TRUTH AND COLLECTION

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Abstract. Answering a question of Kaye, we show that the compositional truth theory with the full collection scheme is conservative over Peano Arithmetic. We demonstrate it by showing that countable models of compositional truth which satisfy the internal induction or collection axioms can be end-extended to models of the respective theory.

§1. Introduction. The area of axiomatic truth theories, studies extensions of foundational axiomatic systems such as Peano arithmetic (PA), elementary arithmetic, or Zermelo–Fränkel set theory with axioms expressing that a fresh unary predicate $T(x)$ defines the set of true sentences.

One of the most basic such theories is the theory of compositional truth over PA, called CT^{-}.[1] Its axioms state that the truth predicate satisfies Tarski's compositional conditions for the arithmetical language. It turns out that the Tarski's axioms themselves constitute a conservative extension of the base theory, even though usually a theory is significantly stronger than a subtheory for which it can formulate a truth predicate.

In a line of research started by [1, 2] and discussed thoroughly in [3] the following question has been explored: What axioms can be added to CT^{-} so as to make the resulting theory nonconservative? The "line" dividing conservative truth principles from the nonconservative ones has been dubbed by Ali Enayat *the Tarski Boundary* and the problem of systematically investigating the conservativity of truth-theoretic extensions has been called *the Tarski Boundary Problem*.

The principles under investigation come roughly in two flavours: either they are of purely truth-theoretic nature, for instance stating that a disjunction of an arbitrary finite size is true iff one of the disjunct is, or they are general principles studied in the context of arithmetical theories, for instance, fragments of the induction scheme.

In the subsequent research these principles turned out to be either conservative or equivalent exactly to the scheme of Δ_0-induction for the full language including the truth predicate. Thus, the Tarski Boundary Problem seems to admit a much more

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[1] In the literature, this theory is also called CT $\restriction$, CT[PA], $\mathrm{PA}^{\mathrm{FT}}$. There are also variants of these names for the related concept of satisfaction.

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structured answer than one could expect. This is a very surprising phenomenon, since *prima facie* the principles under investigation seem completely unrelated.

The topic of axiomatic truth has been classically studied as a part of the theory of models of PA, as a study of so-called *satisfaction classes*. A satisfaction class S in a model M of PA is a subset of M^2 which satisfies Tarski's compositional conditions. They were introduced in the PhD thesis [13], preceded by related ideas in [18]. They are supposed to explore how we can endow nonstandard elements of a model M which are viewed from its point of view as arithmetical formulae with well-defined semantics. From this perspective more important than conservativity of truth-theoretic principles is the question whether in a given model of PA one can find a satisfaction class enjoying additional good properties.

One of the most pressing questions concerning the Tarski Boundary problem which was left open has been posed by Kaye.[2] It asks, whether in any countable recursively saturated model of PA, we can find a full satisfaction class satisfying collection. Equivalently, is CT^- with the full collection scheme for the extended language, but with no induction whatsoever, a conservative extension of PA? Results about the purely arithmetical counterpart of this question supported the intuition that the answer should be positive: If we add the full collection scheme to PA^-, the theory of positive parts of ordered semirings, the resulting extension is Π_1-conservative, as noted in [8, Exercise 7.7].

In this article, we answer Kaye's question in the positive. In fact, following the original conjectural proof strategy, we show that any countable model of PA can be elementarily extended to an ω_1-like model which carries a full satisfaction class (or, equivalently, which expands to a model of CT^-). We achieve this, in turn, by showing that any countable model of CT^- satisfying an additional axiom of the internal induction (or collection) has an end-extension. The proof of this fact, a technical crux of our work, is based on techniques from [15], where a partial answer to Kaye's question has been provided, combined with the copying technique from a novel construction of satisfaction classes presented by Pakhomov in his note [16].[3]

§2. Preliminaries. This article concerns truth theories over (PA). Truth theories result by adding to arithmetic a fresh predicate $T(x)$ with the intended reading "x is the Gödel code of a true (arithmetical) sentence" and axioms guaranteeing that T actually displays truth-like behaviour. Crucially for our purposes, PA can formalise syntax, so actually postulating the existence of such a truth predicate makes sense. The details of the coding of syntax can be found in any textbook on formal arithmetic, for instance in [8]. Across the article, we will use a number of formalised syntactic notions. Sometimes, we introduce them without definition, since most of the notation seems self-explanatory. However, we explain all these notions in the Appendix. A comprehensive introduction to truth theories may be found in [7] and an extensive treatment of the Tarski Boundary problem may be found in [3].

[2] It was stated explicitly in the presentation [9], but we are not aware of the author stating it in print.

[3] The note has not been published, but the ideas contained there were discussed in the presentation [17] available at the Internet address given in the references with a permission of the author.

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In the presentation, we will use a number of conventions to improve readability:

- We will refer to objects which are codes of first-order formulae by variables $\phi, \psi, \ldots$
- We will refer to objects which are codes of terms by $s, t, \ldots$
- We will refer to the codes of sequences of terms by $\bar{s}, \bar{t}, \ldots$
- We will systematically confuse the syntactic operations and the operations on the codes.
- We will use some formulae denoting syntactic notions as if we they were constants denoting sets. So if $A(x)$ is such a formula we will write, for instance, $\exists x \in A\, \phi(x)$ instead of $\exists x \left(A(x) \wedge \phi(x)\right)$.
- Moreover, we will sometimes use provably functional formulae as if they were actual function symbols. Thus, for instance, we will write $A(\phi \wedge \psi)$ instead of "for any z if z is the unique result of joining ϕ and ψ with a conjunction symbol, then $A(z)$."

2.1. Truth theories. Let us introduce the main theories of our interest.

DEFINITION 1. By CT^- (Compositional Truth) we mean the theory in the arithmetical language $\mathcal{L}_{\mathrm{PA}}$ extended with one unary predicate $T(x)$ whose axioms are the axioms of PA along with the following compositional clauses:[4]

1. $\forall x \Big(T(x) \rightarrow \mathrm{Sent}_{\mathcal{L}_{\mathrm{PA}}}(x) \Big)$.
2. $\forall s, t \in \mathrm{ClTerm}_{\mathcal{L}_{\mathrm{PA}}} \Big(T(s = t) \equiv s^\circ = t^\circ \Big)$.
3. $\forall \phi \in \mathrm{Sent}_{\mathcal{L}_{\mathrm{PA}}} \Big(T\neg\phi \equiv \neg T\phi \Big)$.
4. $\forall \phi, \psi \in \mathrm{Sent}_{\mathcal{L}_{\mathrm{PA}}} \Big(T(\phi \wedge \psi) \equiv T\phi \wedge T\psi \Big)$.
5. $\forall \phi \in \mathrm{Form}^{\leq 1}_{\mathcal{L}_{\mathrm{PA}}} \forall v \in \mathrm{Var} \Big(T\exists v \phi(v) \equiv \exists x T\phi(\underline{x}) \Big)$.
6. $\forall \bar{s}, \bar{t} \in \mathrm{ClTermSeq}_{\mathcal{L}_{\mathrm{PA}}} \forall \phi \in \mathrm{Form}_{\mathcal{L}_{\mathrm{PA}}} \Big(\overline{s^\circ} = \overline{t^\circ} \rightarrow T\phi(\bar{s}) \equiv T\phi(\bar{t}) \Big)$.

The last clause, called *the regularity axiom* states that the truth of a sentence does not depend on the exact terms which are used in it, but rather on the values thereof. In the proof, we will actually need a stronger regularity condition which we will discuss in Section 2.4.

One of the fundamental facts about CT^- is that it does not have any arithmetical content extending PA.

THEOREM 2 (Kotlarski–Krajewski–Lachlan). *CT^- is conservative over PA.*

A number of additional conditions can be imposed on the truth predicate so that the resulting theory remains conservative. One of the most notable is that we can assume that every formula $\phi \in \mathrm{Form}_{\mathcal{L}_{\mathrm{PA}}}$, considered separately, satisfies induction.

[4] $\mathrm{Form}_{\mathcal{L}_{\mathrm{PA}}}(x)$ is the formalised notion for the set of arithmetical formulae; $\mathrm{Form}^{\leq 1}_{\mathcal{L}_{\mathrm{PA}}}$ means that there is at most one free variable; the formula $x \in \mathrm{Var}$ means that x is a code of a variable; $\mathrm{Sent}_{\mathcal{L}_{\mathrm{PA}}}(x)$ means that x is a code of an arithmetical sentence; $\mathrm{ClTerm}_{\mathcal{L}_{\mathrm{PA}}}$ refers to the codes of closed arithmetical terms, and $\mathrm{ClTermSeq}_{\mathcal{L}_{\mathrm{PA}}}$ to the sequences of such terms. If s is a closed term, s° denotes its value (which, again, can be formalised in PA for the arithmetised version of terms). Finally, for any x, by $\underline{x}$ we mean (a code for) a canonical term whose value is x, say $\underbrace{S \ldots S}_{x \text{ times}} 0$.

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DEFINITION 3. By the *internal induction* axiom (INT) we mean the following statement:

$$\forall \phi \in \mathrm{Form}^{\leq 1}_{\mathcal{L}_{\mathrm{PA}}} \Big(T\phi(0) \wedge \forall x \big(T\phi(\underline{x}) \rightarrow T\phi(\underline{S(x)}) \big) \rightarrow T \forall x \phi(\underline{x}) \Big).$$

By compositional axioms this is equivalent to saying that all induction axioms are true. Perhaps somewhat surprisingly, this also yields a conservative extension of PA. This result was originally announced in [12]. A proof, by different, model theoretic methods announced in [6], can be found in the longer, unpublished, and privately circulated manuscript [5]. Another argument, now purely proof-theoretic, has been presented in [14] (where it is proved that one can extend CT^- with an arbitrary statement of the form "all instances of the axioms scheme Γ are true" while still keeping the theory in question conservative. It is easy to check that the compositional axioms allow us to derive our version of the internal induction from the statement that all the induction axioms are true).

THEOREM 4 (Kotlarski–Krajewski–Lachlan). *$CT^- + INT$ is conservative over PA.*

2.2. Models of PA. This article will make use of some classical theory of models of PA. Let us now review some basic facts of this area. The standard references are [8] (introductory) and [10] (more advanced), where the proofs of the theorems stated here, and much more, can be found. The first result which we will use repeatedly is the resplendence of recursively saturated models.

DEFINITION 5. We say that a model M is *resplendent* if for any second-order formula $\phi(X)$ with a single second-order variable with all quantifiers ranging over first-order variables and possibly with first-order parametres, if $\phi(X)$ is consistent with the elementary diagram of M (where $\phi(X)$ is viewed as a first-order sentence with a fresh predicate X), then there exists $A \subseteq M$ such that $(M, A) \models \phi(A)$.

The notion of resplendence is very rich in consequences and yet, resplendent models of strong theories are rather easy to find in nature.

THEOREM 6 (Barwise–Schlipf, Ressayre). *Every countable recursively saturated model of PA is resplendent.*

End-extensions of models of truth theories play a crucial role in our article. They are also a very classical thread in the theory of the models of PA.

DEFINITION 7. Let $M \subseteq N$ be models of PA. We say that N is an *end-extension* of M iff for any $c \in N \setminus M$ and an arbitrary $a \in M$, $N \models a < c$. We denote this by $M \subseteq_e N$ (or $M \prec_e N$ if this is in fact an elementary extension). If $M \neq N$, we call the extension *proper*.

We will also use a more sophisticated variant of extensions.

DEFINITION 8. Let $M \prec_e N$. We say that N is a *conservative* extension of M iff for any formula ϕ, possibly with parametres from N, there exists a formula ψ, possibly with parametres from M, such that for any $a \in M$,

$$N \models \phi(a) \text{ iff } M \models \psi(a).$$

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The presence of parametres from N is a crucial requirement in the above definition. Without them, the conclusion would follow trivially by elementarity. One of the key facts from the model theory of PA states that conservative-extensions exist (in fact, conservative extensions automatically have to be end-extensions).

THEOREM 9 (MacDowell–Specker; Gaifman). *Let M be an arbitrary model over a countable signature satisfying the induction scheme for the full language and all axioms of PA. Then M has a proper elementary conservative end-extension N.*

The statement and the proof of the above result can be found in [10, Theorem 2.2.8].[5] An important class of models are κ-like models, of which we will need a special case.

DEFINITION 10. Let $M \models \mathrm{PA}$. We say that M is an ω_1-like model if $|M| = \aleph_1$, but any proper initial segment of M is countable.

The proof of the existence of ω_1-like models is the prototype for the argument presented in this work.

THEOREM 11. *Let $M \models PA$ be an arbitrary countable model. Then there exists an elementary ω_1-like end-extension $M \prec_e N$.*

PROOF. By repeatedly using Theorem 9, we can construct a sequence of countable models $M_\alpha, \alpha < \omega_1$ such that for any $\alpha < \beta$,

$$M_\alpha \prec_e M_\beta.$$

Let $N = \bigcup_{\alpha<\omega_1} M_\alpha$. Then $N \models \mathrm{PA}$ as a union of an elementary chain, N has cardinality $\aleph_1$, and for an arbitrary proper initial segment I, there exists an element $a \in N$, such that $a \notin I$. Let $\alpha < \omega_1$ be any ordinal such that $a \in M_\alpha$. Then, since N is an end-extension of M_α, $I \subset M_\alpha$ and hence it is countable. This shows that N is ω_1-like. ⊣

In our article, we will use a technical condition on cuts of models of PA.

DEFINITION 12. Let $M \models \mathrm{PA}$. We say that a subset I of M is a *cut* of M if the following conditions hold:

- For any $a \in I$ and any $b \leq a$, $b \in M$.
- For any $a \in I$, the successor of a belongs to I.

Cuts in models of arithmetic are a classical subject of intensive study in which a number of their structural properties were isolated. In our argument, we will isolate a new, very weak, regularity condition, whose definition we postpone until Section 4.

2.3. Truth predicates and satisfaction classes. In the literature, there are two competing treatments of the truth-like notions. The first one, more traditional, originates in the theory of models of PA and speaks of satisfaction classes which are treated primarily as subsets in models of arithmetic. The other stems from

[5] Note that the authors use the following convention: "model" means precisely "a model of a theory over a countable signature satisfying the induction scheme for the full language and all axioms of PA" as explained on p.1 of the cited source.

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discussions in philosophical logic and speaks of truth theories which are treated primarily in the axiomatic manner.

Intuitively, a satisfaction class in a model $M \models \mathrm{PA}$, should be a set S of pairs (ϕ, α), where ϕ is an arithmetical formula in the sense of the model and α is a ϕ-assignment such that, collectively, the pairs in S satisfy Tarski's compositional conditions. However, we are often interested in such classes where S is only required to work for some subset of formulae in the model. This, however, makes the notion of a satisfaction class somewhat subtle. In particular, the exact definition of a satisfaction class in not quite consistent between different authors. Below, we present the version from [20] which, we believe, captures various uses of the notion most smoothly.

DEFINITION 13. Let $M \models \mathrm{PA}$, let $S \subset M^2$, and let $\phi \in \mathrm{Form}_{\mathcal{L}_{\mathrm{PA}}}(M)$. We say that S is *compositional* at ϕ if for any $\alpha \in \mathrm{Asn}(\phi)$, $(\phi, \alpha) \in S$ iff one of the following conditions hold:[6]

- There exist $s, t \in \mathrm{Term}_{\mathcal{L}_{\mathrm{PA}}}(M)$ such that $\phi = (s = t)$ and $s^\alpha = t^\alpha$.
- There exists $\psi \in \mathrm{Form}_{\mathcal{L}_{\mathrm{PA}}}(M)$ such that $\phi = \neg\psi$ and $(\psi, \alpha) \notin \phi$.
- There exist $\psi, \eta \in \mathrm{Form}_{\mathcal{L}_{\mathrm{PA}}}(M)$ such that $\phi = \psi \vee \eta$ and $(\psi, \alpha) \in S$ or $(\eta, \alpha) \in S$.
- There exist $\psi, \eta \in \mathrm{Form}_{\mathcal{L}_{\mathrm{PA}}}(M)$ such that $\phi = \psi \wedge \eta$ and both $(\psi, \alpha) \in S$ and $(\eta, \alpha) \in S$.
- There exists $\psi \in \mathrm{Form}_{\mathcal{L}_{\mathrm{PA}}}(M)$ and $v \in \mathrm{Var}$ such that $\phi = \exists v \psi$ and $S(\phi, \alpha)$ holds iff $S(\psi, \beta)$ holds for some $\beta \sim_v \alpha$ i.e., $\beta \in \mathrm{Asn}(\psi)$ such that $\beta(w)$ is equal to $\alpha(w)$ for all w different from v (which is not required to be in the domain of β).
- There exists $\psi \in \mathrm{Form}_{\mathcal{L}_{\mathrm{PA}}}(M)$ and $v \in \mathrm{Var}$ such that $\phi = \forall v \psi$ and $S(\phi, \alpha)$ holds iff $S(\psi, \beta)$ holds for all $\beta \sim_v \alpha$.

We say that a set $S \subset M^2$ is a *satisfaction class* if there exists a set $D \subset \mathrm{Form}_{\mathcal{L}_{\mathrm{PA}}}(M)$ such that

- D is closed under taking direct subformulae.
- For any $\phi \in D$, S is compositional at ϕ.
- For any $\phi \in D$ and any $\alpha \in \mathrm{Asn}(\phi)$, either $(\phi, \alpha) \in S$ or $(\neg\phi, \alpha) \in S$.
- For any $\phi \in \mathrm{Form}_{\mathcal{L}_{\mathrm{PA}}}$, if there exists an α such that $(\phi, \alpha) \in S$ or $(\neg\phi, \alpha) \in S$, then $\phi \in D$.

We call the maximal set D satisfying the above conditions a *domain* of S, denoted $\mathrm{dom}(S)$. If the domain D of S is the whole set $\mathrm{Form}_{\mathcal{L}_{\mathrm{PA}}}(M)$, we say that S is a *full* satisfaction class.

Unfortunately, it turns out that in the absence of some form of induction, the connection between truth predicates and satisfaction classes is not as clear-cut as one could hope. A discussion of that phenomenon may be found in [20]. However, the distinction between truth and satisfaction classes trivialises if we assume that certain regularity properties holds and the truth value of a formula does not depend

[6]The expression $\mathrm{Term}_{\mathcal{L}_{\mathrm{PA}}}$ refers to the arithmetised definition of arithmetical terms; $x \in \mathrm{Asn}(y)$ means that y is a term or formula and x is a y-assignment, i.e., a function whose domain is the set of free variables of y.

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on the choice of specific terms, but rather on their values and it does not depend on the choice of the specific bound variables. We will present the exact assumptions on truth and satisfaction classes in question in the next section.

2.4. Syntactic templates and regularity. As we mentioned before, in the main proof we will need a technical condition on the regularity of the constructed truth and satisfaction classes. We will now spell out those, admittedly tedious, technicalities. Similar considerations played the same role in our previous works, such as [15] or (in a somewhat different formulation) in [20].

DEFINITION 14. Let ϕ be an arbitrary formula. By the *syntactic template* of ϕ, we mean the unique formula $\widehat{\phi}$ such that:

1. $\widehat{\phi}$ differs from ϕ only by term substitution and renaming bound variables.
2. There are no complex terms in $\widehat{\phi}$ (i.e., terms containing function symbols) which contain only free variables.
3. The formula $\widehat{\phi}$ contains no closed terms.
4. Every free variable occurs in $\widehat{\phi}$ at most once.
5. Every bound variable in $\widehat{\phi}$ is quantified over only once.
6. The choice of free and bound variables is such that $\widehat{\phi}$ is the minimal formula satisfying the above conditions.

If $\widehat{\phi} = \widehat{\psi}$, we say that ϕ and ψ are *syntactically similar*. We denote this relation by $\phi \sim \psi$.

In essence, a syntactic template represents the pure syntactic tree of a formula in which all the terms which involve no bound variables were erased and replaced by single variables. Terms which appear under quantifiers are essential for a formula, so we do not modify these.

EXAMPLE 15. Let $\phi := \neg\big(S(x + S(0)) = y \times z\big)$.[7]

Then in $\hat{\phi}$, we replace the term $S(0)$ with a free variable v, and since the term $S(x + v)$ has only free variables, we replace it with a free variable, say v_0. Similarly, we replace $y \times z$ with a free variable, say v_1. Thus, we obtain:

$$\hat{\phi} := \neg(v_0 = v_1).$$

EXAMPLE 16. Let $\phi := \forall x \exists y\ x = y + y$. Then, the syntactic template of ϕ is $\hat{\phi}$:

$$\forall w_0 \exists w_1\ w_0 = w_1 + w_1,$$

where w_0 and w_1 are chosen so as to minimise the formula. Notice that we do not simplify any terms, since they all contain only bound variables.

EXAMPLE 17. Let

$$\phi = \exists x \forall y\big(x + (y \times 0) = S(0) + (x \times (z \times v))\big).$$

Then the $\hat{\phi}$ should satisfy the following conditions:

- The constant 0 and the term $S(0)$ should be replaced with distinct free variables, say w_0, w_1.

[7] Note that in the examples below S below denotes the successor function. We will also use the same symbol for satisfaction classes, but not in a single expression.

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- The term $z \times v$ as a complex term with all variable free should be replaced with a single variable (which we will call w_2).
- The term $x \times (z \times v)$ should be replaced with $x \times w_2$. Note that this is not further replaced with a single variable, since x is not free. Similarly, in $w_1 + x \times (z \times v)$, the addition should not be simplified any further, since one of terms occurring on the right hand side is not free.

Applying these rules, we obtain:

$$\widehat{\phi} = \exists v_0 \forall v_1 \big(v_0 + (v_1 \times w_0) = w_1 + (v_0 \times w_2)\big),$$

where the variables v_i, w_i are chosen so that the resulting formula is minimal.

One of the reasons to introduce syntactic templates is because, as already mentioned, we would like to work with satisfaction classes and the connection between satisfaction and truth classes is not quite as neat, as one could expect and some regularity assumptions seem to be required to actually ensure that actually the two notions coincide. Below, $\phi[\alpha]$ is the sentence obtained by substituting the numeral $\underline{\alpha(v)}$ for every instance of the variable v in the formula ϕ.

DEFINITION 18. Let $M \models \mathrm{PA}$. Let $\phi, \psi \in \mathrm{Form}_{\mathcal{L}_{\mathrm{PA}}}(M)$, let $\alpha \in \mathrm{Asn}(\phi)$, and let $\beta \in \mathrm{Asn}(\psi)$. We say that the pairs (ϕ, α) and (ψ, β) are *syntactically similar* iff ϕ is syntactically similar to ψ and there exist sequences of closed terms $\bar{s}, \bar{t} \in \mathrm{ClTermSeq}_{\mathcal{L}_{\mathrm{PA}}}(M)$ such that:

- $\bar{s}^{\circ} = \bar{t}^{\circ}$ (the terms in both sequences have the same values).
- $\widehat{\phi}(\bar{s})$ differs from $\phi[\alpha]$ only by renaming bound variables.
- $\widehat{\phi}(\bar{t})(= \widehat{\psi}(\bar{t}))$ differs from $\psi[\beta]$ only be renaming bound variables.

If the pairs $(\phi, \alpha), (\psi, \beta)$ are syntactically similar, we denote this fact with

$$(\phi, \alpha) \sim (\psi, \beta).$$

The above notion is actually much simpler than the definition suggests. In essence, we want to say this: Given a formula ϕ, we may look at its pure syntactic tree represented by $\bar{\phi}$. In $\bar{\phi}$, we will see some free variables. If α is a ϕ-assignment, it will actually decide the values of these variables in a unique way. Then the similarity of pairs $(\phi, \alpha) \sim (\psi, \beta)$ means that ϕ and ψ have the same pure syntactic tree and α, β yield the same assignment of the free variables of in the formula representing that tree.

EXAMPLE 19. Let

$$\phi := x = SS(0)$$
$$\psi := SSS(0) = y.$$

Let α be a ϕ-assignment such that

$$\alpha(x) = 3.$$

Let β be a ψ-assignment such that

$$\beta(y) = 2.$$

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Then $(\psi, \alpha) \sim (\psi, \beta)$. Indeed, we have $\hat{\phi} = \hat{\psi}$ equal to:

$$v_0 = v_1$$

and if $\bar{s} = \bar{t} = \langle SSS(0), SS(0)\rangle$, then:

$$\hat{\phi}(\bar{s}) = \phi[\alpha] = \psi[\beta] = \hat{\phi}(\bar{t}).$$

EXAMPLE 20. Let

$$\phi := \exists x \forall y \left(x + (y \times S(S(0))) = z \times (S(0) + S(0))\right)$$
$$\psi := \exists z \forall v \left(z + (v \times (u + w)) = S(S(S(S(0))))\right).$$

Let $\alpha \in \mathrm{Asn}(\phi)$ be an assignment such that

$$\alpha(z) = 2.$$

Let $\beta \in \mathrm{Asn}(\psi)$ be an assignment such that

$$\beta(u) = 1, \beta(w) = 1.$$

Then $(\phi, \alpha) \sim (\psi, \beta)$. Indeed, this is witnessed by $\widehat{\phi} = \widehat{\psi}$ equal to:

$$\exists v_0 \forall v_1 \left(v_0 + v_1 \times w_0 = w_1\right).$$

and sequences $\bar{s}, \bar{t}$ such that:

$$\bar{s} = \langle S(S(0)), S(S(0)) + (S(0) + S(0))\rangle$$
$$\bar{t} = \langle S(0) + S(0), S(S(S(S(0))))\rangle.$$

Then

$$\widehat{\phi}(\bar{s}) = \exists v_0 \forall v_1 \left(v_0 + (v_1 \times S(S(0))) = S(S(0)) + (S(0) + S(0))\right)$$
$$\phi[\alpha] = \exists x \forall y \left(x + (y \times S(S(0)) = S(S(0)) + (S(0) + S(0))\right)$$

which differ only by renaming bound variables. Moreover,

$$\widehat{\psi}(\bar{t}) = \exists v_0 \forall v_1 \left(v_0 + v_1 \times (S(0) + S(0)) = S(S(S(S(0))))\right)$$
$$\psi[\beta] = \exists z \forall v \left(z + (v \times (S(0) + S(0))) = S(S(S(S(0))))\right),$$

which differ only by renaming bound variables.

As we already mentioned, we want to restrict our attention to classes for which good regularity properties hold.

DEFINITION 21. Let (M, S) be a satisfaction class. We say that S is *syntactically regular* if for any $\phi, \psi \in \mathrm{Form}_{\mathcal{L}_{\mathrm{PA}}}(M)$ and $\alpha \in \mathrm{Asn}(\phi), \beta \in \mathrm{Asn}(\psi)$ if $(\phi, \alpha) \sim (\psi, \beta)$, then

$$(\phi, \alpha) \in S \text{ iff } (\psi, \beta) \in S.$$

Let $(M, T) \models \mathrm{CT}^-$. We say that T is *syntactically regular* if for any $\phi, \psi \in \mathrm{Sent}_{\mathcal{L}_{\mathrm{PA}}}(M)$ if $(\phi, \emptyset) \sim (\psi, \emptyset)$ (note that $\emptyset$ is the empty function and can be treated as a trivial assignment),

$$\phi \in T \text{ iff } \psi \in T.$$

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Under these regularity assumptions, we can actually make a straightforward connection between truth predicates and satisfaction classes.

PROPOSITION 22. *Let $M \models PA$ and let S be a full syntactically regular satisfaction class on M. Let*

$$T = \{\phi \in Sent_{\mathcal{L}_{PA}}(M) \mid (\phi, \emptyset) \in S\}.$$

Then $(M, T) \models CT^-$ and, moreover, T is syntactically regular.

Conversely, suppose that $(M, T) \models CT^-$ and that T is syntactically regular. Let

$$S = \{(\phi, \alpha) \in M^2 \mid \phi \in Form_{\mathcal{L}_{PA}}(M), \alpha \in Asn(\phi), \text{ and } \phi[\alpha] \in T\}.$$

Then S is a full syntactically regular satisfaction class.[8]

If S and T are interdefinable in the way postulated in the above proposition, we say that S is a satisfaction class *corresponding* to T and that T is a truth predicate *corresponding* to S. A slightly different statement, in which we used weaker regularity assumptions appeared as [20, Proposition 15]. The definition of syntactic regularity formulated in this work assumes that a syntactically regular satisfaction class is closed under renaming bound variables (equivalently, under α-conversion). This is not needed to obtain the above correspondence, but will be used in the proof of the main theorem. However, *some* regularity assumptions apparently *are* needed in order to obtain the above simple correspondence as discussed in the cited article.

Crucially for our article, we have the following.

PROPOSITION 23. *Let $M \models PA$ be a countable recursively saturated model. Then there exists $T \subset M$ such that $(M, T) \models CT^- + INT$ and T is syntactically regular. Moreover, the same holds for $CT^- + INTColl$.*

This fact appeared as [15, Theorem 23]. The proof of this fact was strictly speaking omitted, but the proof is a (completely straightforward) modification of an argument which appeared there with a precise comment on what modification is needed.

§3. The conservativity of collection. In this section, we present the strategy for the proof of conservativity of the compositional truth predicate with the collection axioms, relegating the demonstration of the crucial technical results to next sections. Let us start with a basic observation already suggested by Kaye as the main tool for the argument we present in this article.

PROPOSITION 24. *Suppose that $M \models PA$ is an ω_1-like model and that $S \subset M$ is an arbitrary subset. Then the expansion (M, S) satisfies the full collection scheme.*

PROOF. Let ϕ be an arbitrary formula in the language $\mathcal{L}_{\mathrm{PA}}$ expanded with the symbol S. Fix any $a \in M$ and suppose that

$$(M, S) \models \forall x < a \exists y\, \phi(x, y).$$

Let $f : M \to M$ be a function such that for any $x \in M$ which is smaller than a,

$$(M, S) \models \phi(x, f(x)).$$

[8] Note that below S refers to the satisfaction class, not the successor function. We will not have to refer explicitly to successor function anymore.

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Since M is ω_1-like model, the interval $[0, a]$ is countable. Therefore, the image $f\big[[0, a]\big]$ cannot be cofinal in M. This means that there exists $b \in M$ such that $f\big[[0, a]\big] \subseteq [0, b]$. In particular,

$$(M, S) \models \forall x < a \exists y < b\ \phi(x, y).$$

Since ϕ was arbitrary, (M, S) is a model of the full collection. ⊣

As a matter of fact, our argument for conservativity of collection follows exactly the path suggested by the above result.

THEOREM 25. *Let $M \models PA$ be an arbitrary countable model. Then, there exists an ω_1-like elementary extension $M' \succ M$ and a full syntactically regular satisfaction class $S \subset M'$.*

The main goal of the article is to prove Theorem 25. One obvious potential strategy for a proof would be to show that for an arbitrary countable model $(M, T) \models \mathrm{CT}^-$, one can find a proper end-extension $(M', T') \succ_e (M, T)$. However, this direct strategy cannot quite work, since as demonstrated by [19], countable models of CT^- do not necessarily have end-extensions.

THEOREM 26 (Smith). *For any countable recursively saturated model $M \models PA$, there exists $T \subset M$ such that $(M, T) \models CT^-$ and there is no $(M', T') \supsetneq_e (M, T)$ satisfying CT^-.*

The proof uses the following result which can be found in [19, Theorem 3.3].

THEOREM 27 (Smith). *Let M be a countable recursively saturated model of PA. Let $A \subseteq M$ be an arbitrary set such that the expansion (M, A) is recursively saturated in the expanded language. Then there exist $T \subset M$ and $\phi(v) \in Form^{\leq 1}_{\mathcal{L}_{PA}}$ such that $(M, T) \models CT^-$ and*

$$A = \{x \in M \mid (M, T) \models T\phi(\underline{x})\}.$$

Now, we can prove theorem 26.

PROOF. Let $M \models \mathrm{PA}$ be countable and recursively saturated. Pick any $a \in M$ in the nonstandard part. Then there exists a bijection between $[0, a]$ and M. We claim that there exist an expansion $(M, T) \models \mathrm{CT}^-$ and a formula $\phi(x, y) \in \mathrm{Form}_{\mathcal{L}_{\mathrm{PA}}}(M)$ such that the following hold:

- $(M, T) \models \forall x < a \exists!\, y\ T\phi(\underline{x}, \underline{y})$.
- $(M, T) \models \forall y \exists!\, x < a\ T\phi(\underline{x}, \underline{y})$.

In order to see that such a T exists, fix an arbitrary $a \in M$ and an arbitrary $A \subseteq M$ which is a bijection between the initial segment $[0, a]$ and the whole model such that (M, A) is recursively saturated (such an A exists by the resplendence of M). Then by Theorem 27, the set A can be captured under the truth predicate by some $\phi \in \mathrm{Form}_{\mathcal{L}_{\mathrm{PA}}}(M)$.

Notice that by the compositional axioms, these conditions can be equivalently rewritten as:

- $(M, T) \models T\big(\forall x < \underline{a} \exists!\, y \phi(x, y)\big)$.
- $(M, T) \models T\big(\forall y \exists!\, x < \underline{a}\ T\phi(x, y)\big)$.

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Now, suppose that $(M', T') \supset_e (M, T)$ satisfies CT^-. Then in (M', T'), ϕ defines a bijection between $[0, a]^{M'}$ and M' under the truth predicate. Moreover, this bijection extends the one defined in (M, T). However this is impossible, since $[0, a]^{M'} = [0, a]^M$, but $M \neq M'$. ⊣

Notice that in the above example, we used the fact that collection was violated for a formula of the form $T\phi(\underline{x}, \underline{y})$. One can wonder whether this is in fact the only possible obstruction to the existence of end-extensions. It turns out that the answer to this question is indeed positive. For simplicity, we will first prove the end-extension result under slightly stronger assumptions, which we will relax in Section 5.

THEOREM 28. *Let $(M, T) \models CT^- + INT$ be a countable model with T syntactically regular. Then there exists a proper end-extension $(M, T) \subset_e (M', T') \models CT^- + INT$ satisfying the syntactic regularity condition.*

The proof of the above result is the heart of this work. We will present it in the next section. Before we do, let us draw our main corollary.

THEOREM 29. *Let $M \models PA$ be an arbitrary countable model. Then there exists an elementary extension $M \prec M'$ such that M' is ω_1-like and there exists $T \subset M'$ for which $(M', T) \models CT^-$.*

PROOF. Fix an arbitrary countable model $M \models \mathrm{PA}$. Let $M_0 \succeq M$ be a countable, recursively saturated model of PA. By resplendence, we can find an expansion of M_0 to a model $(M_0, T_0) \models \mathrm{CT}^- + \mathrm{INT}$ with T_0 syntactically regular. We define inductively a sequence of countable models $(M_\alpha, T_\alpha), \alpha < \omega_1$ satisfying $\mathrm{CT}^- + \mathrm{INT}$ as follows:

- $(M_{\alpha+1}, T_{\alpha+1}) \supset_e (M_\alpha, T_\alpha)$ is an arbitrary proper end-extension satisfying $\mathrm{CT}^- + \mathrm{INT}$.
- $(M_\lambda, T_\lambda) = \bigcup_{\gamma<\lambda}(M_\gamma, T_\gamma)$ for limit λ.

The models at successor steps can be constructed by Theorem 28, so we only have to check that the induction hypotheses can be maintained in the limit steps. However, it can be checked in a straightforward manner that the compositional clauses of CT^- are preserved in the unions of models. Also notice that over CT^-, the internal induction is equivalent to a Π_1-sentence saying "all the instances of the arithmetical induction scheme are true", so it is also preserved at the limit steps. ⊣

We can now complete the main line of the argument.

THEOREM 30. *$CT^- + Coll$ is conservative over PA.*

PROOF. It is enough to show that for an arbitrary arithmetical sentence ϕ, if $\mathrm{PA} + \phi$ is consistent, then $\mathrm{CT}^- + \mathrm{Coll} + \phi$ is consistent.

Suppose that $\mathrm{PA} + \phi$ is consistent and take any countable model $M \models \mathrm{PA} + \phi$. By Theorem 29, there exists an elementary ω_1-like extension $M' \succeq M$ with a subset $T \subset M'$ such that $(M', T) \models \mathrm{CT}^-$. By Proposition 24, (M', T) is actually a model of $\mathrm{CT}^- + \mathrm{Coll}$. By elementarity, $M' \models \phi$, so (M', T) witnesses the consistency of $\mathrm{CT}^- + \mathrm{Coll} + \phi$. ⊣

Actually, since our methods really rely on the countability of the models involved in the construction, it is unclear whether we can show the existence of κ-like models of CT^- for an arbitrary κ.

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QUESTION 31. *Let $M \models PA$ be an arbitrary countable model and let κ be an arbitrary cardinal. Does there exist a κ-like model $M' \succ_e M$ which expands to a model of CT^-?*

§4. End-extensions of satisfaction classes. In this section, we will prove the main result. As we mentioned in the previous section, the key step is to prove that the countable models of $CT^- + INT$ always have end-extensions. For the technical convenience, in this section we will switch to the language of regular satisfaction classes. By Proposition 22, in our framework they directly correspond to the models of CT^- satisfying full regularity.

The structure of argument will be divided into two main parts:

- First we show that given a countable model M of PA carrying a full regular satisfaction class which satisfies the internal induction, we can end-extend it to a model M' with a partial satisfaction class whose domain includes all formulae with the syntactic depth in M and such that M is a nicely behaved cut of M'.
- Then we prove that if a countable model M' has a partial satisfaction class whose domain consists of formulae with the syntactic depth in a certain nicely behaved cut, then in M' we can find a full satisfaction class. Moreover, internal induction can be preserved in this extension.

The next two sections will be devoted to those two main steps of the proof. In particular, wee will make precise the requirements we impose on the cuts in question.

4.1. Stretching Lemma. In this part, we will discuss the first of the main steps in the end-extension theorem. The key ideas of this subsection appeared already in [15]. However, since we will need to extract some additional information from the construction, we will present here the full proof.

As we already mentioned, we will need to impose certain regularity conditions on the cuts arising in our construction. The exact choice of the condition is rather subtle.

DEFINITION 32. Let $I \subset M \models \mathrm{PA}$ be a nonstandard cut. We say that I is *locally semiregular* in M if for any nonstandard $a \in I$ and any function $f : [0, a] \to M$ coded in M, there exist a nonstandard $a' \leq a$ and $b \in I$ such that the following condition holds:

$$f[[0, a']] \cap I \subseteq [0, b].$$

A word of comment is certainly in place. One of the conditions on cuts, classically investigated in the theory of models of PA, is semiregularity. We say that a cut I is *semiregular* in M if for any function f coded in M with the domain $[0, a]$ for some $a \in I$, the set of thhe values $f(i)$ such that $f(i) \in I$ is bounded in I. Local semiregularity wekens this condition, demanding instead that the function f can be restricted to a nonstandard initial segment so that the bounding condition holds.

Admittedly, this is a somewhat technical requirement. However, the choice of this exact condition will be crucial in Section 5, since we were not able to show the main end-extension result from that part which would guarantee any stronger regularity requirements on cuts. On the other hand, any weaker conditions known in the literature do not seem to suffice to perform the copying construction of

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Theorem 34. However, let us remark that we actually do not need the notion of local semiregularity in the context of the internal induction. In Lemma 33, we can obtain a stronger conclusion that the elementary extension we obtain is conservative and we can assume in Theorem 34 that the cut I is a model of PA such that M is its conservative extension. However, this would not really simplify the argument in any substantial way, so we decided to prove it in the greater generality. Nevertheless, the reader who is confused by the notion of local semiregularity may completely ignore it in this section and instead think of conservative extensions. Let also us add that we were also not able to find any condition previously known in the literature which would be equivalent to our new notion.

LEMMA 33 (Stretching Lemma). *Let $M \models PA$ be a countable model and let $S \subset M^2$ be a full regular satisfaction class satisfying the internal induction. Then there exists a proper end extension $(M, S) \subset (M', S')$ such that*:

- *S' is a regular partial satisfaction class.*
- *For any $\phi \in Form_{\mathcal{L}_{PA}}(M')$, $\phi \in dom(S')$ iff $dp(\phi) \in M$.*[9]
- *S' satisfies the INT.*
- *The extension $M \prec_e M'$ is elementary and M is a locally semiregular cut of M'.*

PROOF. Fix a model (M, S) as above. For any $\phi \in \mathrm{Form}_{\mathcal{L}_{\mathrm{PA}}}(M)$, let S_ϕ be the set:

$$S_\phi = \{\alpha \mid (\phi, \alpha) \in S\}.$$

Consider the model $(M, S_\phi)_{\phi \in M}$. By the INT, this model satisfies full induction. Since M is countable, the signature of the expanded model has only countably many symbols. Thus by MacDowell–Specker Theorem, there exists a proper conservative elementary end-extension

$$(M, S_\phi)_{\phi \in M} \prec_e (M', S'_\phi)_{\phi \in M}.$$

Now, let $S' \subset (M')^2$ be defined by the following condition: $(\phi, \alpha) \in S'$ iff there exists a pair (ψ, β) such that:

- $(\psi, \beta) \sim (\phi, \alpha)$.
- $\psi \in M$.
- $S'_\psi(\beta)$ holds.

In other words, we take a union of sets S'_ψ and close it under syntactic similarity. We can then check that by elementarity of the extension $(M', S'_\phi)_{\phi \in M}$ and by the regularity of S, the constructed set S' is a regular satisfaction class satisfying internal induction. The syntactic regularity of S' follows directly by construction.

We have to check that M is a locally semiregular cut of M'. (In fact, we will simply verify that conservativity implies semiregularity). Let

$$f : [0, a] \to M'$$

be a function coded in M'. We want to check that for any $b \in M$, there exists $c \in M$ such that the values of $f \restriction M$ are bounded by c. Since f is coded in M',

[9] If ϕ is a formula in the sense of M, then by $\mathrm{dp}(\phi)$ we mean the height of the syntactic tree of ϕ (where we do no count the complexity of terms).

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by conservativity of the extension $(M, S_\phi)_{\phi\in M} \prec (M', S'_\phi)_{\phi\in M}$, the set $f \cap M$ is definable in the former structure. However, since this model satisfies full induction in the expanded language, the values of $(f \restriction [0, b]) \cap M$ have to be bounded in M. $\dashv$

We will discuss in Section 5 how internal induction can be eliminated from the above argument.

4.2. Copying Lemma. In the previous section, we have shown how to extend a satisfaction class "upwards" so that it is still defined for all the formulae in the original model. In this section, we will discuss the essentially novel part of our argument: we will show how, under additional model-theoretic assumptions, we can extend a satisfaction class defined on a cut of formulae to the whole model. Our argument crucially uses ideas introduced by Fedor Pakhomov in his construction of a satisfaction class presented in his unpublished note [16].[10]

THEOREM 34 (Copying Lemma). *Let $M \models PA$ be a countable model. Let $I \subset M$ be a locally semiregular cut in M. Suppose that there exists a syntactically regular satisfaction class $S \subset M^2$ whose domain consists of formulae with depth in I. Then there exists a full regular satisfaction class $S' \supset S$. Moreover, if S satisfies internal induction, then S' also satisfies it.*

PROOF. Let, M, I, S be as in the assumptions of the theorem. Let D be the set of all formulae in M whose syntactic depth is in I (i.e., D is the domain of S). We will construct a function $f : \mathrm{Form}_{\mathcal{L}_{\mathrm{PA}}}(M) \to D$ satisfying the following conditions for any $\phi, \psi \in \mathrm{Form}_{\mathcal{L}_{\mathrm{PA}}}$ and any $v \in \mathrm{Var}(M)$:

- $f \restriction D = \mathrm{id}$.
- For any ϕ, ψ if $\phi \sim \psi$, then $f(\phi) \sim f(\psi)$.
- $f(\neg\phi) = \neg f(\phi)$.
- $f(\phi \wedge \psi) = f(\phi) \wedge f(\psi)$.
- $f(\phi \vee \psi) = f(\phi) \vee f(\psi)$.
- $f(\exists v\phi) = \exists v f(\phi)$.
- $f(\forall v\phi) = \forall v f(\phi)$.

We will define inductively a sequence of partial functions satisfying the above conditions for a given formula and its subformulae laying at certain depth of the syntactic tree. We will make sure that the functions will be compatible between different formulae and then we will glue them together.

For a formula $\phi \in \mathrm{Form}_{\mathcal{L}_{\mathrm{PA}}}(M)$ and $a \in M$, let $U(\phi, a)$ be the set of formulae ψ such that $\psi \sim \psi'$ for some ψ' located at most at the depth a in the syntactic tree of ϕ.

Fix an enumeration $(\phi_i)_{i\in\omega}$ of $\mathrm{Form}_{\mathcal{L}_{\mathrm{PA}}}(M)$. We will define a sequence of nonstandard elements of M, $a_0 > a_1 > a_2 \ldots$ and functions $f_0, f_1, \ldots$ such that each function f_i satisfies the following conditions:

[10] The construction described in the note was discussed in the Autumn 2020 in the talk [17] seminar "Epistemic and Semantic Commitments of Foundational Theories." The slides and the recording of the talk are available at the seminar webpage given in the bibliography.

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- $f_i : U(\phi_i, a_i) \to \mathrm{Form}_{\mathcal{L}_{\mathrm{PA}}}(M)$.
- If $i \leq k$, then $f_i \restriction U(\phi_i, a_k) \cap U(\phi_k, a_k) = f_k \restriction U(\phi_i, a_k) \cap U(\phi_k, a_k)$.
- f_i commutes with quantifiers and connectives in the sense postulated for f.
- $f_i \restriction D = \mathrm{id}$.
- If $\phi \sim \psi$, then $f_i(\phi) \sim f_i(\psi)$.

Notice that we *do not* require that $f_0 \subset f_1 \subset f_2$ or that the functions f_i agree on the whole common domain. Finally, we will set:

$$f(\phi) = f_i(\phi),$$

where $\phi_i = \widehat{\phi}$ (so i is the index of the template of ϕ in our enumeration).

Let us check that f defined in this way indeed satisfies our conditions. It is enough to check that f preserves syntactic operations, since the other conditions follow directly by assumption on the functions f_i. We will check the condition for conjunction, the others being similar or completely analogous. So let us fix formulae ϕ, ψ. Suppose that $\phi \sim \phi_k, \psi \sim \phi_l, \phi \wedge \psi \sim \phi_m$. Let $n = \max(k, l, m)$. Let $U_1 = U(\phi_m, a_n) \cap U(\phi_k, a_n)$, $U_2 = U(\phi_m, a_n) \cap U(\phi_l, a_n)$. Then

$$f_k \restriction U_1 = f_m \restriction U_1$$

and

$$f_l \restriction U_2 = f_m \restriction U_2.$$

In particular $f(\phi) = f_m(\phi)$, $f(\psi) = f_m(\psi)$ and the claim follows, since f_m preserves the syntactic structure. So it is enough to construct the sequences $(f_i), (a_i)$ as above. We will also construct an auxiliary sequence (c_i) of the elements of M.

Let a_0 be an arbitrary element of the cut I. The construction of f_0 is very similar to the construction of functions f_{i+1} in the successor steps, so we will go directly to that case indicating the (small and obvious) differences whenever they appear in the proof.

Let $a := a_{n+1}$ be an arbitrary nonstandard element of I such that:

$$a_{n+1} 2^{a_{n+1}} < \frac{a_n}{2}.$$

Let $\phi := \phi_{n+1}$. Now, consider the set $U(\phi, a)$. Since $a \in I$, by local semiregularity of this cut we can conclude that (possibly after replacing a with a smaller element a' which we will for simplicity still denote a) there exists $c := c_{n+1} \in I$ such that all subformulae of ϕ which occur at the depth a in its syntactic tree of have themselves syntactic depth either $< c$ or not in I.[11]

Consider the following relation $\trianglelefteq$:

$$\xi \trianglelefteq \eta$$

iff there exists a coded sequence of formulae:

$$\xi_0, \xi_1, \ldots, \xi_p$$

[11] This is the only place where we use local semiregularity. The reader can verify that the existence of a' and c satisfying the above conditions easily follows from the assumption that M is a conservative extension of I and both are models of PA.

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such that $\xi_0 = \xi, \xi_p = \eta$ and for any i, ξ_i is a direct subformula of ξ_{i+1} and $\xi_i \in U(\phi, a)$. In other words, $\xi \trianglelefteq \eta$ means that ξ is a subformula of η as can be witnessed using only formulae from $U(\phi, a)$.

We want to define the function $f := f_{n+1} : U(\phi, a) \to \mathrm{Form}_{\mathcal{L}_{\mathrm{PA}}}(M)$ so that it commutes with the syntactic operations. We actually will need a small technical definition. Let us say that a formula $\zeta \in U(\phi, a)$ is *weakly minimal* with respect to $\trianglelefteq$ if it has a direct subformula which is not in $U(\phi, a)$. Such formulae can be different from $\trianglelefteq$-minimal formulae in the case their main operator is binary, one of the formulae is in $U(\phi, a)$, and the other is not.

It is enough to define the function f on the set weakly minimal formulae from $U(\phi, a)$ and then extend the definition to the whole $U(\phi, a)$ by induction (applied internally in the model). In this manner, we will obtain a function f_{n+1} commuting with all connectives and quantifiers, whenever a formula and all its direct subformulae are in the domain. ⊣

Now pick any weakly $\trianglelefteq$-minimal formula ψ. We consider two cases.

Case I There exists $k < n + 1$ such that $\psi \in \mathrm{dom}(f_k)$. Then we set $f(\psi) = f_k(\psi)$, where k is the maximal such index. (When we construct the function f_0, we simply omit this step.)

Case II Otherwise, let $f(\psi)$ be the formula ψ with every subformula at the level c in the syntactic tree replaced with the sentence $0 = 0$.

As mentioned before, the function f can be then uniquely extended to the set $U(\phi, a)$ by induction on $\trianglelefteq$ performed in the model M. So it is enough check that the sequence of functions (f_n) defined above satisfies our requirements.

It is clear by definition that for all n, f_n is defined on $U(\phi_n, a_n)$ and that it preserves the syntactic operations. We need to check that these functions are the identity when restricted to D, that they are congruent with respect to $\sim$, and that they satisfy the agreement condition.

4.3. The identity condition. We prove by induction on n that $f_n \restriction D = \mathrm{id}$. The initial case will be very similar to the induction step, so we only present the latter.

Fix any formula $\psi \in D \cap U(\phi_{n+1}, a_{n+1})$ and first assume that ψ is $\trianglelefteq$-minimal. If $\psi \in \mathrm{dom}(f_k)$ for some $k \leq n$, then $f_{n+1}(\psi) = f_l(\psi)$, where l is the maximal index for which $\psi \in \mathrm{dom}(f_l)$. Then by induction hypothesis $f_l(\psi) = \psi$ which proves the claim.

If $\psi \notin \mathrm{dom}(f_k)$ for $k \leq n$, then $f_{n+1}(\psi)$ is the formula ψ with any subformula at the syntactic level c_{n+1} replaced with a sentence $0 = 0$. However, by construction any formula with the syntactic depth from I which belongs to $U(\phi_{n+1}, a_{n+1})$ has syntactic depth strictly less than c_{n+1}, so in fact the described substitution is trivial and $f_{n+1}(\psi) = \psi$. Then it is enough to observe that on nonminimal formulae f_{n+1} is defined by induction on $\trianglelefteq$ which clearly preserves the identity condition, since the set D is closed under subformulae.

4.4. The congruence condition. We check by induction on n that for any $\psi, \eta \in \mathrm{dom}(f_n)$ if $\psi \sim \eta$, then $f_n(\psi) \sim f_n(\eta)$.

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Suppose that ψ, η are weakly minimal in $U(\phi_n, a_n)$. First suppose that $\psi \in \mathrm{dom}(f_k)$ for some $k < n$. Notice that the set $U(\phi_k, a_k)$ is closed under $\sim$, so in such case both ϕ and ψ are in $\mathrm{dom}(f_k)$ and by induction hypothesis, $f_n(\psi) \sim f_n(\eta)$.

If, on the other hand, $\psi, \eta \notin \mathrm{dom}(f_k)$ for any $k < n$, then $f_n(\psi)$ and $f_n(\eta)$ are defined by substituting $0 = 0$ for any subformula at the syntactic depth c in these formulae. Notice that if $\psi \sim \eta$, then their syntactic trees were equal up to term substitutions and renaming bound variables. In such a case, those trees with $0 = 0$ substituted for all formulae at the depth c will still be equal.

Now, we can check by induction that f_n is a congruence with respect to $\sim$ on the whole $U(\phi_n, a_n)$ by induction on $\trianglelefteq$, applied internally.

4.5. The coherence condition. We want to check that if $i \leq k$, then

$$f_i \restriction U(\phi_i, a_k) \cap U(\phi_k, a_k) = f_k \restriction U(\phi_i, a_k) \cap U(\phi_k, a_k).$$

We can inductively assume that the desired equality holds for any $j \leq i$:

$$f_i \restriction U(\phi_i, a_j) \cap U(\phi_k, a_j) = f_j \restriction U(\phi_i, a_j) \cap U(\phi_j, a_j).$$

Fix any $\phi \in U(\phi_i, a_k) \cap U(\phi_k, a_k)$. Consider any weakly minimal $\psi \in U(\phi_k, a_k)$ which is $\trianglelefteq$-below ϕ in $U(\phi_k, a_k)$.

Now, observe that any $\trianglelefteq$-chain in $U(\phi_k, a_k)$ has at most

$$a_k 2^{a_k}$$

elements, since for any $\sim$-representative η of ϕ_k, there are at most that many subformulae of η in $U(\phi_k, a_k)$ (since they form a tree with at most binary branches of height at most a_k). Since $a_k 2^{a_k} + a_k < a_i$, all such weakly minimal formulae ψ are in $U(\phi_i, a_i)$ and, in fact in $U(\phi_i, a_j)$ for any $j < k$. Now, by construction, for any such formula ψ we have:

$$f_k(\psi) = f_m(\psi),$$

where $m \geq i$ is the greatest index smaller than k such that $\psi \in U(\phi_m, a_m)$. In particular, by the above remark

$$\psi \in U(\phi_m, a_m) \cap U(\phi_i, a_m).$$

Hence, by induction hypothesis,

$$f_m(\psi) = f_i(\psi),$$

and, consequently,

$$f_k(\psi) = f_m(\psi) = f_i(\psi).$$

Since $f_k(\phi)$ is uniquely determined by the values on weakly minimal formulae which are $\trianglelefteq$-smaller than ϕ in $U(\phi_k, a_k)$ and as we have just argued, all such chains are also contained in $U(\phi_i, a_i)$, we conclude that

$$f_k(\phi) = f_i(\phi).$$

This concludes the proof of the coherence clause and therefore of Theorem 34.

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4.6. The proof of the main theorem. In this section, we will put together the findings of the two previous parts in order to prove Theorem 28, showing that any countable model of $\mathrm{CT}^- + \mathrm{INT}$ has an end-extension.

PROOF OF THEOREM 28. As in the formulation of the theorem, let $(M, T) \models \mathrm{CT}^- + \mathrm{INT}$ be a countable model. Let S be a full regular satisfaction class corresponding to T. By Lemma 33, there exists a proper end-extension

$$(M, S) \subset_e (M', S')$$

such that S' is a regular partial satisfaction class whose domain consists of formulae with depth in M with M being a locally semiregular cut in M'. Morevoer, the obtained satisfaction class satisfies INT.

Now by Theorem 34, there exists $S'' \supset S'$ such that (M', S'') is a full syntactically regular satisfaction class with the internal induction. Let T' be a truth class corresponding to S''. Then $(M', T') \models \mathrm{CT}^- + \mathrm{INT}$ is a desired end-extension. ⊣

§5. Internal collection. In the previous sections, we have shown that for any complete consistent theory in $\mathcal{L}_{\mathrm{PA}}$ extending PA, there exist a model M satisfying that theory which features a full truth class satisfying the full collection scheme.

However, the fact that we needed to use the internal induction as a vehicle for obtaining end-extensions seems to be an artefact of our argument rather than genuine necessity, especially since internal induction does not seem to follow from full collection. Therefore, it is natural to ask, whether an analog of Theorem 28 holds for some weaker theories. A natural candidate for such an analog seems to be the principle of the internal collection.

DEFINITION 35. By the *internal collection* axiom (INTColl), we mean the following principle:

$$\forall \phi \in \mathrm{Form}^{\leq 1}_{\mathcal{L}_{\mathrm{PA}}} \forall a \Big(\forall x < a \exists y \; T\phi(\underline{x}, \underline{y}) \to \exists b \forall x < a \exists y < b \; T\phi(\underline{x}, \underline{y}) \Big).$$

Mimicking the usual proof that induction implies collection, we can show using compositional conditions that the INT entails the INTColl. In particular, both are conservative over PA.

As we already mentioned, internal induction can indeed be replaced with internal collection in our argument.

THEOREM 36. *Let $(M, T) \models CT^- + INTColl$ be a countable model. Then there exists an end-extension $(M, T) \subset_e (M', T') \models CT^- + INTColl$.*

The proof of the theorem is entirely parallel to the argument for internal induction. The only place in our argument in which internal induction was actually used was to assure in the proof of Stretching Lemma 33 that in the end extension $(M, T) \subset_e (M', T')$ we can arrange M to a locally semiregular cut in M'. We actually invoked a significantly stronger fact. By using MacDowell–Specker Theorem, we assured that we produced a *conservative* end-extension of models

$$(M, S_\phi)_{\phi \in M} \prec_e (M', S'_\phi)_{\phi \in M'}$$

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and concluded that M is a semiregular cut in M'. Unfortunately, if (M, S) satisfies only internal collection, in general the model $(M, S_\phi)_{\phi \in M}$ will not satisfy full induction, so MacDowell–Specker theorem cannot be applied. This is where we turn to local semiregularity. It is a weaker condition which can be arranged in the context of INTColl and which fully suffices for our proof.

In the proof of Theorem 36, we will use more than just slicing having an extension. The following result by Smith [19, Theorem 3.1], will play a crucial role.

THEOREM 37. *Let $M \models PA$ and let S be a full satisfaction class on M. Let $\phi \in Form_{\mathcal{L}_{PA}}(M)$. Let*

$$S_\phi = \{\alpha \in Asn(\phi) \mid (\phi, \alpha) \in S\}.$$

Then the expansion (M, S_ϕ) is recursively saturated.

As an immediate corollary, we see that the slicing $(M, S_\phi)_{\phi \in M}$ of a model with a full satisfaction class satisfies a good deal of saturation. Let us give this kind of saturation a handy name.

DEFINITION 38. Let M be an arbitrary model. We say that M is *locally recursively saturated* if any recursive type p in which only finitely many symbols from M are used is realised in M.

COROLLARY 39. *Let M be a countable model with a full satisfaction class S satisfying the internal collection. Then the slicing $(M, S_\phi)_{\phi \in M}$ is a locally recursively saturated structure in a countable language satisfying the full collection scheme.*

PROOF. This is an immediate corollary to Theorem 37. If we expand M with any finite set of predicates $S_{\phi_1}, \ldots, S_{\phi_n}$, the resulting structure is recursively saturated since these predicates can be defined from one predicate S_ψ using the arithmetical coding of tuples. ⊣

It seems that the use of coding was in fact not necessary in the above argument and we could simply reprove Smith's result directly for finite tuples of predicates. Now we are ready to state the analog of MacDowell–Specker Theorem which works in the context of the local collection.

THEOREM 40. *Let M be a countable locally recursively saturated model over a countable signature which satisfies PA and the full collection scheme. Then there exists a proper elementary end-extension $M \prec_e M'$ such that M is a locally semiregular cut of M'.*

In the proof, we will use a simple observation which is very far from new or original. It states the equivalence between the collection scheme and the so-called regularity scheme.

LEMMA 41 (Compression lemma). *Let M be a model in a language with a linear ordering $\leq$ without a largest element, which satisfies the full collection scheme. Let ϕ be a binary formula, let $a \in M$, and suppose that there exist arbitrarily large $y \in M$ such that for some $x < a$, $\phi(x, y)$ holds. Then there exists $x_0 < a$ such that $\phi(x_0, y)$ holds for arbitrarily large $y \in M$.*

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PROOF. Using the same notation as in the statement of the lemma, assume that for an arbitrarily large elements y, there exists an $x < a$ with $\phi(x, y)$. Suppose that for any $x < a$ there exists $z \in M$ such that for any $y > z$, $\neg\phi(x, z)$ holds. Let:

$$\psi(x, z) \equiv \forall y\Big(y > z \to \neg\phi(x, y)\Big).$$

By assumption, for any $x < a$, there exists $z \in M$ such that $M \models \psi(x, z)$. By collection, there exists b such that for any $x < a$, there exists $z < b$ for which $M \models \psi(x, z)$. In particular for any $x < a$ and any $y > b$,

$$M \models \neg\phi(x, y),$$

contrary to our assumptions. ⊣

Now we are ready to prove the main result of this section.

PROOF OF THEOREM 40. The proof is an elaboration of the omitting types argument. Let M be a countable model satisfying full collection and let $a_0, a_1, \ldots$ be an enumeration of its elements. Let c be a fresh constant. We fix an enumeration of all sentences $\phi_0, \phi_1, \ldots$ in the language expanded with constants $c_0, c_1, \ldots$ and $d_0, d_1, \ldots$. We assume that the constants c_i, d_i do not appear in the sentences ϕ_j, for $j < i$. The constants c_i are intended as Henkin constants. The constants d_i will play a slightly different role as bounds which allow us to satisfy the local semiregularity condition.

We construct a chain of theories $T_i, i < \omega$. At each step we will construct auxiliary theories $T_i^1, T_i^2, T_i^3, T_i^4$. Throughout the construction, we assume that all the theories are finite except for the addition of finitely many families of sentences of the form:

$$d > a : a \in M.$$

Each such family includes all parameters $a \in M$ and at each step, we add such families only for finitely many d. In effect, we can state of finitely many elements that they are above M.

Let T_0 be $\mathrm{ElDiag}(M)$ together with all the sentences of the form $c > a$, $a \in M$. Now suppose that we have defined a theory T_i. ⊣

Step 1. In order to construct T_i^1, consider the sentence ϕ_i. If $T_i + \phi_i$ is consistent, we let T_i^1 be that theory. If not, let $T_i^1 = T_i + \neg\phi_i$.

Step 2. In order to construct T_i^2, consider again the sentence ϕ_i. If it was not added to the theory constructed in the previous step, let T_i^2 be equal to T_i^1. If it was, and it has the form $\exists v\psi(v)$, we add the sentence $\psi(c_i)$ to our theory. Note that c_i does not appear in T_i by our bookkeeping assumption.

Step 3. In the third step, we check whether the set of sentences of the form

$$c_i > a, a \in M$$

is consistent with T_i^2. If yes, then we let T_i^3 be obtained from T_i^2 by adding this set. Otherwise, there exists some $b \in M$ such that

$$T_i^2 \vdash c_i < b.$$

We claim that there exists $d \in M$ such that T_i^2 is consistent with the sentence $c_i = d$.

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Suppose otherwise. By assumption T_i^2 is ElDiag(M) together with finitely many types of the form $c_j > a, a \in M$ together with finitely many additional formulae. By considering the minimum of the elements c_j and coding the whole tuple as a single element, we can assume without loss of generality that T_i^2 extends ElDiag(M) by a single formula $\psi(t, c_i)$ and a single theory $t > a : a \in M$. (The constant c_i could be of course also eliminated, but we keep it for the clarity of the rest of the argument.) By assumption $T_i^2 \vdash c_i < b$ for some $b \in M$.

Now, suppose that a theory $t > a : a \in M + \text{ElDiag}(M) + \psi(t, c_i)$ proves for any $d \in M$ that $c_i \neq d$. Notice, however, that this implies that for any $d < b \in M$, there exists $a \in M$ such that:

$$M \models t > a \rightarrow \neg\psi(t, d).$$

Then, by Compression Lemma, there would exists a single element $r \in M$ for which:

$$M \models \forall x > r \forall y < b \; \neg\psi(x, y).$$

However this contradicts our assumption on ψ. Therefore, there exists in M an element $d < b$ such that for any a, we can find $t \in M$ with:

$$M \models \psi(t, d) \wedge t > a.$$

Let us pick any such d and set $T_i^3 := T_i^2 + c_i = d$.

Step 4. Finally, we want to ensure local semiregularity. Suppose that ϕ_i is a sentence expressing that some $c_k, k \leq i$ codes a function with the domain $[0, a]$ for some $a \in M$. Specifically,

$$\phi_i = \forall x \leq a \exists! \, y\Big(\langle x, y\rangle \in c_k\Big).$$

In the rest of the argument, let us denote c_k with f for cleaner presentation. We will also replace $\langle x, y\rangle \in f$ with the more usual notation $f(x) = y$ and treat $f(x)$ as if it were an independent term.

Claim. There exists $a' < a$ and $b \in M$ such that the following theory is consistent (which clearly ensures local semiregularity):

$$T_i^3 + \forall x < a'\Big(\big(f(x) < b \vee f(x) > d_i\big)\Big) + d_i > p : p \in M.$$

We begin with a bit of notation. Recall that T_i^3 extends ElDiag(M) with a sentence $\eta(c_0, \ldots, c_i, d_0, \ldots, d_{i-1})$ and finitely many sets of sentences of the form $c_j, d_j > q, q \in M$. By considering the minimum of all the parameters and via coding of tuples, we may assume that T_i^3 extends ElDiag(M) simply with a theory of the form:

$$\eta(f, t) + t > q : q \in M,$$

where η is a single formula. Notice that by assumption $\eta(f, t)$ implies that f codes a function with the domain $[0, a]$ in the sense explained above.

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Now, let us say that an interval $[0, r]$ *eventually visits* an interval $[b, p]$ if the following holds:

$$\exists N \forall q > N \forall t, f \exists x < r \Big(\eta(t, f) \wedge t > q \rightarrow f(x) \in [b, p] \Big).$$

We denote it with $\xi(r, b, p)$.

Subclaim. There exists a nonstandard $a_0 \in M$ such that for some B, $[0, a_0]$ does not eventually visit any interval $[b, p]$ with $b > B$.

To prove the Subclaim, notice first that for any $n \in \omega$, there exists B for which $\xi(n, b, p)$ does not hold for any $b > B$. Indeed, if no such B exists, then by successively taking intervals which are located higher up in the model, there exists a family of intervals

$$[b_0, p_0] < [b_1, p_1] < \ldots < [b_n, p_n] < [b_{n+1}, p_{n+1}]$$

such that $[0, n]$ eventually visits all these intervals. However, this is impossible by the pigeonhole principle, since the image of $[0, n]$ under f cannot intersect $n + 2$ disjoint sets.

Now, we have just shown that the following set of formulae is a type over M (with the free variable v):

$$\exists B \forall b, p > B \; \neg \xi(v, b, p) \wedge v > \underline{n} : n \in \omega.$$

This type actually uses only finitely many symbols from the signature, so by local recursive saturation, it is realised in M. By fixing a' as an element realising this type, we prove the Subclaim.

We are now ready to prove the claim. Let a' be any element satisfying the subclaim with a bound B. Fix any $p, q \in M$. We claim that there exist $t, f, d_i \in M$ such that:

$$M \models \eta(t, f) \wedge d_i > p \wedge t > q \wedge \forall x < a' \Big(\big(f(x) < B + 1 \vee f(x) > d_i \big) \Big)$$

Indeed, by assumption there is no interval I above b such that $[0, a']$ eventually visits I. In particular, there exist arbitrarily large t such that $\eta(t, f)$ holds and

$$\forall x < a' \; f(x) \notin [B + 1, p + 1].$$

By fixing any such f and $t > q$, and setting $d_i = p + 1$, we find our desired interpretation of these three constants in M such that our fixed finite portion of the theory is satisfied. Let $T_i^4 (=: T_i)$, be the theory:

$$T_i^3 + \forall x < a' \Big(\big(f(x) < b' \vee f(x) > d_i \big) \Big) + d_i > p : p \in M,$$

with a' chosen as in the proof and $b' := B + 1$. (If ϕ_i was not of the specific form considered, of course we set $T_i^4 = T_i^3$.)

The rest of the argument is completely routine. Let $T_\omega = \bigcup_{i \in \omega} T_i$. Consider the Henkin model N given by T_ω. In the step for the theories T_i^3, we have ensured that every element of N is either greater than all the elements of M or one of these elements. In the fourth step, we have ensured that any function on a domain bounded in M which is definable in N can be restricted to one which has all values either

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smaller than a fixed element of M or greater than a fixed element of N which ensures local semiregularity.

In the above proof, we have used the specific facts that our models admit some degree of saturation which also involved some coding of functions. Admittedly, all these assumptions seem somewhat tangential to the main idea of the stated result. In fact, one could imagine another property which makes sense in a more general context. Let M be any model over the signature with a binary symbol $<$ interpreted as a linear order. Let I be an initial segment of M. We say that I has a *gap property* in M if for any formula $\phi(x, y)$ and any $a \in I$, if

$$M \models \forall x < a \exists y\ \phi(x, y),$$

then there exist $b \in I, c \in M$ such that

$$M \models \forall x < a \exists y\ \phi(x, y) \wedge (y < b \vee y > c).$$

It is an interesting question whether any countable model in a signature with a linear order satisfying full collection has an elementary end-extension with a gap property. However, we were unable to obtain our result in this generality.

§6. Questions. There are some natural questions which we were trying to settle in this article, but which we had to leave open. Let us sum them up.

1. We have shown that any countable model of CT^- with internal collection or internal induction end-extends to a model of the same theory. Our argument crucially uses countability of the model. It seems that a more general proof would need to be much more specifically tailored to the case of truth predicates, since MacDowell–Specker Theorem fails to work in this greater generality.

   Let $M \models \mathrm{CT}^- + \mathrm{INT}(\mathrm{INTColl})$ be an arbitrary model. Does it have an end-extension satisfying $\mathrm{CT}^- + \mathrm{INT}(\mathrm{INTColl})$?
2. With the end-extension results limited to countable models, we are only able to produce ω_1-like models of this theory, in contrast to the general case of models of PA. It seems highly doubtful that ω_1 has indeed such a distinguished status in this context.

   Let κ be an arbitrary infinite cardinal. Let $M \models \mathrm{PA}$ be a countable model (alternatively, a model of cardinality $< \kappa$). Does there exist a κ-like model $M' \succ M$ which expands to a model of CT^-?[12]
3. Our main proof in section 5 shows that if M is a countable model of a theory in a countable language which extends PA and features full collection, then it can be end-extended in such a way that it is locally semiregular in the larger model. However, it seems unlikely that this property really depends on anything else but collection (or that we really have to replace semiregularity with local semiregularity). In particular, the use of coding of sets seems to be a technical artefact of our argument. When coding is removed, semiregularity or its local

[12] Ali Enayat has pointed out that a partial answer is provided by [4, Theorem 1.5], based on classical results of Chang and Jensen. If κ is a cardinal such that κ where $\kappa^{<\kappa} = \kappa$ or $\square_\kappa$ holds and U is a consistent theory in a countable language proving full collection, then U has a κ^+ model. By taking the theory $\mathrm{ElDiag}(M) + \mathrm{CT}^- + \mathrm{Coll}$, we see that under certain set-theoretic assumptions for any countable $M \models \mathrm{PA}$, some λ-like elementary extensions will exist for $\lambda > \omega_1$.

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version no longer make sense and one has to reword the whole statement which we propose to do in terms of the gap property. It seems likely that collection itself is sufficient to provide end-extensions in which the original model satisfies this modified claim.

Let M be a countable model in a signature with a linear order satisfying full collection. Does there exist an elementary end-extension $M' \succ_e M$ such that M satisfies the gap property in M'?

Appendix – a glossary of technical notions. In the article, we used a number of formalised syntactic notions. For the convenience of the reader, let us gather them in a single glossary.

- $\mathrm{Asn}(x, y)$ is a formula naturally representing the relation "x is a term or a formula and y is an x-assignment, that is, a function whose domain is the set of free variables of x."
- $\mathrm{ClTerm}_{\mathcal{L}_{\mathrm{PA}}}(x)$ is a formula naturally representing the set of closed terms of $\mathcal{L}_{\mathrm{PA}}$.
- $\mathrm{ClTermSeq}_{\mathcal{L}_{\mathrm{PA}}}(x)$ is a formula naturally representing the set of coded sequences of closed terms of $\mathcal{L}_{\mathrm{PA}}$.
- $\mathrm{dp}(x)$ is a formula naturally representing the syntactic depth of a formula x. It is a binary relational formula which is provably functional and thus written using the fucntional notation, in accordance with our conventions.
- $\mathrm{Form}_{\mathcal{L}_{\mathrm{PA}}}(x)$ is a formula naturally representing the set of arithmetical formulae. By $\mathrm{Form}^{\leq 1}(x)$ we mean the set of formulae with at most one variable free.
- $\mathrm{Sent}_{\mathcal{L}_{\mathrm{PA}}}(x)$ is a formula naturally representing the set of arithmetical sentences.
- t°, t^{α}. If t is (a code of) an arithmetical term and α is a t-assignment, then by t^{α} or $t(\alpha)$ we mean the formally computed value of t under α. If $\mathrm{ClTerm}_{\mathcal{L}_{\mathrm{PA}}}(t)$ holds and α is an empty assignment, we write t° instead.
- $\mathrm{Var}(x)$ is a formula representing the set of first order variables. Since variables have no structure, this means that $\mathrm{Var}(x)$ defines an infinite set which whose members are treated as the Gödel codes of variables under our fixed coding.
- $\alpha \sim_v \beta$. If α and β are two assignments which are equal except, possibly, at a single variable v, we denote this with $\alpha \sim_v \beta$.

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