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An intermediate value theorem for asymptotic values

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1. *Introduction*

We consider asymptotic values of continuous functions u from $\mathbb{R}^m$, or a suitable subset of $\mathbb{R}^m$, to the extended real numbers $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$, with the usual topologies in each case. Building on work of Rippon we obtain some rather unexpected results about such asymptotic values, in particular an intermediate value property.

A *boundary path* in a space X is a path $\Gamma : x(t), 0 \leqslant t < 1$, in X, such that $x(t)$ lies outside any given compact subset K of X, for $t \geqslant t_K$ say. Such a boundary path $\Gamma : x(t), 0 \leqslant t < 1$ is an *asymptotic path* of u for the value α if $u(x(t)) \to \alpha$ as $t \to 1$ and α is called an *asymptotic value* of u. We now state our first main result concerning asymptotic paths.

Theorem 1 (*Intermediate value theorem for asymptotic values*). *If the continuous function* $u : \mathbb{R}^m \to \overline{\mathbb{R}}$ *has paths* Γ_1, Γ_2 *going to* ∞ *such that*

$$\alpha = \limsup_{x \in \Gamma_1, x \to \infty} u(x) < \liminf_{x \in \Gamma_2, x \to \infty} u(x) = \beta, \tag{1·1}$$

and if $\alpha < M < \beta$, *then there is a path* Γ *going to* ∞ *along which* u *has asymptotic value* M; *that is*, u *has an asymptotic path for the value* M.

Theorem 1 leads immediately to a characterisation of the nature of the set of asymptotic values of such a continuous function.

Corollary 1. *If* $u : \mathbb{R}^m \to \overline{\mathbb{R}}$ *is a continuous function, then the set of asymptotic values of* u *is an interval in* $\overline{\mathbb{R}}$.

The simplicity of the set of asymptotic values in this case is in striking contrast to what may occur for continuous complex functions from the open unit disc to the Riemann sphere. For example, the elliptic modular function has $\{0, 1, \infty\}$ for its set of asymptotic values; see [**4**, page 56]. The best positive result known for such functions is that the set of asymptotic values forms an analytic, and hence measurable, set (see [**5**]).

The proof of Theorem 1 is given in Section 2; it uses results about components of sets of the form $\{u < M\}, M \in \mathbb{R}$, which we describe there. In Section 3 we prove results about the *existence* of asymptotic values of continuous functions $u : \mathbb{R}^m \to \overline{\mathbb{R}}$; in particular, we show that for u to have an asymptotic path, it is sufficient that it 'almost' has an asymptotic path. The methods of [**7**] are then used to derive corresponding local versions for asymptotic values in the unit ball. Finally, we deduce an intermediate value theorem for asymptotic values of continuous subharmonic

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functions in the MacLane class $\mathcal{A}$ and provide an example to show that this result does not hold for a general continuous subharmonic function in the unit disc.

2. *Proof of Theorem* 1

To prove our results, we use various concepts introduced in [**6**]. If E is a subset of $\mathbb{R}^m$, then the *solidification* $\tilde{E}$ of E is the union of E with the bounded components of its complement, so $\tilde{E} = \hat{\mathbb{R}}^m \backslash C$, where $\hat{\mathbb{R}}^m = \mathbb{R}^m \cup \{\infty\}$ and C is that component of $\hat{\mathbb{R}}^m \backslash E$ which contains ∞. A set E is then said to be *solid* if $E = \tilde{E}$. This notion is briefly treated by Bourbaki in [**1**, page 157, Exercise 14] where such solid sets are called 'full sets'.

For a continuous function $u: \mathbb{R}^m \to \overline{\mathbb{R}}$, we consider the family $\mathcal{U}_M$ of components of $\{u < M\}$, for $M \in \mathbb{R}$, and the equivalence relation $\sim$ defined on $\mathcal{U}_M$ as follows:

$$U \sim V \text{ if there is some } W \in \mathcal{U}_M \text{ with } U \subseteq \tilde{W} \text{ and } V \subseteq \tilde{W},$$

(intuitively this means that $U \sim V$ if and only if there is some component W in $\mathcal{U}_M$ which 'surrounds' both U and V). Under this equivalence relation, the equivalence class of $U \in \mathcal{U}_M$ is denoted by $[U]$ and we then define

$$\Omega_U = \bigcup \{\tilde{V} : V \in [U]\}, \quad \text{where } U \in \mathcal{U}_M.$$

In [**6**] it is proved that Ω_U is an open, connected, solid set; that $u(x) = M$ for $x \in \partial\Omega_U$; and that the set $\mathcal{U}_M$ of components of $\{u < M\}$ has exactly one of four possible structures, as described in Lemma A below. Note that in [**6**] the codomain of u is $[0, \infty]$, but a simple modification leads to the present situation.

LEMMA A. *Let $u: \mathbb{R}^m \to \overline{\mathbb{R}}$ be continuous and $M \in \mathbb{R}$. Then exactly one of the following holds*:

(*a*) *for some $U \in \mathcal{U}_M$, Ω_U is unbounded but $\Omega_U \neq \mathbb{R}^m$*;
(*b*) *for all $U \in \mathcal{U}_M$, Ω_U is bounded*;
(*c*) *for some $U_M \in \mathcal{U}_M$, $\tilde{U}_M = \mathbb{R}^m$*;
(*d*) *there is a sequence $U_{M,n}$ in $\mathcal{U}_M$ such that each $U_{M,n}$ is bounded,*

$$\tilde{U}_{M,1} \subseteq \tilde{U}_{M,2} \subseteq \cdots \quad \text{and} \quad \bigcup_{n=1}^{\infty} \tilde{U}_{M,n} = \mathbb{R}^m.$$

We now deduce Theorem 1 from Lemma A. For M in (α, β), we consider which case of Lemma A holds when there exist paths Γ_1, Γ_2 satisfying (1·1). We may assume that $u(x) < M$ for all $x \in \Gamma_1$ and $u(x) > M$ for all $x \in \Gamma_2$, since otherwise we may simply discard an initial part of the paths. Then Γ_1 is an unbounded connected set contained in $\{u < M\}$, and hence contained in a single component $U \in \mathcal{U}_M$. Similarly, Γ_2 is an unbounded connected set in the complement of $\{u < M\}$. So $\Omega_U = \tilde{U}$ is unbounded but not equal to $\mathbb{R}^m$. This implies that, for this choice of M, case (a) of Lemma A holds. Thus $\partial\Omega_U$ contains an unbounded component on which $u = M$, and so there must exist an asymptotic path Γ, close to $\partial\Omega_U$, along which u has asymptotic value M; see [**6**, theorem 2] and [**7**, note added in proof]. This completes the proof of Theorem 1.

Theorem 1 implies that if there are paths Γ_1, Γ_2 along which u has asymptotic values α, β then, for any M such that $\alpha < M < \beta$, there must exist a path along which

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u has asymptotic value M. This is precisely the characterisation of an interval in $\overline{\mathbb{R}}$, and we immediately deduce Corollary 1.

Remark. By considering simple examples, it is clear that the interval of asymptotic values in $\overline{\mathbb{R}}$ may be open or closed at either end, a single point or the whole of $\overline{\mathbb{R}}$.

3. *Existence of asymptotic values*

The following result is a strengthened form of [**6**, theorem 3]. It follows from Lemma A by a similar argument to that used in [**6**].

THEOREM 2. *Let $u: \mathbb{R}^m \to \overline{\mathbb{R}}$ be continuous. Then exactly one of the following holds*:

(*a*) *u has an asymptotic value $\alpha \in \overline{\mathbb{R}}$;*

(*b*) *there exist α, β with $-\infty \leqslant \alpha < \beta \leqslant \infty$ such that for every M satisfying $\alpha < M < \beta$ there is a sequence of solid, compact, connected sets*

$$K_1 \subseteq K_2 \subseteq \cdots$$

such that the distances from ∂K_n to K_1 tend to ∞ with n and

$$u(x) = M, \quad \text{for} \quad x \in \bigcup_{n=1}^{\infty} \partial K_n. \tag{3·1}$$

Note that it is case (b) of our Theorem 2 that is somewhat stronger than [**6**, theorem 3, case (b)]. We deduce the following sufficient condition for u to have an asymptotic path. Here we define the *oscillation* of u along a boundary path Γ to be

$$\limsup_{x \in \Gamma, x \to \partial X} u(x) - \liminf_{x \in \Gamma, x \to \partial X} u(x).$$

THEOREM 3. (*Small oscillation theorem*) *Let $u: \mathbb{R}^m \to \overline{\mathbb{R}}$ be continuous. If there exist paths going to ∞ along which u has arbitrarily small oscillation, then u has an asymptotic path.*

To prove Theorem 3 we suppose that $u: \mathbb{R}^m \to \overline{\mathbb{R}}$ is continuous and has no asymptotic path. Let (α, β) be the non-degenerate interval whose existence is asserted in Theorem 2. For M in (α, β) let K_n be the corresponding solid, compact, connected sets associated with M. Any path Γ going to ∞ is an unbounded connected set whcih must meet all but finitely many of the sets ∂K_n, so

$$\liminf_{x \in \Gamma, x \to \infty} u(x) \leqslant M \leqslant \limsup_{x \in \Gamma, x \to \infty} u(x), \tag{3·2}$$

by (3.1). Since this is true for any M in the interval (α, β), we then have

$$\liminf_{x \in \Gamma, x \to \infty} u(x) \leqslant \alpha < \beta \leqslant \limsup_{x \in \Gamma, x \to \infty} u(x). \tag{3·3}$$

These inequalities contradict the existence of paths going to ∞ along which u has arbitrarily small oscillation, and this contradiction proves Theorem 3.

4. *Asymptotic paths in the unit ball*

Applying a simple homeomorphism to the domain, it is easily seen that versions of Theorems 1, 2 and 3 hold for continuous $u: \mathbb{B} \to \overline{\mathbb{R}}$, where $\mathbb{B}$ is the open unit ball in $\mathbb{R}^m$ and the point at ∞ is replaced by $\partial \mathbb{B}$.

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Local versions of these results may be obtained by using a geometric technique from [**7**]. An Apollonian disc in $\partial\mathbb{B}$ is defined by

$$A(\xi,\rho)=\{x\in\partial\mathbb{B}:|x-\xi|\leqslant\rho|x+\xi|\},\quad \xi\in\partial\mathbb{B},\rho>0.$$

We then say that the function u has asymptotic value α *in* $A(\xi,\rho)$ if there is a boundary path $\Gamma: x(t), 0\leqslant t<1$, in $\mathbb{B}$, such that $u(x(t))\to\alpha$ as $t\to 1$, for which the *end* $\overline{\Gamma}\cap\partial\mathbb{B}$ of Γ is a subset of $A(\xi,\rho)$. We also define

$$D(\xi,\rho)=\{x\in\mathbb{B}:|x-\xi|\leqslant\rho|x+\xi|\},\quad \text{for } \xi\in\partial\mathbb{B},\rho>0.$$

By utilising the version of Theorem 1 in which $u:\mathbb{B}\to\overline{\mathbb{R}}$, and using the technique of inverting the function u and paths Γ in Apollonian spheres, we can deduce the following result.

THEOREM 4 (*Local intermediate value theorem for asymptotic values*). *Let* $u:\mathbb{B}\to\overline{\mathbb{R}}$ *be continuous and suppose there are boundary paths* Γ_1,Γ_2 *whose ends are subsets of an Apollonian disc* $A(\xi,\rho)$ *in* $\partial\mathbb{B}$. *If*

$$\alpha=\limsup_{x\in\Gamma_1,|x|\to 1}u(x)<\liminf_{x\in\Gamma_2,|x|\to 1}u(x)=\beta, \tag{4·1}$$

then for any M *satisfying* $\alpha<M<\beta$ *there is an asymptotic path* Γ *whose end is a subset of* $A(\xi,\rho)$ *and along which* u *has asymptotic value* M.

Proof. Let $R_{\xi,\rho}$ denote inversion in the Apollonian sphere

$$S(\xi,\rho)=\{x:|x-\xi|=\rho|x+\xi|\}$$

and put

$$v(x)=\begin{cases}u(x), & x\in D(\xi,\rho),\\ u(R_{\xi,\rho}(x)), & x\in R_{\xi,\rho}(D(\xi,\rho)).\end{cases}$$

Then v is continuous on $\mathbb{B}$, and the existence of the paths Γ_1, Γ_2 satisfying (4.1) implies, by Theorem 4·1, the existence of a path Γ in $\mathbb{B}$ along which v has asymptotic value M. By applying the inversion $R_{\xi,\rho}$ to the parts of Γ which are outside $D(\xi,\rho)$, we obtain an asymptotic path for u for the value M whose end is a subset of $A(\xi,\rho)$.

A corresponding local version of Corollary 1 holds in the ball; namely, that the set of asymptotic values associated with boundary paths whose ends are subsets of a given Apollonian disc in $\partial\mathbb{B}$ forms an interval in $\overline{\mathbb{R}}$.

Using a similar geometric technique, we can obtain local versions of Theorems 2 and 3. For example, here is a local version of Theorem 3. We omit the proof.

THEOREM 5 (*Local small oscillation theorem*). *Let* $u:\mathbb{B}\to\overline{\mathbb{R}}$ *be continuous and let* $A(\xi,\rho)$ *be an Apollonian disc in* $\partial\mathbb{B}$. *If there exist boundary paths in* $\mathbb{B}$ *with ends in* $A(\xi,\rho)$ *along which* u *has arbitrarily small oscillation, then there is an asymptotic path* Γ *with end in* $A(\xi,\rho)$.

5. *An application to the MacLane class*

As an application of the above methods, we prove a result about asymptotic values of subharmonic functions. Let $\mathbb{D}=\{|z|<1\}$ be the open unit disc in the complex plane. We say that a function u has *point asymptotic value* α at ζ in $\partial\mathbb{D}$ if u has

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asymptotic value α along a boundary path in $\mathbb{D}$ whose end is $\{\zeta\}$. The *MacLane class* $\mathcal{A}$ consists of functions which are subharmonic in the open unit disc $\mathbb{D}$ in the complex plane and have asymptotic values at a dense set of points of $\partial\mathbb{D}$. It is a subharmonic analogue, due to Hornblower [**3**], of a class of holomorphic functions studied by MacLane in [**4**].

THEOREM 6. *Suppose that u is a continuous subharmonic function on $\mathbb{D}$ which belongs to the class $\mathcal{A}$. If u has distinct point asymptotic values α, β, then any M with $\alpha < M < \beta$ is a point asymptotic value of u.*

Proof. Suppose that u is such a function and has asymptotic values α, β along paths which end at points of $\partial\mathbb{D}$. Take M such that $\alpha < M < \beta$. Theorem 1 implies the existence of an asymptotic path $\Gamma : z(t)$ along which u has asymptotic value M. If the end of Γ is a single point, then there is nothing further to prove, so we assume that the end of Γ includes a non-trivial open arc $(e^{i\theta}, e^{i\phi})$ in $\partial\mathbb{D}$. Since the set of points of $\partial\mathbb{D}$ at which u has an asymptotic value is dense in $\partial\mathbb{D}$, there must be such a point ζ in $(e^{i\theta}, e^{i\phi})$. Let $\Gamma_\zeta : z_\zeta(t)$ be an asymptotic path whose end is $\{\zeta\}$. Since Γ_ζ must cross Γ at points arbitrarily close to ζ and $u(z(t)) \to M$, we must have $u(z_\zeta(t)) \to M$, so the asymptotic value of u along Γ is M. The proof is complete.

Remarks. First note that the above proof actually applies to any real-valued continuous function which has asymptotic values at a dense set of points of $\partial\mathbb{D}$. Also, Theorem 4 allows us to formulate a local version of Theorem 6. Finally, it follows from Theorem 6 that the set of point asymptotic values of a continuous subharmonic function in the MacLane class $\mathcal{A}$ forms an interval in $\overline{\mathbb{R}}$. It is natural to ask whether the same result is true for general subharmonic functions in the MacLane class. For the larger class of fine continuous functions, the set of point asymptotic values is known only to be an analytic set (see [**8**]).

We also point out that Theorem 6 is false if we drop the hypothesis that u is in the MacLane class. For convenience, we construct a counter-example in the upper half-plane, but a simple domain transformation provides the required counter-example in $\mathbb{D}$.

THEOREM 7. *There is a function u which is continuous and subharmonic in $\mathbb{H} = \{x + iy : y > 0\}$ with the property that*

$$u \text{ has point asymptotic value } x \text{ at } x \in \partial\mathbb{H}, \quad \text{for } |x| \geqslant 1, \tag{5·1}$$

but u takes no value in the interval $(-1, 1)$ as a point asymptotic value.

Proof. To construct this function u in $\mathbb{H}$ we first define the simply-connected region

$$\Omega = \{x + iy : 0 < y < 1, \sin(1/y) < x < \sin(1/y) + y\},$$

as shown in Figure 1.

The region Ω has the property that any path Γ in $\mathbb{H}$ approaching a boundary point x of $\mathbb{H}$ in $(-1, 1)$ must cross Ω infinitely often, in the sense that subpaths of Γ form a sequence of cross-cuts of Ω which approach a point of $(-1, 1)$, as shown in figure 1. These cross-cuts define a prime end of Ω (see [**2**, chapter 9] with impression $[-1, 1]$.

Now let v be a positive harmonic function in Ω with the property that v has the continuous boundary value 0 at points of $\partial\Omega \cap \mathbb{H}$. Such a function v can be

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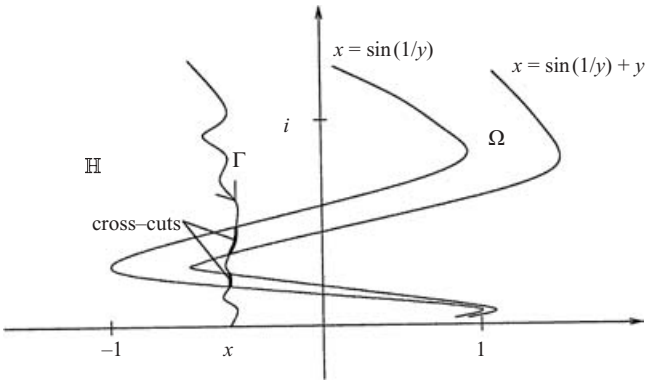


Fig. 1. The region Ω.

constructed by using a Riemann mapping φ from the unit disc $\mathbb{D}$ to Ω which maps $1 \in \partial\mathbb{D}$ to the prime end defined by the above cross-cuts; see [**2**, theorem 9·4]. Indeed, we can take $v(z) = P(\varphi^{-1}(z))$, where P is the Poisson kernel in $\mathbb{D}$ with singularity at 1. By defining v to be 0 in $\mathbb{H}\backslash\Omega$, we obtain a continuous subharmonic function in $\mathbb{H}$, and we then put $u(z) = v(z) + x$, which is also continuous and subharmonic in $\mathbb{H}$.

Now v has point asymptotic value 0 at each x with $|x| \geqslant 1$, so (5·1) certainly holds. Moreover, it is clear that 0 is the *only* possible point asymptotic value of v at these boundary points, so u has no other asymptotic values at points x with $|x| \geqslant 1$.

At boundary points x with $|x| < 1$, there are no point asymptotic values of v or u because for any path Γ in $\mathbb{H}$ which approaches x, we have

$$\liminf_{z\in\Gamma, z\to x} v(z) = 0 \quad \text{and} \quad \limsup_{z\in\Gamma, z\to x} v(z) = \infty. \tag{5·2}$$

These results hold because such a path Γ must cross infinitely often both the boundary of Ω and the curve in Ω which is the image under the Riemann mapping φ of the interval $[0, 1]$ in $\mathbb{D}$. Hence u takes no value in $(-1, 1)$ as a point asymptotic value. Note that, as expected, property (5·2) indicates that u is certainly not in the MacLane class.

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