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PAPER

Asymmetry in mortality volatility and its implications on index-based longevity hedging

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Abstract

Mortality volatility is crucially important to many aspects of index-based longevity hedging, including instrument pricing, hedge calibration and hedge performance evaluation. This paper sets out to develop a deeper understanding of mortality volatility and its implications on index-based longevity hedging. First, we study the potential asymmetry in mortality volatility by considering a wide range of generalised autoregressive conditional heteroskedasticity (GARCH)-type models that permit the volatility of mortality improvement to respond differently to positive and negative mortality shocks. We then investigate how the asymmetry of mortality volatility may impact index-based longevity hedging solutions by developing an extended longevity Greeks framework, which encompasses longevity Greeks for a wider range of GARCH-type models, an improved version of longevity vega, and a new longevity Greek known as "dynamic Delta". Our theoretical work is complemented by two real-data illustrations, the results of which suggest that the effectiveness of an index-based longevity hedge could be significantly impaired if the asymmetry in mortality volatility is not taken into account when the hedge is calibrated.

Keywords: GARCH-type models; Longevity Greeks; S-forwards; Value-at-Risk; The Lee–Carter model

1. Introduction

Various studies have argued that longevity risk exposures borne by pension plans and annuity providers can be shared with capital market investors through standardised mortality-linked instruments (Blake *et al.*, 2013; Biffis & Blake, 2014; Michaelson & Mulholland, 2014). In developing a market for index-based longevity risk transfers, hedgers and capital market investors face a number of technical challenges, which include instrument pricing, hedge calibration and hedge performance evaluation.

Mortality volatility, that is, the level of uncertainty surrounding the trend of mortality improvement, is crucially important to all of the mentioned technical challenges. Methods for pricing mortality-linked instruments often involve an estimate of mortality volatility. For example, the q-forward pricing formula considered by Coughlan *et al.* (2007) and Li & Hardy (2011) determines the forward mortality rate as the corresponding expected mortality rate (under the real-world probability measure) less the product of the assumed Sharpe ratio and an estimate of mortality volatility. When adjusting the real-world Cairns–Blake–Dowd model for pricing purposes, the adjustment amounts to the product of a vector of Sharpe ratios and an estimate of the volatility of the period (time-related) effects driving the evolution of mortality (Cairns *et al.*, 2006). Mortality volatility is also heavily involved in many methods for calibrating longevity hedges. One example

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is the variance-minimising method (Coughlan *et al.*, 2011; Cairns *et al.*, 2014), which leads to an optimal hedge ratio of which both the numerator and denominator depend on an estimate of mortality volatility. Lastly, when evaluating a longevity hedge using typical metrics such as reduction in variance and reduction in Value-at-Risk, an estimate of mortality volatility is needed.

While most studies in stochastic mortality modelling treat mortality volatility as a constant over time, there exists profound empirical evidence supporting the time-varying nature of mortality volatility (Lee & Miller, 2001; Gao & Hu, 2009; Giacometti *et al.*, 2012; Lin *et al.*, 2015). Recently, there has been a growing awareness of this empirical fact, and attempts have been made to extend existing stochastic mortality models to incorporate time-varying mortality volatility. One approach is to borrow methodologies used in the context of interest rate modelling; for instance, Fung *et al.* (2017) employed a discrete version of the Cox–Ingersoll–Ross model to capture the dynamics of mortality volatility. A more common approach is to utilise the family of generalised autoregressive conditional heteroskedasticity (GARCH) models, as seen in the recent contributions by Chai *et al.* (2013), Chen *et al.* (2015), Gao & Hu (2009), Giacometti *et al.* (2012), Lin *et al.* (2015), Wang & Li (2016), and Zhou & Li (2019a).

This paper sets out to obtain a deeper understanding of mortality volatility and its implications on index-based longevity hedging. Our first objective is to investigate, from a global perspective, the potential asymmetry in mortality volatility, which arises when mortality volatility is more (or less) responsive to positive mortality shocks (due to, e.g., wars and pandemics) than negative mortality shocks (due to, e.g., medical breakthroughs). To this end, we apply a range of GARCH models that permit asymmetric volatility, including the E-GARCH model (Nelson, 1991), the GJR-GARCH model (Glosten *et al.*, 1993), the N-GARCH model (Engle & Ng, 1993) and the T-GARCH model (Zakoian, 1994), to mortality data from nine countries that span four continents. While some of these models have previously been used to capture the dynamics of mortality volatility, our study represents the first attempt to compare these models with their symmetric counterpart (the GARCH model with symmetric volatility) with a goal to identify the need to incorporate asymmetric mortality volatility in modelling and applications.

In the context of equity risk, the financial impact of asymmetry in volatility (of equity returns) has been demonstrated by many researchers, including Campbell & Hentschel (1992), Koutmos & Booth (1995), and Bekaert & Wu (2000). It is therefore reasonable to conjecture that asymmetry in mortality volatility, if exists, has a financial impact on participants in the market of longevity risk transfers, and that failing to take such volatility asymmetry into account may result in erroneous index-based longevity hedging solutions.

Our second objective is to study how asymmetric mortality volatility may affect index-based longevity hedging solutions, through a concept known as longevity Greeks. Technically speaking, a longevity Greek measures the sensitivity to changes in a certain parameter in the stochastic process driving the evolution of mortality over time. Longevity Greeks have been proven to be an effective tool for calibrating index-based longevity hedges (Cairns, 2011, 2013; Luciano *et al.*, 2012, 2017; Luciano & Regis, 2014; Liu & Li, 2016; De Rosa *et al.*, 2017; Zhou & Li, 2017; Zhou & Li, 2019a). They have also been shown useful in approximating the values of mortality-linked instruments and liabilities (Cairns, 2011; Zhou & Li, 2017, 2019b), and in assessing different dimensions of longevity risk inherent in pension plans or annuity portfolios (Zhou & Li, 2019a).

In more detail, we make three technical contributions which enable us to link mortality volatility to index-based longevity hedging. First, we extend the work of Zhou & Li (2019a) to the mentioned collection of GARCH models that permit asymmetric mortality volatility. Semi-analytical expressions for three key longevity Greeks – delta, gamma and vega – are derived for each of the models.

Second, we improve the framework of Zhou & Li (2019a) to enable a more robust estimate of longevity vega. As demonstrated in section 4 of this paper, the longevity vega defined by Zhou & Li (2019a) may erroneously take a value of zero, even when volatility is actually time-varying

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and is influential to the value of the liability or instrument under consideration. This problem in turn leads to difficulties in establishing a vega hedge (a hedge that neutralises the longevity vega of the hedger's position). Drawing on the ideas of Engle & Rosenberg (1995) and Badescu *et al*. (2014), in the improved longevity Greeks framework, we treat the liability or instrument under consideration as a function of the conditional volatility at time 1 instead of that at time 0 and take partial derivatives accordingly. We show theoretically that the improved framework can mitigate the "zero problem" found in the original framework.

Third, inspired by the work of Duan (2009), we contribute a new longevity Greek known as "dynamic delta". When the period effect of mortality dynamics is modeled in a stochastic volatility framework, a shock to the period effect has the following two impacts: (1) a direct impact on the level of mortality and (2) an indirect impact on the level of mortality as other parameters in the model are changed due to the shock to the period effect. The original longevity delta can only capture the former impact, because it is defined as the first *partial* derivative with respect to the time-0 period effect, and when the derivative is taken all parameters except the time-0 period effect are held constant. In contrast, defined as the *total* derivative with respect to the time-0 period effect, dynamic delta can capture both direct and indirect impacts. The expression for dynamic delta turns out to be highly intuitive: it is the sum of the original longevity delta (which captures the direct impact) and the product of longevity vega and an adjustment term (which jointly reflect the indirect impact).

Our theoretical contributions are complemented by two real-data illustrations, one of which is based on a value hedge that focuses on the variability of the values of the hedged position at a certain future time point, the other of which is based on a cash flow hedge that focuses on the variability of all cash flows arising from the liability being hedged and the hedging instruments. In line with Solvency II capital requirements, we measure hedge effectiveness in terms of the reduction in the Value-at-Risk at a confidence level of 99.5%. Our empirical results point to the conclusion that if mortality volatility is in fact asymmetric but the asymmetry in mortality volatility is not taken into account when a longevity hedge is calibrated, then the resulting hedge effectiveness could be significantly impaired.

The remainder of the paper is organised as follows. Section 2 describes the GARCH-type models we consider and demonstrates the need of incorporating asymmetric mortality volatility. Section 3 reviews the longevity Greeks defined by Zhou & Li (2019a). Section 4 presents the improved longevity Greeks framework, which encompasses the enhanced version of longevity vega, and the longevity Greeks for a wider class of GARCH models that permit asymmetric mortality volatility. Section 5 introduces the new longevity Greek called dynamic delta. The two real-data illustrations are presented in section 6. Finally, section 7 concludes the paper.

2. Modelling Mortality Volatility

2.1 Data

We consider mortality data from the female populations of nine countries which span four continents. All of the required data are obtained from the Human Mortality Database. The age range we use is 40–89 years, which covers typical ages of annuitants/pensioners and excludes the extreme ages for which the data are extrapolated values instead of raw counts. To better satisfy the data requirements of GARCH-type model and discern the potential asymmetry in mortality volatility, we use all available data (over the chosen age range) from 1900[1]. A summary of the mortality data used is summarised in Table 1.

[1] For some populations, the available data series begins after 1900, and in this situation the beginning point of the calibration window is set to the earliest year for which data are available.

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Table 1. A summary of the mortality data used in this paper.

Continent	Country	Calibration window	Age range	Gender
North America	Canada	1921–2011	40–89	Female
	US	1933–2016	40–89	Female
Europe	Finland	1900–2015	40–89	Female
	France	1900–2015	40–89	Female
	Italy	1900–2014	40–89	Female
	Spain	1908–2014	40–89	Female
	UK	1900–2016	40–89	Female
Oceania	Australia	1921–2014	40–89	Female
Asia	Japan	1947–2016	40–89	Female

2.2 The Lee–Carter structure

Following Zhou & Li (2019a), we assume that mortality follows the Lee–Carter structure:

$$\ln m_{x,t} = a_x + b_x \kappa_t \tag{1}$$

where $m_{x,t}$ is the central death rate at age x and in year t, a_x is an age-specific parameter measuring the average level of mortality at age x, κ_t is the time-varying index (also known as the period effect) capturing the overall level of mortality in year t and b_x is an age-specific parameter reflecting the sensitivity of $m_{x,t}$ to changes in κ_t.

The dynamics of κ_t is specified as

$$\kappa_t = \kappa_{t-1} + \mu + \epsilon_t \tag{2}$$

where μ is the drift term representing the expected rate of change in κ_t, and ϵ_t is the time-t innovation which has a zero mean[2]. While most previous studies in stochastic mortality modelling assume that the variance of ϵ_t is time-invariant, we permit the (conditional) variance of ϵ_t to vary over time[3]. We use σ_t^2 to represent the conditional variance of ϵ_t.

In the rest of this section, we first identify any possible conditional heteroskedasticity, that is, the phenomenon that σ_t^2 is time-varying, for each of the nine populations. We then move to our first research objective to investigate the possibility that σ_t^2 responds asymmetrically to positive and negative shocks (innovations) prior to time t.

2.3 Conditional heteroskedasticity

Using the Poisson maximum likelihood method (Brouhns *et al.*, 2002), we obtain estimates of parameters in equation (1). Figure 1 shows the estimated series of $(\kappa_t - \kappa_{t-1})$ for each of the nine populations under consideration. For all populations except Australia and the US, the variation in $(\kappa_t - \kappa_{t-1})$ clearly varies over time. In particular, the variation seems more pronounced during certain periods, such as 1910s and 1930s. Such a pattern suggests that conditional heteroskedasticity exists.

To verify the existence of conditional heteroskedasticity, we apply Engle's ARCH test to the series of $(\kappa_t - \kappa_{t-1})^2$ for each population. The test results are reported in Table 2. Except for US and Australia, the test results suggest a rejection of the null hypothesis that $(\kappa_t - \kappa_{t-1})^2$ possesses no serial correlation, confirming the existence of conditional heteroskedasticity. We also apply the

[2] It is assumed that $\{\epsilon_t\}$ and hence $\{\kappa_t - \kappa_{t-1}\}$ possess no serial correlation. Despite this assumption, $\{\epsilon_t^2\}$ and $\{(\kappa_t - \kappa_{t-1})^2\}$ are both serially correlated if conditional heteroskedasticity exists.

[3] The condition is the information up to and including time $t-1$.

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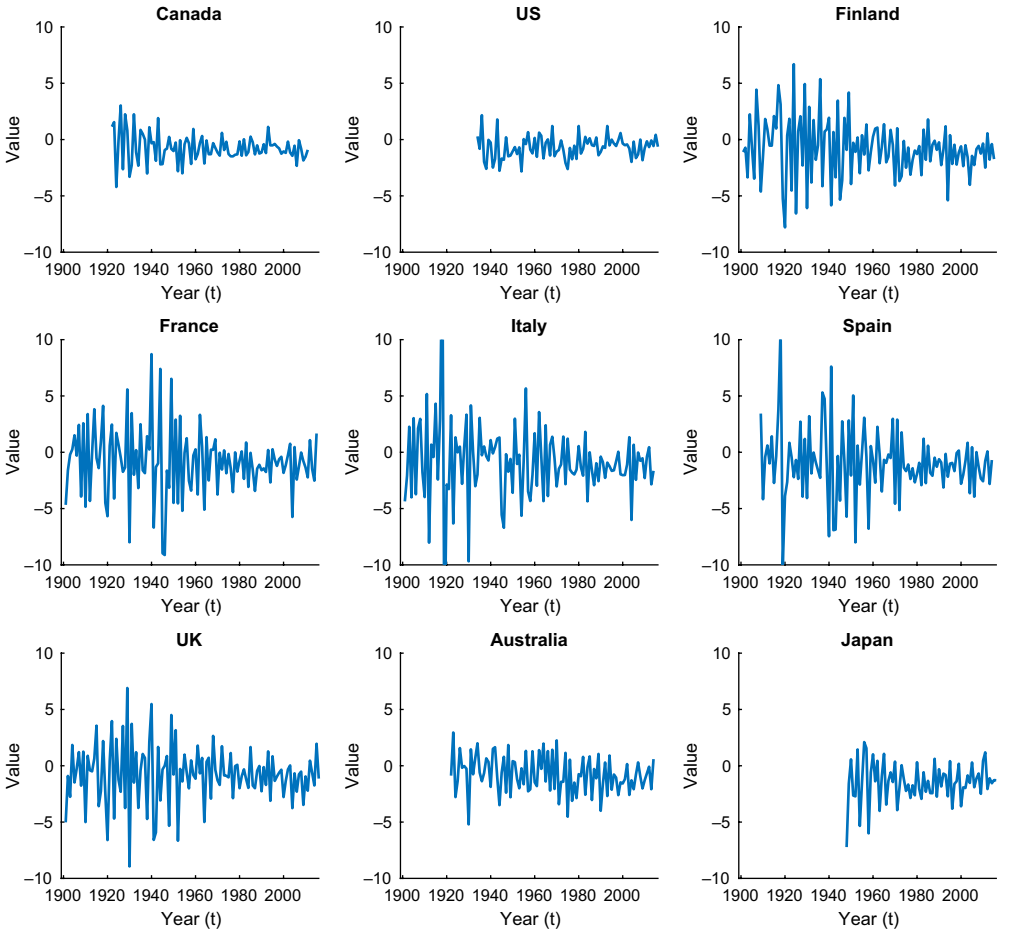


Figure 1. The estimated series of $(\kappa_t - \kappa_{t-1})$ for each of the nine populations under consideration.

Ljung–Box test to the series of $(\kappa_t - \kappa_{t-1})^2$ for each population. The test results (not shown for the sake of space) point to the same conclusion.

In our preliminary modelling work, we estimate a GARCH(1,1) model to the series of ϵ_t (i.e., the mean corrected series of $(\kappa_t - \kappa_{t-1})$) for each of the nine populations under consideration. The GARCH(1,1) model can be expressed as

$$\epsilon_t = \sigma_t \eta_t \tag{3}$$

$$\sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2 \tag{4}$$

where σ_t^2 is the conditional variance of ϵ_t, η_t-s are independent and identically distributed (i.i.d.) standard normal random variables, and ω, α and β are constant parameters. In particular, parameters α and β govern the dependence of the time-t conditional variance on the time-$(t-1)$ squared innovation and the time-$(t-1)$ conditional variance, respectively.

Having fitted the GARCH(1,1) model, the inferred values of σ_t^2 over the calibration window are extracted and are reported in Figure 2. The existence of conditional heteroskedasticity is further supported by the fact that the patterns of the inferred values of σ_t^2 over the calibration window are far from being flat for all of the populations considered, except US and Australia. We then

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Table 2. The test statistic and the *p*-value (in parentheses) of the Engle's ARCH test (at lag 1 to 5) applied to the series of $(\kappa_t - \kappa_{t-1})$ for the each of the nine populations under consideration.

Lag	1	2	3	4	5
Canada	3.9186	34.6104	38.2005	41.9446	39.9087
	(0.0478)	(< 0.0001)	(< 0.0001)	(< 0.0001)	(< 0.0001)
US	0.6037	1.8717	3.5025	4.8071	10.5270
	(0.4372)	(0.3923)	(0.3204)	(0.3077)	(0.0616)
Finland	11.9455	11.7071	12.7023	18.3154	27.5272
	(0.0005)	(0.0029)	(0.0053)	(0.0011)	(< 0.0001)
France	17.9267	18.7987	19.1268	28.9292	30.6836
	(< 0.0001)	(0.0001)	(0.0003)	(< 0.0001)	(< 0.0001)
Italy	25.0178	33.2536	38.0725	39.3564	39.6428
	(< 0.0001)	(< 0.0001)	(< 0.0001)	(< 0.0001)	(< 0.0001)
Spain	24.3091	27.2442	28.3534	28.2829	28.0881
	(< 0.0001)	(< 0.0001)	(< 0.0001)	(< 0.0001)	(< 0.0001)
UK	24.4022	25.0509	24.8998	24.8032	24.8135
	(< 0.0001)	(< 0.0001)	(< 0.0001)	(0.0001)	(0.0002)
Australia	0.0976	0.7170	0.7759	6.3877	7.5213
	(0.7547)	(0.6987)	(0.8552)	(0.1720)	(0.1847)
Japan	0.6633	8.1174	10.0398	10.6085	15.1785
	(0.4154)	(0.0173)	(0.0182)	(0.0313)	(0.0096)

apply Engle's ARCH test to the series of squared standardised residuals[4] for each population. Reported in Table 3, the test results indicate no significant serial correlations in the squared standardised residuals, suggesting that conditional heteroskedasticity is adequately captured by the GARCH(1,1) models. We also apply the Ljung–Box test to the series of squared standardised residuals for each population. The test results (not shown for the sake of space) point to the same conclusion.

While the GARCH(1,1) model appears to have captured conditional heteroskedasticity adequately, it does not capture any asymmetry in volatility. It is clear from equation (4) that a positive shock (innovation) at time $t-1$, say $|\epsilon^*_{t-1}|$, and a negative shock of the same magnitude, that is, $-|\epsilon^*_{t-1}|$, have exactly the same impact on the time-t conditional variance.

2.4 Asymmetry in mortality volatility

To investigate the potential asymmetry in mortality volatility, a wider range of GARCH-type models are considered. The additional models include the E-GARCH model, the GJR-GARCH model, the N-GARCH model and the T-GARCH model. These models differ from the GARCH model in the way in which the dynamics of the conditional variance σ_t^2 is specified. We consider an order of (1,1) for all of the additional models, because the results in the previous subsection suggest that this order can adequately capture the serial correlation in the conditional variances.

Below we describe, for each of the additional GARCH-type models, the specification of the dynamics of σ_t^2 and the way in which the asymmetry in volatility is captured. In all of the models, ω, α, β and γ are constant parameters, and γ is the parameter controlling the asymmetry in volatility.

[4] The standardised residual at time t is defined as the ratio of the residual at time-t to the inferred value of σ_t.

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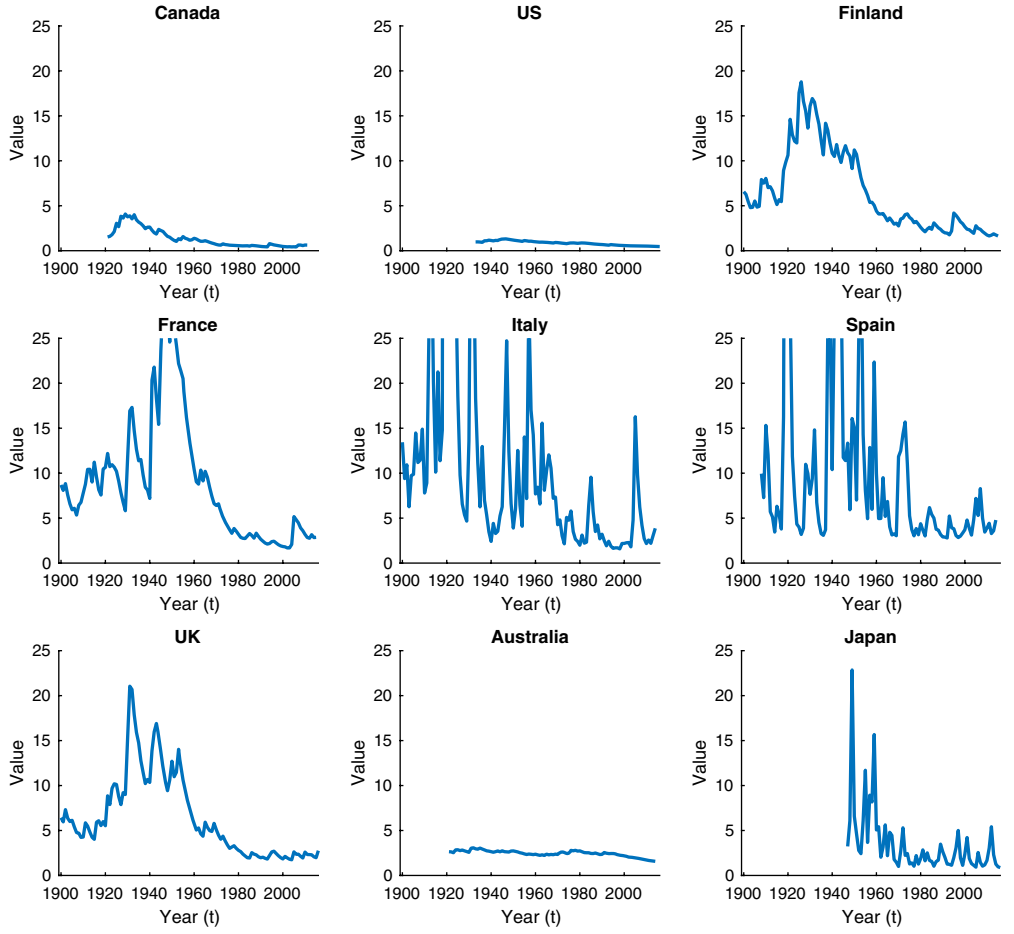


Figure 2. The inferred values of σ_t^2 derived from the estimated GARCH(1,1) models for the nine populations under consideration.

- **E-GARCH**
  The E-GARCH(1,1) model assumes that

$$\ln(\sigma_t^2) = \omega + \gamma\eta_{t-1} + \alpha\left(|\eta_{t-1}| - \sqrt{\frac{2}{\pi}}\right) + \beta\ln\sigma_{t-1}^2 \tag{5}$$

  where $\eta_{t-1} = \epsilon_{t-1}/\sigma_{t-1}$. A positive time-$(t-1)$ innovation with a magnitude of $|\epsilon_{t-1}^*|$ affects $\ln(\sigma_t^2)$ by $\gamma|\epsilon_{t-1}^*|/\sigma_{t-1} + \alpha(|\epsilon_{t-1}^*|/\sigma_{t-1} - \sqrt{2/\pi})$, whereas an otherwise identical negative innovation affects $\ln(\sigma_t^2)$ by $-\gamma|\epsilon_{t-1}^*|/\sigma_{t-1} + \alpha(|\epsilon_{t-1}^*|/\sigma_{t-1} - \sqrt{2/\pi})$.

- **GJR-GARCH**
  The GJR-GARCH(1,1) model assumes that

$$\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 + \gamma\epsilon_{t-1}^2\mathcal{I}_{\{\epsilon_{t-1}<0\}} + \beta\sigma_{t-1}^2 \tag{6}$$

  where $\mathcal{I}_A$ is an indicator function which equals 1 if event A holds true and 0 otherwise. A positive time-$(t-1)$ innovation with a magnitude of $|\epsilon_{t-1}^*|$ affects σ_t^2 by $\alpha(\epsilon_{t-1}^*)^2$, whereas an otherwise identical negative innovation affects σ_t^2 by $(\alpha+\gamma)(\epsilon_{t-1}^*)^2$.

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Table 3. The test statistic and *p*-value (in parentheses) of the Engle's ARCH test (at lag 1 to 5) applied to the squared standardised residuals from the GARCH(1,1) for each of the nine populations under consideration.

Lag	1	2	3	4	5
Canada	0.3250	1.9275	2.0685	2.7649	2.6145
	(0.5686)	(0.3815)	(0.5583)	(0.5979)	(0.7592)
US	0.0613	0.1474	0.4116	3.5730	8.4085
	(0.2597)	(0.2280)	(0.2778)	(0.3262)	(0.3748)
Finland	0.4489	0.6515	0.6034	1.1397	1.0648
	(0.5029)	(0.7220)	(0.8956)	(0.8879)	(0.9572)
France	1.1372	1.8028	3.4347	4.1475	4.0885
	(0.2863)	(0.4060)	(0.3293)	(0.3864)	(0.5367)
Italy	0.1206	0.6007	0.6155	1.2811	1.3904
	(0.7284)	(0.7406)	(0.8929)	(0.8646)	(0.9254)
Spain	1.2405	2.1881	2.9848	2.9233	3.3817
	(0.2654)	(0.3349)	(0.3940)	(0.5707)	(0.6414)
UK	0.2054	0.2797	0.9665	1.6838	1.8774
	(0.6504)	(0.8695)	(0.8094)	(0.7937)	(0.8658)
Australia	0.3451	1.1161	1.1881	5.9202	5.8099
	(0.5569)	(0.5723)	(0.7558)	(0.2052)	(0.3252)
Japan	0.0000	0.7354	2.4198	4.4147	6.1261
	(0.9949)	(0.6923)	(0.4900)	(0.3528)	(0.2941)

- **N-GARCH**
  The N-GARCH(1,1) model assumes that

$$\sigma_t^2 = \omega + \alpha\sigma_{t-1}^2(\eta_{t-1} - \gamma)^2 + \beta\sigma_{t-1}^2 \tag{7}$$

  A positive time-$(t-1)$ innovation with a magnitude of $|\epsilon_{t-1}^*|$ affects σ_t^2 by $\alpha\sigma_{t-1}^2(|\epsilon_{t-1}^*|/\sigma_{t-1} - \gamma)^2$, whereas an otherwise identical negative innovation affects σ_t^2 by $\alpha\sigma_{t-1}^2(-|\epsilon_{t-1}^*|/\sigma_{t-1} - \gamma)^2$.

- **T-GARCH**
  The T-GARCH(1,1) model assumes that

$$\sigma_t = \omega + \alpha(|\epsilon_{t-1}| - \gamma\epsilon_{t-1}) + \beta\sigma_{t-1} \tag{8}$$

  A positive time-$(t-1)$ innovation with a magnitude of $|\epsilon_{t-1}^*|$ affects σ_t by $\alpha(1-\gamma)|\epsilon_{t-1}^*|$, whereas an otherwise identical negative innovation affects σ_t by $\alpha(1+\gamma)|\epsilon_{t-1}^*|$.

We apply the above-mentioned models to all of the nine populations under consideration. To compare the performances of these models as well as the GARCH model and the constant volatility assumption, we consider the Bayesian information criterion (BIC), which is defined such that a lower BIC value indicates a better goodness-of-fit with the number of model parameters being taken into account. The results are reported in Table 4.

In line with the results in section 2.3, the constant volatility assumption produces the lowest BIC values for US and Australia. For all but one of the remaining populations, the best performing model is either the E-GARCH, GJR-GARCH or T-GARCH model, suggesting that there exists asymmetry in mortality volatility. We further examine the estimates of parameter γ for these populations (Table 5). It is found that all of the six estimates of γ are significant at any reasonable level of significance, further confirming the existence of asymmetry in mortality volatility and the need for capturing it. In addition to the BIC, the Akaike information criterion and the Hannan–Quinn

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Table 4. The BIC values resulting from the constant volatility assumption, the GARCH model and four GARCH-type models that incorporate asymmetry in volatility, fitted to the mean-corrected series of $(\kappa_t - \kappa_{t-1})$ for the nine populations under consideration.

	Constant	GARCH	E-GARCH	GJR-GARCH	N-GARCH	T-GARCH
Canada	3.3762	3.1447	3.2016	3.1925	3.1939	3.2452
US	2.9238	2.9668	3.0231	3.0193	3.0199	3.0731
Finland	4.8028	4.6416	4.6211	4.6277	4.6179	4.6461
France	5.0807	4.9103	4.8223	4.8099	4.8088	4.8470
Italy	5.4634	5.0997	5.0547	5.1396	5.0597	5.0550
Spain	5.1629	4.9746	4.8374	4.8123	4.8189	4.9095
UK	5.3489	5.2041	5.1991	5.1961	5.2085	5.1967
Australia	3.9126	3.9971	3.9588	3.9478	3.9693	4.0167
Japan	4.0019	4.0084	3.9396	3.9624	4.0642	4.2027

Table 5. Estimates (and their standard errors and t-values) of parameter γ in the best performing models for Finland, France, Italy, Spain, UK and Japan.

Country	Estimate	Standard error	t-Value
Finland	−3.86888	1.80264	−2.1462
France	−1.96910	0.75138	−2.6206
Italy	0.42944	0.10635	4.0380
Spain	−0.39775	0.00841	−47.3160
UK	−0.41025	0.20227	−2.0282
Japan	0.50631	0.10614	4.7703

information criterion are considered. These two criteria also suggest the existence of asymmetry in mortality volatility for most of the populations considered.

The meaning of the sign of parameter γ depends on how the asymmetric GARCH model is constructed. For the E-GARCH model, a positive γ means that the impact of a positive shock on volatility is more prominent than a negative shock, whereas for the GJR-GARCH, N-GARCH and T-GARCH models, the opposite is true. The estimates of γ reported in Table 5 thus suggest that mortality volatility is more responsive to positive shocks than negative shocks. This property is conceivable, because in the context of mortality modelling, positive shocks that are due to events such as wars and pandemics typically have a more significant and immediate impact compared to negative shocks that arise from medical breakthroughs.

3. A Review of Zhou & Li (2019a)

We study the impact of asymmetry in mortality volatility on index-based longevity hedging through various extensions of existing longevity Greeks. These extensions draw on the work of Zhou & Li (2019a), which we now summarise.

3.1 The key building block

Denote by $q_{x,t}$ the probability that an individual who has survived to age x at time $t-1$ dies between time $t-1$ and t. We have

$$S_{x,t}(T) = \prod_{s=1}^{T} (1 - q_{x+s-1,t+s})$$

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being the *ex post* probability that an individual who has survived to age x at time t would have survived to time $t+T$ for $T=1,2,\ldots$. Assuming a constant force of mortality between consecutive integer ages, which in turn implies that $q_{x,t}=1-\exp(-m_{x,t})$, we can express $S_{x,t}(T)$ as follows:

$$\begin{aligned} S_{x,t}(T) &= \prod_{s=1}^{T} \exp(-m_{x+s-1,t+s}) \\ &= \exp\left(-\sum_{s=1}^{T} \exp(a_{x+s-1}+b_{x+s-1}\kappa_{t+s})\right) \qquad (9) \\ &= \exp\left(-\sum_{s=1}^{T} \exp(Y_{x,t}(s))\right) \\ &= \exp(-W_{x,t}(T)) \qquad (10) \end{aligned}$$

where

$$Y_{x,t}(s) = a_{x+s-1} + b_{x+s-1}\kappa_{t+s} \qquad (11)$$

and

$$W_{x,t}(T) = \sum_{s=1}^{T} \exp(Y_{x,t}(s)) \qquad (12)$$

are defined to make the expressions of longevity Greeks more compact.

We set time 0 to the end point of the calibration window to which the model is fitted. Using equations (2) and (3), we obtain

$$\kappa_t = \kappa_0 + t\mu + \sum_{s=1}^{t} \sigma_s \eta_s, \quad t \geq 1 \qquad (13)$$

so that

$$S_{x,t}(T) = \exp\left(-\sum_{s=1}^{T} \exp\left(a_{x+s-1} + b_{x+s-1}\left(\kappa_0 + (t+s)\mu + \sum_{u=1}^{t+s} \sigma_u \eta_u\right)\right)\right) \qquad (14)$$

Zhou & Li (2019a) assume a GARCH(1,1) model, which implies that

$$\sigma_s^2 = \begin{cases} \omega\left(1+\sum_{u=1}^{s-1}\prod_{v=1}^{u}(\alpha\eta_{s-v}^2+\beta)\right) + (\alpha\epsilon_0^2+\beta\sigma_0^2)\prod_{v=1}^{s-1}(\alpha\eta_{s-v}^2+\beta) & \text{if } s \geq 2 \\ \omega + \alpha\epsilon_0^2 + \beta\sigma_0^2 & \text{if } s = 1 \end{cases} \qquad (15)$$

Let $\mathcal{F}_t$ be the information about the evolution of mortality up to and including time t. It follows from equations (14) and (15) that given $\mathcal{F}_0$, $S_{x,t}(T)$ depends on κ_0 (the time-0 value of the period effect), σ_0^2 (the time-0 value of the conditional volatility) and a sequence of i.i.d. standard normal random variables $\{\eta_s; s=1,\ldots,t+T\}$. It is clear that $S_{x,t}(T)$ given $\mathcal{F}_0$ is a random variable due to its dependence on $\{\eta_s; s=1,\ldots,t+T\}$.

Finally, we arrive at the following key building block of the longevity Greeks framework of Zhou & Li (2019a):

$$p_{x,t}(T,\kappa_0,\sigma_0^2) := \mathrm{E}[S_{x,t}(T)|\mathcal{F}_0]$$

which can be interpreted to mean the probability for an individual aged x at time t to survive to time $t+T$, given the information about the evolution of mortality up to and including time 0.

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3.2 Longevity Greeks for $p_{x,t}(T, \kappa_0, \sigma_0^2)$

The longevity Greeks for $p_{x,t}(T, \kappa_0, \sigma_0^2)$ are obtained by taking the partial derivatives of $p_{x,t}(T, \kappa_0, \sigma_0^2)$ with respect to either κ_0 or σ_0^2.

The longevity delta for $p_{x,t}(T, \kappa_0, \sigma_0^2)$ is

$$\frac{\partial p_{x,t}(T, \kappa_0, \sigma_0^2)}{\partial \kappa_0} = -\sum_{s=1}^{T} b_{x+s-1} \mathrm{E}[\exp(Y_{x,t}(s) - W_{x,t}(T)) | \mathcal{F}_0]$$

a quantity which measures the first-order sensitivity of $p_{x,t}(T, \kappa_0, \sigma_0^2)$ to changes in κ_0.

The longevity gamma for $p_{x,t}(T, \kappa_0, \sigma_0^2)$ is

$$\frac{\partial^2 p_{x,t}(T, \kappa_0, \sigma_0^2)}{\partial \kappa_0^2} = \mathrm{E}\left[\exp(-W_{x,t}(T)) \left(\left(\sum_{s=1}^{T} b_{x+s-1} \exp(Y_{x,t}(s))\right)^2 - \sum_{s=1}^{T} b_{x+s-1}^2 \exp(Y_{x,t}(s))\right) | \mathcal{F}_0\right]$$

a quantity which represents the second-order sensitivity of $p_{x,t}(T, \kappa_0, \sigma_0^2)$ to changes in κ_0.

Finally, the longevity vega for $p_{x,t}(T, \kappa_0, \sigma_0^2)$ is defined as

$$\frac{\partial p_{x,t}(T, \kappa_0, \sigma_0^2)}{\partial \sigma_0^2} = -\sum_{s=1}^{T} b_{x+s-1} \mathrm{E}\left[\exp(Y_{x,t}(s) - W_{x,t}(T)) \left(\frac{\partial \kappa_{t+s}}{\partial \sigma_0^2}\right) | \mathcal{F}_0\right] \tag{16}$$

where

$$\frac{\partial \kappa_{t+s}}{\partial \sigma_0^2} = \sum_{u=1}^{t+s} \frac{\eta_u}{2\sigma_u} \frac{\partial \sigma_u^2}{\partial \sigma_0^2} \tag{17}$$

and

$$\frac{\partial \sigma_t^2}{\partial \sigma_0^2} = \begin{cases} \beta \prod_{v=1}^{t-1} (\alpha \eta_{t-v}^2 + \beta) & \text{if } t \geq 2 \\ \beta & \text{if } t = 1 \end{cases} \tag{18}$$

This longevity Greek measures the first-order sensitivity of $p_{x,t}(T, \kappa_0, \sigma_0^2)$ to changes in σ_0^2. A full derivation of the above formulas can be found in the appendices of Zhou & Li (2019a).

3.3 Longevity Greeks for the values of liabilities and hedging instruments

The values of annuity/liabilities and hedging instruments such as q-forwards and S-forwards can be expressed as a linear combination of $p_{x,t}(T, \kappa_0, \sigma_0^2)$ for different values of x, t and T. As such, the longevity Greeks for the values of annuity/liabilities and hedging instruments can be computed easily as a linear combination of the longevity Greeks of $p_{x,t}(T, \kappa_0, \sigma_0^2)$ for different values of x, t and T.

4. The Enhanced Framework of Longevity Greeks

In this section, we improve the framework of Zhou & Li (2019a) to avoid a technical problem that may happen in practice. We also expand the framework to incorporate the GARCH-type models that permit asymmetry in volatility.

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4.1 Motivations

One problem of the framework of Zhou & Li (2019a) is that it produces a longevity vega that is highly sensitive to parameter β in the GARCH(1,1) model, a parameter that measures the extent of serial dependence in $\{\sigma_t^2\}$. It can be seen from equation (18) that the longevity vega of the value of a liability/instrument must be zero if parameter β equals zero (which happens when the fitted model is an ARCH(1) model), even when volatility is not constant (parameter α is not zero) and the value of the liability/instrument is in fact sensitive to changes in volatility. This zero longevity vega not only leads to a misleading conclusion concerning the liability/instrument's exposure to mortality volatility risk, but also causes difficulties when establishing a vega hedge (i.e., a hedge that neutralises the longevity vega of the hedge's portfolio) due to a division-by-zero problem.

One possible way to circumvent this problem is to create a new Greek that measures the sensitivity to ϵ_0^2 (the squared innovation at time 0). Calculated as a partial derivative of $\mathrm{E}[S_{x,t}(T)|\mathcal{F}_0]$ with respect to ϵ_0^2, this new longevity Greek can be expressed as

$$-\sum_{s=1}^{T} b_{x+s-1}\mathrm{E}\left[\exp\left(Y_{x,t}(s)-W_{x,t}(T)\right)\left(\frac{\partial \kappa_{t+s}}{\partial \epsilon_0^2}\right)|\mathcal{F}_0\right]$$

where

$$\frac{\partial \sigma_t^2}{\partial \epsilon_0^2}=\begin{cases}\alpha\prod_{v=1}^{t-1}\left(\alpha\eta_{t-v}^2+\beta\right) & \text{if } t\geq 2\\ \alpha & \text{if } t=1\end{cases} \tag{19}$$

However, this solution opens up another problem. As implied by equation (19), this new longevity Greek is spuriously zero whenever parameter α (which measures the serial dependence in $\{\epsilon_t^2\}$) in the GARCH(1,1) model is zero even if volatility is in fact not constant (parameter β is non-zero).

Instead of introducing another longevity Greek, we propose to adapt the framework of Zhou & Li (2019a) by considering $\mathrm{E}[S_{x,t}(T)|\mathcal{F}_0]$ as a function of κ_0 and σ_1 rather than κ_0 and σ_0 and redefining longevity vega accordingly.

4.2 Redefining the building block

In developing the improved framework, the first step is to obtain recursive expressions of σ_u^2, $u=2,3,\ldots$, in terms of σ_1^2, using equations (4)–(8). The expressions of σ_u^2, $u=2,3,\ldots$, in terms of σ_1^2 for the GARCH model as well as the four GARCH-type models that permit asymmetry in volatility are summarised in Table 6.

Then, using the expressions in Table 6 and equation (14), we can express $S_{x,t}(T)$ in terms of κ_0 and σ_1 for each of the GARCH-type models we consider. Finally, noting that σ_1^2 is non-random given $\mathcal{F}_0$, we obtain

$$p_{x,t}(T,\kappa_0,\sigma_1^2) \coloneqq \mathrm{E}[S_{x,t}(T)|\mathcal{F}_0] \tag{20}$$

as the key building block of the improved longevity Greeks framework.

4.3 Longevity vega for $p_{x,t}(T,\kappa_0,\sigma_1^2)$

In the improved longevity Greeks framework, the longevity vega for $p_{x,t}(T,\kappa_0,\sigma_1^2)$ is obtained by taking the first-order partial derivative of $p_{x,t}(T,\kappa_0,\sigma_1^2)$ with respect to σ_1^2:

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Table 6. Expression of σ_u^2, $u = 2, 3, \ldots$, in terms of σ_1^2 for all GARCH-type models under consideration.

Model	Expression of σ_u^2 for $u \geq 2$ in terms of σ_1^2
GARCH	$\sigma_u^2 = \begin{cases} \omega\left(1+\sum_{t=1}^{u-2}\prod_{v=1}^{t}(\alpha\eta_{u-v}^2+\beta)\right)+\sigma_1^2\prod_{v=1}^{u-1}(\alpha\eta_{u-v}^2+\beta) & \text{if } u\geq 3 \\ \omega+\sigma_1^2(\alpha\eta_1^2+\beta) & \text{if } u=2 \end{cases}$
E-GARCH	$\ln(\sigma_u^2) = \sum_{t=1}^{u-1}\beta^{t-1}\left(\omega+\gamma\eta_{u-t}+\alpha\left(\lvert\eta_{u-t}\rvert-\sqrt{\frac{2}{\pi}}\right)\right)+\beta^{u-1}\ln(\sigma_1^2)$
GJR-GARCH	$\sigma_u^2 = \begin{cases} \omega\left(1+\sum_{t=1}^{u-2}\prod_{v=1}^{t}(\alpha\eta_{u-v}^2+\gamma\eta_{u-v}^2\mathcal{I}_{\{\eta_{u-v}<0\}}+\beta)\right) & \\ +\sigma_1^2\prod_{v=1}^{u-1}(\alpha\eta_{u-v}^2+\gamma\eta_{u-v}^2\mathcal{I}_{\{\eta_{u-v}<0\}}+\beta) & \text{if } u\geq 3 \\ \omega+\sigma_1^2(\alpha\eta_1^2+\gamma\eta_1^2\mathcal{I}_{\{\eta_1<0\}}+\beta) & \text{if } u=2 \end{cases}$
N-GARCH	$\sigma_u^2 = \begin{cases} \omega\left(1+\sum_{t=1}^{u-2}\prod_{v=1}^{t}(\alpha(\eta_{u-v}-\gamma)^2+\beta)\right) & \\ +\sigma_1^2\prod_{v=1}^{u-1}(\alpha(\eta_{u-v}-\gamma)^2+\beta) & \text{if } u\geq 3 \\ \omega+\sigma_1^2(\alpha(\eta_1-\gamma)^2+\beta) & \text{if } u=2 \end{cases}$
T-GARCH	$\sigma_u = \begin{cases} \omega\left(1+\sum_{t=1}^{u-2}\prod_{v=1}^{t}(\alpha(\lvert\eta_{u-v}\rvert-\gamma\eta_{u-v})+\beta)\right) & \\ +\sigma_1\prod_{v=1}^{u-1}(\alpha(\lvert\eta_{u-v}\rvert-\gamma\eta_{u-v})+\beta) & \text{if } u\geq 3 \\ \omega+\sigma_1(\alpha(\lvert\eta_1\rvert-\gamma\eta_1)+\beta) & \text{if } u=2 \end{cases}$

$$
\begin{aligned}
V_{x,t}(T) &:= \frac{\partial p_{x,t}(T,\kappa_0,\sigma_1^2)}{\partial\sigma_1^2} = \mathrm{E}\left[\frac{\partial S_{x,t}(T)}{\partial\sigma_1^2}\middle|\mathcal{F}_0\right] \\
&= \mathrm{E}\left[\exp\left(-W_{x,t}(T)\right)\left(-\sum_{s=1}^{T}\exp\left(Y_{x,t}(s)\right)\left(b_{x+s-1}\sum_{u=1}^{t+s}\frac{\partial\sigma_u\eta_u}{\partial\sigma_1^2}\right)\right)\middle|\mathcal{F}_0\right] \\
&= -\sum_{s=1}^{T}b_{x+s-1}\mathrm{E}\left[\exp\left(Y_{x,t}(s)-W_{x,t}(T)\right)\left(\sum_{u=1}^{t+s}\frac{\eta_u}{2\sigma_u}\frac{\partial\sigma_u^2}{\partial\sigma_1^2}\right)\middle|\mathcal{F}_0\right] \qquad (21)
\end{aligned}
$$

where $\frac{\partial\sigma_u^2}{\partial\sigma_1^2} = 1$ for $u = 1$, and $\frac{\partial\sigma_u^2}{\partial\sigma_1^2}$ for $u \geq 2$ depends on the chosen GARCH-type model.

For the GARCH, GJR-GARCH and N-GARCH models, the expression of $\frac{\partial\sigma_u^2}{\partial\sigma_1^2}$ for $u \geq 2$ can be easily derived from the corresponding expressions of σ_u^2 for $u \geq 2$ in Table 6. For the E-GARCH model, we use the chain rule to get

$$
\frac{\partial\sigma_t^2}{\partial\sigma_1^2} = \frac{\partial\sigma_t^2}{\partial\ln(\sigma_t^2)}\frac{\partial\ln(\sigma_t^2)}{\partial\ln(\sigma_1^2)}\frac{\partial\ln(\sigma_1^2)}{\partial\sigma_1^2} = \frac{\sigma_t^2}{\sigma_1^2}\frac{\partial\ln(\sigma_t^2)}{\partial\ln(\sigma_1^2)}
$$

where $\frac{\partial\ln(\sigma_u^2)}{\partial\ln(\sigma_1^2)} = \beta^{u-1}$ for $u \geq 2$ can be obtained straightforwardly from the expression of σ_u^2 for $u \geq 2$ for the E-GARCH model as shown in Table 6. Lastly, for the T-GARCH model, the chain rule is used to get

$$
\frac{\partial\sigma_u^2}{\partial\sigma_1^2} = \frac{\partial\sigma_u^2}{\partial\sigma_u}\frac{\partial\sigma_u}{\partial\sigma_1}\frac{\partial\sigma_1}{\partial\sigma_1^2} = \frac{\sigma_u}{\sigma_1}\frac{\partial\sigma_u}{\partial\sigma_1}
$$

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Table 7. Expressions of $\frac{\partial \sigma_u^2}{\partial \sigma_1^2}$, $u = 2, 3, \ldots$, for all GARCH-type models under consideration.

Model	Expression of $\frac{\partial \sigma_u^2}{\partial \sigma_1^2}$ for $u \geq 2$
GARCH	$\frac{\partial \sigma_u^2}{\partial \sigma_1^2} = \prod_{v=1}^{u-1} (\alpha \eta_{u-v}^2 + \beta)$
E-GARCH	$\frac{\partial \sigma_u^2}{\partial \sigma_1^2} = \beta^{u-1} \frac{\sigma_u^2}{\sigma_1^2}$
GJR-GARCH	$\frac{\partial \sigma_u^2}{\partial \sigma_1^2} = \prod_{v=1}^{u-1} (\alpha \eta_{u-v}^2 + \gamma \eta_{u-v}^2 \mathcal{I}_{\{\eta_{u-v}<0\}} + \beta)$
N-GARCH	$\frac{\partial \sigma_u^2}{\partial \sigma_1^2} = \prod_{v=1}^{u-1} (\alpha(\eta_{u-v} - \gamma)^2 + \beta)$
T-GARCH	$\frac{\partial \sigma_u^2}{\partial \sigma_1^2} = \frac{\sigma_u}{\sigma_1} \left(\prod_{v=1}^{u-1} (\alpha(\lvert \eta_{u-v} \rvert - \gamma \eta_{u-v}) + \beta) \right)$

where $\frac{\partial \sigma_u}{\partial \sigma_1} = \prod_{v=1}^{u-1} (\alpha(|\eta_{u-v}| - \gamma \eta_{u-v}) + \beta)$ for $u \geq 2$ can be derived from the expression of σ_u^2 for $u \geq 2$ for the T-GARCH model as shown in Table 6. The resulting expressions of $\frac{\partial \sigma_u^2}{\partial \sigma_1^2}$ for $u \geq 2$ for all GARCH-type models under consideration are presented in Table 7.

The expressions of both $Y_{x,t}(s)$ and $W_{x,t}(T)$ in equation (21) also depend on the chosen GARCH-type model. In particular, they depend on σ_u^2 for $u \geq 2$ (which in turn depends on the GARCH-type model chosen) through equations (11), (12) and (13). The calculation of $Y_{x,t}(s)$ and $W_{x,t}(T)$ is enabled by the expressions provided in Table 6.

Let us revisit the longevity vega under the GARCH model assumption. From Table 7 and equation (21), we observe that in the improved framework the longevity vega under the GARCH model assumption is non-zero unless *both* α and β are zero (which happens only when σ_t^2 is indeed a constant), suggesting that the spurious zero vega problem found in the original longevity Greeks framework of Zhou & Li (2019a) is mitigated. This desirable property also applies to all other GARCH-type models under consideration except the E-GARCH, for which the longevity vega must be zero if $\beta = 0$ regardless of whether or not α is zero.

4.4 Longevity delta and gamma for $p_{x,t}(T, \kappa_0, \sigma_1^2)$

As the partial derivatives of $p_{x,t}(T, \kappa_0, \sigma_1^2)$ and $p_{x,t}(T, \kappa_0, \sigma_0^2)$ with respect to κ_0 are identical, longevity delta and gamma in the improved framework remain the same as those in the original framework of Zhou & Li (2019a). They are reported below for completeness.

The longevity delta for $p_{x,t}(T, \kappa_0, \sigma_1^2)$ is

$$\Delta_{x,t}(T) := \frac{\partial p_{x,t}(T, \kappa_0, \sigma_1^2)}{\partial \kappa_0} = -\sum_{s=1}^{T} b_{x+s-1} \mathrm{E}[\exp(Y_{x,t}(s) - W_{x,t}(T)) | \mathcal{F}_0] \tag{22}$$

and the longevity gamma for $p_{x,t}(T, \kappa_0, \sigma_1^2)$ is

$$\begin{aligned} \Gamma_{x,t}(T) &:= \frac{\partial^2 p_{x,t}(T, \kappa_0, \sigma_1^2)}{\partial \kappa_0^2} \\ &= \mathrm{E}\left[\exp(-W_{x,t}(T)) \left(\left(\sum_{s=1}^{T} b_{x+s-1} \exp(Y_{x,t}(s)) \right)^2 - \sum_{s=1}^{T} b_{x+s-1}^2 \exp(Y_{x,t}(s)) \right) | \mathcal{F}_0 \right] \end{aligned} \tag{23}$$

As discussed in the previous subsection, $Y_{x,t}(s)$ and $W_{x,t}(T)$ in the above expressions depend on the chosen GARCH-type model.

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4.5 Additional remarks

We compute the values of $\Delta_{x,t}(T)$, $\Gamma_{x,t}(T)$ and $V_{x,t}(T)$ with simulations. In particular, using N simulated paths of $\{\eta_s; s = 1, \ldots, t+T\}$, we calculate N realisations of $Y_{x,t}(s)|\mathcal{F}_0$, $W_{x,t}(T)|\mathcal{F}_0$ and $\frac{\partial\sigma_u^2}{\partial\sigma_1^2}|\mathcal{F}_0$ for $u \geq 2$, with which the expectations in equations (1), (22) and (23) can be evaluated.

The values of annuity/pension liabilities and typical mortality-linked instruments can be expressed as linear combinations of $p_{x,t}(T, \kappa_0, \sigma_1^2)$ for different values of x, t and T. As such, their longevity Greeks can be expressed as linear combinations of $\Delta_{x,t}(T)$, $\Gamma_{x,t}(T)$ or $V_{x,t}(T)$ for different values of x, t and T. The calculation of longevity Greeks for the values of liabilities and instruments is illustrated in sections 6.2 and 6.3.

5. A New Longevity Greek: Dynamic Delta

When a GARCH-type model is used, a shock to κ_0 has both direct and indirect impacts on $p_{x,t}(T, \kappa_0, \sigma_1^2)$:

1. Direct impact
   It is clear from equation (14) that a shock to κ_0 directly affects the distribution of $S_{x,t}(T)$, and hence the value of $p_{x,t}(T, \kappa_0, \sigma_1^2)$.
2. Indirect impact
   A shock to κ_0 (represented by the time-0 innovation ϵ_0) alters the value of σ_1^2 through the conditional variance dynamics, thereby leading to changes in the values of σ_u^2 for all $u \geq 2$. These changes affect the distribution of $S_{x,t}(T)$ through equation (14), and thus the value of $p_{x,t}(T, \kappa_0, \sigma_1^2)$.

Longevity delta captures only the direct impact but not the indirect impact, because it is defined as a *partial* derivative which is taken with respect to κ_0 while σ_1^2 is held constant. This limitation makes longevity delta an inadequate measure of the sensitivity to changes in κ_0.

To mitigate this limitation, we propose a new longevity Greek known as "dynamic delta". This new longevity Greek is derived on the basis of a *total* derivative, in which the interrelationship between κ_0 and σ_1^2 is captured through the chain rule. More specifically, the dynamic delta for $p_{x,t}(T, \kappa_0, \sigma_1^2)$ is defined as the first-order *total* derivative of $p_{x,t}(T, \kappa_0, \sigma_1^2)$ with respect to κ_0:

$$\begin{aligned}\Lambda_{x,t}(T) &:= \frac{dp_{x,t}(T, \kappa_0, \sigma_1^2)}{d\kappa_0} \\ &= \frac{d\kappa_0}{d\kappa_0}\frac{\partial p_{x,t}(T, \kappa_0, \sigma_1^2)}{\partial \kappa_0} + \frac{d\sigma_1^2}{d\kappa_0}\frac{\partial p_{x,t}(T, \kappa_0, \sigma_1^2)}{\partial \sigma_1^2} \\ &= \Delta_{x,t}(T) + \frac{d\sigma_1^2}{d\kappa_0} V_{x,t}(T) \end{aligned} \tag{24}$$

Equation (24) says that dynamic delta is the sum of (i) longevity delta and (ii) longevity vega with an adjusting term, $\frac{d\sigma_1^2}{d\kappa_0}$, which represents the sensitivity of σ_1^2 to changes in κ_0. The direct impact of a shock to κ_0 is captured by (i), while the indirect impact is captured by (ii).

To calculate dynamic delta, we need the values of $\Delta_{x,t}(T)$, $V_{x,t}(T)$ and $\frac{d\sigma_1^2}{d\kappa_0}$. The calculation of $\Delta_{x,t}(T)$ and $V_{x,t}(T)$ is detailed in the previous section. The expression for $\frac{d\sigma_1^2}{d\kappa_0}$ is derived below.

First, we use the chain rule to obtain

$$\frac{d\sigma_1^2}{d\kappa_0} = \frac{d\sigma_1^2}{d\epsilon_0}\frac{d\epsilon_0}{d\kappa_0} = \frac{d\sigma_1^2}{d\epsilon_0}$$

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where $\frac{d\epsilon_0}{dk_0} = 1$ according to equation (2). We then derive $\frac{d\sigma_1^2}{d\epsilon_0}$ for each GARCH-type model under consideration:

- **GARCH**
  For the GARCH(1,1) model, we have

$$\begin{aligned}\frac{d\sigma_1^2}{d\epsilon_0} &= \frac{d}{d\epsilon_0}\left(\omega + \alpha\epsilon_0^2 + \beta\sigma_0^2\right)\\ &= 2\alpha\epsilon_0\end{aligned}$$

- **E-GARCH**
  For the E-GARCH(1,1) model, we use the chain rule to get

$$\begin{aligned}\frac{d\sigma_1^2}{d\epsilon_0} &= \frac{d\eta_0}{d\epsilon_0}\frac{d\sigma_1^2}{d\eta_0}\\ &= \frac{1}{\sigma_0}\frac{d}{d\eta_0}\exp\left(\omega + \gamma\eta_0 + \alpha\left(|\eta_0| - \sqrt{\frac{2}{\pi}}\right) + \beta\ln\sigma_0^2\right)\\ &= \frac{\sigma_1^2}{\sigma_0}\left(\gamma + \alpha\frac{|\eta_0|}{\eta_0}\right)\end{aligned}$$

  where $\frac{d\eta_0}{d\epsilon_0} = \frac{1}{\sigma_0}$ according to equation (3).

- **GJR-GARCH**
  If the chosen model is the GJR-GARCH(1,1) model, then we have

$$\begin{aligned}\frac{d\sigma_1^2}{d\epsilon_0} &= \frac{d}{d\epsilon_0}\left(\omega + \alpha\epsilon_0^2 + \gamma\epsilon_0^2\mathcal{I}_{\{\epsilon_0<0\}} + \beta\sigma_0^2\right)\\ &= 2\alpha\epsilon_0 + 2\gamma\epsilon_0\mathcal{I}_{\{\epsilon_0<0\}}\end{aligned}$$

- **N-GARCH**
  For the N-GARCH(1,1) model, we use the chain rule to get

$$\begin{aligned}\frac{d\sigma_1^2}{d\epsilon_0} &= \frac{d\eta_0}{d\epsilon_0}\frac{d\sigma_1^2}{d\eta_0}\\ &= \frac{1}{\sigma_0}\frac{d}{d\eta_0}\left(\omega + \alpha\sigma_0^2(\eta_0 - \gamma)^2 + \beta\sigma_0^2\right)\\ &= 2\alpha\sigma_0(\eta_0 - \gamma)\end{aligned}$$

- **T-GARCH**
  If the T-GARCH(1,1) model is used, we have

$$\begin{aligned}\frac{d\sigma_1^2}{d\epsilon_0} &= \frac{d}{d\epsilon_0}\left(\omega + \alpha(|\epsilon_0| - \gamma\epsilon_0) + \beta\sigma_0\right)^2\\ &= 2\alpha\sigma_1\left(\frac{|\epsilon_0|}{\epsilon_0} - \gamma\right)\end{aligned}$$

Given $\mathcal{F}_0$, $\frac{d\sigma_1^2}{dk_0}$ is non-random and can be analytically calculated. Thus, once the values of $\Delta_{x,t}(T)$ and $V_{x,t}(T)$ are calculated, dynamic delta can be computed easily without any additional simulations.

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Table 8. Parameter estimates (and their standard errors and t-values) of the fitted E-GARCH model for the Italian female population.

Parameter	Estimate	Standard error	t-Value
μ	−0.78690	0.22012	− 3.57489
ω	0.25478	0.11267	2.26134
α	0.34416	0.23105	1.48955
β	0.87076	0.05254	16.5736
γ	0.42944	0.10635	4.03803

6. Illustrations

In this section, we provide two real-data illustrations to demonstrate the impact of asymmetry in mortality volatility on index-based longevity hedging. We begin with a list of assumptions, followed by the derivations of the longevity Greeks for the liability being hedged and the hedging instrument. Hedge calibration is then discussed. Finally, numerical results are presented.

6.1 General assumptions

The following assumptions are used in both illustrations.

- The liability being hedged is a pension plan for pensioners who are all aged 60 years at time 0. The plan pays each pensioner \$1 at the end of each year until death or age 90 years, whichever is the earliest. The mortality experience of the pensioners is identical to that of the Italian female population. The plan contains an infinitely large number of pensioners, so diversifiable risk can be ignored.
- The hedger establishes a Greek longevity hedge for the pension plan using a S-forward at time 0. No adjustment is made to the hedge after time 0.
- At time 0, freshly launched S-forwards with a reference age 60 years and times-to-maturity from 1 to 20 years are available. The hedger uses S-forwards of which the reference population is Italian female, so there is no population basis risk. Further information about S-forwards is given in section 6.3.
- The evaluation of hedge effectiveness is based on the E-GARCH(1,1) model, which, as shown in section 2, is found to be the best fitting model for the Italian female population. Parameter estimates of the E-GARCH(1,1) model fitted to the data from the Italian female population (over ages 40 to 89 and years 1900 to 2014) are reported in Table 8.
- Time 0 is defined as the end of year 2014 (i.e., the end point of the calibration window).
- When discounting future cash flows, an interest rate of $r = 3\%$ per annum for all durations is used.

6.2 Longevity Greeks for the liability being hedged

We now derive the longevity Greeks of the value of the pension plan. As per the assumptions made, we have

$$\mathcal{L} = \sum_{s=1}^{30} (1+r)^{-s} S_{60,0}(s)$$

as the sum of all future cash flows (per pensioner), discounted to time 0. Given $\mathcal{F}_0$, $\mathcal{L}$ is a random variable which depends on the evolution of mortality between time 1 and time 30. It follows that on a per pensioner basis, the time-0 value of the plan is

$$L(\kappa_0, \sigma_1^2) \coloneqq \mathrm{E}[\mathcal{L}|\mathcal{F}_0] = \sum_{s=1}^{30} (1+r)^{-s} p_{60,0}(s, \kappa_0, \sigma_1^2)$$

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As $L(\kappa_0, \sigma_1^2)$ is a linear combination of $p_{60,0}(s, \kappa_0, \sigma_1^2)$ for different values of s, the longevity Greeks of $L(\kappa_0, \sigma_1^2)$ can be calculated straightforwardly using the results in sections 4 and 5. The longevity delta, longevity gamma, longevity vega and dynamic delta of $L(\kappa_0, \sigma_1^2)$ are given by

$$\Delta^{(L)} \coloneqq \frac{\partial L(\kappa_0, \sigma_1^2)}{\partial \kappa_0} = \sum_{s=1}^{30} (1+r)^{-s} \Delta_{60,0}(s)$$

$$\Gamma^{(L)} \coloneqq \frac{\partial^2 L(\kappa_0, \sigma_1^2)}{\partial \kappa_0^2} = \sum_{s=1}^{30} (1+r)^{-s} \Gamma_{60,0}(s)$$

$$V^{(L)} \coloneqq \frac{\partial L(\kappa_0, \sigma_1^2)}{\partial \sigma_1^2} = \sum_{s=1}^{30} (1+r)^{-s} V_{60,0}(s)$$

and

$$\Lambda^{(L)} \coloneqq \frac{dL(\kappa_0, \sigma_1^2)}{d\kappa_0} = \sum_{s=1}^{30} (1+r)^{-s} \Lambda_{60,0}(s)$$

respectively.

6.3 Longevity Greeks for the hedging instrument

Here, we derive the longevity Greeks of the value of a S-forward. A S-forward is a zero-coupon swap with a fixed leg proportional to a forward survival rate S^f that is predetermined at inception, and a floating leg proportional to a (random) realised survival rate. For a S-forward with a time-to-maturity of t^f and a reference age of x^f, the realised survival rate is the one for individuals aged x^f when the S-forward is launched over a period of t^f years. To mitigate longevity risk, a hedger can participate in a S-forward as a fixed-rate payer, so that when future mortality turns out to be lighter than expected (i.e., the realised survival rate turns out to be lower than expected), the hedger will receive from the counterparty a positive net payment, which can be used to offset the correspondingly higher pension/annuity liability.

From the fixed-rate payer's perspective, the pay-off of a S-forward (with a reference age of $x^f = 60$ years and an issue date of $t = 0$) discounted to time 0 is

$$\mathcal{H} = (1+r)^{-t^f} (S_{60,0}(t^f) - S^f)$$

per \$1 notional. Given $\mathcal{F}_0$, $\mathcal{H}$ is a random variable which depends on the evolution of mortality between time 1 and time t^f. It follows that the time-0 value of the S-forward is

$$H(\kappa_0, \sigma_1^2) \coloneqq \mathrm{E}[\mathcal{H}|\mathcal{F}_0] = (1+r)^{-t^f} (p_{60,0}(t^f, \kappa_0, \sigma_1^2) - S^f)$$

per \$1 notional.

As $H(\kappa_0, \sigma_1^2)$ is linearly related to $p_{60,0}(t^f, \kappa_0, \sigma_1^2)$, the longevity Greeks of $H(\kappa_0, \sigma_1^2)$ can be obtained easily using the results in section 4 and 5. The longevity delta, longevity gamma, longevity vega and dynamic delta of $H(\kappa_0, \sigma_1^2)$ are given by

$$\Delta^{(H)} \coloneqq \frac{\partial H(\kappa_0, \sigma_1^2)}{\partial \kappa_0} = (1+r)^{-t^f} \Delta_{60,0}(t^f)$$

$$\Gamma^{(H)} \coloneqq \frac{\partial^2 H(\kappa_0, \sigma_1^2)}{\partial \kappa_0^2} = (1+r)^{-t^f} \Gamma_{60,0}(t^f)$$

$$V^{(H)} \coloneqq \frac{\partial H(\kappa_0, \sigma_1^2)}{\partial \sigma_1^2} = (1+r)^{-t^f} V_{60,0}(t^f)$$

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and

$$\Lambda^{(H)} \coloneqq \frac{dH(\kappa_0, \sigma_1^2)}{d\kappa_0} = (1+r)^{-t^f} \Lambda_{60,0}(t^f)$$

respectively.

The longevity Greeks of a S-forward do not depend on its forward survival rate S^f, because S^f is predetermined at time 0 and is thus treated as a constant. We set S^f to $\mathrm{E}[S_{60,0}(t^f)|\mathcal{F}_0]$, a forward survival rate which implies that no payment exchanges hands at time 0 (i.e., $\mathrm{E}[\mathcal{H}|\mathcal{F}_0] = 0$).

6.4 Hedge calibration

A Greek longevity hedge can be created by matching the longevity Greek(s) of the liability being hedged and the hedging instrument(s). The longevity hedges in both of our illustrations are calibrated by matching dynamic delta, the new longevity Greek proposed in this paper.

Since we are matching one longevity Greek, only one S-forward is needed. When the dynamic deltas of the liability being hedged and the S-forward are matched, the notional amount of the S-forward is calculated at time 0 as

$$u = \frac{\Lambda^{(L)}}{\Lambda^{(H)}}$$

To demonstrate the impact of asymmetry in mortality volatility on the longevity hedges, we calculate two sets of results which respectively represent the situations when asymmetry in mortality volatility is ignored and taken into account:

(I) Asymmetry in mortality volatility is ignored
For this set of results, the longevity Greeks (and hence the notional amount u) are derived from a GARCH(1,1) model, which does not permit volatility to respond asymmetrically to positive and negative shocks. The GARCH(1,1) model is fitted to the Italian female data described in section 6.1. The forward survival rate S^f is also calculated from the GARCH(1,1) model to mimic the situation when participants in the S-forward market ignore any asymmetry in mortality volatility. However, the mortality scenarios used to evaluate hedge effectiveness are generated from the E-GARCH(1,1) model as described in section 6.1 to reflect the fact that asymmetry in mortality volatility exists.

(II) Asymmetry in mortality volatility is taken into account
In line with the model from which the mortality scenarios for evaluating hedge effectiveness are generated, the longevity Greeks (and hence the notional amount u) as well as the forward survival rate S^f for this set of results are calculated from the E-GARCH(1,1) model which permits asymmetry in mortality volatility.

Figure 3 shows the resulting S-forward notional amounts u under both scenarios. For short-term S-forwards, the notional amounts calculated when asymmetry in mortality volatility is ignored are smaller than those calculated when asymmetry in mortality volatility is taken into account. The difference diminishes as the S-forward time-to-maturity increases.

6.5 Illustration 1: value hedge

The first illustration concerns a value hedge, which aims to reduce the variability of the values of the pension plan sponsor's position at time 1. We measure hedge effectiveness by the reduction in the 1-year ahead 99.5% Value-at-Risk over the time-0 value of the unhedged position:

$$1 - \frac{\mathrm{VaR}_{99.5\%}\left(\mathrm{E}[\mathcal{L} - u\mathcal{H}|\mathcal{F}_1]|\mathcal{F}_0\right) - \mathrm{E}[\mathcal{L}|\mathcal{F}_0]}{\mathrm{VaR}_{99.5\%}(\mathrm{E}[\mathcal{L}|\mathcal{F}_1]|\mathcal{F}_0) - \mathrm{E}[\mathcal{L}|\mathcal{F}_0]}$$

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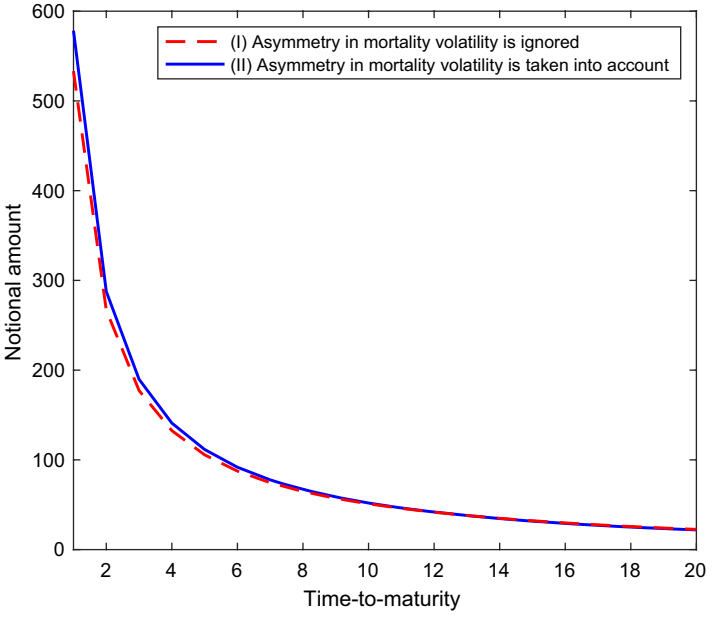


Figure 3. Notional amounts of S-forwards with times-to-maturity ranging from 1 to 20 years, calculated when (I) asymmetry in mortality volatility is ignored (dashed red line) and when (II) asymmetry in mortality volatility is taken into account (solid blue line).

The following comments concerning this metric of hedge effectiveness are made.

- In the expression, $\mathcal{L}$ is the pension plan sponsor's position in the absence of a longevity hedge. Measured in time-0 dollars, the time-0 and time-1 values of the position are $\mathrm{E}[\mathcal{L}|\mathcal{F}_1]$ and $\mathrm{E}[\mathcal{L}|\mathcal{F}_0]$, respectively.
- In the numerator of the expression, $\mathcal{L} - u\mathcal{H}$ is the pension plan sponsor's position when the longevity hedge is in place. Measured in time-0 dollars, the time-1 value of the position is $\mathrm{E}[\mathcal{L} - u\mathcal{H}|\mathcal{F}_1]$.
- Given $\mathcal{F}_0$, $\mathrm{E}[\mathcal{L}|\mathcal{F}_1]$ and $\mathrm{E}[\mathcal{L} - u\mathcal{H}|\mathcal{F}_1]$ are random variables depending on the unknown realisation of $\mathcal{F}_1$, but $\mathrm{E}[\mathcal{L}|\mathcal{F}_0]$ is a known constant.
- If the value hedge is effective, then the value of the metric should be close to one; if the opposite is true, then the value of the metric should be close to zero.

We use nested simulations to calculate the value of the metric. In more detail, we first use the fitted E-GARCH model to generate 10,000 mortality scenarios at time 1 (i.e., 10,000 realisations of $\mathcal{F}_1$). For each of these mortality scenarios, another 10,000 mortality scenarios are generated (also using the fitted E-GARCH model) to compute 10,000 realisations of $\mathrm{E}[\mathcal{L}|\mathcal{F}_1]$ and $\mathrm{E}[\mathcal{H}|\mathcal{F}_1]$, from which empirical distributions of $\mathrm{E}[\mathcal{L} - u\mathcal{H}|\mathcal{F}_1]$ and $\mathrm{E}[\mathcal{L}|\mathcal{F}_1]$ given $\mathcal{F}_0$ are obtained. The 99.5th percentiles of these two empirical distributions give the estimated 99.5% Values-at-Risk of $\mathrm{E}[\mathcal{L} - u\mathcal{H}|\mathcal{F}_1]$ and $\mathrm{E}[\mathcal{L}|\mathcal{F}_1]$, respectively. Lastly, $\mathrm{E}[\mathcal{L}|\mathcal{F}_0]$ is calculated with non-nested simulations.

Figure 4 compares the effectiveness of the value hedges that are calibrated when (I) asymmetry in mortality volatility is ignored and when (II) asymmetry in mortality volatility is taken into account. When asymmetry in mortality volatility is incorporated into the calibration of the value hedge, the hedge effectiveness ranges between 0.9 and 1, indicating that the hedge reduces at least 90% of the 99.5% Value-at-Risk over a horizon of 1 year. If the hedger ignores asymmetry in mortality volatility, then the hedge effectiveness is significantly reduced (by up to 70% points depending on the time-to-maturity of the S-forward used).

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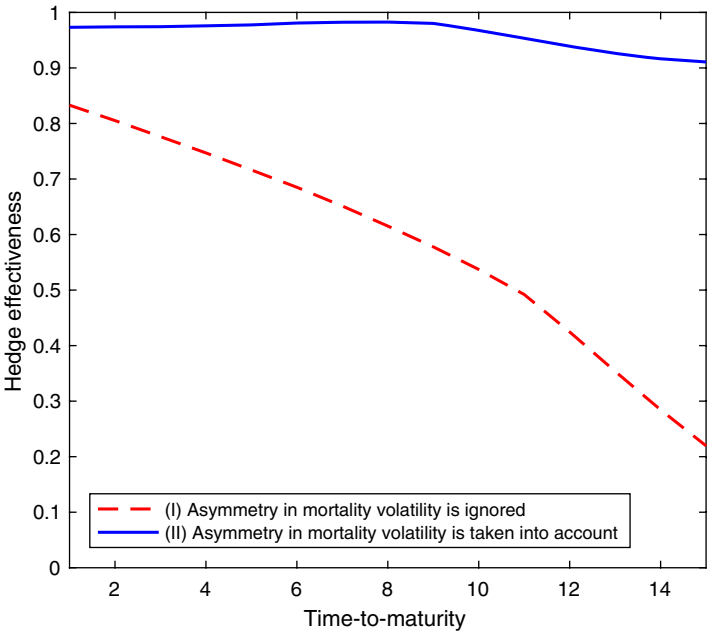


Figure 4. Effectiveness of the value hedges that are calibrated when (I) asymmetry in mortality volatility is ignored (dashed red line) and when (II) asymmetry in mortality volatility is taken into account (solid blue line), for S-forward times-to-maturity ranging from 1 to 15 years.

6.6 Illustration 2: cash flow hedge

The second illustration concerns a cash flow hedge, which aims to reduce the variability of all future cash flows of the pension plan sponsor. In this illustration, we measure hedge effectiveness by the reduction in the 99.5% Value-at-Risk of all future cash flows measured in time-0 dollars over the time-0 value of the liability being hedged:

$$1-\frac{\text{VaR}_{99.5\%}\left(\mathcal{L}-u\mathcal{H}|\mathcal{F}_0\right)-\text{E}[\mathcal{L}|\mathcal{F}_0]}{\text{VaR}_{99.5\%}(\mathcal{L}|\mathcal{F}_0)-\text{E}[\mathcal{L}|\mathcal{F}_0]}$$

where $\mathcal{L}-u\mathcal{H}$ and $\mathcal{L}$ represent the present (time-0) value of all cash flows arising from the hedged and unhedged positions, respectively. If the cash flow hedge is effective, then the value of the metric should be close to one; if the opposite is true, then the value of the metric should be close to zero.

To compute the value of the metric defined above, we use the fitted E-GARCH model to generate 10,000 realisations of $\mathcal{L}$ and $\mathcal{H}$, from which empirical distributions of $\mathcal{L}-u\mathcal{H}$ and $\mathcal{L}$ given $\mathcal{F}_0$ are obtained. The 99.5th percentiles of these two empirical distributions give the estimated 99.5% Values-at-Risk of $\text{E}[\mathcal{L}-u\mathcal{H}|\mathcal{F}_0]$ and $\text{E}[\mathcal{L}|\mathcal{F}_0]$, respectively. As in the first illustration, $\text{E}[\mathcal{L}|\mathcal{F}_0]$ is also calculated with non-nested simulations.

Figure 5 compares the effectiveness of the cash flow hedges that are calibrated when (I) asymmetry in mortality volatility is ignored and when (II) asymmetry in mortality volatility is taken into account. For all S-forward times-to-maturity considered, the effectiveness of the hedge that takes asymmetry in mortality volatility into account is higher compared to the hedge that ignores asymmetry in mortality volatility. The difference in hedge effectiveness ranges from 10% to 40% points, depending on the time-to-maturity of the S-forward used.

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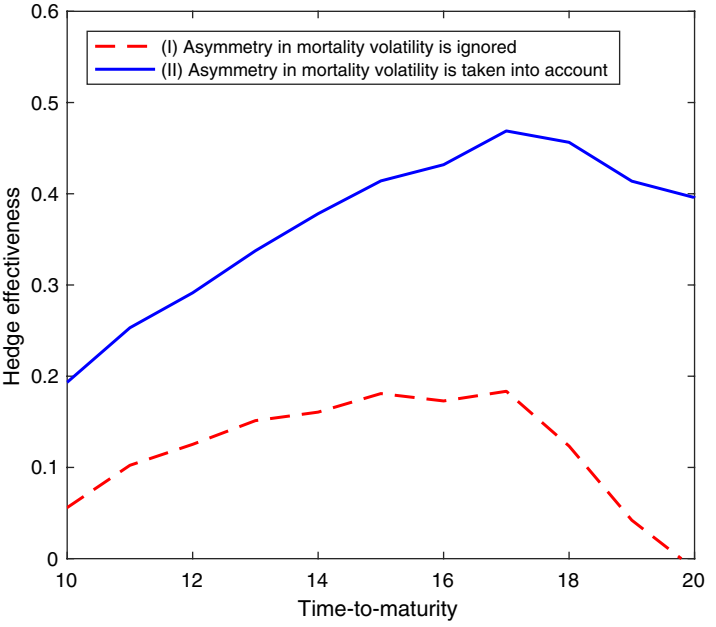


Figure 5. Effectiveness of the cash flow hedges that are calibrated when (I) asymmetry in mortality volatility is ignored (dashed red line) and when (II) asymmetry in mortality volatility is taken into account (solid blue line), for S-forward times-to-maturity ranging from 10 to 20 years.

7. Concluding Remarks

We have investigated the existence of asymmetry in mortality volatility using a range of GARCH-type models that permit volatility to respond asymmetrically to positive and negative shocks. It is found that for six out of the nine populations under consideration, asymmetry in mortality volatility is significant and needs to be modeled.

We have also studied the impact of asymmetry in mortality volatility on index-based longevity hedging, through three technical contributions that are made in this paper.

First, we have improved the framework of Zhou & Li (2019a) by redefining the building block on which longevity Greeks are built. The longevity vega defined in the improved framework works with a wider range of α and β parameters in various GARCH-type models.

Second, we have derived, under the improved framework, semi-analytic expressions for the longevity vega under different GARCH model assumptions. These expressions enable us to calibrate longevity hedges when the assumed model is one that incorporates asymmetry in mortality volatility.

Third, we have introduced a new longevity Greek called dynamic delta. Derived on the basis of total derivative with respect to κ_0, dynamic delta incorporates both the direct and indirect impacts of a shock to the period effect. We have also shown that dynamic delta is a linear combination of longevity delta and longevity vega with an adjusting term that is known at time 0, so that the value of dynamic delta can be obtained easily from the calculated values of longevity delta and longevity vega without any additional simulations.

We have provided two real-data illustrations to demonstrate the impact of asymmetry in mortality volatility on index-based longevity hedging. The results of both illustrations point to the conclusion that if mortality volatility is in fact asymmetric but the asymmetry in mortality volatility is not taken into account when a longevity hedge is calibrated, then the resulting hedge effectiveness could be significantly impaired.

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We stress that not all mortality data exhibit conditional heteroskedasticity and asymmetry in volatility. For the nine mortality data sets considered in this paper, two are not subject to statistically significant conditional heteroskedasticity and one does not show statistically significant asymmetry in volatility. Whether of not these two phenomena are statistically significant also depends on the length of the mortality data series under consideration. A shorter data series (which covers the recent decades when the evolution of mortality has been relatively stable) is less likely to exhibit conditional heteroskedasticity and asymmetry in volatility.

In financial markets, two types of volatility (realised and implied) are considered. In this paper, we have focused on realised volatility, which is estimated from historical data. Implied volatility on the other hand is assembled from market expectation and obtained by back-solving the volatility parameter of a pricing model using observed market prices. The study of implied mortality volatility is impeded by the lack of market price data in today's infantile market for longevity risk transfers. However, it would be interesting to investigate implied mortality volatility when the market becomes more mature.

Finally, we remark that along the lines of dynamic delta, we can define "dynamic gamma" as the second-order total derivative with respect to the time-0 period effect. The derivation of dynamic gamma can be initiated as follows:

$$\begin{aligned}\frac{d^2 p_{x,t}(T,\kappa_0,\sigma_1^2)}{d\kappa_0^2} &= \frac{d}{d\kappa_0}\left(\frac{dp_{x,t}(T,\kappa_0,\sigma_1^2)}{d\kappa_0}\right) = \frac{d}{d\kappa_0}\Lambda_{x,t}(T)\\ &= \frac{d}{d\kappa_0}\left(\Delta_{x,t}(T) + \frac{d\sigma_1^2}{d\kappa_0}V_{x,t}(T)\right)\\ &= \Gamma_{x,t}(T) + \frac{d^2\sigma_1^2}{d\kappa_0^2}\left(V_{x,t}(T)\right) + \frac{d\sigma_1^2}{d\kappa_0}\left(\frac{d}{d\kappa_0}V_{x,t}(T)\right)\end{aligned}$$

Applications of this new longevity Greek are left as a topic for further research.

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