 0 \quad \text{when } y=0;\end{aligned}$$

(iii) $v:\mathbb{R}_+\to\mathbb{R}$ is called a viscosity solution to (24) if it is both a viscosity supersolution and a viscosity subsolution of (24).

We next establish the existence of viscosity solutions to (24). For this purpose, we need the following auxiliary result whose proof is provided in Appendix A.

Lemma 4. *Suppose that assumptions* $(\mathbf{A}_p)$ *and* $(\mathbf{A}_c)$ *hold. Let* $Y^\theta=(Y_t^\theta)_{t\geq 0}$ *be the reflected jump–diffusion process described in (1) under the control strategy* $\theta\in\mathcal{A}_p$ *and* $Y_0=y_0\in\mathbb{R}_+$. *Then*

$$\mathbb{E}\left[\left|Y_t^\theta-y_0\right|^2\right]\leq C(1+y_0^2)t\mathrm{e}^{Ct}\quad\forall t\geq 0,$$

where $C>0$ *is a constant independent of* (θ,t,y_0).

The following lemma shows the existence of viscosity solutions to the ID-HJB equation (24) with Neumann boundary condition. The proof strategy of the lemma bears some resemblance to that used by [32], who considered the quasi-variational inequality.

Lemma 5. *Suppose that assumptions* $(\mathbf{A}_p)$, $(\mathbf{A}_c)$ *and* $(\mathbf{A}_\rho)$ *hold. Then the value function* $u\in C(\mathbb{R}_+)$ *defined by (21) is a viscosity solution of the ID-HJB equation (24) with Neumann boundary condition.*

Proof. We divide the proof of the lemma into two parts.

Step 1: *viscosity subsolution.* Let $y_0\in\mathbb{R}_+$, and let the function $\phi\in C^2(\mathbb{R}_+)$ satisfy

$$0=(u-\phi)(y_0)=\max_{y\in\mathbb{R}_+}(u-\phi)(y). \tag{25}$$

Without loss of generality, we can suppose that the previous supremum is indeed a strict maximum, and we distinguish two cases depending on the location of the maximum y_0.

(i) $y_0>0$. We assume that $\rho u(y_0)>(\mathcal{L}+\mathcal{I})\phi(y_0)$ and work toward a contradiction. Then, using the smoothness of ϕ, there exist some $\delta\in(0,y_0)$ and $\epsilon>0$ such that

$$\min_{y_0-\delta\leq y\leq y_0+\delta}\{\rho\phi(y)-(\mathcal{L}+\mathcal{I})\phi(y)\}>\epsilon. \tag{26}$$

Take a sequence $(y_n)_{n\geq 1}$ satisfying $(y_n)_{n\geq 1}\subset(y_0-\frac{\delta}{2},y_0+\frac{\delta}{2})$, $\gamma_n:=u(y_n)-\phi(y_n)\to 0$ and $y_n\to y_0$ as $n\to\infty$. Then, we can also take a positive sequence $(h_n)_{n\geq 1}$ such that

$$h_n\to 0\quad\text{and}\quad\frac{\gamma_n}{h_n}\to 0\quad\text{as } n\to\infty.$$

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For $n \geq 1$, using the dynamic program, there exists an $\frac{\epsilon h_n}{2}$-optimal control strategy $(\theta_t^{(n)})_{t\geq 0}$ at y_n such that for any $\mathbb{F}$-stopping time τ,

$$u(y_n) - \frac{\epsilon h_n}{2} \leq \mathbb{E}\bigg[- \int_0^\tau \mathrm{e}^{-\rho s} \bigg\{ \Big(c(\theta_s^{(n)}) + \lambda(Y_s^{(n)}) \int_{\mathbb{R}} (\gamma\xi + Y_s^{(n)})^- \nu(\mathrm{d}\xi) \Big) \,\mathrm{d}s + p \,\mathrm{d}L_s^{c,(n)} \bigg\} + \mathrm{e}^{-\rho\tau} u(Y_{\tau-}^n) \bigg]. \tag{27}$$

Here, $Y^{(n)} = (Y_t^{(n)})_{t\geq 0}$ is the associated (controlled) reflected process with $Y_0^{(n)} = y_n \in \mathbb{R}_+$ and $\theta^{(n)} \in \mathcal{A}_p$. In particular, we take the stopping times τ_n and $\tilde{\tau}_n$ with $n \geq 1$ to be, respectively,

$$\tau_n := \inf\{t \geq 0 : |Y_t^{(n)} - y_n| \geq \delta/2\} \quad \text{and} \quad \tilde{\tau}_n := \tau_n \wedge h_n.$$

Hence, we have $Y_s^{(n)} > 0$ for $0 \leq s < \tilde{\tau}_n$ a.s. For $n \geq 1$, take $\tau = \tilde{\tau}_n$ in (27) and apply (25); then we obtain

$$\begin{aligned} &\phi(y_n) + \gamma_n - \frac{\epsilon h_n}{2} \\ &\leq \mathbb{E}\bigg[- \int_0^{\tilde{\tau}_n} \mathrm{e}^{-\rho s} \bigg\{ \Big(c(\theta_s^{(n)}) + \lambda(Y_s^{(n)}) \int_{\mathbb{R}} (\gamma\xi + Y_s^{(n)})^- \nu(\mathrm{d}\xi) \Big) \mathrm{d}s + p \,\mathrm{d}L_s^{c,(n)} \bigg\} \\ &\qquad + \mathrm{e}^{-\rho\tilde{\tau}_n} \phi(Y_{\tilde{\tau}_n-}^n) \bigg] \\ &= \mathbb{E}\bigg[- \int_0^{\tilde{\tau}_n} \mathrm{e}^{-\rho s} \bigg\{ \Big(c(\theta_s^{(n)}) + \lambda(Y_s^{(n)}) \int_{\mathbb{R}} (\gamma\xi + Y_s^{(n)})^- \nu(\mathrm{d}\xi) \Big) \mathrm{d}s \bigg\} \\ &\qquad + \mathrm{e}^{-\rho\tilde{\tau}_n} \phi(Y_{\tilde{\tau}_n-}^n) \bigg]. \end{aligned} \tag{28}$$

Applying Itô's rule to $\mathrm{e}^{-\rho\tilde{\tau}_n}\phi(Y_{\tilde{\tau}_n-}^n)$, the inequality (28) can be read as, for all $n \geq 1$,

$$\frac{\gamma_n}{h_n} - \frac{\epsilon}{2} + \frac{1}{h_n}\mathbb{E}\left[\int_0^{\tilde{\tau}_n} \mathrm{e}^{-\rho\tilde{\tau}_n} \ell(Y_s^{(n)}, \theta_s^{(n)}) \,\mathrm{d}s \right] \leq 0. \tag{29}$$

Here, $\ell : \mathbb{R}_+^2 \to \mathbb{R}$ is defined by

$$\begin{aligned} \ell(y,\theta) := {} & \rho\phi(y) - (\alpha - \beta y)\phi'(y) - \frac{\sigma^2}{2}\phi''(y) - \theta\phi'(y) + c(\theta) \\ & + \lambda(y) \int_{\mathbb{R}} (\gamma\xi + y)^- \nu(\mathrm{d}\xi) - \lambda(y) \int_{\mathbb{R}} \big\{ \phi\big((y + \gamma\xi)^+\big) - \phi(y) \big\} \nu(\mathrm{d}\xi). \end{aligned} \tag{30}$$

With the help of the inequality (26), it follows that for $0 \leq s < \tilde{\tau}_n$,

$$\ell(Y_s^{(n)}, \theta_s^{(n)}) \geq \rho\phi(Y_s^{(n)}) - (\mathcal{L} + \mathcal{I})\phi(Y_s^{(n)}) > \epsilon \ \text{ a.s.} \tag{31}$$

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Note that (29) and (31) jointly imply that

$$\frac{\gamma_n}{h_n} - \epsilon\left(\frac{1}{2} - \frac{1}{h_n}\mathbb{E}\left[\tilde{\tau}_n\right]\right) \le 0. \tag{32}$$

By using Lemma 4 and Lemma 7.2 in [33], we deduce that

$$\mathbb{P}(\tau_n \le h_n) \le \mathbb{P}\left(\sup_{s\in[0,h_n]}|Y_s^{(n)} - y_n| \ge \frac{\delta}{2}\right) \to 0 \quad \text{as } n \to \infty. \tag{33}$$

Moreover, note that $\mathbb{P}(\tau_n > h_n) \le \frac{1}{h_n}\mathbb{E}[\tilde{\tau}_n] \le 1$, which yields that $\frac{1}{h_n}\mathbb{E}[\tilde{\tau}_n]$ converges to 1 when h_n tends to zero. By letting n go to infinity in (32), we get the desired contradiction.

(ii) $y_0 = 0$. We suppose that $\rho u(0) > (\mathcal{L} + \mathcal{I})\phi(0)$ and $p - \phi'(0) > 0$. Then, using the smoothness of ϕ, there exist some $\delta \in (0, y_0)$ and $\epsilon > 0$ such that

$$\min_{y_0-\delta\le y\le y_0+\delta}\{\rho\phi(y) - (\mathcal{L} + \mathcal{I})\phi(y)\} > \epsilon. \tag{34}$$

Noting that $p - \phi'(0) > 0$, following similar lines of proof to the previous case (i), we also get a contradiction.

Step 2: *viscosity supersolution.* Let $y_0 \in \mathbb{R}_+$, and let the function $\phi \in C^2(\mathbb{R}_+)$ satisfy

$$0 = (u - \phi)(y_0) = \min_{y\in\mathbb{R}_+}(u - \phi)(y). \tag{35}$$

(i) $y_0 > 0$. Take a sequence $(y_n)_{n\ge1} \subset (\frac{y_0}{2}, \infty)$ such that

$$y_n \to y_0 \quad \text{and} \quad \gamma_n := u(y_n) - \phi(y_n) \to 0 \quad \text{as } n \to \infty. \tag{36}$$

Choose a constant control strategy $\theta \in [0, M_p]$. Denote by $Y^{(n),\theta} = (Y_t^{(n),\theta})_{t\ge0}$ the associated (controlled) reflected process with $Y_0^{(n),\theta} = y_n$ and this constant strategy $\theta \in [0, M_p]$. Let $(h_n)_{n\ge1}$ be a positive sequence such that $h_n \to 0$ and $\frac{\gamma_n}{h_n} \to 0$ as $n \to \infty$. We introduce a sequence of $\mathbb{F}$-stopping times $(\tau_n)_{n\ge1}$ defined by

$$\tau_n := \inf\{t \ge 0 : |Y_t^{(n),\theta} - y_n| \ge y_0/2\} \wedge h_n.$$

Hence $Y_s^{(n),\theta} > 0$ for $0 \le s < \tilde{\tau}_n$ a.s. Thus, using the dynamic program and (35), it follows that

$$\begin{aligned} u(y_n) &\le \mathbb{E}\Bigg[- \int_0^{\tau_n} \mathrm{e}^{-\rho s}\Big\{\Big(c(\theta) + \lambda(Y_s^{(n),\theta})\int_{\mathbb{R}}(\gamma\xi + Y_s^{(n),\theta})^-\nu(\mathrm{d}\xi)\Big)\,\mathrm{d}s \\ &\qquad + p\,\mathrm{d}L_s^{c,(n)}\Big\} + \mathrm{e}^{-\rho\tau_n}u(Y_{\tau_n-}^{(n),\theta})\Bigg] \\ &= \mathbb{E}\Bigg[- \int_0^{\tau_n} \mathrm{e}^{-\rho s}\left\{\Big(c(\theta) + \lambda(Y_s^{(n),\theta})\int_{\mathbb{R}}(\gamma\xi + Y_s^{(n),\theta})^-\nu(\mathrm{d}\xi)\Big)\,\mathrm{d}s\right\} \\ &\qquad + \mathrm{e}^{-\rho\tau_n}\phi(Y_{\tau_n-}^{(n),\theta})\Bigg]. \end{aligned} \tag{37}$$

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Note that $\gamma_n = u(y_n) - \phi(y_n)$. By applying Itô's rule to $\mathrm{e}^{-\rho\tau_n}\phi(Y^n_{\tau_n})$, we obtain

$$\frac{\gamma_n}{h_n} + \frac{1}{h_n}\mathbb{E}\left[\int_0^{\tau_n} \mathrm{e}^{-\rho\tau_n}\ell(Y_s^{(n),\theta}, \theta)\,\mathrm{d}s\right] \geq 0, \tag{38}$$

where the function $\ell{:}\mathbb{R}^2_+ \to \mathbb{R}$ is defined by (30). For any $\epsilon \in (0, \frac{y_0}{2})$, using Lemma 4 and Lemma 7.2 in [33] again, we have

$$\mathbb{P}\left(\sup_{s\in[0,h_n]}\left|Y_s^{(n),\theta} - y_n\right| > \epsilon\right) \to 0 \quad \text{as } n \to \infty. \tag{39}$$

Now, let us define the event $\Omega_{n,\epsilon} := \{\omega \in \Omega{:}\sup_{s\in[0,h_n]}|Y_s^{(n),\theta} - y_n| \leq \epsilon\}$. Then on $\Omega_{n,\epsilon}$ we have $\tau_n = h_n$, and hence on $\Omega_{n,\epsilon}$ we have

$$\begin{aligned}
&\left|\ell(y_0,\theta) - \frac{1}{h_n}\mathbb{E}\left[\int_0^{\tau_n} \mathrm{e}^{-\rho h_n}\ell(Y_s^{(n),\theta}, \theta)\,\mathrm{d}s\right]\right| \\
&\quad = \left|\ell(y_0,\theta) - \frac{1}{h_n}\mathbb{E}\left[\int_0^{h_n} \mathrm{e}^{-\rho h_n}\ell(Y_s^{(n),\theta}, \theta)\,\mathrm{d}s\right]\right| \\
&\quad \leq \frac{1}{h_n}\mathbb{E}\left[\int_0^{h_n} \left|\ell(y_0,\theta) - \mathrm{e}^{-\rho h_n}\ell(Y_s^{(n),\theta}, \theta)\right|\,\mathrm{d}s\right].
\end{aligned}$$

Since $y \to \ell(y, \theta)$ is continuous, there exists a $y_{n,\epsilon} \in [y_n - \epsilon, y_n + \epsilon]$ such that

$$\left|\ell(y_0,\theta) - \mathrm{e}^{-\rho h_n}\ell(y_{n,\epsilon}, \theta)\right| = \max_{y\in[y_n-\epsilon, y_n+\epsilon]}\left|\ell(y_0,\theta) - \mathrm{e}^{-\rho h_n}\ell(y, \theta)\right|,$$

from which we obtain that as $n \to \infty$ and $\epsilon \to 0$,

$$\left|\ell(y_0,\theta) - \frac{1}{h_n}\mathbb{E}\left[\int_0^{\tau_n} \mathrm{e}^{-\rho h_n}\ell(Y_s^{(n),\theta}, \theta)\,\mathrm{d}s\right]\right| \leq \left|\ell(y_0,\theta) - \mathrm{e}^{-\rho h_n}\ell(y_{n,\epsilon}, \theta)\right| \to 0.$$

On the other hand, on $\Omega^c_{n,\epsilon}$, i.e. the complement of $\Omega_{n,\epsilon}$,

$$\frac{1}{h_n}\mathbb{E}\left[\int_0^{\tau_n} \mathrm{e}^{-\rho\tau_n}\ell(Y_s^{(n),\theta}, \theta)\,\mathrm{d}s\right] \leq \operatorname*{ess\,sup}_{s\in[0,h_n]}\left|\ell(Y_s^{(n),\theta}, \theta)\right|,$$

which is also bounded by similar arguments to those above. Note that a jump at τ_n does not affect the essential supremum. We can now rewrite (38) as

$$\begin{aligned}
\frac{\gamma_n}{h_n} + \frac{1}{h_n}\mathbb{E}\Bigg[&\left(\int_0^{h_n} \mathrm{e}^{-\rho\tau_n}\ell(Y_s^{(n),\theta}, \theta)\,\mathrm{d}s\right)\mathbf{1}_{\Omega_{n,\epsilon}} \\
&+ \left(\int_0^{\tau_n} \mathrm{e}^{-\rho\tau_n}\ell(Y_s^{(n),\theta}, \theta)\,\mathrm{d}s\right)\mathbf{1}_{\Omega^c_{n,\epsilon}}\Bigg] \geq 0.
\end{aligned} \tag{40}$$

Note that as $n \to \infty$, $\mathbb{P}(\Omega_{n,\epsilon}) \to 1$, i.e. $\mathbf{1}_{\Omega_{n,\epsilon}} \to 0$ a.s. for all $\epsilon > 0$. Then, by letting $n \to \infty$ and $\epsilon \to 0$ in (40), we conclude from the dominated convergence theorem that $\ell(y_0, \theta) \geq 0$ for any $\theta \in [0, M_p]$. Thus, $\rho v(y_0) - (\mathcal{L} + \mathcal{I})\phi(y_0) \geq 0$.

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(ii) $y_0 = 0$. In this case, we have $\phi(0) = u(0)$ and $\phi(y) \le u(y)$ for all $y \in \mathbb{R}_+$. Hence we obtain that for any $y \ge 0$,

$$\frac{\phi(y) - \phi(0)}{y - 0} \le \frac{u(y) - u(0)}{y - 0} \le p.$$

Letting $y \to 0$ in the above, we obtain that $\phi'(0) \le p$, i.e. $p - \phi'(0) \ge 0$.

By Steps 1 and 2, the value function $y \to u(y)$ is a viscosity solution of the HJB equation (24). Thus, the proof of the lemma is complete.

Next, we provide an alternative definition of viscosity solutions to the ID-HJB equation (24) with Neumann boundary condition, which is in essence equivalent to Definition 1 (see e.g. [3, 4]).

Definition 2. *(Viscosity solutions of equation (24).)* Let $v : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function. Then we have the following:

(i) $v : \mathbb{R}_+ \to \mathbb{R}$ is called a viscosity supersolution of (24) if for any $\phi \in C^2(\mathbb{R}_+)$ and any $y \ge 0$ at which $v - \phi$ has a local minimum, one has that

$$\begin{aligned} \rho v(y) - \mathcal{L}\phi(y) - \mathcal{I}v(y) &\ge 0 \quad \text{when } y > 0, \\ \max\left\{\rho v(y) - \mathcal{L}\phi(y) - \mathcal{I}v(y),\ p - \phi'(y)\right\} &\ge 0 \quad \text{when } y = 0; \end{aligned}$$

(ii) $v : \mathbb{R}_+ \to \mathbb{R}$ is called a viscosity subsolution of (24) if for any $\phi \in C^2(\mathbb{R}_+)$ and any $y \ge 0$ at which $v - \phi$ has a local maximum, one has that

$$\begin{aligned} \rho v(y) - \mathcal{L}\phi(y) - \mathcal{I}v(y) &\le 0 \quad \text{when } y > 0, \\ \min\left\{\rho v(y) - \mathcal{L}\phi(y) - \mathcal{I}v(y),\ p - \phi'(y)\right\} &\le 0 \quad \text{when } y = 0; \end{aligned}$$

(iii) $v : \mathbb{R}_+ \to \mathbb{R}$ is called a viscosity solution of (24) if it is both a viscosity supersolution and a viscosity subsolution to (24).

Even though Definitions 1 and 2 of viscosity solutions to the ID-HJB equation (24) with Neumann boundary condition are equivalent, we find that it is more convenient to use Definition 2 when proving the uniqueness result below. Based on this consideration, with the value function u defined by (21), let us consider the following equation for the solution $w(y)$: for $y > 0$,

$$\begin{aligned} \rho w(y) = (\alpha - \beta y) w'(y) + \frac{\sigma^2}{2} w''(y) + \Phi_p(w'(y)) - \lambda(y) \int_{\mathbb{R}} (\gamma\xi + y)^- \nu(\mathrm{d}\xi) \\ + \lambda(y) \int_{\mathbb{R}} \{u((y + \gamma\xi)^+) - u(y)\} \nu(\mathrm{d}\xi) \end{aligned} \tag{41}$$

with $w'(0) = p$. Here, similar to the function $y \to \Phi(y)$ defined by (4), the function $y \to \Phi_p(y)$ is defined by

$$\Phi_p(y) := \sup_{\theta \in [0,p]} \{\theta y - c(\theta)\} \quad \forall y \in \mathbb{R}. \tag{42}$$

We observe that (41) can be viewed as a non-homogeneous second-order ODE. However, if a function $w \in C^2(\mathbb{R}_+)$ solves the ODE (41), it does not mean that u also solves the integro-differential sequation (24), unless we can show that (41) admits a unique smooth solution. If the

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uniqueness of solutions to (41) holds true, then $w = u$ on $\mathbb{R}_+$ by using Lemma 5. To establish this, we next prove the comparison result for viscosity solutions to the Neumann problem (41), which can help us to prove the uniqueness of the smooth solution. Under the compact policy space, it is worth noticing that the proof strategy for the uniqueness of the viscosity solution rests on the construction of the penalty function. Here, the challenging task is to construct a polynomial-like penalty function in lieu of the Neumann boundary condition and the linear growth condition satisfied by the value function u given by (21).

Proposition 1. *Suppose that assumptions* $(\mathbf{A}_p)$, $(\mathbf{A}_c)$ *and* $(\mathbf{A}_\rho)$ *hold. Let* $U_1{:}\mathbb{R}_+ \to \mathbb{R}$ *(resp.* $U_2{:}\mathbb{R}_+ \to \mathbb{R}$*) be a viscosity subsolution (resp. supersolution) of (41) satisfying* $|U_1(y)| \vee |U_2(y)| \le C_o(1+y)$ *for all* $y \in \mathbb{R}_+$*, where* $C_o > 0$ *is a constant independent of* y*. Then* $U_1 \le U_2$ *on* $\mathbb{R}_+$.

Proof. We introduce the function $\psi(y){:}\mathbb{R}_+ \to \mathbb{R}$ defined by

$$\psi(y) := \begin{cases} -\left(C_o + \frac{1}{3}\right) y^3 + (3C_o + 1)y^2 - y + C_1, & y \in [0, 1], \\ 3C_o y - C_o - \frac{1}{3} + C_1, & y > 1. \end{cases} \tag{43}$$

Here, $C_1 > 0$ is a positive constant to be determined. It is not difficult to see that $\psi \in C^2(\mathbb{R}_+)$. Moreover, when $y \in [0, 1]$,

$$\begin{aligned}
&\rho\psi(y) - (\alpha - \beta y)\psi'(y) - \frac{\sigma^2}{2}\psi''(y) - \sup_{\theta\in[0,M_p]}\{\theta\psi'(y)\} \\
&= \rho\left[-\left(C_o + \frac{1}{3}\right) y^3 + (3C_o + 1)y^2 - y + C_1\right] - \frac{\sigma^2}{2}\left[-6\left(C_o + \frac{1}{3}\right) + 2(3C_o + 1)\right] \\
&\quad - (\alpha - \beta y)\left[-3\left(C_o + \frac{1}{3}\right) y^2 + 2(3C_o + 1)y - 1\right] \\
&\quad - \sup_{\theta\in[0,M_p]}\left\{\theta\left[-3\left(C_o + \frac{1}{3}\right) y^2 + 2(3C_o + 1)y - 1\right]\right\} \\
&\ge \rho C_1 + \rho\left[-\left(C_o + \frac{1}{3}\right) y^3 + (3C_o + 1)y^2 - y\right] \\
&\quad - \alpha\left[-3\left(C_o + \frac{1}{3}\right) y^2 + 2(3C_o + 1)y - 1\right] \\
&\quad + \beta\left[-3\left(C_o + \frac{1}{3}\right) y^3 + 2(3C_o + 1)y^2 - y\right] - 2M_p(3C_o + 1) \\
&\ge \rho C_1 + \rho\left[-\left(C_o + \frac{1}{3}\right) - 1\right] - |\alpha|\left[3\left(C_o + \frac{1}{3}\right) + 2(3C_o + 1) + 1\right] \\
&\quad + \beta\left[-3\left(C_o + \frac{1}{3}\right) 1 - 1\right] - 2M_p(3C_o + 1) \\
&\ge \rho C_1 - \rho\left(C_o + \frac{4}{3}\right) - |\alpha|(9C_o + 4) - \beta\left[3\left(C_o + \frac{1}{3}\right) + 1\right] - 2M_p(3C_o + 1) \\
&=: K_1(C_1),
\end{aligned} \tag{44}$$

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while for $y > 1$ we have

$$\begin{aligned}
&\rho\psi(y) - (\alpha - \beta y)\psi'(y) - \frac{\sigma^2}{2}\psi''(y) - \sup_{\theta\in[0,M_p]}\left\{\theta\psi'(y)\right\} \\
&\quad = \rho\left(3C_o y - C_o - \frac{1}{3} + C_1\right) - 3C_o(\alpha - \beta y) - \sup_{\theta\in[0,M_p]}\{3C_o\theta\} \\
&\quad \geq \rho C_1 - \rho\left(C_o + \frac{1}{3}\right) - 3C_o|\alpha| - 3C_o M_p =: K_2(C_1). \qquad (45)
\end{aligned}$$

Note that both $K(C_1)$ and $K_2(C_1)$ are linear functions of C_1 with gradient $\rho > 0$. Take the constant $C_1 > 0$ large enough so that $K_1(C_1) > 0$ and $K_2(C_1) > 0$. Then, for any $\epsilon > 0$, from (44), (45) and the fact that $\psi'(0) = 1$, it follows that $U_2^\epsilon := U_2 + \epsilon\psi$ is a viscosity supersolution of the equation

$$\begin{cases} \rho v(y) - \mathcal{L}v(y) + \mathcal{I}u(y) = 0, \quad y > 0; \\ \qquad p - \epsilon - v'(0) = 0. \end{cases}$$

Thus, it is sufficient to prove that, for all $\epsilon > 0$,

$$U_1 \leq U_2^\epsilon \quad \text{on } \mathbb{R}_+. \qquad (46)$$

Next, we fix an arbitrary $\epsilon > 0$ and prove (46) by contradiction. Assume that $M := \max_{y\in\mathbb{R}_+}(U_1(y) - U_2^\epsilon(y)) > 0$. Using the linear growth of U_1 and U_2 and (43), there exists some constant $K > 0$ such that

$$M = \max_{y\in[0,K)}\{U_1(y) - U_2^\epsilon(y)\} = \max_{y\in\mathbb{R}_+}\{U_1(y) - U_2^\epsilon(y)\} > 0.$$

Now, for any $n \geq 1$, we introduce the function given by, for all $(x, y) \in \mathbb{R}_+^2$,

$$\Phi_n(x, y) := U_1(x) - U_2^\epsilon(y) - \phi_n(x, y) \qquad (47)$$

with $\phi_n(x, y) := n|x - y|^2 + (p - \frac{\epsilon}{2})(x - y)$. Note that Φ_n is a continuous function on $\mathbb{R}_+^2$. Then Φ_n has a maximum on the compact set $[0, K]^2$. Let $M_n = \max_{[0,K]^2}\Phi_n = \Phi_n(x_n, y_n)$ with $(x_n, y_n) \in [0, K]^2$. Then there exists a subsequence of the bounded sequence $(x_n, y_n)_{n\geq1}$ (still denoted by $(x_n, y_n)_{n\geq1}$) that converges to some $(\bar{x}, \bar{y}) \in [0, K]^2$.

For $n \geq 1$, we have that

$$\begin{aligned}
M \leq M_n &= U_1(x_n) - U_2^\epsilon(y_n) - n|x_n - y_n|^2 - \left(p - \frac{\epsilon}{2}\right)(x_n - y_n) \qquad (48) \\
&\leq U_1(x_n) - U_2^\epsilon(y_n) - \epsilon(x_n - y_n) \qquad (49)
\end{aligned}$$

Since the sequence $\{U_1(x_n) - U_2^\epsilon(y_n) - \epsilon(x_n - y_n)\}_{n\geq1}$ is bounded, we see from (48) that the sequence $\{n|x_n - y_n|^2\}_{n\geq1}$ is also bounded, which yields $\bar{x} = \bar{y}$. Then, letting $n \to \infty$ in (48), it follows that $M \leq U_1(\bar{x}) - U_2^\epsilon(\bar{x}) \leq M$. Thus, we get $\bar{x} = \bar{y} \in [0, K)$, $M_n \to M$ and $n|x_n - y_n|^2 \to 0$ as $n \to \infty$. Since (x_n, y_n) tends to $(\bar{x}, \bar{x}) \in [0, K)^2$ as $n \to \infty$, there exists an $N > 0$ such that $\{(x_n, y_n)\}_{n\geq N} \subset [0, K)^2$. For simplicity, we assume $\{(x_n, y_n)\}_{n\geq1} \subset [0, K)^2$ without loss of generality. Hence,

$$\begin{aligned}
&x_n \text{ is a local maximum of } x \to U_1(x) - \phi_n(x, y_n) \text{ on } \mathbb{R}_+, \\
&y_n \text{ is a local minimum of } y \to U_2^\epsilon(y) + \phi_n(x_n, y) \text{ on } \mathbb{R}_+.
\end{aligned}$$

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By applying Lemmas 4.45 and 4.46 from [27], for $n \geq 1$ there exist $a, b \in \mathbb{R}$ with $a \leq b$ such that

$$\min\left\{\rho U_1(x_n) - \mathcal{L}^a\phi_n(x, y_n) - \mathcal{I}u(x), p - \frac{\partial \phi_n}{\partial x}(x, y_n)\right\}\Big|_{x=x_n} \leq 0,$$
$$\max\left\{\rho U_2^{\epsilon}(y_n) - \mathcal{L}^b(-\phi_n)(x_n, y) - \mathcal{I}u(y), p - \epsilon - \left(-\frac{\partial \phi_n}{\partial y}\right)(x_n, y)\right\}\Big|_{y=y_n} \geq 0.$$

Here, the operator $\mathcal{L}^c$ with $c \in \{a, b\}$ is defined by, for all $f \in C^2(\mathbb{R}_+)$,

$$\mathcal{L}^c f(y) := (\alpha - \beta y) f'(y) + \frac{\sigma^2}{2} c + \sup_{\theta \in [0, M_p]} \{\theta f'(y) - c(\theta)\}$$
$$- \lambda(y) \int_{\mathbb{R}} (\gamma \xi + y)^- \nu(\mathrm{d}\xi) \quad \forall y \in \mathbb{R}_+.$$

Recall ϕ_n given in (47); for all $n \geq 1$, we have that

$$p - \frac{\partial \phi_n}{\partial x}(0, y_n) = 2ny_n + \frac{\epsilon}{2} > 0,$$
$$p - \epsilon - (-\frac{\partial \phi_n}{\partial y})(x_n, 0) = -2nx_n - \frac{\epsilon}{2} < 0.$$

This implies that

$$\left(\rho U_1(x_n) - \mathcal{L}^a\phi_n(x, y_n) - \mathcal{I}u(x)\right)\big|_{x=x_n} \leq 0,$$
$$\left(\rho U_2^{\epsilon}(y_n) - \mathcal{L}^b(-\phi_n)(x_n, y) - \mathcal{I}u(y)\right)\big|_{y=y_n} \geq 0.$$

By subtracting the above two inequalities, we deduce that

$$\begin{aligned}\rho\left(U_1(x_n) - U_2^{\epsilon}(y_n)\right) &\leq \mathcal{L}^a\phi_n(x, y_n)\big|_{x=x_n} - \mathcal{L}^b(-\phi_n)(x_n, y)\big|_{y=y_n} \\ &\leq -\beta(x_n - y_n)\left(2n(x_n - y_n) + p - \frac{\epsilon}{2}\right) \\ &= -2n\beta(x_n - y_n)^2 - \left(p - \frac{\epsilon}{2}\right)(x_n - y_n). \end{aligned} \tag{50}$$

Letting $n \to \infty$ in (50), we conclude that $\rho M \leq 0$, which yields the contradiction. Thus, we complete the proof of the proposition.

5. Well-posedness of the ID-HJB equation and the verification theorem

In this section we first study the existence of classical solutions $y \to w(y)$ to (41) with $y \to \mathcal{I}u(y)$ (recall that $y \to u(y)$ is the value function defined by (21)) viewed as a given term. We then characterize the optimal (feedback) inventory control strategy to the primal optimal inventory control problem (3).

The following result provides the existence of classical solutions to (41).

Theorem 1. *Suppose that assumptions* $(\mathbf{A}_p)$, $(\mathbf{A}_c)$ *and* $(\mathbf{A}_\rho)$ *hold. Then (41) admits a classical solution* $w \in C^2((0, \infty)) \cap C(\mathbb{R}_+)$ *satisfying the linear growth condition.*

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To prove Theorem 1, we need some auxiliary lemmas. Before introducing these auxiliary lemmas, we define the following function for all $y \in \mathbb{R}_+$:

$$F^u(y) := -\lambda(y) \int_{\mathbb{R}} (\gamma\xi + y)^- \nu(\mathrm{d}\xi) + \lambda(y) \int_{\mathbb{R}} \{u((y+\gamma\xi)^+) - u(y)\}\nu(\mathrm{d}\xi), \tag{51}$$

where we recall that $y \to u(y)$ is the value function defined by (21). The following lemma shows the Lipschitz-continuous property of the function $y \to F^u(y)$, which plays an important role in our later analysis.

Lemma 6. *Suppose that assumptions* $(\mathbf{A}_p)$ *and* $(\mathbf{A}_\rho)$ *hold. Then there exists a positive constant* $M_F > 0$ *such that* $|F^u(y_1) - F^u(y_2)| \le M_F|y_1 - y_2|$ *for all* $y_1, y_2 \in \mathbb{R}_+$.

The proof of Lemma 6 is relegated to Appendix A. Next, we rewrite the HJB equation (41) with Neumann boundary condition in the following form:

$$\begin{cases} (\alpha - \beta y)w'(y) + \dfrac{\sigma^2}{2}w''(y) + \Phi_p(w'(y)) + F^u(y) = \rho w(y) \quad \forall y > 0; \\ w'(0) = p. \end{cases} \tag{52}$$

Define $W(y) := w'(y)$ for $y \in \mathbb{R}_+$. Then (52) becomes

$$\begin{cases} (\alpha - \beta y)W(y) + \dfrac{\sigma^2}{2}W'(y) + \Phi_p(W(y)) + F^u(y) = \rho w(y) \quad \forall y > 0; \\ W(0) = p. \end{cases} \tag{53}$$

Taking $y = 0$ in the above equation, this yields $\rho w(0) = \alpha p + \frac{\sigma^2}{2}W'(0) + \Phi_p(p) + F^u(0)$. Integrating $\rho w'(y) = \rho W(y)$ from 0 to $y > 0$, we obtain that, for all $y \in \mathbb{R}_+$,

$$\begin{aligned} \rho w(y) &= \rho w(0) + \rho \int_0^y W(x)\,\mathrm{d}x \\ &= \alpha p + \frac{\sigma^2}{2}W'(0) + \Phi_p(p) + F^u(0) + \rho \int_0^y W(x)\,\mathrm{d}x. \end{aligned} \tag{54}$$

Plugging (54) into (53), we next study the initial boundary value problem

$$\begin{cases} (\alpha - \beta y)W(y) + \dfrac{\sigma^2}{2}W'(y) + \Phi_p(W(y)) + F^u(y) \\ \qquad = \alpha p + \dfrac{\sigma^2}{2}q + \Phi_p(p) + F^u(0) + \rho \displaystyle\int_0^y W(x)\,\mathrm{d}x \quad \forall y > 0; \\ W(0) = p, \quad W'(0) = q, \end{cases} \tag{55}$$

where $q \in \mathbb{R}$ is a constant to be determined later. In fact, (55) can be rewritten in the following integral form: for $y \in \mathbb{R}_+$,

$$\begin{aligned} \frac{\sigma^2}{2}W(y) = \frac{\sigma^2}{2}p + \left(\alpha p + \frac{\sigma^2}{2}q + \Phi_p(p) + F^u(0)\right)y + \int_0^y (\beta x - \alpha)W(x)\,\mathrm{d}x \\ - \int_0^y \Phi_p(W(x))\,\mathrm{d}x - \int_0^y F^u(x)\,\mathrm{d}x + \rho \int_0^y \int_0^x W(u)\,\mathrm{d}u\,\mathrm{d}x. \end{aligned} \tag{56}$$

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Using Lemma 6 and a standard Picard iteration argument, we can establish that Eq. (55) admits a unique solution $W_q \in C^1([0, x_q))$ on the interval $[0, x_q)$, where $x_q > 0$ is the explosion level of the solution W_q (see e.g. Chapter 2 of [23]). To prove Theorem 1, motivated by [35], we aim to find some real constant q such that the initial boundary value problem (55) has a classical solution $y \to W_q(y)$ satisfying

$$x_q = +\infty, \quad W_q(0) = p, \quad W_q'(0) = q. \tag{57}$$

Thus, we can use (54) to get the desired solution $y \to w(y)$ of (41) in terms of $y \to W_q(y)$.

The next few lemmas can help us to find the appropriate q satisfying (57). In the proofs of the following lemmas, we fix the value of $p > 0$ and analyze the family of solutions $(W_q)_{q\in\mathbb{R}}$ to (55). The next lemma verifies the monotonicity of $q \to W_q(y)$ for fixed $y > 0$.

Lemma 7. *Let $q_1, q_2 \in \mathbb{R}$ with $q_1 > q_2$. Then for all $0 \le y < x_{q_1} \wedge x_{q_2}$,*

$$W_{q_1}(y) \ge W_{q_2}(y) + (q_1 - q_2) \exp\left(\frac{\beta y^2 - 2(\alpha + M_p)y}{\sigma^2}\right) \int_0^y \exp\left(\frac{-\beta x^2 + 2(\alpha + M_p)x}{\sigma^2}\right) \mathrm{d}x.$$

In particular, we have $W_{q_1}(y) > W_{q_2}(y)$ for all $0 < y < x_{q_1} \wedge x_{q_2}$.

Proof. We first prove that $W_{q_1}(y) > W_{q_2}(y)$ for all $0 < y < x_{q_1} \wedge x_{q_2}$ by contradiction. Assume that there exist some $y_0 \in (0, x_{q_1} \wedge x_{q_2})$ such that $W_{q_1}(y_0) \le W_{q_2}(y_0)$. Note that $W_{q_1}(0) = W_{q_2}(0) = p$ and $W'_{q_1}(0) = q_1 > W'_{q_2}(0) = q_2$. Then there is a constant $y_1 \in (0, y_0]$ such that $W_{q_1}(y) > W_{q_2}(y)$ for all $y \in (0, y_1)$ and $W_{q_1}(y_1) = W_{q_2}(y_1)$. This also yields $W'_{q_1}(y_1) \le W'_{q_2}(y_1)$. We can conclude from (55) that

$$\frac{\sigma^2}{2}\left(W'_{q_1}(y_1) - W'_{q_2}(y_1)\right) = \frac{\sigma^2}{2}(q_1 - q_2) + \rho \int_0^{y_1} \left(W_{q_1}(x) - W_{q_2}(x)\right) \mathrm{d}x > 0,$$

which gives the contradiction. Thus, we have $W_{q_1}(y) > W_{q_2}(y)$ for all $0 < y < x_{q_1} \wedge x_{q_2}$. Using (55) again, we have, for $0 \le y < x_{q_1} \wedge x_{q_2}$,

$$\begin{aligned}\frac{\sigma^2}{2}\left(W'_{q_1}(y) - W'_{q_2}(y)\right) + \left(\Phi_p(W_{q_1}(y)) - \Phi_p(W_{q_2}(y))\right) - (\beta y - \alpha)\left(W_{q_1}(y) - W_{q_2}(y)\right) \\ = \frac{\sigma^2}{2}(q_1 - q_2) + \rho \int_0^y \left(W_{q_1}(x) - W_{q_2}(x)\right) \mathrm{d}x.\end{aligned} \tag{58}$$

Recall that $\Phi_p(y) = \sup_{\theta\in[0,M_p]}\{\theta y - c(\theta)\}$ for $y \in \mathbb{R}$ and $W_{q_1}(y) > W_{q_2}(y)$ for all $y \in (0, x_{q_1} \wedge x_{q_2})$. We have, for all $y \in (0, x_{q_1} \wedge x_{q_2})$,

$$\begin{aligned}\Phi_p(W_{q_1}(y)) - \Phi_p(W_{q_2}(y)) &= \sup_{\theta\in[0,M_p]}\{\theta W_{q_1}(y) - c(\theta)\} - \sup_{\theta\in[0,M_p]}\{\theta W_{q_2}(y) - c(\theta)\} \\ &\le \sup_{\theta\in[0,M_p]}\{\theta W_{q_1}(y) - \theta W_{q_2}(y)\} \\ &\le M_p\left(W_{q_1}(y) - W_{q_2}(y)\right).\end{aligned} \tag{59}$$

Then it follows from (58) and (59) that

$$\frac{\sigma^2}{2}\left(W'_{q_1}(y) - W'_{q_2}(y)\right) - (\beta y - \alpha - M_p)\left(W_{q_1}(y) - W_{q_2}(y)\right) \ge \frac{\sigma^2}{2}(q_1 - q_2). \tag{60}$$

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This implies that, for all $y \in (0, x_{q_1} \wedge x_{q_2})$,

$$\begin{aligned}
&\frac{\mathrm{d}(W_{q_1}(y) - W_{q_2}(y))}{dy} - \frac{2(\beta y - \alpha - M_p)}{\sigma^2}\,(W_{q_1}(y) - W_{q_2}(y)) \geq (q_1 - q_2),\\
&\quad\Longrightarrow \frac{\mathrm{d}}{\mathrm{dy}}\left[\big(W_{q_1}(y) - W_{q_2}(y)\big)\exp\left(\frac{-\beta y^2 + 2(\alpha + M_p)y}{\sigma^2}\right)\right]\\
&\qquad\geq (q_1 - q_2)\exp\left(\frac{-\beta y^2 + 2(\alpha + M_p)y}{\sigma^2}\right).
\end{aligned}$$

Thus, the desired result follows upon integrating both sides of the above inequality.

Next, we discuss the local extrema of $y \to W_q(y)$ on $y \in \mathbb{R}_+$.

Lemma 8. *Consider the solution $y \to W_q(y)$ of (55) on the interval $[0, x_q)$. Assume that there is a constant $y_0 \in (0, x_q)$ such that $W_q'(y_0) = 0$. Then the following hold.*

(i) If $W_q(y_0) > \frac{M_F}{\beta+\rho}$, then $y = y_0$ is a strict local minimum of $y \to W_q(y)$.

(ii) If $W_q(y_0) < -\frac{M_F}{\beta+\rho}$, then $y = y_0$ is a strict local maximum of $y \to W_q(y)$.

Here, $M_F > 0$ is the Lipschitz coefficient of the function $y \to F^u(y)$ obtained in Lemma 6.

Proof. We only show assertion (i), since the proof of claim (ii) is similar. It is sufficient to show that there exists some $\delta > 0$ such that $W_q'(y) < 0$ for all $y \in (y_0 - \delta, y_0) \cap [0, x_q)$ and $W_q'(y) > 0$ for all $y \in (y_0, y_0 + \delta) \cap [0, x_q)$. It follows from (55) that, for all $y \in [0, x_q)$,

$$\begin{aligned}
&\frac{\sigma^2}{2}\frac{\big(W_q'(y) - W_q'(y_0)\big)}{y - y_0}\\
&\quad= -\frac{\Phi_p(W_q(y)) - \Phi_p(W_q(y_0))}{y - y_0} + \frac{(\beta y - \alpha)W_q(y) - (\beta y_0 - \alpha)W_q(y_0)}{y - y_0}\\
&\qquad- \frac{F^u(y) - F^u(y_0)}{y - y_0} + \frac{\rho}{y - y_0}\int_{y_0}^{y} W_q(x)\,\mathrm{d}x.
\end{aligned} \tag{61}$$

A direct calculation yields

$$\begin{aligned}
&\lim_{y\to y_0}\frac{\Phi_p(W_q(y)) - \Phi_p(W_q(y_0))}{y - y_0} = \big[\Phi_p(W_q(y))\big]'\Big|_{y=y_0} = W_q'(y_0)\Phi_p'(W_q(y_0)) = 0,\\
&\lim_{y\to y_0}\frac{(\beta y - \alpha)W_q(y) - (\beta y_0 - \alpha)W_q(y_0)}{y - y_0} = \big[(\beta y - \alpha)W_q(y)\big]'\Big|_{y=y_0} = \beta W_q(y_0),\\
&\lim_{y\to y_0}\frac{\rho}{y - y_0}\int_{y_0}^{y} W_q(x)\,\mathrm{d}x = \rho W_q(y_0).
\end{aligned}$$

We deduce from Lemma 6 that

$$\begin{aligned}
\liminf_{y\to y_0}\frac{\sigma^2}{2}\frac{W_q'(y) - W_q'(y_0)}{y - y_0} &\geq (\beta + \rho)W_q(y_0) + \liminf_{y\to y_0}\left(-\frac{F^u(y) - F^u(y_0)}{y - y_0}\right)\\
&\geq (\beta + \rho)W_q(y_0) - M_F > 0.
\end{aligned} \tag{62}$$

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Then we conclude that there exists some $\delta > 0$ such that $W_q'(y) > 0$ for all $y \in (y_0, y_0 + \delta) \cap [0, x_q)$. Otherwise, for any $n \geq 1$, there exists $y_n \in (y_0, y_0 + \frac{1}{n}) \cap [0, x_q)$ such that $W_q'(y_n) \leq 0$. This implies that

$$\liminf_{y \to y_0} \frac{W_q'(y) - W_q'(y_0)}{y - y_0} \leq \liminf_{n \to \infty} \frac{W_q'(y_n) - W_q'(y_0)}{y_n - y_0} \leq 0,$$

which yields a contradiction to (62). In a similar fashion, we can show that $W_q'(y) < 0$ for all $y \in (y_0 - \delta, y_0) \cap [0, x_q)$. Thus, we complete the proof of this lemma.

The following lemma shows that the explosion level of any bounded solution $y \to W_q(y)$ to (55) is indeed infinite. The proof is exactly the same as that of Lemma 6.11 in [35], and hence we omit it here.

Lemma 9. *Consider the solution $y \to W_q(y)$ of (55) on the interval $[0, x_q)$. If $y \to W_q(y)$ is bounded, then $x_q = +\infty$.*

Thus, we have the following result.

Lemma 10. *There exists at most one real number $q \in \mathbb{R}$ such that (55) admits a (global) bounded solution $y \to W_q(y)$ on $\mathbb{R}_+$ with $W_q(0) = p$ and $W_q'(0) = q$.*

Proof. We prove the lemma by contradiction. We assume that for $i = 1, 2$, $y \to W_{q_i}(y)$ is a bounded solution of (55) with $W_{q_i}(0) = p$ and $W_{q_i}'(0) = q_i$ for some $q_i \in \mathbb{R}$ (Lemma 9 implies that $x_{q_i} = +\infty$). Without loss of generality, we assume that $q_1 > q_2$. Since $x_{q_i} = +\infty$, Lemma 7 gives $W_{q_1}(y) > W_{q_2}(y)$ for all $y > 0$ and

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}y}\left[\big(W_{q_1}(y) - W_{q_2}(y)\big) \exp\left(\frac{-\beta y^2 + 2(\alpha + M_p)y}{\sigma^2}\right)\right] \\ &\quad \geq (q_1 - q_2) \exp\left(\frac{-\beta y^2 + 2(\alpha + M_p)y}{\sigma^2}\right) > 0. \end{aligned}$$

Thus, we deduce that, for all $y > 1$,

$$\begin{aligned} &\big(W_{q_1}(y) - W_{q_2}(y)\big) \exp\left(\frac{-\beta y^2 + 2(\alpha + M_p)y}{\sigma^2}\right) \\ &\quad \geq \big(W_{q_1}(1) - W_{q_2}(1)\big) \exp\left(\frac{-\beta + 2(\alpha + M_p)}{\sigma^2}\right) \\ &\Longrightarrow \big(W_{q_1}(y) - W_{q_2}(y)\big) \geq C \exp\left(\frac{\beta y^2 - 2(\alpha + M_p)y}{\sigma^2}\right) \\ &\Longrightarrow \lim_{y \to +\infty} \big(W_{q_1}(y) - W_{q_2}(y)\big) = +\infty, \end{aligned}$$

where the constant $C := (W_{q_1}(1) - W_{q_2}(1)) \exp((-\beta + 2(\alpha + M_p))/\sigma^2) > 0$. However, both $y \to W_{q_1}(y)$ and $y \to W_{q_2}(y)$ are bounded solutions, and thus $y \to W_{q_1}(y) - W_{q_2}(y)$ is also bounded, which yields the contradiction. This completes the proof of the lemma.

In the following, we study the boundedness and asymptotic properties of $y \to W_q(y)$ of (55).

Lemma 11. *For given fixed $p > 0$ and $M > \max\{p, \frac{M_F}{\beta + \rho}\}$, there exists a real number pair (q_1, q_2) satisfying $q_1 > q_2$ such that the following statements hold.*

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(i) For any $q \geq q_1$, let $y \to W_q(y)$ be the solution to (55) on the interval $[0, x_q)$ with $W_q(0) = p$ and $W_q'(0) = q$. Then $W_q(y) > M$ for some $y \in [0, x_q)$, while $W_q(y) \geq -M$ for all $y \in [0, x_q)$. Moreover, $y \to W_q(y)$ is non-decreasing for large y for all $y > N$ with some $N > 0$ and $\lim_{y \to x_q} W_q(y) = +\infty$.

(ii) For any $q \leq q_2$, let $y \to W_q(y)$ be the solution to (55) on the interval $[0, x_q)$ with $W_q(0) = p$ and $W_q'(0) = q$. Then $W_q(y) < -M$ for some $y \in [0, x_q)$, while $W_q(y) \leq M$ for all $y \in [0, x_q)$. Moreover, $y \to W_q(y)$ is non-increasing for large y for all $y > N$ with some $N > 0$ and $\lim_{y \to x_q} W_q(y) = -\infty$.

Proof. We prove only assertion (i), since the proof of (ii) is very similar. We first conclude that there exists some $q_1 \in \mathbb{R}$ such that $W_{q_1}(y) > M$ for some $y > 0$. In fact, it follows from Lemma 10 that there exists some $\hat{q} \in \mathbb{R}$ such that the solution $y \to W_{\hat{q}}(y)$ is unbounded on $(0, x_{\hat{q}})$. If $W_{\hat{q}}(y) > M$ for some $y > 0$, then we can take $q_1 = \hat{q}$ with the help of Lemma 7; otherwise, it must be true that $W_{\hat{q}}(y_0) < -M$ for some $y_0 > 0$. By applying Lemma 7, we can always find a real number $q_1 \in \mathbb{R}$ large enough that

$$\begin{aligned} W_{q_1}(y_0) &\geq W_{\hat{q}}(y_0) \\ &\quad + (q_1 - \hat{q}) \exp\left(\frac{\beta y_0^2 - 2(\alpha + M_p) y_0}{\sigma^2}\right) \int_0^{y_0} \exp\left(\frac{-\beta x^2 + 2(\alpha + M_p) x}{\sigma^2}\right) \mathrm{d}x \\ &> M. \end{aligned}$$

If the explosion level x_{q_1} of the solution $y \to W_{q_1}(y)$ satisfies $x_{q_1} > y_0$, then one can find a desired $q_1 \in \mathbb{R}$ with $W_{q_1}(y_0) > M$. However, if $x_{q_1} \leq y_0$ and $W_{q_1}(y) \leq M$ for all $y \in [0, x_{q_1})$, we have from Lemma 7 that $W_{q_1}(y) > W_{\hat{q}}(y)$ for $0 < y < x_{q_1} \leq y_0$. Hence, $y \to W_{q_1}(y)$ is a bounded solution to (55) on the interval $[0, x_{q_1})$. Thanks to Lemma 9, we know that $y_0 \geq x_{q_1} = +\infty$, which yields the contradiction. Thus, there exists some $q_1 \in \mathbb{R}$ such that $W_{q_1}(y) > M$ for some $y > 0$.

We next show that $y \to W_{q_1}(y)$ is non-decreasing for large y for all $y > N$ with some $N > 0$ and $W_{q_1}(y) \geq -M$ for all $y \in [0, x_{q_1})$. For the case of $W_{q_1}'(y_0) \leq 0$ with y_0 obtained above, using the continuity of $y \to W_{q_1}(y)$ and $M > p$, there exists some $\hat{y} \in (0, y_0]$ such that $W_{q_1}(\hat{y}) = \sup_{y \in [0, y_0]} W_{q_1}(y)$. Note that

$$W_{q_1}(\hat{y}) \geq W_{q_1}(y_0) > M \geq \frac{M_F}{\beta + \rho}.$$

It follows from Lemma 8 that $\hat{y}$ is a strict local minimum of $y \to W_q(y)$, which is in conflict with the definition of $\hat{y}$. Thus, we deduce that $W_{q_1}'(y_0) > 0$. Moreover, $y \to W_{q_1}(y)$ is non-decreasing for $y \in [y_0, x_{q_1})$; otherwise, the set $\inf\{y \geq y_0 : W_{q_1}'(y) < 0\}$ is non-empty, and we let $\hat{y} = \inf\{y \geq y_0 : W_{q_1}'(y) < 0\}$. Using the continuity of $y \to W_{q_1}(y)$ and $y \to W_{q_1}'(y)$, it follows that $W_{q_1}'(\hat{y}) = 0$ and $W_{q_1}(\hat{y}) \geq W_{q_1}(y_0) > M$. However, Lemma 8 tells us that $\hat{y}$ is a strict local minimum of $y \to W_q(y)$ and $\inf\{y \geq y_0 : W_{q_1}'(y) < 0\} < \hat{y}$. Furthermore, it must be true that $W_{q_1}(y) \geq -M$ for all $y \in [0, x_{q_1}]$. We show this by contradiction. Assume that there exists some $\hat{y} \in [0, x_{q_1}]$ such that $W_{q_1}(\hat{y}) \geq -M$. Then we can see that $\hat{y} \in (0, y_0)$, since $W_{q_1}(y_0) > M$ and $y \to W_{q_1}(y)$ is non-decreasing for $y \in [y_0, x_{q_1})$. Let $\tilde{y}$ be the strict minimum point of $y \to W_{q_1}(y)$ on $y \in (0, y_0)$, so that $W_{q_1}'(\tilde{y}) = 0$ and $W_{q_1}(\tilde{y}) < -M$. However, with Lemma 8, $\hat{y}$ is a strict maximum point, which yields the contradiction. Thus, we conclude that $y \to W_{q_1}(y)$ is non-decreasing for all $y > N$ with some $N > 0$ and $W_{q_1}(y) \geq -M$ for all $y \in [0, x_{q_1})$.

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We then deal with the case of $q > q_1$. If the explosion level $x_q \geq x_{q_1}$, then $W_q(y) > W_{q_1}(y_0) > M$. If $x_q < x_{q_1}$, then $y \to W_q(y)$ is an unbounded solution to (55) on $(0, x_q)$. Since $W_q(y) > W_{q_1}(y) \geq -M$ for all $y \in [0, x_q)$, we have $W_q(y) > M$ for some $y \in [0, x_q)$. In both cases, by a similar argument to that for the case of deriving q_1, we can show that $y \to W_q(y)$ is non-decreasing for large y and $W_q(y) \geq -M$ for all $y \geq 0$.

Toward this end, it remains to show that, for all $q \geq q_1$, we have $\lim_{y \to x_q} W_q(y) = +\infty$. In fact, for any $q \geq q_1$, using the monotonicity of $y \to W_q(y)$ for large y, the limit $\lim_{y \to x_q} W_q(y)$ exists, and it is either finite or infinity. For the case with $\lim_{y \to x_q} W_q(y) < +\infty$, $y \to W_q(y)$ is bounded on $[0, x_q)$, and Lemma 9 gives $x_q = +\infty$. But, by applying Lemma 10, there is at most one real number q such that $y \to W_q(y)$ is a bounded solution on $\mathbb{R}_+$. Thus, we have $\lim_{y \to x_q} W_q = +\infty$.

We are now in a position to prove Theorem 1. Our proof scheme is to extend and develop the arguments used by [36], who dealt with the homogeneous case to establish existence of global (classical) solutions to the non-homogeneous second-order ODE (41) by overcoming two difficulties: (i) the unknown regularity of the non-homogeneous term; and (ii) the unknown monotonicity of the value function u. Let us mention that when reduced to the case of $\gamma = 0$ (without jumps) or the case where the intensity of the jump is independent of the inventory state process (i.e. jumps are not state-dependent), it can be verified that the value function $y \to u(y)$ is non-decreasing. This gives a lower bound for u'(y) if u is smooth (i.e. $u'(y) \geq 0$ for all $y \in \mathbb{R}_+$). Correspondingly, one only needs to establish a single set S_1^M in (63) to get an upper bound. However, in our setting with state-dependent jumps, the monotonicity of the value function $y \to u(y)$ does not hold true. To overcome this difficulty, we establish an additional set S_2^M in (63) to show that u' admits a lower bound.

Proof of Theorem 1. To prove the existence of classical solutions to (41), it suffices to show that there exists some $q \in \mathbb{R}$ such that (55) has a solution with $W_q(0) = p$, $W_q'(0) = q$ and explosion level $x_q = +\infty$. For any $M > \max\{p, \frac{M_F}{\beta+\rho}\}$, we introduce

$$S_1^M = \big\{q \in \mathbb{R}: \sup_{0 \leq y < x_q} W_q(y) > M\big\}, \quad S_2^M = \big\{q \in \mathbb{R}: \inf_{0 \leq y < x_q} W_q(y) < -M\big\}. \tag{63}$$

By applying Lemma 11, we can see that the set S_1^M has a lower bound, while the set S_2^M has a upper bound. Therefore

$$\eta_1 := \inf S_1^M \quad \text{and} \quad \eta_2 := \sup S_2^M$$

are well-defined. We next show that $\eta_2 \leq \eta_1$. In fact, for any $\hat{q} \in S_1^M$ we have that $\sup_{0 \leq y < x_{\hat{q}}} W_{\hat{q}}(y) > M$. Then there exists some $y_0 \in (0, x_{\hat{q}})$ such that $W_{\hat{q}}(y_0) > M$. It follows from Lemma 11 that, for all $q \geq \hat{q}$, $W_q(y) \geq -M$ for all $y \in (0, x_q)$. Hence, we have $\eta_2 \leq \hat{q}$ for all $\hat{q} \in S_1^M$. Thus, we get $\eta_2 \leq \eta_1$. Take $q \in [\eta_2, \eta_1]$ and consider the solution $y \to W_q(y)$ with $W_q(0) = p$ and $W_q'(0) = q$. By the theory of differential equations, one has that $q \to W_q(y)$ is continuous (see e.g. Chapter 5 in [21]). Then, if there exists some real number y_0 such that $W_q(y_0) > M$, by the continuity of $q \to W_q(y)$ there also exists a real number $\hat{q} < q$ such that $W_{\hat{q}}(y_0) > M$. Then $\eta_1 \leq \hat{q} < q$, which is in conflict with $q \in [\eta_2, \eta_1]$. Similarly, if there exists some real number y_0 such that $W_q(y_0) < -M$, by the continuity of $q \to W_q(y)$ there also exists a real number $\hat{q} > q$ such that $W_{\hat{q}}(y_0) < -M$. Then we have $\eta_2 \geq \hat{q} > q$, which is in conflict with $q \in [\eta_2, \eta_1]$. Thus, we conclude that $y \to W_q(y)$ is bounded by M, and Lemma 9 gives $x_q = +\infty$. Then the function $y \to w(y)$ given by (54) is a classical solution to (41). Moreover, note that $|w'(y)| = |W(y)| \leq M$ for all $y \in \mathbb{R}_+$, so $y \to w(y)$ satisfies the linear growth, which completes the proof of the theorem.

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With Theorem 1, we can characterize the optimal (feedback) inventory control strategy for the primal optimal inventory control problem (3).

Theorem 2. (Verification result.) *Suppose that assumptions* $(\mathbf{A}_p)$, $(\mathbf{A}_\rho)$ *and* $(\mathbf{A}_c)$ *hold. Then the following hold.*

(i) The (primal) ID-HJB equation (17) with Neumann boundary condition admits a unique classical solution $v \in C^2((0,\infty)) \cap C(\mathbb{R}_+)$ satisfying the linear growth.

(ii) For the classical solution v obtained in (i), we define the so-called optimal (feedback) control function

$$\theta^*(y) := \Psi(-v'(y)) \quad \forall y \in \mathbb{R}_+, \tag{64}$$

where $\Psi(y) = (c')^{-1}(y)\mathbf{1}_{y>c'(0)}$, with $\Phi{:}\mathbb{R} \to \mathbb{R}_+$ as defined by (4). Consider the (controlled) reflected netflow process $Y^* = (Y_t^*)_{t\geq 0}$ described in (1) with $\theta_s^* := \theta^*(Y_s^*)$ replaced by θ_s. Then $\theta^* = (\theta_t^*)_{t\in\mathbb{R}_+}$ is an optimal (admissible) inventory strategy. In other words, $\theta^* \in \mathcal{A}$, and for all $\theta \in \mathcal{A}$ we have that

$$J(y;\theta) = \mathbb{E}_y\left[\int_0^\infty \mathrm{e}^{-\rho s}\big(c(\theta_s)\,\mathrm{d}s + p\,\mathrm{d}L_s\big)\right] \geq v(y) = V(y) = J(y;\theta^*) \quad \forall y \in \mathbb{R}_+.$$

Proof. First of all, it follows from Theorem 1 that (41) has a classical solution $w(y) \in C^2((0,\infty)) \cap C(\mathbb{R}_+)$, and obviously it is also a viscosity solution to (41). Since the value function $y \to u(y)$ given by (21) is a (continuous) viscosity solution to (41), we get from Proposition 4 that $u(y) = w(y)$ for all $y \in \mathbb{R}_+$. This yields $u(y) \in C^2((0,\infty)) \cap C(\mathbb{R}_+)$ which satisfies (24). We next show that the value function $y \to u(y)$ is also bounded on $\mathbb{R}_+$, i.e. $|u(y)| \leq M_b(\beta)$ for all $y \in \mathbb{R}_+$, where $M_b(\beta){:}\mathbb{R}_+ \to \mathbb{R}_+$ is given by

$$M_b(\beta) := \begin{cases} \dfrac{|\alpha|}{\beta} + 3\dfrac{\sigma}{\beta} + \gamma M_\lambda \displaystyle\int_{\mathbb{R}} |\xi|\nu(\mathrm{d}\xi), & \beta > 0; \\ \dfrac{1}{\rho}\left(|\alpha| + 3\sigma + \gamma M_\lambda \displaystyle\int_{\mathbb{R}} |\xi|\nu(\mathrm{d}\xi)\right), & \beta = 0. \end{cases} \tag{65}$$

In fact, let $Y^y = (Y_t^y)_{t\geq 0}$ be the (controlled) reflected process (1) with $Y_0^y = y \in \mathbb{R}_+$. Denote by $L^y = (L_t^y)_{t\geq 0}$ the regulator process of the reflected process Y^y with $L_0^y = 0$. For $y \in \mathbb{R}_+$, consider the constant inventory control $\hat{\theta}_t \equiv 0$ for $t \geq 0$ (clearly $\hat{\theta} \in \mathcal{A}_p$). It follows from (3), (7) and assumption $(\mathbf{A}_c)$ that

$$\begin{aligned} |u(y)| = -u(y) \leq \tilde{J}(y;\hat{\theta}) &= \mathbb{E}_y\left[\int_0^\infty \mathrm{e}^{-\rho s} p\,\mathrm{d}L_s^y\right] = \lim_{T\to\infty} \mathbb{E}_y\left[\int_0^T \mathrm{e}^{-\rho s} p\,\mathrm{d}L_s^y\right] \\ &= p \lim_{T\to\infty} \mathbb{E}_y\left[\mathrm{e}^{-(\rho+\beta)T}\hat{L}_T^y + \rho\int_0^T \mathrm{e}^{-(\rho+\beta)s}\hat{L}_s^y\,\mathrm{d}s\right], \end{aligned} \tag{66}$$

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with $\hat{L}_t^y := \int_0^t \mathrm{e}^{\beta s}\,\mathrm{d}L_s^y$ for all $t \geq 0$. We deduce from (9) that for any $s \geq 0$,

$$\begin{aligned}\hat{L}_s^y &= \sup_{0\leq u\leq s}\left(y + \int_0^u \mathrm{e}^{\beta v}\{(\alpha+\hat{\theta}_v)\,\mathrm{d}v + \sigma\,\mathrm{d}B_v\} + \hat{I}_u(Y^y)\right)^- \\ &\leq \int_0^s \mathrm{e}^{\beta v}|\alpha|\,\mathrm{d}v + \sigma\sup_{0\leq u\leq s}\left|\int_0^u \mathrm{e}^{\beta v}\,\mathrm{d}B_v\right| \\ &\quad + \sup_{0\leq u\leq s}\int_0^u \mathrm{e}^{\beta v}\int_{\mathbb{R}_+\times\mathbb{R}} \gamma|\xi|\mathbf{1}_{z\leq\lambda(Y_{v-})}\pi(\mathrm{d}v,\mathrm{d}z,\mathrm{d}\xi) \qquad (67)\\ &\leq \int_0^s \mathrm{e}^{\beta v}|\alpha|\,\mathrm{d}v + \sigma\sup_{0\leq u\leq s}\left|\int_0^u \mathrm{e}^{\beta v}\,\mathrm{d}B_v\right| + \int_0^s \mathrm{e}^{\beta v}\int_{\mathbb{R}_+\times\mathbb{R}} \gamma|\xi|\mathbf{1}_{z\leq\lambda(Y_{v-})}\pi(\mathrm{d}v,\mathrm{d}z,\mathrm{d}\xi),\end{aligned}$$

where $\hat{I}_t(Y^y) := \int_0^t \mathrm{e}^{\beta s}\,\mathrm{d}I_s(Y^y)$ for all $t \in \mathbb{R}_+$. Taking the expectation on both sides of the inequality (67), we obtain the following.

- In the case with $\beta > 0$, by applying assumption $(\mathbf{A}_p)$ and the Burkholder–Davis–Gundy inequality, we have

$$\mathbb{E}\left[\hat{L}_s^y\right] \leq \left(\frac{|\alpha|}{\beta} + 3\frac{\sigma}{\sqrt{\beta}} + \frac{\gamma M_\lambda}{\beta}\int_{\mathbb{R}}|\xi|\nu(\mathrm{d}\xi)\right)\mathrm{e}^{\beta s} \quad \forall s \geq 0. \qquad (68)$$

  It follows from (66) and (68) that

$$|u(y)| \leq \frac{|\alpha|}{\beta} + 3\frac{\sigma}{\sqrt{\beta}} + \frac{\gamma M_\lambda}{\beta}\int_{\mathbb{R}}|\xi|\nu(\mathrm{d}\xi) \quad \forall y \in \mathbb{R}_+.$$

- In the case with $\beta = 0$, we obtain that

$$\mathbb{E}\left[\hat{L}_s^y\right] \leq \left(|\alpha| + 3\sigma + \gamma M_\lambda\int_{\mathbb{R}}|\xi|\nu(\mathrm{d}\xi)\right)s \quad \forall s \geq 0. \qquad (69)$$

  Then it follows that

$$|u(y)| \leq \frac{1}{\rho}\left(|\alpha| + 3\sigma + \gamma M_\lambda\int_{\mathbb{R}}|\xi|\nu(\mathrm{d}\xi)\right) \quad \forall y \in \mathbb{R}_+.$$

This yields the boundedness of $y \to u(y)$ on $\mathbb{R}_+$ dominated by $M_b(\beta)$. Hence, the value function $y \to u(y)$ satisfies the linear growth condition.

Let $v(y) = -u(y)$ for all $y \in \mathbb{R}_+$. From the Lipschitz-continuous property of $y \to u(y)$ (cf. (23)), it follows that

$$|v(y_1) - v(y_2)| \leq 2p|y_1 - y_2| \quad \forall y_1, y_2 \in \mathbb{R}_+.$$

This yields $|v'(y)| \leq 2p$ for all $y \in \mathbb{R}_+$. Also, $y \to v(y)$ is a bounded classical solution of the (primal) ID-HJB equation (17) with Neumann boundary condition. Using (64) and (4) under assumption $(\mathbf{A}_c)$, we have $\theta^*(y) = \Psi(-v'(y)) \in [0, M_p]$ for all $y \in \mathbb{R}_+$. Hence, for $\theta_t^* = \theta^*(Y_t^*)$ with $t \in \mathbb{R}_+$, it is not difficult to verify that $\theta^* \in \mathcal{A}$. Next, for any admissible $\theta \in \mathcal{A}$, let $Y = (Y_t)_{t\geq 0}$ be the controlled state process (1) under this control θ. Then, for an arbitrary $T > 0$, with the help of Itô's rule applied to $\mathrm{e}^{-\rho T}v(Y_T)$, we obtain

$$\begin{aligned}\mathrm{e}^{-\rho T}v(Y_T) = v(y) &+ \int_0^T v'(Y_t)\big\{(\alpha-\beta Y_t)\,\mathrm{d}t + \theta_t\,\mathrm{d}t + \sigma\,\mathrm{d}B_t + \mathrm{d}L_t^c\big\} + \frac{\sigma^2}{2}\int_0^T v''(Y_t)\,\mathrm{d}t \\ &+ \int_0^T\int_{\mathbb{R}_+}\int_{\mathbb{R}}\big\{v(\max\{Y_{t-} + \gamma\xi\mathbf{1}_{z\leq\lambda(Y_{t-})}, 0\}) - v(Y_{t-})\big\}\pi(\mathrm{d}t,\mathrm{d}z,\mathrm{d}\xi).\end{aligned}$$

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Taking the expectation on both sides of the above equality, it follows from the HJB equation (17) satisfied by $y \to v(y)$ that

$$\mathbb{E}_y\left[\mathrm{e}^{-\rho T} v(Y_T)\right] \geq v(y) - \mathbb{E}_y\left[\int_0^T \mathrm{e}^{-\rho s}(c(\theta_s)\,\mathrm{d}s + p\,\mathrm{d}L_s)\right] \quad \forall y \in \mathbb{R}_+. \tag{70}$$

Since $y \to v(y)$ is bounded on $y \in \mathbb{R}_+$, we have that

$$\lim_{T\to\infty} \mathbb{E}\left[\mathrm{e}^{-\rho T} v(Y_T)\right] = 0.$$

Then, letting $T \to \infty$ on both sides of the equality (70), we deduce from the monotone convergence theorem that, for all $\theta \in \mathcal{A}$,

$$\mathbb{E}_y\left[\int_0^T \mathrm{e}^{-\rho s}\big(c(\theta_s)\,\mathrm{d}s + p\,\mathrm{d}L_s\big)\right] \geq v(y) = V(y) \quad \forall y \in \mathbb{R}_+,$$

where the equality holds when $\theta = \theta^*$.

Toward this end, let $v_1, v_2{:}\mathbb{R}_+ \to \mathbb{R}$ be two distinct classical solutions of (17) satisfying the linear growth under $v_1'(0) = v_2'(0) = -p$. By a similar argument to that above, we can conclude that for $i = 1, 2$,

$$v_i(y) = V(y) = \inf_{\theta\in\mathcal{A}} \mathbb{E}_y\left[\int_0^\infty \mathrm{e}^{-\rho s}\big(c(\theta_s)\,\mathrm{d}s + p\,\mathrm{d}L_s\big)\right] \quad \forall y \in \mathbb{R}_+.$$

This implies that $v_1 = v_2$ on $\mathbb{R}_+$, and hence the uniqueness of classical solutions to (17) holds true. Thus, the proof of the verification theorem is complete.

Appendix A. Proofs of auxiliary results

This appendix collects the proofs of some auxiliary results that are used to prove the main results of the paper.

Proof of Lemma 1. Let $\hat{Y}_s := \mathrm{e}^{\beta s} Y_s$ for all $s \geq 0$. Then, for all $s \geq 0$,

$$\begin{aligned}\hat{Y}_s &= Y_0 + \int_0^s \mathrm{e}^{\beta v}\,\mathrm{d}Y_v + \beta\int_0^s \mathrm{e}^{\beta v} Y_v\,\mathrm{d}v \\ &= \underbrace{y + \int_0^s \mathrm{e}^{\beta v}\{(\alpha+\theta_v)\,\mathrm{d}v + \sigma\,\mathrm{d}B_v\} + \hat{I}_s(\hat{Y})}_{=:X_s} + \int_0^s \mathrm{d}\hat{L}_v.\end{aligned} \tag{71}$$

Here, the processes $\hat{I}_s(\hat{Y}) := \int_0^s \mathrm{e}^{\beta v}\,\mathrm{d}I_v(\mathrm{e}^{-\beta\cdot}\hat{Y})$ and $\hat{L}_s := \int_0^s \mathrm{e}^{\beta v}\,\mathrm{d}L_v$ for all $s \geq 0$. Hence, $s \to \hat{L}_s$ is a non-decreasing and RCLL process satisfying

$$\int_0^s \mathbf{1}_{\hat{Y}_v=0}\,\mathrm{d}\hat{L}_v = \int_0^s \mathrm{e}^{\beta v}\mathbf{1}_{Y_v=0}\,\mathrm{d}L_v = \int_0^s \mathrm{e}^{\beta v}\,\mathrm{d}L_v = \int_0^s \mathrm{d}\hat{L}_v.$$

That is, $\hat{Y}_s = X_s + \hat{L}_s$ with $s \geq 0$ is an SP. It is enough to show that the SDE (71) has a unique strong solution. The proof of existence is done by using the classical Picard iteration. Toward this end, we introduce the sequence of processes as $\hat{Y}_t^{(0)} = y \in \mathbb{R}_+$ and $\hat{L}_t^{(0)} = 0$ for all $t \geq 0$, and then construct the process $\hat{Y}^{(n+1)}$ in terms of $\hat{Y}^{(n)}$ in the following way:

$$\hat{Y}_t^{(n+1)} := y + \int_0^s \mathrm{e}^{\beta v}\{(\alpha+\theta_v)\,\mathrm{d}v + \sigma\,\mathrm{d}B_v\} + \hat{I}_s(\hat{Y}^{(n)}) + \int_0^s \mathrm{d}\hat{L}_v^{(n+1)}, \tag{72}$$

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where the process $\hat{L}^{(n+1)} = (\hat{L}_t^{(n+1)})_{t\geq 0}$ is given by

$$\hat{L}_t^{(n+1)} = \sup_{0\leq u\leq t}\left(y + \int_0^u \mathrm{e}^{\beta v}\{(\alpha+\theta_v)\,\mathrm{d}v + \sigma\,\mathrm{d}B_v\} + \hat{I}_u(\hat{Y}^{(n)})\right)^- \quad \forall t \geq 0. \tag{73}$$

It follows from (72), (73) and the SP that $\mathbb{P}$-a.s.,

$$\int_0^t \hat{Y}_s^{(n+1)}\,\mathrm{d}\hat{L}_s^{(n+1)} = 0 \quad \forall t \geq 0. \tag{74}$$

Fix $t \geq 0$. We have that, for all $n \geq 1$ and $0 \leq s \leq t$,

$$\begin{aligned}
\left|\hat{L}_s^{(n+1)} - \hat{L}_s^{(n)}\right| &\leq \sup_{0\leq u\leq s}\left|\hat{I}_u(Y^{(n)}) - \hat{I}_u(Y^{(n-1)})\right| \\
&= \gamma \sup_{0\leq u\leq s}\left|\int_0^u \mathrm{e}^{\beta r}\int_{\mathbb{R}_+\times\mathbb{R}} \xi\left(\mathbf{1}_{z\leq\lambda(\mathrm{e}^{-\beta r}\hat{Y}_{r-}^{(n)})} - \mathbf{1}_{z\leq\lambda(\mathrm{e}^{-\beta r}\hat{Y}_{r-}^{(n-1)})}\right)\pi(\mathrm{d}r,\mathrm{d}z,\mathrm{d}\xi)\right| \\
&\leq \gamma \sup_{0\leq u\leq s}\int_0^u \mathrm{e}^{\beta r}\int_{\mathbb{R}_+\times\mathbb{R}} |\xi|\left|\mathbf{1}_{z\leq\lambda(\mathrm{e}^{-\beta r}\hat{Y}_{r-}^{(n-1)})} - \mathbf{1}_{z\leq\lambda(\mathrm{e}^{-\beta r}\hat{Y}_{r-}^{(n-1)})}\right|\pi(\mathrm{d}r,\mathrm{d}z,\mathrm{d}\xi) \\
&\leq \gamma \int_0^s \mathrm{e}^{\beta r}\int_{\mathbb{R}_+\times\mathbb{R}} |\xi|\left|\mathbf{1}_{z\leq\lambda(\mathrm{e}^{-\beta r}\hat{Y}_{r-}^{(n)})} - \mathbf{1}_{z\leq\lambda(\mathrm{e}^{-\beta r}\hat{Y}_{r-}^{(n-1)})}\right|\pi(\mathrm{d}r,\mathrm{d}z,\mathrm{d}\xi) \\
&\leq \gamma \int_0^t \mathrm{e}^{\beta r}\int_{\mathbb{R}_+\times\mathbb{R}} |\xi|\left|\mathbf{1}_{z\leq\lambda(\mathrm{e}^{-\beta r}\hat{Y}_{r-}^{(n)})} - \mathbf{1}_{z\leq\lambda(\mathrm{e}^{-\beta r}\hat{Y}_{r-}^{(n-1)})}\right|\pi(\mathrm{d}r,\mathrm{d}z,\mathrm{d}\xi).
\end{aligned} \tag{75}$$

We obtain from (75) that, for all $t \geq 0$,

$$\begin{aligned}
&\sup_{s\in[0,t]}\left|\hat{L}_s^{(n+1)} - \hat{L}_s^{(n)}\right| \\
&\quad\leq \gamma \int_0^t \mathrm{e}^{\beta r}\int_{\mathbb{R}_+\times\mathbb{R}} |\xi|\left|\mathbf{1}_{z\leq\lambda(\mathrm{e}^{-\beta r}\hat{Y}_{r-}^{(n)})} - \mathbf{1}_{z\leq\lambda(\mathrm{e}^{-\beta r}\hat{Y}_{r-}^{(n-1)})}\right|\pi(\mathrm{d}r,\mathrm{d}z,\mathrm{d}\xi).
\end{aligned} \tag{76}$$

Taking the expectation on both sides of the inequality (76) and using assumption $(\mathbf{A}_p)$, we arrive at

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,t]}\left|\hat{L}_s^{(n+1)} - \hat{L}_t^{(n)}\right|\right] \\
&\quad\leq \gamma\mathbb{E}\left[\int_0^t \mathrm{e}^{\beta r}\int_{\mathbb{R}_+}\int_{\mathbb{R}} |\xi|\left|\mathbf{1}_{z\leq\lambda(\mathrm{e}^{-\beta r}\hat{Y}_{r-}^{(n)})} - \mathbf{1}_{z\leq\lambda(\mathrm{e}^{-\beta r}\hat{Y}_{r-}^{(n-1)})}\right|\nu(\mathrm{d}\xi)\,\mathrm{d}z\,\mathrm{d}r\right] \\
&\quad= \gamma\int_{\mathbb{R}} |\xi|\nu(\mathrm{d}\xi)\int_0^t \mathrm{e}^{\beta r}\mathbb{E}\left[\int_{\mathbb{R}_+}\left|\mathbf{1}_{z\leq\lambda(\mathrm{e}^{-\beta r}\hat{Y}_{r-}^{(n)})} - \mathbf{1}_{z\leq\lambda(\mathrm{e}^{-\beta r}\hat{Y}_{r-}^{(n-1)})}\right|\mathrm{d}z\right]\mathrm{d}r \\
&\quad= \gamma\int_{\mathbb{R}} |\xi|\nu(\mathrm{d}\xi)\int_0^t \mathrm{e}^{\beta r}\mathbb{E}\left[\left|\lambda(\mathrm{e}^{-\beta r}\hat{Y}_{r-}^{(n)}) - \lambda(\mathrm{e}^{-\beta r}\hat{Y}_{r-}^{(n-1)})\right|\right]\mathrm{d}r \\
&\quad\leq \frac{\rho_0}{2}\int_0^t \mathbb{E}\left[\left|\hat{Y}_{r-}^{(n)} - \hat{Y}_{r-}^{(n-1)}\right|\right]\mathrm{d}r \leq \frac{\rho_0}{2}t\mathbb{E}\left[\sup_{s\in[0,t]}\left|\hat{Y}_s^{(n)} - \hat{Y}_s^{(n-1)}\right|\right]
\end{aligned} \tag{77}$$

with the constant $\rho_0 > 0$ given in assumption $(\mathbf{A}_\rho)$. By applying a similar argument to derive (75) and (77), it follows that, for all $t \geq 0$,

$$\mathbb{E}\left[\sup_{s\in[0,t]}\left|\hat{I}_s(\hat{Y}^{(n+1)}) - \hat{I}_s(\hat{Y}^{(n)})\right|\right] \leq \frac{\rho_0}{2}t\,\mathbb{E}\left[\sup_{s\in[0,t]}\left|\hat{Y}_s^{(n)} - \hat{Y}_s^{(n-1)}\right|\right]. \tag{78}$$

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It follows from (72), (77) and (78) that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,t]}\left|\hat{Y}_s^{(n+1)}-\hat{Y}_s^{(n)}\right|\right]\\
&\quad\le \mathbb{E}\left[\sup_{s\in[0,t]}\left|\hat{I}_s(\hat{Y}^{(n+1)})-\hat{I}_s(\hat{Y}^{(n)})\right|\right]+\mathbb{E}\left[\sup_{s\in[0,t]}\left|\hat{L}_s^{(n+1)}-\hat{L}_s^{(n)}\right|\right]\\
&\quad\le \rho_0 t\,\mathbb{E}\left[\sup_{s\in[0,t]}\left|\hat{Y}_s^{(n)}-\hat{Y}_s^{(n-1)}\right|\right].
\end{aligned}\tag{79}$$

Choose $t_1>0$ such that $\rho_0 t_1<\frac{1}{2}$; we then deduce from (79) that

$$\mathbb{E}\left[\sup_{s\in[0,t_1]}\left|\hat{Y}_s^{(n+1)}-\hat{Y}_s^{(n)}\right|\right]\le 2^{-n}\,\mathbb{E}\left[\sup_{s\in[0,t_1]}\left|\hat{Y}_s^{(1)}-\hat{Y}_s^{(0)}\right|\right].\tag{80}$$

We deduce from (72) and (73) that, for all $s\in[0,\,t_1]$,

$$\begin{aligned}
\left|\hat{Y}_s^{(1)}-\hat{Y}_s^{(0)}\right|&=\left|\int_0^s \mathrm{e}^{\beta v}\{(\alpha+\theta_v)\,\mathrm{d}v+\sigma\,\mathrm{d}B_v\}+\hat{I}_s(\hat{Y}^{(0)})+\hat{L}_s^{(1)}\right|\\
&\le\left|\int_0^s \mathrm{e}^{\beta v}\{(\alpha+\theta_v)\,\mathrm{d}v+\sigma\,\mathrm{d}B_v\}+\hat{I}_s(\hat{Y}^{(0)})\right|\\
&\quad+\left|\sup_{0\le u\le s}\left(y+\int_0^u \mathrm{e}^{\beta v}\{(\alpha+\theta_v)\,\mathrm{d}v+\sigma\,\mathrm{d}B_v\}+\hat{I}_u(\hat{Y}^{(0)})\right)^-\right|\\
&\le y+2\sup_{0\le u\le s}\left|\int_0^u \mathrm{e}^{\beta v}\{(\alpha+\theta_v)\,\mathrm{d}v+\sigma\,\mathrm{d}B_v\}+\hat{I}_u(\hat{Y}^{(0)})\right|\\
&\le y+2\int_0^{t_1}\mathrm{e}^{\beta v}(|\alpha|+\theta_v)\,\mathrm{d}v+\sup_{0\le u\le t_1}|\sigma B_u|\\
&\quad+\int_0^{t_1}\mathrm{e}^{\beta v}\int_{\mathbb{R}_+\times\mathbb{R}}\gamma|\xi|\mathbf{1}_{z\le\lambda(\mathrm{e}^{-\beta v}y)}\pi(\mathrm{d}v,\,\mathrm{d}z,\,\mathrm{d}\xi).
\end{aligned}\tag{81}$$

Consequently, we also have

$$\begin{aligned}
\sup_{s\in[0,t_1]}\left|\hat{Y}_s^{(1)}-\hat{Y}_s^{(0)}\right|&\le y+2\int_0^{t_1}\mathrm{e}^{\beta v}(|\alpha|+\theta_v)\,\mathrm{d}v+\sup_{0\le u\le t_1}|\sigma B_u|\\
&\quad+\int_0^{t_1}\mathrm{e}^{-\beta v}\int_{\mathbb{R}_+\times\mathbb{R}}\gamma|\xi|\mathbf{1}_{z\le\lambda(\mathrm{e}^{-\beta v}y)}\pi(\mathrm{d}v,\,\mathrm{d}z,\,\mathrm{d}\xi).
\end{aligned}\tag{82}$$

Taking the expectation on both sides of the inequality (82) and using assumption $(\mathbf{A}_p)$, we deduce that

$$\mathbb{E}\left[\sup_{s\in[0,t_1]}\left|\hat{Y}_s^{(1)}-Y_s^{(0)}\right|\right]<+\infty.\tag{83}$$

Thus, the estimates (77), (80) and (83) imply that $(\hat{Y}^{(n)},\,\hat{L}^{(n)})_{n\ge0}$ converges to some stochastic process $(\hat{Y},\,\hat{L})\in\mathbb{S}_{t_1}\times\mathbb{S}_{t_1}$. Moreover, we can see from (73) and (74) that $\hat{Y}=(\hat{Y})_{s\in[0,t_1]}$ is non-negative, $\hat{L}=(\hat{L})_{s\in[0,t_1]}$ is non-decreasing and they also satisfy that, for all $s\in[0,\,t_1]$,

$$\int_0^s \hat{Y}_r\,\mathrm{d}\hat{L}_r=0\quad\mathbb{P}\text{-a.s.}$$

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Then we can prove that $(\hat{Y}, \hat{L})$ is a solution of the SDE (71) on $\mathbb{R}_+$ by proceeding iteratively over successive intervals $[t_1, 2t_1], [2t_1, 3t_1], \ldots$.

Next, we deal with the uniqueness of solutions. Let $(\hat{Y}, \hat{L}), (\tilde{Y}, \tilde{L}) \in \mathbb{S}_\infty \times \mathbb{S}_\infty$ be two strong solutions to the SDE (71) with initial condition $\hat{Y}_0 = \tilde{Y}_0 = y$. In a similar fashion to (77), (78) and (79), we obtain

$$\max\left\{\mathbb{E}\left[\sup_{s\in[0,t]}\left|\hat{L}_s^{(n+1)} - \hat{L}_t^{(n)}\right|\right], \mathbb{E}\left[\sup_{s\in[0,t]}\left|\hat{Y}_s - \tilde{Y}_s\right|\right]\right\} \leq 2Ct\,\mathbb{E}\left[\sup_{s\in[0,t]}\left|\hat{Y}_s - \tilde{Y}_s\right|\right].$$

Again, choosing $t_1 > 0$ such that $2Ct_1 < \frac{1}{2}$, we get the uniqueness of solutions on $[0, t_1]$. Then we can repeat this argument on intervals $[t_1, 2t_1], [2t_1, 3t_1], \ldots$ and iterate up to any finite $t > 0$. Thus, the proof of the lemma is complete.

Proof of Lemma 4. It follows from applying Itô's rule to $|Y_t^\theta - y_0|^2$ that, for all $\theta \in \mathcal{A}_p$ and $t \geq 0$,

$$\begin{aligned}\left|Y_t^\theta - y_0\right|^2 &= \int_0^t 2(Y_s^\theta - y_0)\left(\alpha + \frac{\sigma^2}{2} - \beta Y_s^\theta + \theta_s\right)\mathrm{d}s + \int_0^t 2\sigma(Y_s^\theta - y_0)\,\mathrm{d}B_s \\ &\quad + \int_0^t 2(Y_s^\theta - y_0)\,\mathrm{d}L_s + \int_0^t\int_{\mathbb{R}_+}\int_{\mathbb{R}}\{|Y_s^\theta - y_0 + \gamma\xi|^2 - |Y_s^\theta - y_0|^2\}\pi(\mathrm{d}s, \mathrm{d}z, \mathrm{d}\xi).\end{aligned} \tag{84}$$

Taking the expectation on both sides of the equality (84), we deduce from assumptions $(\mathbf{A}_p)$ and $(\mathbf{A}_c)$ that

$$\begin{aligned}&\mathbb{E}\left[|Y_t^\theta - y_0|^2\right] \\ &\leq C\int_0^t\left(1 + y_0^2 + \mathbb{E}\left[|Y_s^\theta - y_0|^2\right]\right)\mathrm{d}s + C\int_0^t\int_{\mathbb{R}}\mathbb{E}\left[|Y_s^\theta - y_0 + \gamma\xi|^2\right]\nu(\mathrm{d}\xi)\,\mathrm{d}s \\ &\leq C\int_0^t\{1 + y_0^2 + \mathbb{E}[|Y_s^\theta - y_0|^2]\}\,\mathrm{d}s,\end{aligned} \tag{85}$$

where $C > 0$ is a constant independent of (t, θ, y_0) that may be different from line to line. Then the desired result follows from Gronwall's lemma.

Proof of Lemma 6. Let $y_1, y_2 \in \mathbb{R}_+$. Without loss of generality, we assume that $y_1 \geq y_2$. Then, from assumption $(\mathbf{A}_p)$ and Lemma 2, it follows that

$$\begin{aligned}&|F^u(y_1) - F^u(y_2)| \\ &\leq \left|\lambda(y_1)\int_{\mathbb{R}}(\gamma\xi + y_1)^-\nu(\mathrm{d}\xi) - \lambda(y_2)\int_{\mathbb{R}}(\gamma\xi + y_2)^-\nu(\mathrm{d}\xi)\right| \\ &\quad + \left|\lambda(y_1)\int_{\mathbb{R}}\{u((y_1 + \gamma\xi)^+) - u(y_1)\}\nu(\mathrm{d}\xi) - \lambda(y_2)\int_{\mathbb{R}}\{u((y_2 + \gamma\xi)^+) - u(y_2)\}\nu(\mathrm{d}\xi)\right|\end{aligned}$$

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$$
\begin{aligned}
&\le \lambda(y_1)\left|\int_{\mathbb{R}}(\gamma\xi+y_1)^-\nu(\mathrm{d}\xi)-\int_{\mathbb{R}}(\gamma\xi+y_2)^-\nu(\mathrm{d}\xi)\right|+|\lambda(y_1)-\lambda(y_2)|\int_{\mathbb{R}}(\gamma\xi+y_2)^-\nu(\mathrm{d}\xi)\\
&\quad+\lambda(y_1)\left|\int_{\mathbb{R}}\{u((y_1+\gamma\xi)^+)-u((y_2+\gamma\xi)^+)\}\nu(\mathrm{d}\xi)\right|+\lambda(y_1)\left|\int_{\mathbb{R}}\{u((y_1)-u(y_2)\}\nu(\mathrm{d}\xi)\right|\\
&\quad+|\lambda(y_1)-\lambda(y_2)|\left|\int_{\mathbb{R}}\{u((y_2+\gamma\xi)^+)-u(y_2)\}\nu(\mathrm{d}\xi)\right|\\
&\le C|y_1-y_2|+y_2\,|\lambda(y_1)-\lambda(y_2)|+C\,|y_1-y_2|\left|\int_{\mathbb{R}}\{(y_2+\gamma\xi)^+-y_2\}\nu(\mathrm{d}\xi)\right|\\
&\le C|y_1-y_2|+C\,|y_1-y_2|\int_{\mathbb{R}}\gamma|\xi|\nu(\mathrm{d}\xi)\\
&\le C|y_1-y_2|,
\end{aligned}
$$

where $C>0$ is a constant independent of (y_1, y_2) that may be different from line to line. Thus, we complete the proof of the lemma.

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There were no competing interests to declare which arose during the preparation or publication process of this article.

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