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NON-WELL-FOUNDED PROOFS FOR THE GRZEGORCZYK MODAL LOGIC

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Abstract. We present a sequent calculus for the Grzegorczyk modal logic Grz allowing cyclic and other non-well-founded proofs and obtain the cut-elimination theorem for it by constructing a continuous cut-elimination mapping acting on these proofs. As an application, we establish the Lyndon interpolation property for the logic Grz proof-theoretically.

§1. Introduction. A non-well-founded proof is usually defined as a possibly infinite tree of formulas (sequents) that is constructed according to inference rules of a proof system, that satisfies possible further conditions on infinite branches. A cyclic, or circular, proof can be defined as a finite directed graph of formulas (sequents) whose unraveling yields a non-well-founded proof. These proofs turn out to be an interesting alternative to traditional proofs for logics with inductive and co-inductive definitions, fixed-point operators and similar features. For example, proof systems allowing cyclic proofs can be defined for the modal μ-calculus [1], the Lambek calculus with iteration [13] and for Peano arithmetic [21]. In the last case, these proofs can be understood as a formalization of the concept of proof by infinite descent.

Structural proof theory of deductive systems allowing cyclic and non-well-founded proofs seems to be underdeveloped. In [11], Fortier and Santocanale considered the case of the μ-calculus with additive connectives. They presented a procedure for eliminating all applications of the cut-rule from a cyclic proof that results in an infinite tree of sequents. However they didn't show that this tree satisfies the necessary conditions for non-well-founded proofs in the μ-calculus.[1] In [3], Doumane et al. consider non-well-founded proofs for a variant of the μ-calculus called μMALL, but their proof of the cut elimination theorem seems to rely on a semantic argument. The only paper the authors know where cut elimination is established purely syntactically for a system with non-well-founded proofs is the paper by Das and Pous [7], where they consider such a system for the logic of $*$-continuous action lattices. In our paper we propose an approach for syntactically proving cut elimination for such systems using the Grzegorczyk modal logic as an example case.[2]

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[1] After submission of the paper the authors were contacted by Jérôme Fortier who informed us that he addressed this problem in his thesis on page 144 https://tel.archives-ouvertes.fr/tel-01154972.

[2] This article is an extended version of a conference paper [18].

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The Grzegorczyk modal logic Grz is a well-known modal logic [14], which can be characterized by reflexive partially ordered Kripke frames without infinite ascending chains. This logic is complete w.r.t. the arithmetical semantics, where the modal connective $\Box$ corresponds to the strong provability operator *"... is true and provable in Peano arithmetic."* There is a translation from Grz into the Gödel–Löb provability logic GL such that

$$\mathsf{Grz} \vdash A \Longleftrightarrow \mathsf{GL} \vdash A^*,$$

where A^* is obtained from A by replacing all subformulas of the form $\Box B$ by $B \wedge \Box B$.

Recently a new proof-theoretic presentation for the Gödel–Löb provability logic GL in the form of a sequent calculus allowing non-well-founded proofs was given in [12, 20]. We wonder whether cyclic and, more generally, non-well-founded proofs can be fruitfully considered in the case of Grz. We consider a sequent calculus allowing non-well-founded proofs for the Grzegorczyk modal logic and present the cut-elimination theorem for the given system. In order to avoid nested co-inductive and inductive reasoning, we adopt an approach from denotational semantics of computer languages, where program types are interpreted as ultrametric spaces and fixed-point combinators are encoded using the Banach fixed-point theorem (see [4], [10], [22]). We consider the set of non-well-founded proofs of Grz and various sets of operations acting on these proofs as ultrametric spaces and define our cut-elimination operator following the lines of the Prieß-Crampe fixed-point theorem[3] (see [17]), which is a strengthening of Banach's theorem. We believe that this approach to cut elimination can also be applied to more complicated systems such as various logics with fixed-point operators. As an application of our cut elimination result for the logic Grz, we establish that it is sufficient to consider only unravellings of cut-free cyclic proofs instead of arbitrary non-well-founded proofs in order to obtain all provable sequents. As another application, we prove the Lyndon interpolation property for the logic Grz proof-theoretically.

Recall that the Craig interpolation property for a logic L says that if A implies B, then there is an interpolant, that is, a formula I containing only common variables of A and B such that A implies I and I implies B. The Lyndon interpolation property is a strengthening of the Craig interpolation that also takes into consideration negative and positive occurrences of the shared propositional variables; that is, the variables occurring in I positively (negatively) must also occur both in A and B positively (negatively).

Though the Grzegorczyk logic has the Lyndon interpolation property [15], there were seemingly no syntactic proofs for this result. It is unclear how Lyndon interpolation can be obtained from previously introduced sequent systems for Grz [2, 5, 9] by direct proof-theoretic arguments because these systems contain inference rules in which a polarity change occurs under the passage from the principal formula in the conclusion to its immediate ancestors in the premises. In this article, we give a syntactic proof of Lyndon interpolation for the Grzegorczyk modal logic.

The paper is organized as follows. In §2 we recall a standard sequent calculus for Grz. In §3 we introduce the proof system Grz_∞ that allows non-well-founded proofs and prove its equivalence to the standard one. In §4 we recall basic notions of the theory of ultrametric spaces and consider several relevant examples. In §5 we state admissibility of several rules for our system that will be used later. In §6 we establish

[3] This fixed-point result was also obtained by Petalas and Vidalis in [16].

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$$\Gamma, A \Rightarrow A, \Delta\ , \qquad \Gamma, \bot \Rightarrow \Delta\ ,$$

$$\to_{\mathrm{L}} \frac{\Gamma, B \Rightarrow \Delta \qquad \Gamma \Rightarrow A, \Delta}{\Gamma, A \to B \Rightarrow \Delta}\ , \qquad \to_{\mathrm{R}} \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \to B, \Delta}\ ,$$

$$\mathrm{refl}\ \frac{\Gamma, B, \Box B \Rightarrow \Delta}{\Gamma, \Box B \Rightarrow \Delta}\ , \qquad \Box_{\mathrm{Grz}} \frac{\Box\Pi, \Box(A \to \Box A) \Rightarrow A}{\Gamma, \Box\Pi \Rightarrow \Box A, \Delta}\ .$$

Fig. 1. The system $\mathsf{Grz}_{\mathsf{Seq}}$.

the cut elimination result for the system Grz_∞ syntactically. In §7 we prove the Lyndon interpolation property for the logic Grz. Finally, in §8 we establish that every provable sequent of Grz_∞ has a cyclic proof.

§2. Preliminaries. In this section we recall the Grzegorczyk modal logic Grz and define an ordinary sequent calculus for it.

Formulas of Grz, denoted by A, B, C, are built up as follows:

$$A ::= \bot \mid p \mid (A \to A) \mid \Box A\ ,$$

where p stands for atomic propositions. We treat other boolean connectives and the modal operator $\Diamond$ as abbreviations:

$$\neg A := A \to \bot, \qquad \top := \neg\bot, \qquad A \wedge B := \neg(A \to \neg B),$$
$$A \vee B := (\neg A \to B), \qquad \Diamond A := \neg\Box\neg A.$$

The Hilbert-style axiomatization of Grz is given by the following axioms and inference rules (see [6], §3.8):

Axioms:

(i) Boolean tautologies;
(ii) $\Box(A \to B) \to (\Box A \to \Box B)$; and
(iii) $\Box(\Box(A \to \Box A) \to A) \to A$.

Rules: modus ponens, $A/\Box A$.

Notice that transitivity and reflexivity of the modal operator $\Box$ are derivable in Grz, i.e., $\mathsf{Grz} \vdash \Box A \to \Box\Box A$ and $\mathsf{Grz} \vdash \Box A \to A$ for any formula A.

Now we define an ordinary sequent calculus for Grz. A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of formulas. For a multiset of formulas $\Gamma = A_1, \ldots, A_n$, we define $\Box\Gamma$ to be $\Box A_1, \ldots, \Box A_n$.

The system $\mathsf{Grz}_{\mathsf{Seq}}$, which is a variant of the sequent calculus from [2], is defined by the following initial sequents and inference rules:

The cut rule has the form

$$\mathsf{cut}\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ ,$$

where A is called the *cut formula* of the given inference.

Lemma 2.1 (cf. Theorem 1 from [2].) $\mathsf{Grz}_{\mathsf{Seq}} + \mathsf{cut} \vdash \Gamma \Rightarrow \Delta$ *if and only if* $\mathsf{Grz} \vdash \bigwedge \Gamma \to \bigvee \Delta$.

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$$\Gamma, p \Rightarrow p, \Delta\ , \qquad \Gamma, \bot \Rightarrow \Delta\ ,$$

$$\to_{\mathrm{L}} \frac{\Gamma, B \Rightarrow \Delta \qquad \Gamma \Rightarrow A, \Delta}{\Gamma, A \to B \Rightarrow \Delta}\ , \qquad \to_{\mathrm{R}} \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \to B, \Delta}\ ,$$

$$\mathrm{refl}\ \frac{\Gamma, B, \Box B \Rightarrow \Delta}{\Gamma, \Box B \Rightarrow \Delta}\ , \qquad \Box\ \frac{\Gamma, \Box\Pi \Rightarrow A, \Delta \qquad \Box\Pi \Rightarrow A}{\Gamma, \Box\Pi \Rightarrow \Box A, \Delta}\ .$$

Fig. 2. The system Grz_∞.

This lemma is completely standard, so we omit the proof.

THEOREM 2.2. *If* $\mathsf{Grz}_{\mathsf{Seq}} + \mathsf{cut} \vdash \Gamma \Rightarrow \Delta$, *then* $\mathsf{Grz}_{\mathsf{Seq}} \vdash \Gamma \Rightarrow \Delta$.

A syntactic cut-elimination for Grz was obtained by Borga and Gentilini in [5]. In this paper, we will give another proof of this cut-elimination theorem in the next sections.

§3. Non-well-founded proofs. In this section we introduce a sequent calculus for Grz allowing non-well-founded proofs and define two translations that connect the ordinary and non-well-founded sequent systems.

Inference rules and initial sequents of the sequent calculus Grz_∞ have the following form:

The system $\mathsf{Grz}_\infty + \mathsf{cut}$ is defined by adding the rule (cut) to the system Grz_∞. An ∞-*proof* in Grz_∞ ($\mathsf{Grz}_\infty + \mathsf{cut}$) is a (possibly infinite) tree whose nodes are marked by sequents and whose leaves are marked by initial sequents and that is constructed according to the rules of the sequent calculus. In addition, every infinite branch in an ∞-proof must pass through a right premise of the rule ($\Box$) infinitely many times. A sequent $\Gamma \Rightarrow \Delta$ is *provable* in Grz_∞ ($\mathsf{Grz}_\infty + \mathsf{cut}$) if there is an ∞-proof in Grz_∞ ($\mathsf{Grz}_\infty + \mathsf{cut}$) with the root marked by $\Gamma \Rightarrow \Delta$.

The *n-fragment* of an ∞-proof is a finite tree obtained from the ∞-proof by cutting every branch at the nth from the root right premise of the rule ($\Box$), so that this premise is removed. The 1-fragment of an ∞-proof is also called its *main fragment*. We define the *local height* $|\pi|$ of an ∞-proof π as the length of the longest branch in its main fragment. An ∞-proof only consisting of an initial sequent has height 0.

For instance, consider an ∞-proof of the sequent $\Box(\Box(p \to \Box p) \to p) \Rightarrow p$:

$$\dfrac{\dfrac{\mathrm{Ax}}{F, p \Rightarrow p} \qquad \dfrac{\dfrac{\dfrac{\mathrm{Ax}}{F, p \Rightarrow \Box p, p}}{F \Rightarrow p \to \Box p, p} \to_{\mathrm{R}} \qquad \dfrac{\dfrac{\mathrm{Ax}}{p, F \Rightarrow p} \qquad \dfrac{\pi}{F \Rightarrow p}}{p, F \Rightarrow \Box p} \Box}{F \Rightarrow p \to \Box p} \to_{\mathrm{R}}}{F \Rightarrow \Box(p \to \Box p), p} \Box}{\dfrac{\Box(p \to \Box p) \to p, F \Rightarrow p}{F \Rightarrow p}\ \mathrm{refl}}\ \to_{\mathrm{L}}\ ,$$

where $F = \Box(\Box(p \to \Box p) \to p)$ and the subtree π is isomorphic to the whole ∞-proof. The local height of this ∞-proof equals 4 and its main fragment has the form

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$$\dfrac{\dfrac{\dfrac{}{F,p \Rightarrow p}\,\mathrm{Ax} \qquad \dfrac{\dfrac{\dfrac{}{F,p \Rightarrow \Box p, p}\,\mathrm{Ax}}{F \Rightarrow p \to \Box p, p}\,{\to_{\mathrm{R}}}}{F \Rightarrow \Box(p \to \Box p), p}\,\Box}{\Box(p \to \Box p) \to p, F \Rightarrow p}\,{\to_{\mathrm{L}}}}{F \Rightarrow p}\,\mathrm{refl}.$$

We denote the set of all ∞-proofs in the system $\mathsf{Grz}_\infty + \mathsf{cut}$ by $\mathcal{P}$. For $\pi, \tau \in \mathcal{P}$, we write $\pi \sim_n \tau$ if n-fragments of these ∞-proofs coincide. For any $\pi, \tau \in \mathcal{P}$, we also set $\pi \sim_0 \tau$.

Now we define two translations that connect the ordinary and non-well-founded sequent calculi for Grz.

LEMMA 3.3. *For any sequent* $\Gamma \Rightarrow \Delta$ *and any formula* A *we have* $\mathsf{Grz}_\infty \vdash \Gamma, A \Rightarrow A, \Delta$.

Proof. Standard induction on the structure of A. □

LEMMA 3.4. *For any formula* A *we have* $\mathsf{Grz}_\infty \vdash \Box(\Box(A \to \Box A) \to A) \Rightarrow A$.

Proof. Consider an example of ∞-proof for the sequent $\Box(\Box(p \to \Box p) \to p) \Rightarrow p$ given above. We transform this example into an ∞-proof for $\Box(\Box(A \to \Box A) \to A) \Rightarrow A$ by replacing p with A and adding required ∞-proofs instead of initial sequents using Lemma 3.3. □

Recall that an inference rule is called admissible (in a given proof system) if, for any instance of the rule, the conclusion is provable whenever all premises are provable.

LEMMA 3.5. *The rule*

$$\mathrm{weak}\,\frac{\Gamma \Rightarrow \Delta}{\Pi, \Gamma \Rightarrow \Delta, \Sigma}$$

is admissible in the systems $\mathsf{Grz}_{\mathsf{Seq}}$ *and* $\mathsf{Grz}_\infty + \mathsf{cut}$.

Proof. Standard induction on the structure (local height) of a proof of $\Gamma \Rightarrow \Delta$. □

THEOREM 3.6. *If* $\mathsf{Grz}_{\mathsf{Seq}} + \mathsf{cut} \vdash \Gamma \Rightarrow \Delta$, *then* $\mathsf{Grz}_\infty + \mathsf{cut} \vdash \Gamma \Rightarrow \Delta$.

Proof. Assume π is a proof of $\Gamma \Rightarrow \Delta$ in $\mathsf{Grz}_{\mathsf{Seq}} + \mathsf{cut}$. By induction on the size of π we prove $\mathsf{Grz}_\infty + \mathsf{cut} \vdash \Gamma \Rightarrow \Delta$.

If $\Gamma \Rightarrow \Delta$ is an initial sequent of $\mathsf{Grz}_{\mathsf{Seq}} + \mathsf{cut}$, then it is either an initial sequent of $\mathsf{Grz}_\infty + \mathsf{cut}$, or is provable in $\mathsf{Grz}_\infty + \mathsf{cut}$ by Lemma 3.3. Otherwise, consider the last application of an inference rule in π.

The only nontrivial case is when the proof π has the form

$$\Box_{\mathrm{Grz}}\,\frac{\begin{matrix}\pi'\\ \Box\Pi, \Box(A \to \Box A) \Rightarrow A\end{matrix}}{\Sigma, \Box\Pi \Rightarrow \Box A, \Lambda},$$

where $\Sigma, \Box\Pi = \Gamma$ and $\Box A, \Lambda = \Delta$. By the induction hypothesis there is an ∞-proof ξ of $\Box\Pi, \Box(A \to \Box A) \Rightarrow A$ in $\mathsf{Grz}_\infty + \mathsf{cut}$.

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We have the following ∞-proof λ of $\Box\Pi \Rightarrow A$ in $\mathsf{Grz}_\infty + \mathsf{cut}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\xi'}{\Box\Pi, \Box(A \to \Box A) \Rightarrow A, A}\,{\to_{\mathrm{R}}}}{\Box\Pi \Rightarrow G, A}\,\Box \quad \dfrac{\dfrac{\xi}{\Box\Pi, \Box(A \to \Box A) \Rightarrow A}\,{\to_{\mathrm{R}}}}{\Box\Pi \Rightarrow G}\,\Box}{\Box\Pi \Rightarrow \Box G, A}\,\mathrm{cut} \qquad \dfrac{\theta}{\Box\Pi, \Box G \Rightarrow A}}{\Box\Pi \Rightarrow A}\,\mathrm{cut},$$

where $G = \Box(A \to \Box A) \to A$, ξ' is an ∞-proof of $\Box\Pi, \Box(A \to \Box A) \Rightarrow A, A$ obtained from ξ by Lemma 3.5, and θ is an ∞-proof of $\Box\Pi, \Box G \Rightarrow A$, which exists by Lemma 3.4 and Lemma 3.5.

The required ∞-proof for $\Sigma, \Box\Pi \Rightarrow \Box A, \Lambda$ has the form

$$\dfrac{\dfrac{\lambda'}{\Sigma, \Box\Pi \Rightarrow A, \Lambda} \qquad \dfrac{\lambda}{\Box\Pi \Rightarrow A}}{\Sigma, \Box\Pi \Rightarrow \Box A, \Lambda}\,\Box,$$

where λ' is an ∞-proof of the sequent $\Sigma, \Box\Pi \Rightarrow A, \Lambda$ obtained from the ∞-proof λ by Lemma 3.5.

The cases of other inference rules being last in π are straightforward, so we omit them. □

For a sequent $\Gamma \Rightarrow \Delta$, let $Sub(\Gamma \Rightarrow \Delta)$ be the set of all subformulas of the formulas from $\Gamma \cup \Delta$. For a finite set of formulas Λ, let Λ^* be the set $\{\Box(A \to \Box A) \mid A \in \Lambda\}$.

THEOREM 3.7. *If* $\mathsf{Grz}_\infty \vdash \Gamma \Rightarrow \Delta$, *then* $\mathsf{Grz}_{\mathsf{Seq}} \vdash \Gamma \Rightarrow \Delta$.

Proof. First we will prove a more general statement: if $\mathsf{Grz}_\infty \vdash \Gamma \Rightarrow \Delta$, then $\mathsf{Grz}_{\mathsf{Seq}} \vdash \Lambda^*, \Gamma \Rightarrow \Delta$ for any finite set of formulas Λ. The statement of the theorem immediately follows by assuming the set Λ to be empty.

Assume π is an ∞-proof of the sequent $\Gamma \Rightarrow \Delta$ in Grz_∞ and Λ is a finite set of formulas. By induction on the number of elements in the finite set $Sub(\Gamma \Rightarrow \Delta) \backslash \Lambda$ with a subinduction on $|\pi|$, we prove $\mathsf{Grz}_{\mathsf{Seq}} \vdash \Lambda^*, \Gamma \Rightarrow \Delta$.

If $|\pi| = 0$, then $\Gamma \Rightarrow \Delta$ is an initial sequent. We see that the sequent $\Lambda^*, \Gamma \Rightarrow \Delta$ is an initial sequent and it is provable in $\mathsf{Grz}_{\mathsf{Seq}}$. Otherwise, consider the last application of an inference rule in π.

Case 1. Suppose that π has the form

$$\dfrac{\dfrac{\pi'}{\Gamma, A \Rightarrow B, \Sigma}\,{\to_{\mathrm{R}}}}{\Gamma \Rightarrow A \to B, \Sigma},$$

where $A \to B, \Sigma = \Delta$. Notice that $|\pi'| < |\pi|$. By the induction hypothesis for π' and Λ, the sequent $\Lambda^*, \Gamma, A \Rightarrow B, \Sigma$ is provable in $\mathsf{Grz}_{\mathsf{Seq}}$. Applying the rule $(\to_{\mathrm{R}})$ to it, we obtain that the sequent $\Lambda^*, \Gamma \Rightarrow \Delta$ is provable in $\mathsf{Grz}_{\mathsf{Seq}}$.

Case 2. Suppose that π has the form

$$\dfrac{\dfrac{\pi'}{\Sigma, B \Rightarrow \Delta} \qquad \dfrac{\pi''}{\Sigma \Rightarrow A, \Delta}}{\Sigma, A \to B \Rightarrow \Delta}\,{\to_{\mathrm{L}}},$$

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where $\Sigma, A \to B = \Gamma$. We see that $|\pi'| < |\pi|$. By the induction hypothesis for π' and Λ, the sequent $\Lambda^*, \Sigma, B \Rightarrow \Delta$ is provable in $\mathsf{Grz}_{\mathsf{Seq}}$. Analogously, we have $\mathsf{Grz}_{\mathsf{Seq}} \vdash \Lambda^*, \Sigma \Rightarrow A, \Delta$. Applying the rule $(\to_{\mathsf{L}})$, we obtain that the sequent $\Lambda^*, \Sigma, A \to B \Rightarrow \Delta$ is provable in $\mathsf{Grz}_{\mathsf{Seq}}$.

Case 3. Suppose that π has the form

$$\mathrm{refl}\,\frac{\begin{array}{c}\pi'\\ \Sigma, A, \Box A \Rightarrow \Delta\end{array}}{\Sigma, \Box A \Rightarrow \Delta},$$

where $\Sigma, \Box A = \Gamma$. We see that $|\pi'| < |\pi|$. By the induction hypothesis for π' and Λ, the sequent $\Lambda^*, \Sigma, A, \Box A \Rightarrow \Delta$ is provable in $\mathsf{Grz}_{\mathsf{Seq}}$. Applying the rule (refl), we obtain $\mathsf{Grz}_{\mathsf{Seq}} \vdash \Lambda^*, \Sigma, \Box A \Rightarrow \Delta$.

Case 4. Suppose that π has the form

$$\Box\,\frac{\begin{array}{cc}\pi' & \pi''\\ \Phi, \Box\Pi \Rightarrow A, \Sigma & \Box\Pi \Rightarrow A\end{array}}{\Phi, \Box\Pi \Rightarrow \Box A, \Sigma},$$

where $\Phi, \Box\Pi = \Gamma$ and $\Box A, \Sigma = \Delta$.

Subcase 4.1: the formula A belongs to Λ. We see that $|\pi'| < |\pi|$. By the induction hypothesis for π' and Λ, the sequent $\Lambda^*, \Phi, \Box\Pi \Rightarrow A, \Sigma$ is provable in $\mathsf{Grz}_{\mathsf{Seq}}$. Then we see

$$\to_{\mathsf{L}}\,\frac{\begin{array}{cc}\mathrm{Ax} & \mathrm{weak}\,\dfrac{\Lambda^*, \Phi, \Box\Pi \Rightarrow A, \Sigma}{\Lambda^*, \Phi, \Box\Pi \Rightarrow A, \Box A\ \ \Sigma}\\ \Lambda^*, \Box A, \Phi, \Box\Pi \Rightarrow \Box A, \Sigma & \end{array}}{\mathrm{refl}\,\dfrac{(\Lambda\backslash\{A\})^*, A \to \Box A, \Box(A \to \Box A), \Phi, \Box\Pi \Rightarrow \Box A, \Sigma}{(\Lambda\backslash\{A\})^*, \Box(A \to \Box A), \Phi, \Box\Pi \Rightarrow \Box A, \Sigma}},$$

where the rule (weak) is admissible by Lemma 3.5.

Subcase 4.2: the formula A doesn't belong to Λ. We have that the number of elements in $Sub(\Box\Pi \Rightarrow A)\backslash(\Lambda \cup \{A\})$ is strictly less than the number of elements in $Sub(\Phi, \Box\Pi \Rightarrow \Box A, \Sigma)\backslash\Lambda$. Therefore, by the induction hypothesis for π'' and $\Lambda \cup \{A\}$, the sequent $\Lambda^*, \Box(A \to \Box A), \Box\Pi \Rightarrow A$ is provable in $\mathsf{Grz}_{\mathsf{Seq}}$. Then we have

$$\Box_{\mathrm{Grz}}\,\frac{\Lambda^*, \Box(A \to \Box A), \Box\Pi \Rightarrow A}{\Lambda^*, \Phi, \Box\Pi \Rightarrow \Box A, \Sigma}.$$

□

§4. Ultrametric spaces. In this section we recall basic notions of the theory of ultrametric spaces (cf. [19]) and consider several examples concerning ∞-proofs.

An *ultrametric space* (M, d) is a metric space that satisfies a stronger version of the triangle inequality: for any $x, y, z \in M$

$$d(x, z) \leqslant \max\{d(x, y), d(y, z)\}.$$

For $x \in M$ and $r \in (0, +\infty)$, the set $B_r(x) = \{y \in M \mid d(x, y) \leqslant r\}$ is called the *closed ball* with *center* x and *radius* r. An ultrametric space (M, d) is called *spherically complete* if each descending sequence of closed balls

$$B_{r_0}(x_0) \supset B_{r_1}(x_1) \supset B_{r_2}(x_2) \supset \cdots$$

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has a common point. We recall that a metric space (M,d) is *complete* if any descending sequence of closed balls, with radii tending to 0, has a common point.

Let us remark that not every complete ultrametric space is spherically complete. In particular, consider an ultrametric space $(\mathbb{N},d_{\mathbb{N}})$, where

$$d_{\mathbb{N}}(m,n) = \begin{cases} 0 & \text{if } m = n, \\ 1 + \max\{\frac{1}{m+1}, \frac{1}{n+1}\} & \text{otherwise.} \end{cases}$$

We see that $B_r(x) = \{x\}$ if $r \leqslant 1$. Consequently, for any descending sequence of closed balls

$$B_{r_0}(x_0) \supset B_{r_1}(x_1) \supset B_{r_2}(x_2) \supset \cdots$$

with radii tending to 0, there is a natural number N such that $B_{r_m}(x_m) = B_{r_N}(x_N) = \{x_N\}$ for any $m \geqslant N$ and x_N is a common point of the balls. The ultrametric space $(\mathbb{N},d_{\mathbb{N}})$ is complete.

Note also $B_{1+\frac{1}{1+n}}(n) = \{m \in \mathbb{N} \mid m \geqslant n\}$. We have a descending sequence of closed balls

$$B_2(0) \supset B_{1\frac{1}{2}}(1) \supset \cdots \supset B_{1+\frac{1}{1+n}}(n) \supset \cdots$$

without a common point. Therefore the ultrametric space $(\mathbb{N},d_{\mathbb{N}})$ is not spherically complete.

In an ultrametric space (M,d), a function $f : M \to M$ is called (*strictly*) *contractive* if $d(f(x),\ f(y)) < d(x,y)$ whenever $x \neq y$.

THEOREM 4.8 (Prieß-Crampe [17], Petalas and Vidalis [16]). *Let (M,d) be a nonempty spherically complete ultrametric space. Then every strictly contractive mapping $f : M \to M$ has a unique fixed-point.*

Now consider the set $\mathcal{P}$ of all ∞-proofs of the system $\mathsf{Grz}_\infty + \mathsf{cut}$. We can define an ultrametric $d_{\mathcal{P}} : \mathcal{P} \times \mathcal{P} \to [0,1]$ on $\mathcal{P}$ by putting

$$d_{\mathcal{P}}(\pi,\tau) = \inf\{\frac{1}{2^n} \mid \pi \sim_n \tau\}.$$

We see that $d_{\mathcal{P}}(\pi,\tau) \leqslant 2^{-n}$ if and only if $\pi \sim_n \tau$. Thus, the ultrametric $d_{\mathcal{P}}$ can be considered as a measure of similarity between ∞-proofs.

PROPOSITION 4.9. *The ultrametric space $(\mathcal{P},d_{\mathcal{P}})$ is complete.*

Proof. Assume, in the given space, we a have descending sequence of closed balls

$$B_{r_0}(\pi_0) \supset B_{r_1}(\pi_1) \supset B_{r_2}(\pi_2) \supset \cdots$$

with radii tending to 0. Consider a subsequence of this sequence

$$B_{r'_0}(\pi'_0) \supset B_{r'_1}(\pi'_1) \supset B_{r'_2}(\pi'_2) \supset \cdots$$

such that $r'_i \leqslant 2^{-i}$ for all $i \in \mathbb{N}$. Trivially, we have $\bigcap_{i=0}^{\infty} B_{r_i}(\pi_i) = \bigcap_{i=0}^{\infty} B_{r'_i}(\pi'_i)$. Note also that

$$\pi'_0 \sim_0 \pi'_1 \sim_1 \pi'_2 \sim_2 \cdots.$$

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Consequently there exists an ∞-proof π of the system $\mathsf{Grz}_\infty + \mathsf{cut}$ such that $\pi \sim_i \pi'_i$ for any $i \in \mathbb{N}$. We shall check that $\pi \in B_{r'_i}(\pi'_i)$. For any $i,k \in \mathbb{N}$, we have

$$d_{\mathcal{P}}(\pi'_i,\pi) \leqslant \max\{d_{\mathcal{P}}(\pi'_i,\pi'_{i+k}), d_{\mathcal{P}}(\pi'_{i+k},\pi)\} \leqslant \max\{r'_i, d_{\mathcal{P}}(\pi'_{i+k},\pi)\}.$$

Since $\lim\limits_{n\to\infty} d_{\mathcal{P}}(\pi'_n,\pi) = 0$, we obtain $d_{\mathcal{P}}(\pi'_i,\pi) \leqslant r'_i$ for any $i \in \mathbb{N}$. It follows that $\pi \in \bigcap\limits_{i=0}^{\infty} B_{r'_i}(\pi'_i) = \bigcap\limits_{i=0}^{\infty} B_{r_i}(\pi_i)$. □

In an ultrametric space (M,d), a function $f\colon M \to M$ is called *nonexpansive* if $d(f(x),f(y)) \leqslant d(x,y)$ for all $x,y \in M$. For ultrametric spaces (M,d_M) and (N,d_N), the Cartesian product $M \times N$ can be also considered as an ultrametric space with the metric

$$d_{M\times N}((x_1,y_1),(x_2,y_2)) = \max\{d_M(x_1,x_2), d_N(y_1,y_2)\}.$$

Hence, for any $m \in \mathbb{N}$, we have an ultrametric on the set M^m.

Let us consider another example. For $m \in \mathbb{N}$, let $\mathcal{F}_m$ denote the set of all nonexpansive functions from $\mathcal{P}^m$ to $\mathcal{P}$. Note that any function $\mathsf{u}\colon \mathcal{P}^m \to \mathcal{P}$ is nonexpansive if and only if for any tuples $\vec{\pi}$ and $\vec{\pi}'$, and any $n \in \mathbb{N}$ we have

$$\pi_1 \sim_n \pi'_1, \ldots, \pi_m \sim_n \pi'_m \Rightarrow \mathsf{u}(\vec{\pi}) \sim_n \mathsf{u}(\vec{\pi}').$$

Now we introduce an ultrametric for $\mathcal{F}_m$. For $\pi \in \mathcal{P}$, recall that $|\pi|$ denotes the length of the longest branch in the main fragment of π. For $\mathsf{a},\mathsf{b} \in \mathcal{F}_m$, we write $\mathsf{a} \sim_{n,k} \mathsf{b}$ if $\mathsf{a}(\vec{\pi}) \sim_n \mathsf{b}(\vec{\pi})$ for any $\vec{\pi} \in \mathcal{P}^m$ and, in addition, $\mathsf{a}(\vec{\pi}) \sim_{n+1} \mathsf{b}(\vec{\pi})$ whenever $\sum\limits_{i=1}^{m} |\pi_i| < k$.[4] An ultrametric l_m on $\mathcal{F}_m$ is defined by

$$l_m(\mathsf{a},\mathsf{b}) = \frac{1}{2}\inf\left\{\frac{1}{2^n} + \frac{1}{2^{n+k}} \mid \mathsf{a} \sim_{n,k} \mathsf{b}\right\}.$$

We see that $l_m(\mathsf{a},\mathsf{b}) \leqslant 2^{-n-1} + 2^{-n-k-1}$ if and only if $\mathsf{a} \sim_{n,k} \mathsf{b}$.

Notice that any operator $\mathsf{U}\colon \mathcal{F}_m \to \mathcal{F}_m$ is strictly contractive if and only if for any $\mathsf{a},\mathsf{b} \in \mathcal{F}_m$, and any $n,k \in \mathbb{N}$ we have

$$\mathsf{a} \sim_{n,k} \mathsf{b} \Rightarrow \mathsf{U}(\mathsf{a}) \sim_{n,k+1} \mathsf{U}(\mathsf{b}).$$

PROPOSITION 4.10. *Every strictly contractive mapping* $\mathsf{U}\colon \mathcal{F}_m \to \mathcal{F}_m$ *has a unique fixed-point*.

This proposition can be derived from Theorem 4.8 by proving that the space $(\mathcal{F}_m,l_m)$ is spherically complete. However we give a direct proof of this result in order to show an explicit construction of the required fixed-point.

Proof of Proposition 4.10. Since the set $\mathcal{F}_m$ is nonempty, there is a nonexpansive function $\mathsf{a} \in \mathcal{F}_m$. For any ordinal $\alpha < \omega^2$, we recursively define a function $\mathsf{a}_\alpha\colon \mathcal{P}^m \to \mathcal{P}$ in the following way. We put $\mathsf{a}_0 = \mathsf{a}$ and $\mathsf{a}_{\alpha+1} = \mathsf{U}(\mathsf{a}_\alpha)$. Additionally, in the limit case, we set $\mathsf{a}_{\omega\cdot(n+1)}(\vec{\pi}) = \mathsf{a}_{\omega\cdot n+(\sum_{s=1}^{m}|\pi_s|)+1}(\vec{\pi})$ for any $\vec{\pi} \in \mathcal{P}^m$ and any $n \in \mathbb{N}$.

[4] This definition is inspired by [8, Subsection 2.1]. It reveals our intention to construct mappings on the set of ∞-proofs by co-induction with subinduction on the sum of local heights of the arguments.

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Now we check by transfinite induction up to the ordinal ω^2 that all functions a_α are nonexpansive. The function $\mathsf{a}_0 = \mathsf{a}$ is nonexpansive. The successor case is trivial. It remains to check the limit case. Consider a function $\mathsf{a}_{\omega\cdot(n+1)}$. For all tuples $\vec{\pi}$ and $\vec{\pi}'$ we have $\mathsf{a}_{\omega\cdot(n+1)}(\vec{\pi}) \sim_0 \mathsf{a}_{\omega\cdot(n+1)}(\vec{\pi}')$ by definition. If for tuples $\vec{\pi}$ and $\vec{\pi}'$ and some $h \geqslant 1$ we have $\pi_1 \sim_h \pi'_1, \ldots, \pi_m \sim_h \pi'_m$, then $\sum_{s=1}^m |\pi_s| = \sum_{s=1}^m |\pi'_s| = t$ and $\mathsf{a}_{\omega\cdot(n+1)}(\vec{\pi}) = \mathsf{a}_{\omega\cdot n+t+1}(\vec{\pi}) \sim_h \mathsf{a}_{\omega\cdot n+t+1}(\vec{\pi}') = \mathsf{a}_{\omega\cdot(n+1)}(\vec{\pi}')$, because the function $\mathsf{a}_{\omega\cdot n+t+1}$ is nonexpansive by the induction hypothesis. Therefore all functions a_α are nonexpansive.

In this proof we will write $\mathsf{c} \simeq_\alpha \mathsf{d}$ if $\mathsf{c} \sim_{n,k} \mathsf{d}$ and $\alpha = \omega\cdot n + k$. We claim that for any ordinal $\alpha < \omega^2$ and any ordinal $\beta < \alpha$ we have $\mathsf{a}_\beta \simeq_\beta \mathsf{a}_\alpha$. We check this claim by transfinite induction on α. The case $\alpha = 0$ is trivial.

Suppose $\alpha = \gamma + 1$. By the induction hypothesis, we have $\mathsf{a}_\beta \simeq_\beta \mathsf{a}_\gamma$ for any ordinal $\beta < \gamma$. If $\gamma = 0$, then $\mathsf{a}_0 \simeq_0 \mathsf{a}_1$ and the statement holds. If $\gamma = \gamma' + 1$, then we have $\mathsf{a}_{\gamma'} \simeq_{\gamma'} \mathsf{a}_\gamma$ by the induction hypothesis. Since U is a contractive mapping, we have $\mathsf{a}_\gamma = \mathsf{U}(\mathsf{a}_{\gamma'}) \simeq_{\gamma'+1} \mathsf{U}(\mathsf{a}_\gamma) = \mathsf{a}_\alpha$. In addition, for any $\beta < \gamma$ we see $\mathsf{a}_\beta \simeq_\beta \mathsf{a}_\gamma \simeq_\gamma \mathsf{a}_\alpha$. Hence $\mathsf{a}_\beta \simeq_\beta \mathsf{a}_\alpha$ and the statement holds. If $\gamma = \omega\cdot(n+1)$, then $\mathsf{a}_{\omega\cdot n+k} \simeq_{\omega\cdot n+k} \mathsf{a}_{\omega\cdot(n+1)}$ for any $k \in \mathbb{N}$. We see $\mathsf{a}_{\omega\cdot n+k+1} = \mathsf{U}(\mathsf{a}_{\omega\cdot n+k}) \simeq_{\omega\cdot n+k+1} \mathsf{U}(\mathsf{a}_{\omega\cdot(n+1)}) = \mathsf{a}_\alpha$ for any $k \in \mathbb{N}$. Thus $\mathsf{a}_{\omega\cdot(n+1)}(\vec{\pi}) = \mathsf{a}_{\omega\cdot n+\sum_{s=1}^m |\pi_s|+1}(\vec{\pi}) \sim_{n+1} \mathsf{a}_\alpha(\vec{\pi})$ for any tuple $\vec{\pi} \in \mathcal{P}^m$. Consequently $\mathsf{a}_{\omega\cdot(n+1)} \simeq_{\omega\cdot(n+1)} \mathsf{a}_\alpha$. For any $\beta < \gamma$ we have $\mathsf{a}_\beta \simeq_\beta \mathsf{a}_\gamma \simeq_\gamma \mathsf{a}_\alpha$. Hence $\mathsf{a}_\beta \simeq_\beta \mathsf{a}_\alpha$ and the statement holds.

Now suppose $\alpha = \omega\cdot(n+1)$ for some $n \in \mathbb{N}$. From the induction hypothesis, we have $\mathsf{a}_{\omega\cdot n} \simeq_{\omega\cdot n} \mathsf{a}_{\omega\cdot n+k}$ for any $k \in \mathbb{N}$. Hence $\mathsf{a}_{\omega\cdot(n+1)}(\vec{\pi}) = \mathsf{a}_{\omega\cdot n+\sum_{s=1}^m |\pi_s|+1}(\vec{\pi}) \sim_n \mathsf{a}_{\omega\cdot n}(\vec{\pi})$ for any tuple $\vec{\pi} \in \mathcal{P}^m$ and $\mathsf{a}_{\omega\cdot n} \simeq_{\omega\cdot n} \mathsf{a}_\alpha$. Now consider any $\beta < \alpha$. If $\beta < \omega\cdot n$, then $\mathsf{a}_\beta \simeq_\beta \mathsf{a}_{\omega\cdot n}$ by the induction hypothesis. We see $\mathsf{a}_\beta \simeq_\beta \mathsf{a}_{\omega\cdot n} \simeq_{\omega\cdot n} \mathsf{a}_\alpha$ and $\mathsf{a}_\beta \simeq_\beta \mathsf{a}_\alpha$. Otherwise, $\beta = \omega\cdot n + k$ for some $k \in \mathbb{N}$. It remains to check that $\mathsf{a}_{\omega\cdot n+k} \simeq_{\omega\cdot n+k} \mathsf{a}_\alpha$. Consider any tuple $\vec{\tau} \in \mathcal{P}^m$. We see $\mathsf{a}_{\omega\cdot(n+1)}(\vec{\tau}) = \mathsf{a}_{\omega\cdot n+\sum_{s=1}^m |\tau_s|+1}(\vec{\tau}) \sim_n \mathsf{a}_{\omega\cdot n}(\vec{\tau}) \sim_n \mathsf{a}_{\omega\cdot n+k}(\vec{\tau})$. If $\sum_{s=1}^m |\tau_s| < k$, then $\mathsf{a}_{\omega\cdot n+\sum_{s=1}^m |\tau_s|+1} \simeq_{\omega\cdot n+\sum_{s=1}^m |\tau_s|+1} \mathsf{a}_{\omega\cdot n+k}$ and $\mathsf{a}_{\omega\cdot(n+1)}(\vec{\tau}) = \mathsf{a}_{\omega\cdot n+\sum_{s=1}^m |\tau_s|+1}(\vec{\tau}) \sim_{n+1} \mathsf{a}_{\omega\cdot n+k}(\vec{\tau})$. We have checked that for any ordinal $\alpha < \omega^2$ and any ordinal $\beta < \alpha$ we have $\mathsf{a}_\beta \simeq_\beta \mathsf{a}_\alpha$.

Now we define the required fixed-point a_{ω^2}. We see that for any tuple $\vec{\pi} \in \mathcal{P}^m$

$$\mathsf{a}_0(\vec{\pi}) \sim_0 \mathsf{a}_\omega(\vec{\pi}) \sim_1 \mathsf{a}_{\omega\cdot 2}(\vec{\pi}) \sim_2 \mathsf{a}_{\omega\cdot 3}(\vec{\pi}) \sim_3 \cdots.$$

In other words, we have a descending sequence of closed balls

$$B_1(\mathsf{a}_0(\vec{\pi})) \supset B_{\frac{1}{2}}(\mathsf{a}_\omega(\vec{\pi})) \supset B_{\frac{1}{4}}(\mathsf{a}_{\omega\cdot 2}(\vec{\pi})) \supset B_{\frac{1}{8}}(\mathsf{a}_{\omega\cdot 3}(\vec{\pi})) \supset \cdots$$

with radii tending to 0. By Proposition 4.9, there is an ∞-proof κ such that $\kappa \in \bigcap_{i=0}^{\infty} B_{2^{-i}}(\mathsf{a}_{\omega\cdot i}(\vec{\pi}))$, i.e., $\kappa \sim_i \mathsf{a}_{\omega\cdot i}(\vec{\pi})$ for all $i \in \mathbb{N}$. This ∞-proof is unique since radii of the balls tend to 0. We define $\mathsf{a}_{\omega^2}(\vec{\pi}) = \kappa$. Let us check that the mapping a_{ω^2} is nonexpansive. If for tuples $\vec{\pi}$ and $\vec{\pi}'$ we have $\pi_1 \sim_h \pi'_1, \ldots, \pi_m \sim_h \pi'_m$ for some $h \in \mathbb{N}$, then we can choose i such that $i > h$. We have $\mathsf{a}_{\omega^2}(\vec{\pi}) \sim_h \mathsf{a}_{\omega\cdot i}(\vec{\pi}) \sim_h \mathsf{a}_{\omega\cdot i}(\vec{\pi}') \sim_h \mathsf{a}_{\omega^2}(\vec{\pi}')$. Therefore $\mathsf{a}_{\omega^2}(\vec{\pi}) \sim_h \mathsf{a}_{\omega^2}(\vec{\pi}')$ and the mapping a_{ω^2} is nonexpansive.

Now we claim that $\mathsf{a}_{\omega^2} = \mathsf{U}(\mathsf{a}_{\omega^2})$. It is sufficient to show that $\mathsf{a}_{\omega^2} \simeq_{\omega\cdot i} \mathsf{U}(\mathsf{a}_{\omega^2})$ for any $i \in \mathbb{N}$. By the definition of a_{ω^2}, we see $\mathsf{a}_{\omega^2} \simeq_{\omega\cdot i} \mathsf{a}_{\omega\cdot i}$ for any $i \in \mathbb{N}$. It follows that $\mathsf{a}_{\omega^2} \simeq_{\omega\cdot i} \mathsf{a}_{\omega\cdot i} \simeq_{\omega\cdot i} \mathsf{a}_{\omega\cdot i+1} = \mathsf{U}(\mathsf{a}_{\omega\cdot i}) \simeq_{\omega\cdot i+1} \mathsf{U}(\mathsf{a}_{\omega^2})$. Hence $\mathsf{a}_{\omega^2} \simeq_{\omega\cdot i} \mathsf{U}(\mathsf{a}_{\omega^2})$ for any $i \in \mathbb{N}$ and $\mathsf{a}_{\omega^2} = \mathsf{U}(\mathsf{a}_{\omega^2})$. □

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§5. Admissible rules. In this section, for the system $\mathsf{Grz}_\infty + \mathsf{cut}$, we state admissibility of auxiliary inference rules, which will be used in the proof of the cut-elimination theorem.

Recall that the set $\mathcal{P}$ of all ∞-proofs of the system $\mathsf{Grz}_\infty + \mathsf{cut}$ can be considered as an ultrametric space with the metric $d_{\mathcal{P}}$.

By $\mathcal{P}_n$ we denote the set of all ∞-proofs that do not contain applications of the cut rule in their n-fragments. We also set $\mathcal{P}_0 = \mathcal{P}$.

A mapping $\mathsf{u} : \mathcal{P}^m \to \mathcal{P}$ is called *adequate* if for any $n \in \mathbb{N}$ we have $\mathsf{u}(\pi_1, \ldots, \pi_m) \in \mathcal{P}_n$, whenever $\pi_i \in \mathcal{P}_n$ for all $i \leqslant m$.

In $\mathsf{Grz}_\infty + \mathsf{cut}$, we call a single-premise inference rule *strongly admissible* if there is a nonexpansive adequate mapping $\mathsf{u}\colon \mathcal{P} \to \mathcal{P}$ that maps any ∞-proof of the premise of the rule to an ∞-proof of the conclusion. The mapping u must also satisfy one additional condition: $|\mathsf{u}(\pi)| \leqslant |\pi|$ for any $\pi \in \mathcal{P}$.

In the following lemmata, nonexpansive mappings are defined in a standard way by induction on the local heights of ∞-proofs for the premises. We give a detailed proof for one case of Lemma 5.12 and omit further details.

LEMMA 5.11. *For any finite multisets of formulas* Π *and* Σ, *the inference rule*

$$\mathrm{wk}_{\Pi;\Sigma}\ \frac{\Gamma \Rightarrow \Delta}{\Pi, \Gamma \Rightarrow \Delta, \Sigma}$$

is strongly admissible in $\mathsf{Grz}_\infty + \mathsf{cut}$.

LEMMA 5.12. *For any formulas* A *and* B, *the rules*

$$\mathrm{li}_{A\to B}\ \frac{\Gamma, A \to B \Rightarrow \Delta}{\Gamma, B \Rightarrow \Delta} \qquad \mathrm{ri}_{A\to B}\ \frac{\Gamma, A \to B \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}$$

$$\mathrm{i}_{A\to B}\ \frac{\Gamma \Rightarrow A \to B, \Delta}{\Gamma, A \Rightarrow B, \Delta} \qquad \mathrm{i}_{\bot}\ \frac{\Gamma \Rightarrow \bot, \Delta}{\Gamma \Rightarrow \Delta} \qquad \mathrm{li}_{\Box A}\ \frac{\Gamma \Rightarrow \Box A, \Delta}{\Gamma \Rightarrow A, \Delta}$$

are strongly admissible in $\mathsf{Grz}_\infty + \mathsf{cut}$.

LEMMA 5.13. *For any atomic proposition* p, *the rules*

$$\mathrm{acl}_p\ \frac{\Gamma, p, p \Rightarrow \Delta}{\Gamma, p \Rightarrow \Delta} \qquad \mathrm{acr}_p\ \frac{\Gamma \Rightarrow p, p, \Delta}{\Gamma \Rightarrow p, \Delta}$$

are strongly admissible in $\mathsf{Grz}_\infty + \mathsf{cut}$.

Proof of Lemma 5.12. We define the required nonexpansive adequate mapping only for the last inference rule of Lemma 5.12 and omit other cases. Assume we have a formula A and an ∞-proof π of a sequent $\Sigma \Rightarrow \Upsilon$. If $\Box A \notin \Upsilon$, then we put $\mathrm{li}_{\Box A}(\pi) = \pi$. Otherwise, we define $\mathrm{li}_{\Box A}(\pi)$ by induction on $|\pi|$.

If $|\pi| = 0$, then $\Sigma \Rightarrow \Upsilon$ is an initial sequent. We see that $\Sigma \Rightarrow A, \Upsilon \backslash \{\Box A\}$ is also an initial sequent and define $\mathrm{li}_{\Box A}(\pi)$ as the ∞-proof consisting only of this sequent.

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Suppose $|\pi| > 0$. We define $\mathsf{li}_{\Box A}(\pi)$ according to the last application of an inference rule in π:

$$\rightarrow_{\mathrm{L}} \frac{\begin{array}{cc}\pi_0 & \pi_1\end{array}}{\Gamma, C \to D \Rightarrow \Box A, \Delta} \begin{array}{cc}\Gamma, D \Rightarrow \Box A, \Delta & \Gamma \Rightarrow C, \Box A, \Delta\end{array} \longmapsto \rightarrow_{\mathrm{L}} \frac{\begin{array}{cc}\mathsf{li}_{\Box A}(\pi_0) & \mathsf{li}_{\Box A}(\pi_1)\end{array}}{\Gamma, C \to D \Rightarrow A, \Delta} \begin{array}{cc}\Gamma, D \Rightarrow A, \Delta & \Gamma \Rightarrow C, A, \Delta\end{array},$$

$$\rightarrow_{\mathrm{R}} \frac{\pi_0}{\Gamma \Rightarrow C \to D, \Box A, \Delta} \begin{array}{c}\Gamma, C \Rightarrow D, \Box A, \Delta\end{array} \longmapsto \rightarrow_{\mathrm{R}} \frac{\mathsf{li}_{\Box A}(\pi_0)}{\Gamma \Rightarrow C \to D, A, \Delta} \begin{array}{c}\Gamma, C \Rightarrow D, A, \Delta\end{array},$$

$$\mathrm{refl} \frac{\pi_0}{\Gamma, \Box C \Rightarrow \Box A, \Delta} \begin{array}{c}\Gamma, C, \Box C \Rightarrow \Box A, \Delta\end{array} \longmapsto \mathrm{refl} \frac{\mathsf{li}_{\Box A}(\pi_0)}{\Gamma, \Box C \Rightarrow A, \Delta} \begin{array}{c}\Gamma, C, \Box C \Rightarrow A, \Delta\end{array},$$

$$\mathrm{cut} \frac{\begin{array}{cc}\pi_0 & \pi_1\end{array}}{\Gamma \Rightarrow \Box A, \Delta} \begin{array}{cc}\Gamma \Rightarrow C, \Box A, \Delta & \Gamma, C \Rightarrow \Box A, \Delta\end{array} \longmapsto \mathrm{cut} \frac{\begin{array}{cc}\mathsf{li}_{\Box A}(\pi_0) & \mathsf{li}_{\Box A}(\pi_1)\end{array}}{\Gamma \Rightarrow A, \Delta} \begin{array}{cc}\Gamma \Rightarrow C, A, \Delta & \Gamma, C \Rightarrow A, \Delta\end{array}.$$

In these cases ∞-proofs $\mathsf{li}_{\Box A}(\pi_0)$ (and $\mathsf{li}_{\Box A}(\pi_1)$) are defined since $|\pi_0| < |\pi|$ (and $|\pi_1| < |\pi|$).

Suppose the last inference in π is an application of the rule $(\Box)$ with the principle formula being not $\Box A$. Then we define $\mathsf{li}_{\Box A}(\pi)$ as follows:

$$\Box \frac{\begin{array}{cc}\pi_0 & \pi_1\end{array}}{\Gamma, \Box\Pi \Rightarrow \Box \boldsymbol{C}, \Box \boldsymbol{A}, \Delta} \begin{array}{cc}\Gamma, \Box\Pi \Rightarrow \boldsymbol{C}, \Box \boldsymbol{A}, \Delta & \Box\Pi \Rightarrow \boldsymbol{C}\end{array} \longmapsto \Box \frac{\begin{array}{cc}\mathsf{li}_{\Box A}(\pi_0) & \pi_1\end{array}}{\Gamma, \Box\Pi \Rightarrow \Box C, A, \Delta} \begin{array}{cc}\Gamma, \Box\Pi \Rightarrow C, A, \Delta & \Box\Pi \Rightarrow C\end{array}.$$

It remains to consider the case when the last inference in π has the form:

$$\Box \frac{\begin{array}{cc}\pi_0 & \pi_1\end{array}}{\Gamma, \Box\Pi \Rightarrow \Box A, \Delta} \begin{array}{cc}\Gamma, \Box\Pi \Rightarrow A, \Delta & \Box\Pi \Rightarrow A\end{array}.$$

In this case we put $\mathsf{li}_{\Box A}(\pi) = \pi_0$. The mapping $\mathsf{li}_{\Box A}$ is well-defined.

Now we claim that $\mathsf{li}_{\Box A}$ is nonexpansive. We shall check that for any ∞-proofs π and π', and any $n \in \mathbb{N}$, we have

$$\pi \sim_n \pi' \Rightarrow \mathsf{li}_{\Box A}(\pi) \sim_n \mathsf{li}_{\Box A}(\pi').$$

Assume we have two ∞-proofs π and π' such that $\pi \sim_n \pi'$. If $n = 0$, then trivially $\mathsf{li}_{\Box A}(\pi) \sim_n \mathsf{li}_{\Box A}(\pi')$. If $n > 0$, then the main fragments of π and π' are identical. Suppose the root of π is marked by a sequent $\Sigma \Rightarrow \Upsilon$, where $\Box A \notin \Upsilon$. Then the same condition holds for the ∞-proof π'. In this case, we have

$$\mathsf{li}_{\Box A}(\pi) = \pi \sim_n \pi' = \mathsf{li}_{\Box A}(\pi').$$

Otherwise, the statement $\mathsf{li}_{\Box A}(\pi) \sim_n \mathsf{li}_{\Box A}(\pi')$ is proved by induction on $|\pi| = |\pi'|$ with the case analysis according to the definition of $\mathsf{li}_{\Box A}$. If $|\pi| = |\pi'| = 0$, then $\pi = \pi'$ and $\mathsf{li}_{\Box A}(\pi) \sim_n \mathsf{li}_{\Box A}(\pi')$.

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If the last inference in π is an application of the rule $(\to_{\mathsf{R}})$, then π and π' have the following forms

$$\to_{\mathsf{R}} \dfrac{\begin{matrix}\pi_0\\ \Gamma, C \Rightarrow D, \Box A, \Delta\end{matrix}}{\Gamma \Rightarrow C \to D, \Box A, \Delta} \qquad \to_{\mathsf{R}} \dfrac{\begin{matrix}\pi'_0\\ \Gamma, C \Rightarrow D, \Box A, \Delta\end{matrix}}{\Gamma \Rightarrow C \to D, \Box A, \Delta},$$

where $\pi_0 \sim_n \pi'_0$. By the induction hypothesis for $|\pi_0| = |\pi'_0|$, we have $\mathsf{li}_{\Box A}(\pi_0) \sim_n \mathsf{li}_{\Box A}(\pi'_0)$. From the definition of the mapping $\mathsf{li}_{\Box A}$, we obtain $\mathsf{li}_{\Box A}(\pi) \sim_n \mathsf{li}_{\Box A}(\pi')$.

Consider the case when the last inference in π is an application of the rule $(\Box)$ with the principle formula $\Box A$. We have that π and π' have the following forms

$$\Box \dfrac{\begin{matrix}\pi_0 & \pi_1\\ \Gamma, \Box\Pi \Rightarrow A, \Delta & \Box\Pi \Rightarrow A\end{matrix}}{\Gamma, \Box\Pi \Rightarrow \Box A, \Delta} \qquad \Box \dfrac{\begin{matrix}\pi'_0 & \pi'_1\\ \Gamma, \Box\Pi \Rightarrow A, \Delta & \Box\Pi \Rightarrow A\end{matrix}}{\Gamma, \Box\Pi \Rightarrow \Box A, \Delta},$$

where $\pi_0 \sim_n \pi'_0$ and $\pi_1 \sim_{n-1} \pi'_1$. From the definition of the mapping $\mathsf{li}_{\Box A}$, we obtain $\mathsf{li}_{\Box A}(\pi) = \pi_0 \sim_n \pi'_0 = \mathsf{li}_{\Box A}(\pi')$.

All the other cases are treated similarly, so we omit them. We have that the mapping $\mathsf{li}_{\Box A}$ is nonexpansive. Now we check that $|\mathsf{li}_{\Box A}(\pi)| \leqslant |\pi|$ for any $\pi \in \mathcal{P}$.

Assume we have an ∞-proof π. Suppose the root of π is marked by a sequent $\Sigma \Rightarrow \Upsilon$, where $\Box A \notin \Upsilon$. In this case, we have $\mathsf{li}_{\Box A}(\pi) = \pi$ and $|\mathsf{li}_{\Box A}(\pi)| \leqslant |\pi|$.

Otherwise, the assertion $|\mathsf{li}_{\Box A}(\pi)| \leqslant |\pi|$ is proved by induction on $|\pi|$ with the case analysis according to the definition of $\mathsf{li}_{\Box A}$. If $|\pi| = 0$, then $\Sigma \Rightarrow \Upsilon$ is an initial sequent. We see that $\Sigma \Rightarrow A, \Upsilon \backslash \{\Box A\}$ is also an initial sequent and $\mathsf{li}_{\Box A}(\pi)$ was defined as the ∞-proof consisting only of this sequent. We have $|\mathsf{li}_{\Box A}(\pi)| = 0$ and $|\mathsf{li}_{\Box A}(\pi)| \leqslant |\pi|$.

If the last inference in π is an application of the rule $(\to_{\mathsf{R}})$, then π has the form

$$\to_{\mathsf{R}} \dfrac{\begin{matrix}\pi_0\\ \Gamma, C \Rightarrow D, \Box A, \Delta\end{matrix}}{\Gamma \Rightarrow C \to D, \Box A, \Delta}.$$

By the induction hypothesis for $|\pi_0|$, we have $|\mathsf{li}_{\Box A}(\pi_0)| \leqslant |\pi_0|$. From the definition of the mapping $\mathsf{li}_{\Box A}$, we see $|\mathsf{li}_{\Box A}(\pi)| = |\mathsf{li}_{\Box A}(\pi_0)| + 1 \leqslant |\pi_0| + 1 = |\pi|$.

Consider the case when the last inference in π is an application of the rule $(\Box)$ with the principle formula $\Box A$. We see that π has the form

$$\Box \dfrac{\begin{matrix}\pi_0 & \pi_1\\ \Gamma, \Box\Pi \Rightarrow A, \Delta & \Box\Pi \Rightarrow A\end{matrix}}{\Gamma, \Box\Pi \Rightarrow \Box A, \Delta},$$

From the definition of the mapping $\mathsf{li}_{\Box A}$, we obtain $|\mathsf{li}_{\Box A}(\pi)| = |\pi_0| \leqslant |\pi|$.

Cases of other inferences are treated similarly, so we omit them. We have that $|\mathsf{li}_{\Box A}(\pi)| \leqslant |\pi|$ for any $\pi \in \mathcal{P}$. It remains to check that the mapping $\mathsf{li}_{\Box A}$ is adequate.

Assume we have an ∞-proof π such that $\pi \in \mathcal{P}_n$. We claim that $\mathsf{li}_{\Box A}(\pi) \in \mathcal{P}_n$. If $n = 0$, then trivially $\mathsf{li}_{\Box A}(\pi) \in \mathcal{P}_0 = \mathcal{P}$. Suppose the root of π is marked by a sequent $\Sigma \Rightarrow \Upsilon$, where $\Box A \notin \Upsilon$. Then we have $\mathsf{li}_{\Box A}(\pi) = \pi \in \mathcal{P}_n$.

Otherwise, the assertion $\mathsf{li}_{\Box A}(\pi) \in \mathcal{P}_n$ is proved by induction on $|\pi|$ with the case analysis according to the definition of $\mathsf{li}_{\Box A}$. If $|\pi| = 0$, then $\Sigma \Rightarrow \Upsilon$ is an initial sequent. We see that $\Sigma \Rightarrow A, \Upsilon \backslash \{\Box A\}$ is also an initial sequent and $\mathsf{li}_{\Box A}(\pi)$ was defined as the ∞-proof consisting only of this sequent. We have $\mathsf{li}_{\Box A}(\pi) \in \mathcal{P}_n$.

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If the last inference in π is an application of the rule $(\to_{\mathsf{R}})$, then π has the form

$$\to_{\mathsf{R}} \frac{\begin{matrix}\pi_0\\ \Gamma, C \Rightarrow D, \Box A, \Delta\end{matrix}}{\Gamma \Rightarrow C \to D, \Box A, \Delta},$$

where $\pi_0 \in \mathcal{P}_n$. By the induction hypothesis for $|\pi_0|$, we have $\mathsf{li}_{\Box A}(\pi_0) \in \mathcal{P}_n$. From the definition of the mapping $\mathsf{li}_{\Box A}$, we see $\mathsf{li}_{\Box A}(\pi) \in \mathcal{P}_n$.

Consider the case when the last inference in π is an application of the rule $(\Box)$ with the principle formula $\Box A$. We see that π has the form

$$\Box \frac{\begin{matrix}\pi_0 & \pi_1\\ \Gamma, \Box\Pi \Rightarrow A, \Delta & \Box\Pi \Rightarrow A\end{matrix}}{\Gamma, \Box\Pi \Rightarrow \Box A, \Delta},$$

where $\pi_0 \in \mathcal{P}_n$ and $\pi_1 \in \mathcal{P}_{n-1}$. From the definition of the mapping $\mathsf{li}_{\Box A}$, we obtain $\mathsf{li}_{\Box A}(\pi) = \pi_0 \in \mathcal{P}_n$.

Notice that the last inference in π differs from an application of the rule (cut). The remaining cases of other inferences are treated in a similar way to the case of $(\to_{\mathsf{R}})$, so we omit them. □

§6. Cut elimination. In this section, we construct a continuous function from $\mathcal{P}$ to $\mathcal{P}$, which maps any ∞-proof of the system $\mathsf{Grz}_\infty + \mathsf{cut}$ to a cut-free ∞-proof of the same sequent.

Let us call a pair of ∞-proofs (π, τ) a *cut pair* if π is an ∞-proof of the sequent $\Gamma \Rightarrow \Delta, A$ and τ is an ∞-proof of the sequent $A, \Gamma \Rightarrow \Delta$ for some Γ, Δ, and A. For a cut pair (π, τ), we call the sequent $\Gamma \Rightarrow \Delta$ its *cut result* and the formula A its cut formula.

For a modal formula A, a nonexpansive mapping u from $\mathcal{P} \times \mathcal{P}$ to $\mathcal{P}$ is called *A-removing* if it maps every cut pair (π, τ) with the cut formula A to an ∞-proof of its cut result. By $\mathcal{R}_A$, let us denote the set of all A-removing mappings. Now we consider the set $\mathcal{R}_A$ together with the ultrametric l_2, which was defined in 4

LEMMA 6.14. *In the ultrametric space $(\mathcal{R}_A, l_2)$, any contractive operator $\mathsf{U} : \mathcal{R}_A \to \mathcal{R}_A$ has a unique fixed-point.*

Proof. Let us check that the set $\mathcal{R}_A$ is nonempty. Consider the mapping $\mathsf{u}_{cut} : \mathcal{P}^2 \to \mathcal{P}$ that is defined as follows. For a cut pair (π, τ) with the cut formula A, it joins the ∞-proofs π and τ with an appropriate instance of the rule (cut) . For all other pairs, the mapping u_{cut} returns the first argument. Clearly, u_{cut} is nonexpansive and therefore lies in $\mathcal{R}_A$.

The rest of the proof is completely analogous to the proof of Proposition 4.10 □

In what follows, we use the nonexpansive adequate mappings $\mathsf{wk}_{\Pi;\Sigma}$, $\mathsf{li}_{A\to B}$, $\mathsf{ri}_{A\to B}$, $\mathsf{i}_{A\to B}$, $\mathsf{i}_\bot$, $\mathsf{li}_{\Box A}$, acl_p, acr_p from Lemma 5.11, Lemma 5.12, and Lemma 5.13

LEMMA 6.15. *For any atomic proposition p, there exists an adequate p-removing mapping* re_p.

Proof. Given a cut pair with the cut formula p, consider the ∞-proof obtained by applying the corresponding instance of the cut rule. This instance of cut can be eliminated by "moving" it upwards within the main fragment of the ∞-proof.

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Therefore we can define the required p-removing mapping re_p following the standard cut reductions for cuts with atomic cut formulas.

Assume we have two ∞-proofs π and τ. If the pair (π,τ) is not a cut pair or is a cut pair with the cut formula being not p, then we put $\mathsf{re}_p(\pi,\tau)=\pi$. Otherwise, we define $\mathsf{re}_p(\pi,\tau)$ by induction on $|\pi|$. Let the cut result of the pair (π,τ) be $\Gamma\Rightarrow\Delta$.

If $|\pi|=0$, then $\Gamma\Rightarrow\Delta,p$ is an initial sequent. Suppose that $\Gamma\Rightarrow\Delta$ is also an initial sequent. Then $\mathsf{re}_p(\pi,\tau)$ is defined as the ∞-proof consisting only of the sequent $\Gamma\Rightarrow\Delta$. If $\Gamma\Rightarrow\Delta$ is not an initial sequent, then Γ has the form p,Φ, and τ is an ∞-proof of the sequent $p,p,\Phi\Rightarrow\Delta$. Applying the nonexpansive adequate mapping acl_p from Lemma 5.13, we put $\mathsf{re}_p(\pi,\tau):=\mathsf{acl}_p(\tau)$.

Now suppose that $|\pi|>0$. We define $\mathsf{re}_p(\pi,\tau)$ according to the last application of an inference rule in π:

$$\left(\to_{\mathrm{R}}\dfrac{\begin{matrix}\pi\\ \Gamma,B\Rightarrow C,\Sigma,p\end{matrix}}{\Gamma\Rightarrow B\to C,\Sigma,p},\tau\right)\longmapsto \to_{\mathrm{R}}\dfrac{\begin{matrix}\mathsf{re}_p(\pi_0,\mathrm{i}_{B\to C}(\tau))\\ \Gamma,B\Rightarrow C,\Sigma\end{matrix}}{\Gamma\Rightarrow B\to C,\Sigma},$$

$$\left(\to_{\mathrm{L}}\dfrac{\begin{matrix}\pi_0 & \pi_1\\ \Sigma,C\Rightarrow\Delta,p & \Sigma\Rightarrow B,\Delta,p\end{matrix}}{\Sigma,B\to C\Rightarrow\Delta,p},\tau\right)\longmapsto \to_{\mathrm{L}}\dfrac{\begin{matrix}\mathsf{re}_p(\pi_0,\mathrm{li}_{B\to C}(\tau)) & \mathsf{re}_p(\pi_1,\mathrm{ri}_{B\to C}(\tau))\\ \Sigma,C\Rightarrow\Delta & \Sigma\Rightarrow B,\Delta\end{matrix}}{\Sigma,B\to C\Rightarrow\Delta},$$

$$\left(\mathrm{refl}\dfrac{\begin{matrix}\pi_0\\ \Sigma,B,\Box B\Rightarrow\Delta,p\end{matrix}}{\Sigma,\Box B\Rightarrow\Delta,p},\tau\right)\longmapsto \mathrm{refl}\dfrac{\begin{matrix}\mathsf{re}_p(\pi_0,\mathrm{wk}_{B;\varnothing}(\tau)\\ \Sigma,B,\Box B\Rightarrow\Delta\end{matrix}}{\Sigma,\Box B\Rightarrow\Delta},$$

$$\left(\mathrm{cut}\dfrac{\begin{matrix}\pi_0 & \pi_1\\ \Gamma\Rightarrow B,\Delta,p & \Gamma,B\Rightarrow\Delta,p\end{matrix}}{\Gamma\Rightarrow\Delta,p},\tau\right)\longmapsto \mathrm{cut}\dfrac{\begin{matrix}\mathsf{re}_p(\pi_0,\mathrm{wk}_{\varnothing;B}(\tau)) & \mathsf{re}_p(\pi_1,\mathrm{wk}_{B;\varnothing}(\tau))\\ \Gamma\Rightarrow B,\Delta & \Gamma,B\Rightarrow\Delta\end{matrix}}{\Gamma\Rightarrow\Delta}.$$

In these cases ∞-proofs $\mathsf{re}_p(\pi_0,\cdot)$ (and $\mathsf{re}_p(\pi_1,\cdot)$) are defined since $|\pi_0|<|\pi|$ (and $|\pi_1|<|\pi|$). Consider the last case when the last inference in π is an application of the rule $(\Box)$. We set

$$\left(\Box\dfrac{\begin{matrix}\pi_0 & \pi_1\\ \Phi,\Box\Pi\Rightarrow B,\Sigma,p & \Box\Pi\Rightarrow B\end{matrix}}{\Phi,\Box\Pi\Rightarrow\Box B,\Sigma,p},\tau\right)\longmapsto \Box\dfrac{\begin{matrix}\mathsf{re}_p(\pi_0,\mathrm{li}_{\Box B}(\tau)) & \pi_1\\ \Phi,\Box\Pi\Rightarrow B,\Sigma & \Box\Pi\Rightarrow B\end{matrix}}{\Phi,\Box\Pi\Rightarrow\Box B,\Sigma},$$

where $\mathsf{re}_p(\pi_0,\mathrm{li}_{\Box B}(\tau))$ is defined since $|\pi_0|<|\pi|$. The mapping re_p is well-defined.

Now we claim that re_p is nonexpansive. It sufficient to check that for any pairs (π,τ) and (π',τ'), and any $n\in\mathbb{N}$ we have

$$\pi\sim_n\pi',\ \tau\sim_n\tau'\Rightarrow\mathsf{re}_p(\pi,\tau)\sim_n\mathsf{re}_p(\pi',\tau').$$

Assume we have two pairs of ∞-proofs (π,τ) and (π',τ') such that $\pi\sim_n\pi'$, $\tau\sim_n\tau'$. If $n=0$, then trivially $\mathsf{re}_p(\pi,\tau)\sim_n\mathsf{re}_p(\pi',\tau')$. If $n>0$, then the main fragments of π and π' (τ and τ') are identical. Suppose that (π,τ) is not a cut pair or is a cut pair with

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the cut formula not being p. Then the same condition holds for the pair (π', τ'). In this case, we have

$$\mathsf{re}_p(\pi, \tau) = \pi \sim_n \pi' = \mathsf{re}_p(\pi', \tau').$$

Otherwise, (π, τ) and (π', τ') are cut pairs with the cut formula p. Now the statement $\mathsf{re}_p(\pi, \tau) \sim_n \mathsf{re}_p(\pi', \tau')$ is proved by induction on $|\pi| = |\pi'|$ with the case analysis according to the definition of re_p. Let the cut result of the pair (π, τ) (and (π', τ')) be $\Gamma \Rightarrow \Delta$.

If $|\pi| = |\pi'| = 0$, then $\Gamma \Rightarrow \Delta, p$ is an initial sequent. Suppose that $\Gamma \Rightarrow \Delta$ is an initial sequent. Then $\mathsf{re}_p(\pi, \tau) = \mathsf{re}_p(\pi', \tau')$. If $\Gamma \Rightarrow \Delta$ is not an initial sequent, then $\mathsf{re}_p(\pi, \tau) = \mathsf{acl}_p(\tau) \sim_n \mathsf{acl}_p(\tau') = \mathsf{re}_p(\pi', \tau')$ since the mapping acl_p is nonexpansive.

If the last inference in π is an application of the rule $(\to_{\mathrm{R}})$, then π and π' have the following forms

$$\to_{\mathrm{R}} \frac{\begin{matrix}\pi_0 \\ \Gamma, B \Rightarrow C, \Sigma, p\end{matrix}}{\Gamma \Rightarrow B \to C, \Sigma, p} \qquad \to_{\mathrm{R}} \frac{\begin{matrix}\pi'_0 \\ \Gamma, B \Rightarrow C, \Sigma, p\end{matrix}}{\Gamma \Rightarrow B \to C, \Sigma, p},$$

where $\pi_0 \sim_n \pi'_0$. Since $\mathsf{i}_{B \to C}$ is nonexpansive, we have $\mathsf{i}_{B \to C}(\tau) \sim_n \mathsf{i}_{B \to C}(\tau')$. Consequently, by the induction hypothesis for pairs $(\pi_0, \mathsf{i}_{B \to C}(\tau))$ and $(\pi'_0, \mathsf{i}_{B \to C}(\tau'))$, we have $\mathsf{re}_p(\pi_0, \mathsf{i}_{B \to C}(\tau)) \sim_n \mathsf{re}_p(\pi'_0, \mathsf{i}_{B \to C}(\tau'))$. Thus we obtain $\mathsf{re}_p(\pi, \tau) \sim_n \mathsf{re}_p(\pi', \tau')$.

Consider the case when the last inference in π is an application of the rule $(\Box)$. We have that π and π' have the following forms

$$\Box \frac{\begin{matrix}\pi_0 & \pi_1 \\ \Phi, \Box\Pi \Rightarrow B, \Sigma, p & \Box\Pi \Rightarrow B\end{matrix}}{\Phi, \Box\Pi \Rightarrow \Box B, \Sigma, p} \qquad \Box \frac{\begin{matrix}\pi'_0 & \pi'_1 \\ \Phi, \Box\Pi \Rightarrow B, \Sigma, p & \Box\Pi' \Rightarrow B\end{matrix}}{\Phi, \Box\Pi \Rightarrow \Box B, \Sigma, p},$$

where $\pi_0 \sim_n \pi'_0$ and $\pi_1 \sim_{n-1} \pi'_1$. Since $\mathsf{li}_{\Box B}$ is nonexpansive, we have $\mathsf{li}_{\Box B}(\tau) \sim_n \mathsf{li}_{\Box B}(\tau')$. Consequently, by the induction hypothesis for pairs $(\pi_0, \mathsf{li}_{\Box B}(\tau))$ and $(\pi'_0, \mathsf{li}_{\Box B}(\tau'))$, we have $\mathsf{re}_p(\pi_0, \mathsf{li}_{\Box B}(\tau)) \sim_n \mathsf{re}_p(\pi'_0, \mathsf{li}_{\Box B}(\tau'))$. Thus we obtain $\mathsf{re}_p(\pi, \tau) \sim_n \mathsf{re}_p(\pi', \tau')$.

All the other cases are treated similarly, so we omit them. We have that the mapping re_p is nonexpansive. It remains to check that the mapping re_p is adequate.

Assume we have a pair of ∞-proofs (π, τ) such that $\pi \in \mathcal{P}_n$ and $\tau \in \mathcal{P}_n$. We claim that $\mathsf{re}_p(\pi, \tau) \in \mathcal{P}_n$. If $n = 0$, then trivially $\mathsf{re}_p(\pi, \tau) \in \mathcal{P}_0 = \mathcal{P}$. If the pair (π, τ) is not a cut pair or is a cut pair with the cut formula being not p, then we have $\mathsf{re}_p(\pi, \tau) = \pi \in \mathcal{P}_n$.

Now suppose (π, τ) is a cut pair with the cut formula p and $n > 0$. We proceed by induction on $|\pi|$ with the case analysis according to the definition of re_p. Let the cut result of the pair (π, τ) be $\Gamma \Rightarrow \Delta$.

If $|\pi| = 0$, then $\Gamma \Rightarrow \Delta, p$ is an initial sequent. Suppose that $\Gamma \Rightarrow \Delta$ is an initial sequent. Then $\mathsf{re}_p(\pi, \tau) \in \mathcal{P}_n$ by the definition of re_p. If $\Gamma \Rightarrow \Delta$ is not an initial sequent, then $\mathsf{re}_p(\pi, \tau) = \mathsf{acl}_p(\tau) \in \mathcal{P}_n$ since the mapping acl_p is adequate.

Now let us consider the cases of inference rules $(\to_{\mathrm{L}})$ and $(\Box)$. If the last inference in π is an application of the rule $(\to_{\mathrm{L}})$, then π has the following form

$$\to_{\mathrm{L}} \frac{\begin{matrix}\pi_0 & \pi_1 \\ \Sigma, C \Rightarrow \Delta, p & \Sigma \Rightarrow B, \Delta, p\end{matrix}}{\Sigma, B \to C \Rightarrow \Delta, p},$$

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where $\pi_0,\pi_1 \in \mathcal{P}_n$. Since $\mathsf{li}_{B\to C}$ and $\mathsf{ri}_{B\to C}$ are adequate, we have $\mathsf{li}_{B\to C}(\tau) \in \mathcal{P}_n$ and $\mathsf{ri}_{B\to C}(\tau) \in \mathcal{P}_n$. Consequently, by the induction hypothesis for the pairs $(\pi_0,\mathsf{li}_{B\to C}(\tau))$ and $(\pi_1,\mathsf{ri}_{B\to C}(\tau))$, we have $\mathsf{re}_p(\pi_0,\mathsf{li}_{B\to C}(\tau)) \in \mathcal{P}_n$ and $\mathsf{re}_p(\pi_1,\mathsf{ri}_{B\to C}(\tau)) \in \mathcal{P}_n$. Thus we obtain $\mathsf{re}_p(\pi,\tau) \in \mathcal{P}_n$.

Consider the case when the last inference in π is an application of the rule $(\Box)$. Then π has the form

$$\Box\,\frac{\begin{matrix}\pi_0 & \pi_1\\ \Phi,\Box\Pi \Rightarrow B,\Sigma,p & \Box\Pi \Rightarrow B\end{matrix}}{\Phi,\Box\Pi \Rightarrow \Box B,\Sigma,p}\,,$$

where $\pi_0 \in \mathcal{P}_n$ and $\pi_1 \in \mathcal{P}_{n-1}$. Since $\mathsf{li}_{\Box B}$ is adequate, we have $\mathsf{li}_{\Box B}(\tau) \in \mathcal{P}_n$. Consequently, by the induction hypothesis for the pair $(\pi_0,\mathsf{li}_{\Box B}(\tau))$, we have $\mathsf{re}_p(\pi_0,\mathsf{li}_{\Box B}(\tau)) \in \mathcal{P}_n$. Thus we obtain $\mathsf{re}_p(\pi,\tau) \in \mathcal{P}_n$.

Notice that the last inference in π differs from an application of the rule (cut). The remaining cases of inference rules $(\to_{\mathsf{R}})$ and (refl) are treated in a similar way to the case of $(\to_{\mathsf{L}})$, so we omit them. □

LEMMA 6.16. *Given an adequate B-removing mapping re_B, there exists an adequate $\Box B$-removing mapping $\mathsf{re}_{\Box B}$.*

Proof. Given a cut pair with the cut formula $\Box B$, consider the ∞-proof obtained by applying the corresponding instance of the cut rule. By ordinary cut reductions we can "move" it upwards within the ∞-proof arbitrarily high through the fragments. The desired result of applying the $\Box B$-removing operator can be viewed as the limit of this procedure. To do it accurately we define the mapping $\mathsf{re}_{\Box B}$ as the fixed-point of a contractive operator $\mathsf{G}_{\Box B}\colon \mathcal{R}_{\Box B} \to \mathcal{R}_{\Box B}$.

For a mapping $\mathsf{u} \in \mathcal{R}_{\Box B}$ and a pair of ∞-proofs (π,τ), the ∞-proof $\mathsf{G}_{\Box B}(\mathsf{u})(\pi,\tau)$ is defined as follows. If (π,τ) is not a cut pair or a cut pair with the cut formula being not $\Box B$, then we put $\mathsf{G}_{\Box B}(\mathsf{u})(\pi,\tau) = \pi$.

Now let (π,τ) be a cut pair with the cut formula $\Box B$ and the cut result $\Gamma \Rightarrow \Delta$. If $|\pi| = 0$ or $|\tau| = 0$, then $\Gamma \Rightarrow \Delta$ is an initial sequent. In this case, we define $\mathsf{G}_{\Box B}(\mathsf{u})(\pi,\tau)$ as the ∞-proof consisting only of the sequent $\Gamma \Rightarrow \Delta$.

Suppose that $|\pi| > 0$ and $|\tau| > 0$. We define $\mathsf{G}_{\Box B}(\mathsf{u})(\pi,\tau)$ according to the last application of an inference rule in π:

$$\left(\to_{\mathrm{R}}\frac{\begin{matrix}\pi_0\\ \Gamma,C \Rightarrow D,\Sigma,\Box B\end{matrix}}{\Gamma \Rightarrow C \to D,\Sigma,\Box B},\tau\right) \longmapsto \to_{\mathrm{R}}\frac{\begin{matrix}\mathsf{u}(\pi_0,\mathsf{i}_{C\to D}(\tau))\\ \Gamma,C \Rightarrow D,\Sigma\end{matrix}}{\Gamma \Rightarrow C \to D,\Sigma}\,,$$

$$\left(\to_{\mathrm{L}}\frac{\begin{matrix}\pi_0 & \pi_1\\ \Sigma,D \Rightarrow \Delta,\Box B & \Sigma \Rightarrow C,\Delta,\Box B\end{matrix}}{\Sigma,C \to D \Rightarrow \Delta,\Box B},\tau\right) \longmapsto \to_{\mathrm{L}}\frac{\begin{matrix}\mathsf{u}(\pi_0,\mathsf{li}_{C\to D}(\tau)) & \mathsf{u}(\pi_1,\mathsf{ri}_{C\to D}(\tau))\\ \Sigma,D \Rightarrow \Delta & \Sigma \Rightarrow C,\Delta\end{matrix}}{\Sigma,C \to D \Rightarrow \Delta}\,,$$

$$\left(\mathsf{refl}\,\frac{\begin{matrix}\pi_0\\ \Sigma,C,\Box C \Rightarrow \Delta,\Box B\end{matrix}}{\Sigma,\Box C \Rightarrow \Delta,\Box B},\tau\right) \longmapsto \mathsf{refl}\,\frac{\begin{matrix}\mathsf{u}(\pi_0,\mathsf{wk}_{C;\varnothing}(\tau))\\ \Sigma,C,\Box C \Rightarrow \Delta\end{matrix}}{\Sigma,\Box C \Rightarrow \Delta}\,,$$

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$$\left(\mathrm{cut}\ \frac{\begin{matrix}\pi_0 & \pi_1\\ \Gamma\Rightarrow C,\Delta,\Box B & \Gamma,C\Rightarrow\Delta,\Box B\end{matrix}}{\Gamma\Rightarrow\Delta,\Box B},\tau\right)\longmapsto \mathrm{cut}\ \frac{\begin{matrix}\mathsf{u}(\pi_0,\mathrm{wk}_{\varnothing;C}(\tau)) & \mathsf{u}(\pi_1,\mathrm{wk}_{C;\varnothing}(\tau))\\ \Gamma\Rightarrow C,\Delta & \Gamma,C\Rightarrow\Delta\end{matrix}}{\Gamma\Rightarrow\Delta},$$

$$\left(\Box\ \frac{\begin{matrix}\pi_0 & \pi_1\\ \Phi,\Box\Pi\Rightarrow C,\Sigma,\Box B & \Box\Pi\Rightarrow C\end{matrix}}{\Phi,\Box\Pi\Rightarrow\Box C,\Sigma,\Box B},\tau\right)\longmapsto \Box\ \frac{\begin{matrix}\mathsf{u}(\pi_0,\mathrm{li}_{\Box C}(\tau)) & \pi_1\\ \Phi,\Box\Pi\Rightarrow C,\Sigma & \Box\Pi\Rightarrow\end{matrix}}{\Phi,\Box\Pi\Rightarrow\Box C,\Sigma}.$$

Consider the case when π has the form

$$\Box\ \frac{\begin{matrix}\pi_0 & \pi_1\\ \Phi,\Box\Pi\Rightarrow B,\Sigma & \Box\Pi\Rightarrow B\end{matrix}}{\Phi,\Box\Pi\Rightarrow\Box B,\Sigma}.$$

We define $\mathsf{G}_{\Box B}(\mathsf{u})(\pi,\tau)$ according to the last application of an inference rule in τ. If the last inference is an application of the rule (refl) with the principal formula being not $\Box B$ or is an application of the rule $(\Box)$ without the formula $\Box B$ in the right premise, then $\mathsf{G}_{\Box B}(\mathsf{u})(\pi,\tau)$ is defined similarly to the previous cases of (refl) and $(\Box)$. Cases of inference rules $(\to_{\mathsf{L}})$, $(\to_{\mathsf{R}})$, and (cut) are also completely similar to the previous cases of $(\to_{\mathsf{L}})$, $(\to_{\mathsf{R}})$, and (cut).

If the last application of an inference rule in τ is an application of the rule (refl) with the principal formula $\Box B$, then we put

$$\mathsf{G}_{\Box B}(\mathsf{u}):\left(\pi,\ \mathsf{refl}\ \frac{\begin{matrix}\tau_0\\ \Gamma,B,\Box B\Rightarrow\Delta\end{matrix}}{\Gamma,\Box B\Rightarrow\Delta}\right)\longmapsto \mathrm{re}_B(\pi_0,\mathsf{u}(\mathrm{wk}_{B;\varnothing}(\pi),\tau_0)).$$

It remains to consider the crucial case when τ has the form

$$\Box\ \frac{\begin{matrix}\tau_0 & \tau_1\\ \Phi',\Box B,\Box\Pi'\Rightarrow C,\Sigma' & \Box B,\Box\Pi'\Rightarrow C\end{matrix}}{\Phi',\Box B,\Box\Pi'\Rightarrow\Box C,\Sigma'}.$$

We define $\mathsf{G}_{\Box B}(\mathsf{u})(\pi,\tau)$ as

$$\Box\ \frac{\begin{matrix}\mathsf{u}(\mathrm{li}_{\Box C}(\pi),\tau_0) & \mathsf{u}\big(\pi',\mathrm{wk}_{\Box\Pi\setminus\Box\Pi';\varnothing}(\tau_1)\big)\\ \Phi',\Box\Pi'\Rightarrow C,\Sigma' & \Box\Pi\setminus\Box\Pi',\Box\Pi'\Rightarrow C\end{matrix}}{\Phi',\Box\Pi'\Rightarrow\Box C,\Sigma'},$$

where π' equals

$$\Box\ \frac{\begin{matrix}\mathrm{wk}_{\Box\Pi'\setminus\Box\Pi;C}(\pi_1) & \pi_1\\ \Box\Pi'\setminus\Box\Pi,\Box\Pi\Rightarrow B,C & \Box\Pi\Rightarrow B\end{matrix}}{\Box\Pi'\setminus\Box\Pi,\Box\Pi\Rightarrow\Box B,C}.$$

Notice that $\Phi',\Box\Pi'=\Phi,\Box\Pi$ and $\Box C,\Sigma'=\Sigma$, and $\Box\Pi'\setminus\Box\Pi,\Box\Pi=\Box\Pi\setminus\Box\Pi',\Box\Pi'$.

Now the operator $\mathsf{G}_{\Box B}$ is well-defined. By the case analysis according to the definition of $\mathsf{G}_{\Box B}$, we see that $\mathsf{G}_{\Box B}(\mathsf{u})$ is nonexpansive and belongs to $\mathcal{R}_{\Box B}$ whenever $\mathsf{u}\in\mathcal{R}_{\Box B}$.

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We claim that $\mathsf{G}_{\Box B}$ is contractive. It sufficient to check that for any $\mathsf{u},\mathsf{v} \in \mathcal{R}_{\Box B}$ and any $n,k \in \mathbb{N}$ we have

$$\mathsf{u} \sim_{n,k} \mathsf{v} \Rightarrow \mathsf{G}_{\Box B}(\mathsf{u}) \sim_{n,k+1} \mathsf{G}_{\Box B}(\mathsf{v}).$$

Assume there are two $\Box B$-removing mappings u and v such that $\mathsf{u} \sim_{n,k} \mathsf{v}$. Consider an arbitrary pair of ∞-proofs (π,τ). By the case analysis according to the definition of $\mathsf{G}_{\Box B}$, we prove that $\mathsf{G}_{\Box B}(\mathsf{u})(\pi,\tau) \sim_n \mathsf{G}_{\Box B}(\mathsf{v})(\pi,\tau)$. In addition, we check that if $|\pi|+|\tau|<k+1$, then $\mathsf{G}_{\Box B}(\mathsf{u})(\pi,\tau) \sim_{n+1} \mathsf{G}_{\Box B}(\mathsf{v})(\pi,\tau)$.

If the pair (π,τ) is not a cut pair or is a cut pair with the cut formula being not $\Box B$, then we have $\mathsf{G}_{\Box B}(\mathsf{u})(\pi,\tau) = \pi = \mathsf{G}_{\Box B}(\mathsf{v})(\pi,\tau)$. Otherwise, consider last inferences in π and τ.

Suppose that the ∞-proof π has the form

$$\mathrm{refl}\,\dfrac{\overset{\pi_0}{\Sigma, C, \Box C \Rightarrow \Delta, \Box B}}{\Sigma, \Box C \Rightarrow \Delta, \Box B}\,.$$

Then we have $\mathsf{u}(\pi_0, \mathsf{wk}_{C;\emptyset}(\tau)) \sim_n \mathsf{v}(\pi_0, \mathsf{wk}_{C;\emptyset}(\tau))$. Hence, $\mathsf{G}_{\Box B}(\mathsf{u})(\pi,\tau) \sim_n \mathsf{G}_{\Box B}(\mathsf{v})(\pi,\tau)$.

In addition, if $|\pi|+|\tau|<k+1$, then $|\pi_0|+|\mathsf{wk}_{C;\emptyset}(\tau)|<k$. Thus, $\mathsf{u}(\pi_0, \mathsf{wk}_{C;\emptyset}(\tau)) \sim_{n+1} \mathsf{v}(\pi_0, \mathsf{wk}_{C;\emptyset}(\tau))$. We obtain that $\mathsf{G}_{\Box B}(\mathsf{u})(\pi,\tau) \sim_{n+1} \mathsf{G}_{\Box B}(\mathsf{v})(\pi,\tau)$.

Now let us consider the case when π and τ have the forms

$$\Box\,\dfrac{\overset{\pi_0}{\Phi, \Box\Pi \Rightarrow B, \Sigma} \qquad \overset{\pi_1}{\Box\Pi \Rightarrow B}}{\Phi, \Box\Pi \Rightarrow \Box B, \Sigma}\,, \quad \mathrm{refl}\,\dfrac{\overset{\tau_0}{\Gamma, B, \Box B \Rightarrow \Delta}}{\Gamma, \Box B \Rightarrow \Delta}\,.$$

We see that $\mathsf{u}(\mathsf{wk}_{B;\emptyset}(\pi),\tau_0) \sim_n \mathsf{v}(\mathsf{wk}_{B;\emptyset}(\pi),\tau_0)$. Since the mapping re_B is nonexpansive, we have

$$\mathsf{G}_{\Box B}(\mathsf{u})(\pi,\tau) = \mathsf{re}_B(\pi_0, \mathsf{u}(\mathsf{wk}_{B;\emptyset}(\pi),\tau_0)) \sim_n \mathsf{re}_B(\pi_0, \mathsf{v}(\mathsf{wk}_{B;\emptyset}(\pi),\tau_0)) = \mathsf{G}_{\Box B}(\mathsf{v})(\pi,\tau).$$

In addition, if $|\pi|+|\tau|<k+1$, then $|\mathsf{wk}_{B;\emptyset}(\pi)|+|\tau_0|<k$. Consequently, $\mathsf{u}(\mathsf{wk}_{B;\emptyset}(\pi),\tau_0) \sim_{n+1} \mathsf{v}(\mathsf{wk}_{B;\emptyset}(\pi),\tau_0)$. Thus, we have

$$\mathsf{G}_{\Box B}(\mathsf{u})(\pi,\tau) = \mathsf{re}_B(\pi_0, \mathsf{u}(\mathsf{wk}_{B;\emptyset}(\pi),\tau_0)) \sim_{n+1} \mathsf{re}_B(\pi_0, \mathsf{v}(\mathsf{wk}_{B;\emptyset}(\pi),\tau_0)) = \mathsf{G}_{\Box B}(\mathsf{v})(\pi,\tau).$$

Let us consider the main case when π and τ have the forms

$$\Box\,\dfrac{\overset{\pi_0}{\Phi, \Box\Pi \Rightarrow B, \Sigma} \qquad \overset{\pi_1}{\Box\Pi \Rightarrow B}}{\Phi, \Box\Pi \Rightarrow \Box B, \Sigma}\,, \quad \Box\,\dfrac{\overset{\tau_0}{\Phi', \Box B, \Box\Pi' \Rightarrow C, \Sigma'} \qquad \overset{\tau_1}{\Box B, \Box\Pi' \Rightarrow C}}{\Phi', \Box B, \Box\Pi' \Rightarrow \Box C, \Sigma'}\,.$$

We have $\mathsf{u}\big(\pi', \mathsf{wk}_{\Box\Pi\setminus\Box\Pi';\emptyset}(\tau_1)\big) \sim_n \mathsf{v}\big(\pi', \mathsf{wk}_{\Box\Pi\setminus\Box\Pi';\emptyset}(\tau_1)\big)$, and $\mathsf{u}(\mathsf{li}_{\Box C}(\pi),\tau_0) \sim_n \mathsf{v}(\mathsf{li}_{\Box C}(\pi),\tau_0)$. Hence, $\mathsf{G}_{\Box B}(\mathsf{u})(\pi,\tau) \sim_n \mathsf{G}_{\Box B}(\mathsf{v})(\pi,\tau)$.

If $|\pi|+|\tau|<k+1$, then $\mathsf{u}\big(\pi', \mathsf{wk}_{\Box\Pi\setminus\Box\Pi';\emptyset}(\tau_1)\big) \sim_n \mathsf{v}\big(\pi', \mathsf{wk}_{\Box\Pi\setminus\Box\Pi';\emptyset}(\tau_1)\big)$ and $\mathsf{u}(\mathsf{li}_{\Box C}(\pi),\tau_0) \sim_{n+1} \mathsf{v}(\mathsf{li}_{\Box C}(\pi),\tau_0)$, because $|\mathsf{li}_{\Box C}(\pi)|+|\tau_0|<k$. We obtain $\mathsf{G}_{\Box B}(\mathsf{u})(\pi,\tau) \sim_{n+1} \mathsf{G}_{\Box B}(\mathsf{v})(\pi,\tau)$.

All the other cases of lowermost inferences in π and τ are treated similarly, so we omit them. We have that the operator $\mathsf{G}_{\Box\mathsf{B}}$ is contractive.

Now we define the required $\Box B$-removing mapping $\mathsf{re}_{\Box B}$ as the fixed-point of the the operator $\mathsf{G}_{\Box B}\colon \mathcal{R}_{\Box B} \to \mathcal{R}_{\Box B}$, which exists by Lemma 6.14

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It remains to check that the mapping $\mathsf{re}_{\Box B}$ is adequate. For any $n,k \in \mathbb{N}$, let us call a mapping $\mathsf{u} \in \mathcal{R}_{\Box B}(n,k)$-adequate if it satisfies the following two conditions: $\mathsf{u}(\pi,\tau) \in \mathcal{P}_i$ for any $i \leqslant n$ and any $\pi,\tau \in \mathcal{P}_i$; $\mathsf{u}(\pi,\tau) \in \mathcal{P}_{n+1}$ whenever $\pi,\tau \in \mathcal{P}_{n+1}$ and $|\pi|+|\tau| < k$.

We claim that $\mathsf{G}_{\Box B}(\mathsf{u})$ is $(n,k+1)$-adequate whenever u is a (n,k)-adequate mapping from $\mathcal{R}_{\Box B}$. Consider an arbitrary pair of ∞-proofs (π,τ) and an (n,k)-adequate mapping $\mathsf{u} \in \mathcal{R}_{\Box B}$. By the case analysis according to the definition of $\mathsf{G}_{\Box \mathsf{B}}$, we prove that $\mathsf{G}_{\Box B}(\mathsf{u})(\pi,\tau) \in \mathcal{P}_i$ if $i \leqslant n$ and $\pi,\tau \in \mathcal{P}_i$. In addition, we check that $\mathsf{G}_{\Box B}(\mathsf{u})(\pi,\tau) \in \mathcal{P}_{n+1}$ if $|\pi|+|\tau| < k+1$ and $\pi,\tau \in \mathcal{P}_{n+1}$.

If the pair (π,τ) is not a cut pair or is a cut pair with the cut formula being not $\Box B$, then we have $\mathsf{G}_{\Box B}(\mathsf{u})(\pi,\tau) = \pi$. In this case, the required statements trivially hold. Otherwise, consider last inferences in π and τ.

Suppose that the ∞-proof π has the form

$$\rightarrow_{\mathrm{R}} \dfrac{\overset{\pi_0}{\Gamma, C \Rightarrow D, \Sigma, \Box B}}{\Gamma \Rightarrow C \to D, \Sigma, \Box B}\,.$$

Since the mapping $\mathsf{i}_{C\to D}$ is adequate, we have $\mathsf{u}(\pi_0, \mathsf{i}_{C\to D}(\tau)) \in \mathcal{P}_i$ whenever $i \leqslant n$ and $\pi,\tau \in \mathcal{P}_i$. In this case, we see that $\mathsf{G}_{\Box B}(\mathsf{u})(\pi,\tau) \in \mathcal{P}_i$.

If $|\pi|+|\tau| < k+1$, then $|\pi_0|+|\mathsf{i}_{C\to D}(\tau)| < k$. Thus, $\mathsf{u}(\pi_0, \mathsf{i}_{C\to D}(\tau)) \in \mathcal{P}_{n+1}$ whenever $\pi,\tau \in \mathcal{P}_{n+1}$. We obtain that $\mathsf{G}_{\Box B}(\mathsf{u})(\pi,\tau) \in \mathcal{P}_{n+1}$.

Suppose that π and τ have the forms

$$\Box \dfrac{\overset{\pi_0}{\Phi, \Box\Pi \Rightarrow B, \Sigma} \qquad \overset{\pi_1}{\Box\Pi \Rightarrow B}}{\Phi, \Box\Pi \Rightarrow \Box B, \Sigma}\,, \quad \Box \dfrac{\overset{\tau_0}{\Phi', \Box B, \Box\Pi' \Rightarrow C, \Sigma'} \qquad \overset{\tau_1}{\Box B, \Box\Pi' \Rightarrow C}}{\Phi', \Box B, \Box\Pi' \Rightarrow \Box C, \Sigma'}\,.$$

Trivially, $\mathsf{G}_{\Box B}(\mathsf{u})(\pi,\tau) \in \mathcal{P}_0 = \mathcal{P}$. Thus consider the case when $\pi,\tau \in \mathcal{P}_i$ for some $0 < i \leqslant n$. Then we see that $\pi_0,\tau_0 \in \mathcal{P}_i$ and $\pi_1,\tau_1 \in \mathcal{P}_{i-1}$. Since the mapping $\mathsf{wk}_{\Box\Pi\setminus\Box\Pi';\emptyset}$ and $\mathsf{li}_{\Box C}$ are adequate, we have $\mathsf{u}(\mathsf{li}_{\Box C}(\pi),\tau_0) \in \mathcal{P}_i$ and $\mathsf{u}\left(\pi', \mathsf{wk}_{\Box\Pi\setminus\Box\Pi';\emptyset}(\tau_1)\right) \in \mathcal{P}_{i-1}$. Hence, $\mathsf{G}_{\Box B}(\mathsf{u})(\pi,\tau) \in \mathcal{P}_i$.

If $|\pi|+|\tau| < k+1$ and $\pi,\tau \in \mathcal{P}_{n+1}$, then $|\mathsf{li}_{\Box C}(\pi)|+|\tau_0| < k$ and $\mathsf{u}(\mathsf{li}_{\Box C}(\pi),\tau_0) \in \mathcal{P}_{n+1}$. Also, we see that $\pi' \in \mathcal{P}_n$ and $\mathsf{wk}_{\Box\Pi\setminus\Box\Pi';\emptyset}(\tau_1) \in \mathcal{P}_n$. Hence, $\mathsf{u}\left(\pi', \mathsf{wk}_{\Box\Pi\setminus\Box\Pi';\emptyset}(\tau_1)\right) \in \mathcal{P}_n$. We obtain $\mathsf{G}_{\Box B}(\mathsf{u})(\pi,\tau) \in \mathcal{P}_{n+1}$.

We omit other cases of lowermost inferences in π and τ, because they are treated similarly.

We established that $\mathsf{G}_{\Box B}(\mathsf{u})$ is $(n,k+1)$-adequate for any (n,k)-adequate $\mathsf{u} \in \mathcal{R}_{\Box B}$. Notice that if a mapping u is (n,k)-adequate for all $k \in \mathbb{N}$, then it is also $(n+1,0)$-adequate. Now by induction on n with a subinduction on k, we immediately obtain that the mapping $\mathsf{re}_{\Box B}$, which is a fixed-point of $\mathsf{G}_{\Box B}$, is (n,k)-adequate for all $n,k \in \mathbb{N}$. Therefore the mapping $\mathsf{re}_{\Box B}$ is adequate. □

LEMMA 6.17. *For any formula A, there exists an adequate A-removing mapping re_A.*

Proof. We define re_A by induction on the structure of the formula A.

Case 1: A has the form p. In this case, re_p is defined by Lemma 6.15.

Case 2: A has the form $\bot$. Then we put $\mathsf{re}_\bot(\pi,\tau) := \mathsf{i}_\bot(\pi)$, where $\mathsf{i}_\bot$ is the nonexpansive adequate mapping from Lemma 5.12.

Case 3: A has the form $B \to C$. Then we put

$$\mathsf{re}_{B\to C}(\pi,\tau) := \mathsf{re}_C(\mathsf{re}_B(\mathsf{wk}_{\emptyset;C}(\mathsf{ri}_{B\to C}(\tau)), \mathsf{i}_{B\to C}(\pi)), \mathsf{li}_{B\to C}(\tau)),$$

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where $\mathsf{ri}_{B\to C}$, $\mathsf{i}_{B\to C}$, $\mathsf{li}_{B\to C}$ are the nonexpansive adequate mappings from Lemma 5.12 and $\mathsf{wk}_{\emptyset;C}$ is the nonexpansive adequate mapping from Lemma 5.11.

Case 4: A has the form $\Box B$. By the induction hypothesis, there is an adequate B-removing mapping re_B. The required $\Box B$-removing mapping $\mathsf{re}_{\Box B}$ exists by Lemma 6.16. □

A mapping $\mathsf{u}\colon \mathcal{P}\to\mathcal{P}$ is called *root-preserving* if it maps ∞-proofs to ∞-proofs of the same sequents. Let $\mathcal{T}$ denote the set of all root-preserving nonexpansive mappings from $\mathcal{P}$ to $\mathcal{P}$.

LEMMA 6.18. *In the ultrametric space* $(\mathcal{T},l_1)$, *any contractive operator* $\mathsf{U}:\mathcal{T}\to\mathcal{T}$ *has a unique fixed-point*.

Proof. The space is obviously nonempty, since the identity function lies in $\mathcal{T}$. The proof is analogous to the proof of Proposition 4.10 □

THEOREM 6.19 (cut-elimination). *If* $\mathsf{Grz}_\infty+\mathsf{cut}\vdash\Gamma\Rightarrow\Delta$, *then* $\mathsf{Grz}_\infty\vdash\Gamma\Rightarrow\Delta$.

Proof. We construct the required cut-elimination mapping ce so it commutes with every application of inference rules except (cut) and satisfies the following condition:

$$\mathsf{ce}\left(\mathsf{cut}\,\frac{\overset{\pi_0}{\Gamma\Rightarrow A,\Delta}\qquad\overset{\pi_1}{\Gamma,A\Rightarrow\Delta}}{\Gamma\Rightarrow\Delta}\right)=\mathsf{re}_A(\mathsf{ce}(\pi_0),\mathsf{ce}(\pi_1)).$$

In order to do this, we define a contractive operator $\mathsf{F}\colon\mathcal{T}\to\mathcal{T}$ and obtain the mapping ce as the fixed-point of F.

For a mapping $\mathsf{u}\in\mathcal{T}$ and an ∞-proof π, the ∞-proof $\mathsf{F}(\mathsf{u})(\pi)$ is defined as follows. If $|\pi|=0$, then we define $\mathsf{F}(\mathsf{u})(\pi)$ to be π.

Otherwise, we define $\mathsf{F}(\mathsf{u})(\pi)$ according to the last application of an inference rule in π:

$$\to_{\mathrm{L}}\frac{\overset{\pi_0}{\Gamma,B\Rightarrow\Delta}\qquad\overset{\pi_1}{\Gamma\Rightarrow A,\Delta}}{\Gamma,A\to B\Rightarrow\Delta}\longmapsto\to_{\mathrm{L}}\frac{\overset{\mathsf{u}(\pi_0)}{\Gamma,B\Rightarrow\Delta}\qquad\overset{\mathsf{u}(\pi_1)}{\Gamma\Rightarrow A,\Delta}}{\Gamma,A\to B\Rightarrow\Delta},$$

$$\to_{\mathrm{R}}\frac{\overset{\pi_0}{\Gamma,A\Rightarrow B,\Delta}}{\Gamma\Rightarrow A\to B,\Delta}\longmapsto\to_{\mathrm{R}}\frac{\overset{\mathsf{u}(\pi_0)}{\Gamma,A\Rightarrow B,\Delta}}{\Gamma\Rightarrow A\to B,\Delta},$$

$$\mathsf{refl}\,\frac{\overset{\pi_0}{\Gamma,A,\Box A\Rightarrow\Delta}}{\Gamma,\Box A\Rightarrow\Delta}\longmapsto\mathsf{refl}\,\frac{\overset{\mathsf{u}(\pi_0)}{\Gamma,A,\Box A\Rightarrow\Delta}}{\Gamma,\Box A\Rightarrow\Delta},$$

$$\Box\,\frac{\overset{\pi_0}{\Gamma,\Box\Pi\Rightarrow A,\Delta}\qquad\overset{\pi_1}{\Box\Pi\Rightarrow A}}{\Gamma,\Box\Pi\Rightarrow\Box A,\Delta}\longmapsto\Box\,\frac{\overset{(\pi_0)}{\Gamma,\Box\Pi\Rightarrow A,\Delta}\qquad\overset{(\pi_1)}{\Box\Pi\Rightarrow A}}{\Gamma,\Box\Pi\Rightarrow\Box A,\Delta},$$

$$\mathsf{cut}\,\frac{\overset{\pi_0}{\Gamma\Rightarrow A,\Delta}\qquad\overset{\pi_1}{\Gamma,A\Rightarrow\Delta}}{\Gamma\Rightarrow\Delta}\longmapsto\mathsf{re}_A(\mathsf{u}(\pi_0),\mathsf{u}(\pi_1)).$$

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Now the operator F is well-defined. By the case analysis according to the definition of F, we see that $\mathsf{F}(\mathsf{u})$ is nonexpansive and belongs to $\mathcal{T}$ whenever $\mathsf{u} \in \mathcal{T}$.

We claim that F is contractive. It sufficient to check that for any $\mathsf{u},\mathsf{v} \in \mathcal{T}$ and any $n,k \in \mathbb{N}$ we have

$$\mathsf{u} \sim_{n,k} \mathsf{v} \Rightarrow \mathsf{F}(\mathsf{u}) \sim_{n,k+1} \mathsf{F}(\mathsf{v}).$$

Assume there are mappings u and v from $\mathcal{T}$ such that $\mathsf{u} \sim_{n,k} \mathsf{v}$. Consider an arbitrary ∞-proof π. By the case analysis according to the definition of F, we prove that $\mathsf{F}(\mathsf{u})(\pi) \sim_n \mathsf{F}(\mathsf{v})(\pi)$. In addition, we check that if $|\pi| < k+1$, then $\mathsf{F}(\mathsf{u})(\pi) \sim_{n+1} \mathsf{F}(\mathsf{v})(\pi)$.

Let us consider the case when π has the form

$$\Box \dfrac{\overset{\pi_0}{\Gamma, \Box\Pi \Rightarrow A, \Delta} \qquad \overset{\pi_1}{\Box\Pi \Rightarrow A}}{\Gamma, \Box\Pi \Rightarrow \Box A, \Delta}.$$

We have $\mathsf{u}(\pi_0) \sim_n \mathsf{v}(\pi_0)$ and $\mathsf{u}(\pi_1) \sim_n \mathsf{v}(\pi_1)$. Thus, $F(\mathsf{u})(\pi) \sim_n F(\mathsf{v})(\pi)$.

If $|\pi| < k+1$, then $|\pi_0| < k$. We have $\mathsf{u}(\pi_0) \sim_{n+1} \mathsf{v}(\pi_0)$ and $\mathsf{u}(\pi_1) \sim_n \mathsf{v}(\pi_1)$. Hence, $F(\mathsf{u})(\pi) \sim_{n+1} F(\mathsf{v})(\pi)$.

Now consider the case when π has the form

$$\mathrm{cut} \dfrac{\overset{\pi_0}{\Gamma \Rightarrow A, \Delta} \qquad \overset{\pi_1}{\Gamma, A \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}.$$

We have $\mathsf{u}(\pi_0) \sim_n \mathsf{v}(\pi_0)$ and $\mathsf{u}(\pi_1) \sim_n \mathsf{v}(\pi_1)$. Since the mapping re_A is nonexpansive, we see that

$$F(\mathsf{u})(\pi) = \mathsf{re}_A(\mathsf{u}(\pi_0),\mathsf{u}(\pi_1)) \sim_n \mathsf{re}_A(\mathsf{v}(\pi_0),\mathsf{v}(\pi_1)) = F(\mathsf{v})(\pi).$$

Moreover, if $|\pi| < k+1$, then $|\pi_0| < k$ and $|\pi_1| < k$. From the assumption $\mathsf{u} \sim_{n,k} \mathsf{v}$, we obtain $\mathsf{u}(\pi_0) \sim_{n+1} \mathsf{v}(\pi_0)$ and $\mathsf{u}(\pi_1) \sim_{n+1} \mathsf{v}(\pi_1)$. Hence,

$$F(\mathsf{u})(\pi) = \mathsf{re}_A(\mathsf{u}(\pi_0),\mathsf{u}(\pi_1)) \sim_{n+1} \mathsf{re}_A(\mathsf{v}(\pi_0),\mathsf{v}(\pi_1)) = F(\mathsf{v})(\pi).$$

Other cases are straightforward, so we omit them. We have that the operator F is contractive.

Now we define the required cut-elimination mapping ce as the fixed-point of the the operator $\mathsf{F}\colon \mathcal{T} \to \mathcal{T}$, which exists by Lemma 6.18

For any $n,k \in \mathbb{N}$, let us call a mapping $\mathsf{u} \in \mathcal{T}(n,k)$-free if it satisfies the following two conditions: $\mathsf{u}(\pi) \in \mathcal{P}_n$ for any $\pi \in \mathcal{P}$; $\mathsf{u}(\pi) \in \mathcal{P}_{n+1}$ whenever $|\pi| < k$.

We claim that $F(\mathsf{u})$ is $(n,k+1)$-free whenever u is an (n,k)-free mapping from $\mathcal{T}$. Consider an arbitrary ∞-proof π and an (n,k)-free mapping $\mathsf{u} \in \mathcal{T}$. By the case analysis according to the definition of F, we prove that $F(\mathsf{u})(\pi) \in \mathcal{P}_n$. In addition, we check that $\mathsf{F}(\mathsf{u})(\pi) \in \mathcal{P}_{n+1}$ if $|\pi| < k+1$.

Let us consider the case when π has the form

$$\Box \dfrac{\overset{\pi_0}{\Gamma, \Box\Pi \Rightarrow A, \Delta} \qquad \overset{\pi_1}{\Box\Pi \Rightarrow A}}{\Gamma, \Box\Pi \Rightarrow \Box A, \Delta}.$$

We have $\mathsf{u}(\pi_0), \mathsf{u}(\pi_1) \in \mathcal{P}_n$. Hence, $F(\mathsf{u})(\pi) \in \mathcal{P}_n$.

If $|\pi| < k+1$, then $|\pi_0| < k$. We have $\mathsf{u}(\pi_0) \in \mathcal{P}_{n+1}$ and $\mathsf{u}(\pi_1) \in \mathcal{P}_n$. Hence, $F(\mathsf{u})(\pi) \in \mathcal{P}_{n+1}$.

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Now we consider the case when π has the form

$$\mathrm{cut}\,\frac{\overset{\pi_0}{\Gamma \Rightarrow A, \Delta} \qquad \overset{\pi_1}{\Gamma, A \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}\,.$$

We have $\mathsf{u}(\pi_0), \mathsf{u}(\pi_1) \in \mathcal{P}_n$. Since the mapping re_A is adequate, we see that $F(\mathsf{u})(\pi) = \mathsf{re}_A(\mathsf{u}(\pi_0), \mathsf{u}(\pi_1)) \in \mathcal{P}_n$.

If $|\pi| < k+1$, then $|\pi_0| < k$ and $|\pi_1| < k$. We obtain $\mathsf{u}(\pi_0), \mathsf{u}(\pi_1) \in \mathcal{P}_{n+1}$. Hence, $F(\mathsf{u})(\pi) = \mathsf{re}_A(\mathsf{u}(\pi_0), \mathsf{u}(\pi_1)) \in \mathcal{P}_{n+1}$.

We omit the other cases of the lowermost inferences in π, because they are trivial.

We established that $\mathsf{F}(\mathsf{u})$ is $(n, k+1)$-free for any (n,k)-free $\mathsf{u} \in \mathcal{T}$. Notice that if a mapping u is (n,k)-free for all $k \in \mathbb{N}$, then it is also $(n+1, 0)$-free. Now by induction on n with a subinduction on k, we immediately obtain that the mapping ce, which is a fixed-point of F, is (n,k)-free for all $n,k \in \mathbb{N}$. Therefore, for any ∞-proof π, the ∞-proof $\mathsf{ce}(\pi)$ does not contain instances of the rule (cut).

Now assume $\mathsf{Grz}_\infty + \mathsf{cut} \vdash \Gamma \Rightarrow \Delta$. Take an ∞-proof of the sequent $\Gamma \Rightarrow \Delta$ in the system $\mathsf{Grz}_\infty + \mathsf{cut}$ and apply the mapping ce to it. We obtain an ∞-proof of the same sequent in the system Grz_∞. □

Theorem 2.2 is now established as a direct consequence of Theorem 3.6, Theorem 6.19, and Theorem 3.7.

COROLLARY 6.20. *For any finite multisets of formulas* Γ *and* Δ, *the following are equivalent*:

1. $\mathsf{Grz} \vdash \bigwedge \Gamma \to \bigvee \Delta$,
2. $\mathsf{Grz}_{\mathsf{Seq}} \vdash \Gamma \Rightarrow \Delta$,
3. $\mathsf{Grz}_{\mathsf{Seq}} + \mathsf{cut} \vdash \Gamma \Rightarrow \Delta$,
4. $\mathsf{Grz}_\infty \vdash \Gamma \Rightarrow \Delta$,
5. $\mathsf{Grz}_\infty + \mathsf{cut} \vdash \Gamma \Rightarrow \Delta$.

§7. Lyndon interpolation. The Lyndon interpolation property for the logic Grz was established in [14] on the basis of Kripke semantics. Here we present a proof-theoretic argument for the same result.

For a formula or a multiset of formulas X, let $pos(X)$ be the set of atomic propositions that have positive occurrences in X and let $neg(X)$ be the set of atomic propositions with negative occurrences in X. For a sequent $\Gamma \Rightarrow \Delta$, let $pos(\Gamma \Rightarrow \Delta) = pos(\Delta) \cup neg(\Gamma)$ and $neg(\Gamma \Rightarrow \Delta) = neg(\Delta) \cup pos(\Gamma)$.

Recall that for a finite set of formulas Λ, the set Λ^* is $\{\Box(A \to \Box A) \mid A \in \Lambda\}$.

LEMMA 7.21. *For any finite sets of formulas* Λ_1, Λ_2, *if* $\mathsf{Grz}_\infty \vdash \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$, *then there exists a formula* I *called an interpolant of this sequent, such that* $pos(I) \subset neg(\Gamma_1 \Rightarrow \Delta_1) \cap pos(\Gamma_2 \Rightarrow \Delta_2)$, $neg(I) \subset pos(\Gamma_1 \Rightarrow \Delta_1) \cap neg(\Gamma_2 \Rightarrow \Delta_2)$, *and*

$$\mathsf{Grz} \vdash \bigwedge \Lambda_1^* \cup \Gamma_1 \to \bigvee \Delta_1 \cup \{I\} \quad \textit{and} \quad \mathsf{Grz} \vdash \bigwedge \Lambda_2^* \cup \{I\} \cup \Gamma_2 \to \bigvee \Delta_2\,.$$

Proof. Assume π is an ∞-proof of the sequent $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ in Grz_∞ and Λ_1, Λ_2 are finite sets of formulas. We prove the statement of the lemma by induction on the sum of cardinalities $card(Sub(\Gamma_1 \Rightarrow \Delta_1) \backslash \Lambda_1) + card(Sub(\Gamma_2 \Rightarrow \Delta_2) \backslash \Lambda_2)$ with a subinduction on $|\pi|$.

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Suppose $|\pi| = 0$. Then $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ is an initial sequent. If $\Gamma_1 \Rightarrow \Delta_1$ ($\Gamma_2 \Rightarrow \Delta_2$) is an initial sequent, then we put $I := \bot$ ($I := \top$). Otherwise, we have an atomic proposition p such that $p \in \Gamma_1$ and $p \in \Delta_2$ ($p \in \Gamma_2$ and $p \in \Delta_1$). In this case, we put $I := p$ ($I := \neg p$).

Now consider the last application of an inference rule in π.

Case 1. Suppose that π has the form

$$\rightarrow_{\mathrm{R}} \frac{\begin{array}{c}\pi' \\ \Gamma_1, \Gamma_2, A \Rightarrow B, \Sigma\end{array}}{\Gamma_1, \Gamma_2 \Rightarrow A \rightarrow B, \Sigma},$$

where $A \rightarrow B, \Sigma = \Delta_1, \Delta_2$. Notice that $|\pi'| < |\pi|$. If $A \rightarrow B \in \Delta_1$ ($A \rightarrow B \in \Delta_2$), then we put $\Gamma_1' := \Gamma_1 \cup \{A\}$; $\Delta_1' := (\Delta_1 \backslash \{A \rightarrow B\}) \cup \{B\}$; $\Gamma_2' := \Gamma_2$; $\Delta_2' := \Delta_2$ ($\Gamma_2' := \Gamma_2 \cup \{A\}$; $\Delta_2' := (\Delta_2 \backslash \{A \rightarrow B\}) \cup \{B\}$; $\Gamma_1' := \Gamma_1$; $\Delta_1' := \Delta_1$). We see that π' is an ∞-proof of the sequent $\Gamma_1', \Gamma_2' \Rightarrow \Delta_1', \Delta_2'$ in Grz_∞. By the induction hypothesis for π', Λ_1 and Λ_2, there is a corresponding interpolant I'. We set $I := I'$.

Case 2. Suppose that π has the form

$$\rightarrow_{\mathrm{L}} \frac{\begin{array}{cc}\pi' & \pi'' \\ \Sigma, B \Rightarrow \Delta_1, \Delta_2 & \Sigma \Rightarrow A, \Delta_1, \Delta_2\end{array}}{\Sigma, A \rightarrow B \Rightarrow \Delta_1, \Delta_2},$$

where $\Sigma, A \rightarrow B = \Gamma_1, \Gamma_2$. We see that $|\pi'| < |\pi|$. If $A \rightarrow B \in \Gamma_1$ ($A \rightarrow B \in \Gamma_2$), then we put $\Gamma_1' := (\Gamma_1 \backslash \{A \rightarrow B\}) \cup \{B\}$; $\Gamma_1'' := \Gamma_1 \backslash \{A \rightarrow B\}$; $\Delta_1' := \Delta_1$; $\Delta_1'' := \Delta_1 \cup \{A\}$; $\Gamma_2' := \Gamma_2$; $\Gamma_2'' := \Gamma_2$; $\Delta_2' := \Delta_2$; $\Delta_2'' := \Delta_2$ ($\Gamma_2' := (\Gamma_2 \backslash \{A \rightarrow B\}) \cup \{B\}$; $\Gamma_2'' := \Gamma_2 \backslash \{A \rightarrow B\}$; $\Delta_2' := \Delta_2$; $\Delta_2'' := \Delta_2 \cup \{A\}$; $\Gamma_1' := \Gamma_1$; $\Gamma_1'' := \Gamma_1$; $\Delta_1' := \Delta_1$; $\Delta_1'' := \Delta_1$). We see that π' and π'' are ∞-proofs of $\Gamma_1', \Gamma_2' \Rightarrow \Delta_1', \Delta_2'$ and $\Gamma_1'', \Gamma_2'' \Rightarrow \Delta_1'', \Delta_2''$ in Grz_∞. By the induction hypothesis for π', Λ_1 and Λ_2, there is an interpolant I'. Analogously, by the induction hypothesis for π'', Λ_1 and Λ_2, there is an interpolant I''. If $A \rightarrow B \in \Gamma_1$, then we set $I := I' \vee I''$. Otherwise, when $A \rightarrow B \in \Gamma_2$, we set $I := I' \wedge I''$.

Case 3. Suppose that π has the form

$$\mathrm{refl} \frac{\begin{array}{c}\pi' \\ \Sigma, A, \Box A \Rightarrow \Delta_1, \Delta_2\end{array}}{\Sigma, \Box A \Rightarrow \Delta_1, \Delta_2},$$

where $\Sigma, \Box A = \Gamma_1, \Gamma_2$. If $\Box A \in \Gamma_1$ ($\Box A \in \Gamma_2$), then we put $\Gamma_1' := \Gamma_1 \cup \{A\}$; $\Delta_1' := \Delta_1$; $\Gamma_2' := \Gamma_2$; $\Delta_2' := \Delta_2$ ($\Gamma_2' := \Gamma_2 \cup \{A\}$; $\Delta_2' := \Delta_2$; $\Gamma_1' := \Gamma_1$; $\Delta_1' := \Delta_1$). We see that π' is an ∞-proof of the sequent $\Gamma_1', \Gamma_2' \Rightarrow \Delta_1', \Delta_2'$ in Grz_∞. By the induction hypothesis for π', Λ_1 and Λ_2, there is a corresponding interpolant I'. We set $I := I'$.

Case 4. Suppose that π has the form

$$\Box \frac{\begin{array}{cc}\pi' & \pi'' \\ \Phi, \Box\Pi \Rightarrow A, \Sigma & \Box\Pi \Rightarrow A\end{array}}{\Phi, \Box\Pi \Rightarrow \Box A, \Sigma},$$

where $\Phi, \Box\Pi = \Gamma_1, \Gamma_2$ and $\Box A, \Sigma = \Delta_1, \Delta_2$.

Subcase 4.1: the formula $\Box A \in \Delta_1$ and $A \in \Lambda_1$ ($\Box A \in \Delta_2$ and $A \in \Lambda_2$). We see that $|\pi'| < |\pi|$. We put $\Gamma_1' := \Gamma_1$; $\Delta_1' := (\Delta_1 \backslash \{\Box A\}) \cup \{A\}$; $\Gamma_2' := \Gamma_2$; $\Delta_2' := \Delta_2$ ($\Gamma_2' := \Gamma_2$; $\Delta_2' := (\Delta_2 \backslash \{\Box A\}) \cup \{A\}$; $\Gamma_1' := \Gamma_1$; $\Delta_1' := \Delta_1$). We see that π' is an ∞-proof of the sequent $\Gamma_1', \Gamma_2' \Rightarrow \Delta_1', \Delta_2'$ in Grz_∞. By the induction hypothesis for π', Λ_1 and Λ_2, there is a corresponding interpolant I' such that

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$$\mathsf{Grz} \vdash \bigwedge \Lambda_1^* \cup \Gamma_1' \to \bigvee \Delta_1' \cup \{I'\} \quad \text{and} \quad \mathsf{Grz} \vdash \bigwedge \Lambda_2^* \cup \{I'\} \cup \Gamma_2' \to \bigvee \Delta_2'.$$

If $\Box A \in \Delta_1$ and $A \in \Lambda_1$, then $\Box(A \to \Box A) \in \Lambda_1^*$. Thus we have $\mathsf{Grz} \vdash \bigwedge \Lambda_1^* \to (A \to \Box A)$. It follows that we can replace the formula A in Δ_1' by $\Box A$ and obtain

$$\mathsf{Grz} \vdash \bigwedge \Lambda_1^* \cup \Gamma_1' \to \bigvee \Delta_1 \cup \{I'\}.$$

Note also that $\Gamma_1' := \Gamma_1$, $\Gamma_2' := \Gamma_2$, $\Delta_2' := \Delta_2$. Hence,

$$\mathsf{Grz} \vdash \bigwedge \Lambda_1^* \cup \Gamma_1 \to \bigvee \Delta_1 \cup \{I'\} \quad \text{and} \quad \mathsf{Grz} \vdash \bigwedge \Lambda_2^* \cup \{I'\} \cup \Gamma_2 \to \bigvee \Delta_2.$$

We set $I := I'$. The case $\Box A \in \Delta_2$ and $A \in \Lambda_2$ is analogous.

Subcase 4.2: the formula $\Box A \in \Delta_2$ and $A \notin \Lambda_2$ ($\Box A \in \Delta_1$ and $A \notin \Lambda_1$). We have that $\Gamma_1, \Gamma_2 = \Phi, \Box\Pi$ and $\Delta_1, \Delta_2 = \Box A, \Sigma$. We split $\Box\Pi$ into two multisets $\Box\Pi_1 \subset \Gamma_1$ and $\Box\Pi_2 \subset \Gamma_2$ such that $\Box\Pi = \Box\Pi_1, \Box\Pi_2$. Put $\Gamma_1'' := \Box\Pi_1$; $\Delta_1'' := \emptyset$; $\Gamma_2'' := \Box\Pi_2$; $\Delta_2'' := \{A\}$ ($\Gamma_1'' := \Box\Pi_1$; $\Delta_1'' := \{A\}$; $\Gamma_2'' := \Box\Pi_2$; $\Delta_2'' := \emptyset$). We see that π'' is an ∞-proof of the sequent $\Gamma_1'', \Gamma_2'' \Rightarrow \Delta_1'', \Delta_2''$ in Grz_∞.

If $\Box A \in \Delta_2$ and $A \notin \Lambda_2$, then $card(Sub(\Box\Pi_1 \Rightarrow) \backslash \Lambda_1) + card(Sub(\Box\Pi_2 \Rightarrow A) \backslash (\Lambda_2 \cup \{A\})$ is strictly less than $card(Sub(\Gamma_1 \Rightarrow \Delta_1) \backslash \Lambda_1) + card(Sub(\Gamma_2 \Rightarrow \Delta_2) \backslash \Lambda_2)$. Hence by the induction hypothesis for π'', Λ_1 and $\Lambda_2 \cup \{A\}$ there exists an interpolant I'' such that

$$\mathsf{Grz} \vdash \bigwedge \Lambda_1^* \cup \Box\Pi_1 \to I'' \quad \text{and} \quad \mathsf{Grz} \vdash \bigwedge \Lambda_2^* \cup \{\Box(A \to \Box A)\} \cup \{I''\} \cup \Box\Pi_2 \to A.$$

Since $\mathsf{Grz} \vdash \Box C \to \Box\Box C$ for any formula C, from the left condition we obtain

$$\mathsf{Grz} \vdash \bigwedge \Lambda_1^* \cup \Box\Pi_1 \to \Box I''.$$

Hence,

$$\mathsf{Grz} \vdash \bigwedge \Lambda_1^* \cup \Gamma_1 \to \bigvee \Delta_1 \cup \{\Box I''\}.$$

From the right condition we see

$$\mathsf{Grz} \vdash \bigwedge \Lambda_2^* \cup \{I''\} \cup \Box\Pi_2 \to (\Box(A \to \Box A) \to A).$$

Thus,

$$\mathsf{Grz} \vdash \bigwedge \Lambda_2^* \cup \{\Box I''\} \cup \Box\Pi_2 \to \Box(\Box(A \to \Box A) \to A).$$

Applying the axiom (v) of Grz we see that

$$\mathsf{Grz} \vdash \bigwedge \Lambda_2^* \cup \{\Box I''\} \cup \Box\Pi_2 \to \Box A.$$

Therefore,

$$\mathsf{Grz} \vdash \bigwedge \Lambda_2^* \cup \{\Box I''\} \cup \Gamma_2 \to \bigvee \Delta_2.$$

Now we can set $I := \Box I''$. In the case $\Box A \in \Delta_1$, $A \notin \Lambda_1$ we can analogously set $I := \Diamond I''$. □

THEOREM 7.22 (Lyndon interpolation). *If* $\mathsf{Grz} \vdash A \to B$, *then there exists a formula* C, *called an interpolant of* $A \to B$, *such that* $pos(C) \subset pos(A) \cap pos(B)$, $neg(C) \subset neg(A) \cap neg(B)$, *and*

$$\mathsf{Grz} \vdash A \to C, \qquad \mathsf{Grz} \vdash C \to B.$$

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Proof. Assume $\mathsf{Grz} \vdash A \to B$. By Lemma 2.1, we have $\mathsf{Grz}_{\mathsf{Seq}} + \mathsf{cut} \vdash A \Rightarrow B$. Applying Theorem 3.6 and Theorem 6.19 we obtain $\mathsf{Grz}_\infty \vdash A \Rightarrow B$. Applying the previous lemma with $\Lambda_1 = \Lambda_2 = \emptyset$, $\Gamma_1 = \{A\}$, $\Delta_2 = \{B\}$, $\Gamma_2 = \Delta_1 = \emptyset$, we find an interpolant for $A \to B$. □

§8. Cyclic proofs. There exists a simple class of ∞-proofs that is sufficient to derive all theorems of Grz_∞. An ∞-proof is called *regular* if it contains only finitely many nonisomorphic subtrees.

Regular ∞-proofs have useful finite representations called cyclic (circular) proofs. A *cyclic proof* of a sequent $\Gamma \Rightarrow \Delta$ is a pair (κ, d), where κ is a finite tree of sequents constructed according to the rules of Grz_∞ with the root marked by $\Gamma \Rightarrow \Delta$, and d is a function with the following properties: the function d is defined on the set of all leaves of κ that are not marked by initial sequents; the image $d(a)$ of a leaf a lies on the path from the root of κ to the leaf a; a and $d(a)$ are marked by the same sequent; and there is a right premise of the rule $(\Box)$ between a and $d(a)$ (this premise can be a, but it can not be $d(a)$). If the function d is defined at a leaf a, then we say that the nodes a and $d(a)$ are joined by a back-link. Here is an example of a cyclic proof for the sequent $\Box(\Box(p \to \Box p) \to p) \Rightarrow p$:

$$\dfrac{\dfrac{\mathsf{Ax}\ \dfrac{}{F, p \Rightarrow p} \qquad \dfrac{\dfrac{\mathsf{Ax}\ \dfrac{}{F, p \Rightarrow \Box p, p}}{F \Rightarrow p \to \Box p, p}\,{\to}\mathrm{R}}{F \Rightarrow \Box(p \to \Box p), p}\,\Box}{\Box(p \to \Box p) \to p, F \Rightarrow p}\,{\to}\mathrm{L}}{F \Rightarrow p}\,\mathrm{refl}$$

$$\dfrac{\dfrac{\mathsf{Ax}\ \dfrac{}{p, F \Rightarrow p} \qquad F \Rightarrow p}{p, F \Rightarrow \Box p}\,{\to}R}{F \Rightarrow p \to \Box p}\,\Box$$

where $F = \Box(\Box(p \to \Box p) \to p)$.

The notion of cyclic proof determines the same provability relation as the notion of regular ∞-proof. Obviously, each cyclic proof can be unravelled into a regular ∞-proof. The converse is also true.

Proposition 8.23 cf. [20], Proposition 3.1. *Any regular ∞-proof of Grz_∞ can be obtained by unraveling a cyclic proof.*

In the rest of the section we establish that any sequent provable in Grz_∞ has a cyclic proof.

Lemma 8.24. *For any formula A, the rules*

$$\mathrm{cl}_A\ \frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad \mathrm{cr}_A\ \frac{\Gamma \Rightarrow A, A, \Delta}{\Gamma \Rightarrow A, \Delta}$$

are admissible in Grz_∞.

Proof. Assume π and τ are ∞-proofs of the sequents $\Gamma, A, A \Rightarrow \Delta$ and $\Gamma \Rightarrow A, A, \Delta$ in the system Grz_∞. Let ξ be an ∞-proof of the sequent $\Gamma, A \Rightarrow A, \Delta$ in Grz_∞, which exists by Lemma 3.3.

The required ∞-proofs of the sequents $\Gamma, A \Rightarrow \Delta$ and $\Gamma \Rightarrow A, \Delta$ are defined by setting $\mathsf{cl}_A(\pi) = \mathsf{re}_A(\xi, \pi)$ and $\mathsf{cr}_A(\tau) = \mathsf{re}_A(\tau, \xi)$, where re_A is an A-removing mapping from

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Lemma 6.17 Since the mapping re_A is adequate, the ∞-proofs $\mathsf{cl}_A(\pi)$ and $\mathsf{cr}_A(\tau)$ do not contain applications the rule (cut). □

Let $\mathcal{T}^*$ denote the set of all root-preserving mappings from the set of ∞-proofs of Grz_∞ to itself. On the set $\mathcal{T}^*$ we can define the relation $\sim^*_{n,k}$ and the ultrametric l_1^* the same way as $\sim_{n,k}$ and l_1.

LEMMA 8.25. *In the ultrametric space* $(\mathcal{T}^*, l_1^*)$, *any contractive operator* $\mathsf{U} : \mathcal{T}^* \to \mathcal{T}^*$ *has a unique fixed-point*.

Proof. The space is obviously nonempty, since the identity function lies in $\mathcal{T}^*$. The proof is analogous to the proof of Proposition 4.10, but we do not need to check the nonexpansiveness here. □

An application of the modal rule $(\Box)$

$$\Box\,\frac{\Gamma, \Box\Pi \Rightarrow A, \Delta \qquad \Box\Pi \Rightarrow A}{\Gamma, \Box\Pi \Rightarrow \Box A, \Delta}\,.$$

is called *slim* if the multiset Π here is a set. An ∞-proof is called *slim* if every application of the rule $(\Box)$ in it is slim.

LEMMA 8.26. *If* $\mathsf{Grz}_\infty \vdash \Gamma \Rightarrow \Delta$, *then the sequent* $\Gamma \Rightarrow \Delta$ *has a slim* ∞*-proof in* Grz_∞.

Proof. We construct a mapping $\mathsf{slim} \in \mathcal{T}^*$ that maps an ∞-proof to a slim ∞-proof of the same sequent. This mapping is defined as the fixed-point of a contractive operator $\mathsf{H} \colon \mathcal{T}^* \to \mathcal{T}^*$.

For a mapping $\mathsf{u} \in \mathcal{T}^*$ and an ∞-proof π , the ∞-proof $\mathsf{H}(\mathsf{u})(\pi)$ is defined as follows. If $|\pi| = 0$, then we put $\mathsf{H}(\mathsf{u})(\pi) = \pi$.

Otherwise, we define $\mathsf{H}(\mathsf{u})(\pi)$ according to the last application of an inference rule in π:

$$\to\!\mathrm{L}\,\frac{\overset{\pi_1}{\Gamma, B \Rightarrow \Delta} \qquad \overset{\pi_2}{\Gamma \Rightarrow A, \Delta}}{\Gamma, A \to B \Rightarrow \Delta} \longmapsto \to\!\mathrm{L}\,\frac{\overset{\mathsf{u}(\pi_1)}{\Gamma, B \Rightarrow \Delta} \qquad \overset{\mathsf{u}(\pi_2)}{\Gamma \Rightarrow A, \Delta}}{\Gamma, A \to B \Rightarrow \Delta}\,,$$

$$\mathrm{refl}\,\frac{\overset{\pi_0}{\Gamma, A, \Box A \Rightarrow \Delta}}{\Gamma, \Box A \Rightarrow \Delta} \longmapsto \mathrm{refl}\,\frac{\overset{\mathsf{u}(\pi_0)}{\Gamma, A, \Box A \Rightarrow \Delta}}{\Gamma, \Box A \Rightarrow \Delta}\,.$$

$$\mathrm{refl}\,\frac{\overset{\pi_0}{\Gamma, A, \Box A \Rightarrow \Delta}}{\Gamma, \Box A \Rightarrow \Delta} \longmapsto \mathrm{refl}\,\frac{\overset{\mathsf{u}(\pi_0)}{\Gamma, A, \Box A \Rightarrow \Delta}}{\Gamma, \Box A \Rightarrow \Delta}\,.$$

For a multiset Π, we denote its underlying set by Π^S. The case of the modal rule is as follows:

$$\Box\,\frac{\overset{\pi_1}{\Gamma, \Box\Pi \Rightarrow A, \Delta} \qquad \overset{\pi_2}{\Box\Pi \Rightarrow A}}{\Gamma, \Box\Pi \Rightarrow \Box A, \Delta} \longmapsto \Box\,\frac{\overset{\mathsf{u}(\pi_1)}{\Gamma, \Box\Pi \Rightarrow A, \Delta} \qquad \overset{\mathsf{u}(\pi_2)}{\Box\Pi^S \Rightarrow A}}{\Gamma, \Box\Pi \Rightarrow \Box A, \Delta}\,,$$

where π_2' is an ∞-proof of the sequent $\Box\Pi^S \Rightarrow A$ in Grz_∞. This ∞-proof exists by the admissibility of contraction rules obtained in Lemma 8.24.

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Now it can be easily shown, that if $\mathsf{u} \sim^*_{n,k} \mathsf{v}$ for some $\mathsf{u},\mathsf{v} \in \mathcal{T}^*$ and $n,k \in \mathbb{N}$, then $\mathsf{H}(\mathsf{u}) \sim^*_{n,k+1} \mathsf{H}(\mathsf{v})$. Therefore the mapping H is a contractive operator. Thus, by Lemma 8.25 it has a fixed-point, which we denote by slim.

In an analogous way to the proofs of Lemma 6.16 and Theorem 6.19, we see that $\mathsf{slim}(\pi)$ is a slim ∞-proof for any π. □

THEOREM 8.27. *If* $\mathsf{Grz}_\infty \vdash \Gamma \Rightarrow \Delta$, *then there is a cyclic proof for the given sequent*.

Proof. Let π be a slim ∞-proof of the sequent $\Gamma \Rightarrow \Delta$ obtained in the previous lemma. Note that all formulas from π are subformulas of the formulas from $\Gamma \cup \Delta$. Consequently, the ∞-proof π contains only finitely many different sequents that occur as right premises of the rule $(\Box)$ in it. By k, we denote the number of these sequents.

Let ξ be the $(k+2)$-fragment of the ∞-proof π. Consider any path in ξ starting at the root and passing through $k+1$ right premises of the rule $\Box$. This branch contains a pair of different nodes a and b that are both right premises of the rule $\Box$ and are marked by the same sequent. Assuming that b is further from the root of ξ than a, we remove the subtree with the root b and connect b, which has become a leaf, to a by a back-link. Applying the same operation to all other such paths in ξ, we turn the $(k+2)$-fragment of the ∞-proof π into a cyclic proof of the sequent $\Gamma \Rightarrow \Delta$. □

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BIBLIOGRAPHY

[1] Afshari, B. & G. Leigh (2017). Cut-free completeness for modal mu-calculus. In *LICS '17: Proceedings of the 32nd Annual ACM/IEEE Symposium on Logic in Computer Science*. IEEE Press, 1-12.

[2] Avron, A. (1984). On modal systems having arithmetical interpretations. *Journal of Symbolic Logic*, **49**(3), 935–942.

[3] Baedel, D., A. Doumane, & A. Saurin (2016). Infinitary proof theory: the multiplicative additive case. In Talbot, J. M., and Regnier, L., editors. *25th EACSL Annual Conference on Computer Science Logic*. Dagstuhl: Schloss Dagstuhl, pp. 42: 1–42:17.

[4] Baier, C. & M. E. Majster-Cederbaum (1994). Denotational semantics in the cpo and metric approach. *Theoretical Computer Science*, **135**, 171–220.

[5] Borga, M. & P. Gentilini. (1986). On the proof theory of the modal logic Grz. *Mathematical Logic Quarterly*, **32**(10–12), 145–148.

[6] Chagrov, A. & M. Zakharyaschev. (1997). *Modal Logic*. Oxford: Oxford University Press.

[7] Das, A. & D. Pous. (2018). Non-wellfounded proof theory for (Kleene+action) (algebras+lattices). In Ghica, D. R., and Jung, A., editors. *27th EACSL Annual Conference on Computer Science Logic* 2018. Dagstuhl: Schloss Dagstuhl, pp. 19: 1–19:18.

https://doi.org/10.1017/S1755020319000510 Published online by Cambridge University Press

[8] Di Gianantonio, P. & M. Miculan (2002). A unifying approach to recursive and co-recursive definitions. In GeuversH., & Wiedijk F., editors. *TYPES* 2002: *Types for Proofs and Programs*. Lecture Notes in Computer Science, Vol. 2646. Berlin: Springer, pp. 148–161.

[9] Dyckhoff, R. & S. Negri (2016). A cut-free sequent system for Grzegorczyk logic, with an application to the Gödel–McKinsey–Tarski embedding. *Journal of Logic and Computation*, **26**(1), 169–187

[10] Escardó, M. H. (1998). A metric model of PCF. Laboratory for Foundations of Computer Science, The University of Edinburgh.

[11] Fortier, J. & L. Santocanale (2013). Cuts for circular proofs: semantics and cut-elimination. In Della Rocca, S. R., editor. *Computer Science Logic* 2013. Dagstuhl: Schloss Dagstuhl, pp. 248–262.

[12] Iemhoff, R. (2016). Reasoning in circles. In van Eijck, J, Iemhoff, R., and Joosten, J. J., editors. *Liber Amicorum Alberti. A Tribute to Albert Visser*. London: College Publications, pp. 165–176.

[13] Kuznetsov, S. (2017). The Lambek calculus with Iteration: two Variants. In Kennedy, J., & de Queiroz, R., editors. *Logic, Language, Information, and Computation*. Berlin: Springer, pp. 182–198.

[14] Maksimova, L. L. (2008). On modal Grzegorczyk logic, *Fundamenta Informaticae*, **81**(1–3), 203–210.

[15] ———. (2014). The Lyndon property and uniform interpolation over the Grzegorczyk logic. *Siberian Mathematical Journal*, **55**(1), 118–124.

[16] Petalas, C. & T. Vidalis (1993). A fixed point theorem in non-Archimedean vector spaces. *Proceedings of the American Mathematical Society*, **118**(3), 819–821.

[17] Prieß-Crampe, S. (1990). Der Banachsche Fixpunktsatz für ultrametrische Räume. *Results in Mathematics*, **18**(1–2), 178–186.

[18] Savateev, Y. & D. Shamkanov (2017). Cut-elimination for the modal Grzegorczyk logic via non-well-founded proofs. In Kennedy, J., & de Queiroz, R., editors. *Logic, Language, Information, and Computation*. Berlin: Springer, pp. 321–335.

[19] Schörner, E. (2003). Ultrametric fixed point theorems and applications. In Kuhlmann, VF. V., Kuhlmann, S., and Marshall, M., editors. *Valuation Theory and Its Applications, Volume II*. Providence, RI: American Mathematical Society, pp. 353–359.

[20] Shamkanov, D. S. (2014). Circular proofs for the Gödel-Löb provability logic. *Mathematical Notes*, **96**(3), 575–585.

[21] Simpson, A. (2017). Cyclic arithmetic is equivalent to peano arithmetic. In Esparza, J., & Murawski, A., editors. *Foundations of Software Science and Computation Structures*. Berlin: Springer, pp. 283–300.

[22] van Breugel, F. (2001). An introduction to metric semantics: operational and denotational models for programming and specfication languages. *Theoretical Computer Science*, **258**, 1–98.

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