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Weakly compact subsets of symmetric operator spaces

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Abstract

Under natural conditions it is shown that the rearrangement invariant hull of a weakly compact subset of a properly symmetric Banach space of measurable operators affiliated with a semi-finite von Neumann algebra is again relatively weakly compact.

0. *Introduction*

If the usual duality for spaces of type $L^1(\mu)$, $L^\infty(\mu)$ is interpreted in the sense of Köthe, then it is well known that classical criteria for weak compactness in spaces of type $L^1(\mu)$ [**16**] admit natural extensions to very general classes of function spaces, and even more generally to the setting of vector lattices [**12**], [**1**]. These same classical criteria admit natural extensions to the preduals of general von Neumann algebras [**2**], [**25**] where additional technical difficulty is encountered due to the lack of the lattice structure which plays a fundamental role in the (commutative) function space setting. On the other hand the duality between a semi-finite von Neumann algebra and its predual may be viewed as a very special case of a more general non-commutative theory of Köthe duality. Such a theory, which reduces to that for rearrangement invariant Banach function spaces [**18**], [**17**], [**13**] when specialized to the commutative setting, was initiated in the case of trace ideals by Garling [**15**] and developed in the more general setting of spaces of measurable operators by Yeadon [**27**] and by the present authors [**7**]. The central theme of [**15**], in which the present paper finds its motivation, is the study of special properties of weakly compact subsets of trace ideals which reflect the symmetric structure. Under natural conditions, it was shown by Garling [**15**] that the rearrangement invariant hull of a weakly compact subset in a trace ideal is again relatively weakly compact by showing that weakly compact subsets in a trace ideal could be characterized in terms of the corresponding set of sequences of singular values. Subsequently, the analogue of Garling's work in a general (commutative) function space setting with the role of the singular value sequence of a compact operator played by the decreasing rearrangement of a measurable function was given by Fremlin [**13**], whose work was based on vector lattice techniques. It is our intention in this paper to characterize weakly compact sets in the more general setting of rearrangement invariant spaces

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of measurable operators affiliated with a semi-finite von Neumann algebra equipped with a distinguished trace, thereby unifying the work of Garling and Fremlin. The present approach is based on the characterization of the Köthe dual of a rearrangement invariant operator space given in [7]. Our principal characterization of weak compactness (Proposition 2·8) is of Dunford–Pettis type and for preduals of von Neumann algebras is due, in part, to Akemann [2]. The main consequences of our characterization are given in Corollary 2·9 and Proposition 2·10 and these results contain as special cases those of Garling [15] and Fremlin [13] referred to earlier.

In the function space and vector lattice setting, the study of weak compactness has led to various characterizations of certain topological properties in terms of the geometric and order structures. In the third section we show that familiar Banach lattice characterizations of weak sequential completeness and reflexivity, as given for example in [1], continue to hold in the non-commutative setting, despite the lack, in general, of any lattice structure.

1. *Preliminaries*

Throughout this paper, $\mathscr{H}$ will denote a Hilbert space, with inner product $(,)$, $\mathscr{L}(\mathscr{H})$ the linear space of all bounded operators in $\mathscr{H}$ and $\mathscr{M} \subseteq \mathscr{L}(\mathscr{H})$ a von Neumann algebra equipped with a normal, faithful semi-finite trace τ. We denote by $\mathscr{M}^p$ the set of orthogonal projections in $\mathscr{M}$. A closed densely defined linear operator x in $\mathscr{H}$ is said to be *affiliated with* $\mathscr{M}$ if and only if $u^*xu = x$ for all unitary u which belong to the commutant $\mathscr{M}'$ of $\mathscr{M}$. If x is affiliated with $\mathscr{M}$ then x is called τ-*measurable* if and only if there exists a number $s \geqslant 0$ such that $\tau(\chi_{(s,\infty)}(|x|)) < \infty$, where the projection $\chi_{(s,\infty)}(|x|)$ is defined via the usual functional calculus for self-adjoint operators. We denote by $\tilde{\mathscr{M}}$ the set of all τ-measurable operators. Sum and product in $\tilde{\mathscr{M}}$ are defined as the respective closures of the algebraic sum and product. For $x \in \tilde{\mathscr{M}}$, the *decreasing rearrangement* (or *generalized singular value function*) $\mu(x)$ of x is defined by

$$\mu_t(x) = \inf\{s \geqslant 0 : \tau(\chi_{(s,\infty)}(|x|)) \leqslant t\}, \quad t \geqslant 0.$$

It follows simply that $\mu(x)$ is a decreasing, right-continuous function on the half-line $[0, \infty)$. For basic properties of decreasing rearrangements of measurable operators, we refer to [11]. We denote by $\tilde{\mathscr{M}}_0$ the set of all $x \in \tilde{\mathscr{M}}$ such that $\mu_t(x) \to 0$ as $t \to \infty$.

The topology defined by the translation invariant metric d on $\tilde{\mathscr{M}}$ obtained by setting

$$d(x, y) = \inf\{t \geqslant 0 : \mu_t(x-y) \leqslant t\}, \quad \text{for } x, y \in \tilde{\mathscr{M}},$$

is called the *measure topology*. It is not difficult to see that the measure topology has a neighbourhood basis of zero, consisting of all sets of the form

$$N(\epsilon, \delta) = \{x \in \tilde{\mathscr{M}} : \exists e \in \mathscr{M}^p \text{ such that } \|xe\| \leqslant \epsilon \text{ and } \tau(1-e) \leqslant \delta\}$$

where ϵ, δ are positive numbers, which is the definition given by Nelson [19]. It is shown in [19] and [26] that $\tilde{\mathscr{M}}$ equipped with the measure topology is a complete, Hausdorff, topological *-algebra in which $\mathscr{M}$ is dense. Moreover, if x, $x_n \in \tilde{\mathscr{M}}$, $n = 1, 2, \ldots$ and $d(x_n, x) \to_n 0$ then $\mu(x_n) \to_n \mu(x)$ a.e. on $\mathbb{R}^+$ with respect to Lebesgue measure. See for example [11].

With the partial ordering defined on $\tilde{\mathscr{M}}$ by setting $x \geqslant 0$ if and only if $(x\xi, \xi) \geqslant 0$ for all ξ in the domain of x, it may be shown [7] that $(\tilde{\mathscr{M}}, \leqslant)$ is an ordered vector

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space, that the positive cone $\tilde{\mathcal{M}}_+$ is closed in $\tilde{\mathcal{M}}$ for the measure topology and that $\tilde{\mathcal{M}}$ is order complete in the sense that every increasing order bounded net in $\tilde{\mathcal{M}}_+$ has a supremum in $\tilde{\mathcal{M}}$. If $\{x_\alpha\}$ is an increasing net in $\tilde{\mathcal{M}}_+$ and if $x = \sup x_\alpha$ holds in $\tilde{\mathcal{M}}$, then $\mu(x_\alpha) \uparrow_\alpha \mu(x)$ holds in the space $L^0(\mathbb{R}^+)$ of all Lebesgue measurable functions on $[0, \infty)$. If $\{x_\alpha\} \subseteq \tilde{\mathcal{M}}_0$ and $x_\alpha \downarrow_\alpha 0$ holds in $\tilde{\mathcal{M}}$, then $\mu(x_\alpha) \downarrow_\alpha 0$ in $L^0(\mathbb{R}^+)$. The trace τ can be extended to an additive, positively homogeneous map $\tau : \tilde{\mathcal{M}}_+ \to [0, \infty]$, which is unitarily invariant and normal in the sense that $x_\alpha \uparrow x$ in $\tilde{\mathcal{M}}_+$ implies $\tau(x) = \sup_\alpha \tau(x_\alpha)$.

If $x, y \in \tilde{\mathcal{M}}$, we say that x is *submajorized by* y, written $x \prec\prec y$, if and only if

$$\int_0^\alpha \mu_t(x)\, dt \leqslant \int_0^\alpha \mu_t(y)\, dt, \quad \text{for all } \alpha \geqslant 0.$$

A normed linear subspace $E \subseteq \tilde{\mathcal{M}}$ is called *rearrangement invariant* if and only if $x \in \tilde{\mathcal{M}}$, $y \in E$ and $\mu(x) \leqslant \mu(y)$ implies $x \in E$ and $\|x\|_E \leqslant \|y\|_E$; *symmetric* if and only if $x, y \in E$ and $x \prec\prec y$ implies $\|x\|_E \leqslant \|y\|_E$; *fully symmetric* if and only if $x \in \tilde{\mathcal{M}}$, $y \in E$ and $x \prec\prec y$ implies $x \in E$ and $\|x\|_E \leqslant \|y\|_E$. If $E \subseteq \tilde{\mathcal{M}}$ is rearrangement invariant, then E is order complete in the sense that, if $\{x_\alpha\} \subseteq E$ and $0 \leqslant x_\alpha \uparrow_\alpha \leqslant x \in E$ then also $\sup_\alpha x_\alpha \in E$.

We identify $L^\infty(\mathbb{R}^+)$ throughout as a commutative von Neumann algebra acting by multiplication on $L^2(\mathbb{R}^+)$ with trace given by integration with respect to Lebesgue measure. A Banach space $E(\mathbb{R}^+)$ of almost everywhere finite, measurable functions on $\mathbb{R}^+$ will be called a *rearrangement invariant* (*symmetric*, *fully symmetric*) *Banach function space* on $\mathbb{R}^+$ if the corresponding conditions above hold with respect to the von Neumann algebra $L^\infty(\mathbb{R}^+)$. While the above terminology differs from that of [**17**], this should cause no confusion in the sequel.

If $E(\mathbb{R}^+)$ is a rearrangement invariant symmetric Banach function space on $\mathbb{R}^+$, we denote by $E(\mathcal{M})$ the linear subspace of those $x \in \tilde{\mathcal{M}}$ for which $\mu(x) \in E(\mathbb{R}^+)$ with norm defined by setting $\|x\|_{E(\mathcal{M})} = \|\mu(x)\|_{E(\mathbb{R}^+)}$, $x \in E(\mathcal{M})$. As is shown in [**5**], [**6**] (see also [**24**]), $E(\mathcal{M})$ is a Banach space and it now immediately follows that $E(\mathcal{M})$ is a rearrangement invariant symmetric operator space. It should also be observed that if $E(\mathbb{R}^+)$ is one of the familiar L^p-spaces, $1 \leqslant p < \infty$, then the spaces $L^p(\mathcal{M})$ given by the preceding construction coincide with those defined by Nelson [**19**]; the equality $L^\infty(\mathcal{M}) = \mathcal{M}$ holds with equality of norms, and it has been shown, for example in [**21**], [**11**] that the equality $(L^1 + L^\infty)(\mathcal{M}) = L^1(\mathcal{M}) + \mathcal{M}$ holds with equality of norms. For notational convenience we will write $G(\mathcal{M})$, $H(\mathcal{M})$ for the Banach spaces $L^1(\mathcal{M}) + \mathcal{M}$, $L^1(\mathcal{M}) \cap \mathcal{M}$ respectively. A Banach space $E \subseteq \tilde{\mathcal{M}}$ will be called *properly symmetric* if E is rearrangement invariant, symmetric and intermediate for the Banach couple $(L^1(\mathcal{M}), \mathcal{M})$ in the sense that $H(\mathcal{M}) \subseteq E \subseteq G(\mathcal{M})$ with continuous embeddings. Since each rearrangement invariant Banach function space $E(\mathbb{R}^+)$ is necessarily intermediate for the pair $(L^1(\mathbb{R}^+), L^\infty(\mathbb{R}^+))$ [**17**], it follows that if in addition $E(\mathbb{R}^+)$ is symmetric then the operator space $E(\mathcal{M})$ is properly symmetric.

Suppose now that $\mathcal{N}$ is a semi-finite von Neumann algebra with faithful normal semi-finite trace σ. The following result is proved in [**7**], proposition 4·11.

PROPOSITION 1·1. *If $x \in G(\mathcal{M})$ and $y \in G(\mathcal{N})$ then*

$$\int_{[0,\infty)} \mu_t(x)\, \mu_t(y)\, dt = \sup\{|\tau(xz)| : z \in H(\mathcal{M}),\ z \prec\prec y\}$$
$$= \sup\{\tau(|xz|) : z \in H(\mathcal{M}),\ z \prec\prec y\}.$$

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We now gather some of the essential properties [7] of Köthe duality needed in what follows. If $E \subseteq \tilde{\mathcal{M}}$ is a properly symmetric Banach space, the Köthe dual $E^\times$ consists of those $x \in \tilde{\mathcal{M}}$ such that $xy \in L^1(\mathcal{M})$ for all $y \in E$. Equipped with the norm defined by setting

$$\|x\|_{E^\times} = \sup\{\tau(|xy|) : y \in E,\ \|y\|_E \leqslant 1\}, \quad x \in E^\times,$$

it may be shown that $(E^\times, \|\ \|_{E^\times})$ is a properly symmetric, fully symmetric Banach space for which the natural embedding into the Banach dual E^* is an isometry, and that the equality

$$\|x\|_{E^\times} = \sup\left\{\int_{[0,\infty)} \mu_t(x)\,\mu_t(y)\,dt : y \in H(\mathcal{M}),\ \|y\|_E \leqslant 1\right\}$$

holds for all $x \in E^\times$. If $E(\mathbb{R}^+)$ is a properly symmetric Banach function space on $\mathbb{R}^+$, then the equality $E^\times(\mathcal{M}) = E(\mathcal{M})^\times$ holds in the sense of Banach spaces. The following result identifies the Köthe dual as a linear subspace of the Banach dual E^* in terms of order properties ([7], proposition 5·11).

PROPOSITION 1·2. *If $E \subseteq \tilde{\mathcal{M}}$ is a properly symmetric Banach space and if $\psi \in E^*$ then the following are equivalent.*

(i) *$x_\alpha \downarrow_\alpha 0 \subseteq E$ implies $\psi(x_\alpha) \to 0$.*

(ii) *$e_\alpha \downarrow_\alpha 0 \subseteq \mathcal{M}^p$ implies $\max\{|\psi(xe_\alpha)|, |\psi(e_\alpha x)|\} \to 0$ for all $x \in E$.*

(iii) *There exists $a \in E^\times$ such that $\psi(x) = \tau(ax)$ for all $x \in E$.*

Throughout the remainder of this paper, it will be assumed that $E \subseteq \tilde{\mathcal{M}}$ is a properly symmetric Banach space. If the natural injection of E into the Köthe bidual $E^{\times\times}$ is a surjection then E will be called *perfect*. If E embeds isometrically onto $E^{\times\times}$ then E will be called *maximal*.

2. *Characterizations of weak compactness*

We begin with an extension of [15], theorem 10 and corollary.

PROPOSITION 2·1. *Let T be the locally convex topology on E defined by the family of norms $\{\|\,.\,\|_y : 0 \neq y \in E^\times\}$ where*

$$\|x\|_y = \int_{[0,\infty)} \mu_t(x)\,\mu_t(y)\,dt, \quad x \in E.$$

(i) *If E is perfect then E is T-complete.*

(ii) *If $E^{\times\times} \subseteq \tilde{\mathcal{M}}_0$ and if E is T-complete, then E is perfect.*

(iii) *If $E \subseteq \tilde{\mathcal{M}}_0$ then the dual of E for T is precisely $E^\times$.*

Proof. (i) If $y \in E^\times$, we set

$$E_y(\mathbb{R}^+) = \left\{f \in L^0(\mathbb{R}^+) : \int_{[0,\infty)} \mu_t(f)\,\mu_t(y)\,dt < \infty\right\}.$$

Equipped with the norm defined by setting

$$\|f\|_{E_y(\mathbb{R}^+)} = \int_{[0,\infty)} \mu_t(f)\,\mu_t(y)\,dt, \quad f \in E_y(\mathbb{R}^+),$$

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each space $E_y(\mathbb{R}^+)$ is a maximal Banach function space on $\mathbb{R}^+$. Moreover, the embedding of E into each space $E_y(\mathcal{M})$ is continuous, and perfectness of E implies that

$$E = \bigcap \{E_y(\mathcal{M}) : 0 \leqslant y \in E^\times\}.$$

If $\{x_\alpha\} \subseteq E$ is a T-Cauchy net, there exists $x_y \in E_y(\mathcal{M})$ such that $x_\alpha \to x_y$ in $E_y(\mathcal{M})$. For each $z \in H(\mathcal{M})$ and for each $0 \leqslant y \in E^\times$, the map

$$x \mapsto \tau(xz), \quad x \in E_y(\mathcal{M}),$$

is continuous. Consequently, if $0 \leqslant y, y' \in E^\times$, then the equality

$$\tau(x_y z) = \tau(x_{y'} z)$$

holds for each $z \in H(\mathcal{M})$. It follows that there exists

$$x \in \bigcap \{E_y(\mathcal{M}) : 0 \leqslant y \in E^\times\} = E$$

such that $x_\alpha \to x$ holds in each space $E_y(\mathcal{M})$, $0 \leqslant y \in E^\times$. Consequently E is T-complete.

(ii) Assume now that $E^{\times\times} \subseteq \tilde{\mathcal{M}}_0$ and let $0 \leqslant x \in E^{\times\times}$. There exists $\{x_\alpha\} \subseteq H(\mathcal{M})$ with $0 \leqslant x_\alpha \uparrow_\alpha x$ in $E^{\times\times}$, and since $E^{\times\times} \subseteq \tilde{\mathcal{M}}_0$, it follows that $\mu(x - x_\alpha) \downarrow_\alpha 0$. This implies that $\{x_\alpha\}$ is T-Cauchy, hence convergent, in E and consequently $x \in E$.

(iii) If ϕ is a T-continuous linear functional on E, then there exists $0 \leqslant y \in E^\times$ such that

$$|\phi(x)| \leqslant \int_{[0,\infty)} \mu_t(x)\,\mu_t(y)\,dt \leqslant \|x\|_E \|y\|_{E^\times}, \quad \text{for all } x \in E.$$

It follows that $\phi \in E^*$, and since $E \subseteq \tilde{\mathcal{M}}_0$, it follows from the Lebesgue monotone convergence theorem and the implication (i) $\Rightarrow$ (iii) of Proposition 1·2 that there exists $y' \in E^\times$ such that

$$\phi(x) = \tau(xy') \quad (x \in E),$$

and this completes the proof of the Proposition. ∎

The proof of the preceding proposition shows that if $E^{\times\times} \subseteq \tilde{\mathcal{M}}_0$, then $E^{\times\times}$ may be identified with the T-completion of E.

PROPOSITION 2·2. *Let $K \subseteq E^\times$ be bounded. If E is perfect and if $E^\times \subseteq \tilde{\mathcal{M}}_0$, then the following statements are equivalent.*

(i) *K is relatively $\sigma(E^\times, E)$-compact.*

(ii) *K is relatively countably $\sigma(E^\times, E)$-compact.*

Proof. It follows from Proposition 2·1, applied to $E^\times$, that $E^\times$ is complete for the Mackey topology $\tau(E^\times, E)$ and the stated equivalence is now a consequence of Eberlein's Theorem ([16], theorem 5·3·2). ∎

We remark that in the case of trace ideals the preceding Proposition 2·2 is due to Garling [15], theorem 11.

PROPOSITION 2·3. *If $K \subseteq E^\times$ is bounded and if either*

(A) $\sup\{|\tau(yx_\alpha)| : y \in K\} \to_\alpha 0$ *for every family $\{x_\alpha\} \subseteq E$ with $x_\alpha \downarrow_\alpha 0$, or*

(B) $\sup\{\max(|\tau(ye_\alpha x)|, |\tau(xe_\alpha y)|) : y \in K\} \to_\alpha 0$ *for every family $\{e_\alpha\} \subseteq \mathcal{M}^p$ with $e_\alpha \downarrow_\alpha 0$ and for every $0 \leqslant x \in E$, or*

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(C) $\sup\{\max(|\tau(ye_n x)|, |\tau(xe_n y)|) : y \in K\} \to_n 0$ *for every mutually orthogonal sequence* $\{e_n\} \subseteq \mathscr{M}^p$ *and every* $0 \leqslant x \in E$,

then K *is relatively* $\sigma(E^\times, E)$*-compact.*

Proof. It should first be noted that an argument essentially the same as that of [9], lemma 3·2 shows that the assertion of (B) is equivalent to that given by (C). To show that K is relatively $\sigma(E^\times, E)$-compact, it suffices to show that each $\sigma(E^*, E)$-accumulation point of K in E^* belongs to $E^\times$. This however follows from either of the equicontinuity conditions (A), (B) and Proposition 1·2. ∎

If $x \in G(\mathscr{M})$, we set

$$\Omega(x) = \{y \in \tilde{\mathscr{M}} : y \prec\prec x\}.$$

It should be noted that if E is fully symmetric, in particular if E is perfect, and if $x \in E$ then also $\Omega(x) \subseteq E$. It is shown in [7], corollary 4·3 and proposition 4·7 that $\Omega(x)$ is convex and $\sigma(G(\mathscr{M}), H(\mathscr{M}))$-compact.

PROPOSITION 2·4. *If* $E^\times \subseteq \tilde{\mathscr{M}}_0$, *if* E *is fully symmetric and if* $x \in E$ *then* $\Omega(x)$ *is* $\sigma(E, E^\times)$*-compact.*

Proof. If $0 \leqslant x_\alpha \downarrow_\alpha 0 \subseteq E^\times$ and if $y \in \Omega(x)$ then it follows from the inequality

$$|\tau(x_\alpha y)| \leqslant \int_{[0,\infty)} \mu_t(x_\alpha)\, \mu_t(x)\, dt,$$

and Proposition 2·3 that $\Omega(x)$ is relatively $\sigma(E^{\times\times}, E^\times)$-compact. However, $\Omega(x)$ is $\sigma(G(\mathscr{M}), H(\mathscr{M}))$-compact hence $\sigma(G(\mathscr{M}), H(\mathscr{M}))$-closed in $G(\mathscr{M})$; consequently, $\Omega(x)$ is $\sigma(E^{\times\times}, E^\times)$-compact. The assertion now follows since E is fully symmetric. ∎

In the special case that $E = L^1[0, 1]$, the preceding Proposition 2·4 may be found in Ryff [22], theorem 2, while an extension to more general function spaces may be found in Luxemburg [18], theorem 15·3.

LEMMA 2·5. (i) *If* $x \in G(\mathscr{M})$, *then* $\Omega(x)$ *is closed for the measure topology.*
(ii) *If* E *is fully symmetric,* $x \in E$ *and if* $E^\times \subseteq \tilde{\mathscr{M}}_0$, *then*

$$\int_{[0,\infty)} \mu_t(z_n)\, \mu_t(y)\, dt \to_n 0,$$

for each $y \in E^\times$, *whenever* $\{z_n\} \subseteq \Omega(x)$ *converges to zero for the measure topology. In particular, for each* $y \in E^\times$, *the functional*

$$z \to \tau(zy), \quad z \in \Omega(x),$$

is continuous on $\Omega(x)$ *for the measure topology.*

Proof. (i) If $\{z_n\} \subseteq \Omega(x)$ and $z_n \to z \in \tilde{\mathscr{M}}$ for the measure topology then $\mu(z_n) \to \mu(z)$ a.e. and since

$$\int_{[0,\alpha)} \mu_t(z_n)\, dt \leqslant \int_{[0,\alpha)} \mu_t(x)\, dt$$

holds for all $n = 1, 2, \ldots$ and all $\alpha > 0$, it follows from Fatou's Lemma that

$$\int_{[0,\alpha)} \mu_t(z)\, dt \leqslant \int_{[0,\alpha)} \mu_t(x)\, dt$$

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holds for all $\alpha > 0$ and it follows from the stated assumption that $z \in \Omega(x)$.

(ii) If $y \in E^\times$ and if $\{e_m\} \subseteq [0, \infty)$ is any decreasing sequence of measurable subsets with void intersection, then $\chi_{e_m} \mu(y) \downarrow_m 0$, where χ denotes the usual indicator function, and since $y \in \tilde{\mathscr{M}}_0$, it follows that $\mu(\chi_{e_m} \mu(y)) \downarrow_m 0$. Since

$$\int_{[0,\infty)} \mu_t(z_n)\, \chi_{e_m}(t)\, \mu_t(y)\, dt \leqslant \int_{[0,\infty)} \mu_t(x)\, \mu_t(\chi_{e_m} \mu(y))\, dt,$$

for $n, m = 1, 2, \ldots$, it follows from [**10**], IV 8·9 that the sequence $\{\mu(z_n)\}$ is relatively weakly compact in $L^1(\mu_t(y)\, dt)$. Since $\mu(z_n) \to 0$ a.e. the assertion of (ii) now follows from the Vitali convergence theorem [**10**], IV 8·9, 10·9. ∎

PROPOSITION 2·6. *Assume that E is fully symmetric, that $E^\times \subseteq \tilde{\mathscr{M}}_0$ and that $K \subseteq E^\times$ is relatively $\sigma(E^\times, E)$-compact. If $x \in E$ and if $\{x_n\} \subseteq \Omega(x)$ converges to zero for the measure topology, then*

$$\sup\left\{ \int_{[0,\infty)} \mu_t(x_n)\, \mu_t(y)\, dt : y \in K \right\} \to_n 0.$$

Proof. If the assertion of the Proposition is not valid, it may be assumed that there exists $\epsilon > 0$ and a sequence $\{y_n\} \subseteq K$ such that

$$\int_{[0,\infty)} \mu_t(x_n)\, \mu_t(y_n)\, dt > 2\epsilon, \quad n = 1, 2, \ldots.$$

By Proposition 1·1, there exists a sequence $\{z_n\} \subseteq E$ with

$$z_n \prec\prec x_n, \quad n = 1, 2, \ldots,$$

and

$$|\tau(z_n y_n)| > 2\epsilon, \quad n = 1, 2, \ldots.$$

The inequalities

$$\int_{[0,1)} \mu_t(z_n)\, dt \leqslant \int_{[0,1)} \mu_t(x_n)\, dt \leqslant \int_{[0,1)} \mu_t(x)\, dt, \quad n = 1, 2, \ldots$$

imply that $z_n \to 0$ in $G(\mathscr{M})$ and hence $\{z_n\}$ converges to zero for the measure topology.

As follows from Proposition 2·4, $\Omega(x)$ is $\sigma(E, E^\times)$-compact and so Šmulian's theorem ([**16**], 5.3.1) implies that $\{y_n\}$ contains a subsequence which converges pointwise on $\Omega(x)$. By a suitable change of notation and Lemma 2·5(ii), it may be assumed that $\{y_n\}$ converges pointwise on $\Omega(x)$ and that

$$|\tau(z_{n+1} y_n)| < \epsilon, \quad n = 1, 2, \ldots.$$

Setting

$$w_n = y_n - y_{n-1}, \quad n = 2, 3, \ldots,$$

it follows that $\{w_n\}$ converges pointwise to zero on $\Omega(x)$ and

$$|\tau(z_n w_n)| > \epsilon, \quad n = 2, 3, \ldots.$$

From Lemma 2·5(i), it follows that $\Omega(x)$ equipped with the measure topology is a complete metric space and from Lemma 2·5(ii) it follows that the functions

$$f_n : z \to \tau(z w_n), \quad z \in \Omega(x), \quad n \geqslant 2,$$

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are continuous on $\Omega(x)$ for the measure topology and the sequence $\{f_n\}$ converges pointwise to zero on $\Omega(x)$. By Baire's theorem, there exists a neighbourhood N of zero for the measure topology in $\Omega(x)$ and an index n_0 such that

$$|\tau(zw_n)| \leqslant \epsilon, \quad n \geqslant n_0, \quad z \in N.$$

Consequently, there exists a natural number m_0 such that

$$|\tau(z_m w_n)| \leqslant \epsilon, \quad m \geqslant m_0, \quad n \geqslant n_0,$$

and this contradiction suffices to complete the proof of the assertion. ∎

We mention the following Corollary due to Sakai [**23**], lemma 2 on which the preceding Proposition 2·6 is patterned.

COROLLARY 2·7. *If $\tau(1) < \infty$ and if $\{x_n\} \subseteq \mathcal{M}$ is a bounded sequence for which $\tau(x_n^* x_n) \to 0$ as $n \to \infty$, then $\{x_n\}$ converges to zero uniformly on each relatively weakly compact subset of $L^1(\mathcal{M})$.*

Our principal result characterizing weak compactness now follow. It reduces to a classical criterion of Dunford and Pettis [**10**] in the commutative setting and is easily seen to contain as a special case a characterization of uniform integrability in L^1-spaces in terms of decreasing rearrangements due to Chong [**4**]. In the non-commutative setting, it extends the criterion given for trace ideals in [**9**] and for the case that $E = \mathcal{M}$, it contains a characterization due to Akemann [**2**].

PROPOSITION 2·8. *Let K be a bounded subset of $E^\times$ and consider the following statements.*

(i) *K is relatively $\sigma(E^\times, E)$-compact.*

(ii) *For every system $\{x_\alpha\} \subseteq E$ with $x_\alpha \downarrow_\alpha 0$,*

$$\sup\{|\tau(yx_\alpha)| : y \in K\} \to_\alpha 0.$$

(iii) *For every mutually orthogonal sequence $\{e_n\} \subseteq \mathcal{M}^p$ and every $0 \leqslant x \in E$,*

$$\sup\{\max(|\tau(ye_n x)|, |\tau(xe_n y)|) : y \in K\} \to_n 0.$$

(iv) *For every $x \in E$ and $\{x_n\} \subseteq \Omega(x)$ with $x_n \to 0$ for the measure topology,*

$$\sup\left\{\int_{[0,\infty)} \mu_t(x_n)\,\mu_t(y)\,dt : y \in K\right\} \to_n 0.$$

(v) *For every mutually orthogonal sequence $\{e_n\} \subseteq \mathcal{M}^p$ and every $0 \leqslant x \in E$,*

$$\sup\left\{\int_{[0,\infty)} \mu_t(y)\,\mu_t(xe_n)\,dt : y \in K\right\} \to_n 0.$$

(a) If E is fully symmetric and $E^\times \subseteq \tilde{\mathcal{M}}_0$ then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

(b) If E is fully symmetric, if $E^\times \subseteq \tilde{\mathcal{M}}_0$ and $E \subseteq \tilde{\mathcal{M}}_0$ then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v).

Proof. (*a*) The implications (iii) $\Rightarrow$ (i), (ii) $\Rightarrow$ (i) are given in Proposition 2·3.

(i) $\Rightarrow$ (ii). Assume that there exists $\epsilon > 0$ and a directed system $\{x_\alpha\} \subseteq E$ with $x_\alpha \downarrow_\alpha 0$, and such that

$$\sup\{|\tau(yx_\alpha)| : y \in K\} > \epsilon, \quad \text{for all indices } \alpha.$$

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Since $K \subseteq E^\times$, there exist sequences $\{y_n\} \subseteq K$, $\{x_{\alpha_n}\} \subseteq \{x_\alpha\}$ such that $x_{\alpha_n} \downarrow_n$ and such that

$$|\tau(y_n x_{\alpha_n})| > \epsilon$$

and
$$|\tau(y_j x_{\alpha_{n+1}})| < \tfrac{1}{4}\epsilon, \quad 1 \leqslant j \leqslant n, \quad n = 1, 2, \ldots.$$

As in the proof of Proposition 2·6, it may be assumed, via Šmulian's theorem, that the sequence $\{y_n\}$ converges pointwise on $\Omega(x_{\alpha_1})$. For $n = 1, 2, \ldots$, we set

$$z_n = x_{\alpha_n} - x_{\alpha_{n+1}},$$

and observe that if F is any finite subset of $\mathbb{N}$, then

$$0 \leqslant \sum_{j \in F} z_j \leqslant x_{\alpha_1}.$$

Consequently, $\sum_{j \in S} z_j \in \Omega(x_{\alpha_1})$ for each subset S of $\mathbb{N}$. For $n \geqslant 2$ and $S \in 2^{\mathbb{N}}$, define

$$\nu_n(S) = \tau((y_n - y_{n-1}) \sum_{j \in S} z_j).$$

It is easily checked that $\{\nu_n\}$ is a sequence of bounded, additive measures on the Boolean algebra $2^{\mathbb{N}}$ with $\nu_n(S) \to 0$ for each $S \in 2^{\mathbb{N}}$. From Phillips' Lemma [8] it follows that $\nu_n(\{n\}) \to 0$ as $n \to \infty$. However,

$$\begin{aligned} |\nu_n(\{n\})| &= |\tau((y_n - y_{n-1})(x_{\alpha_n} - x_{\alpha_{n+1}}))| \\ &\geqslant |\tau(y_n x_{\alpha_n})| - |\tau(y_{n-1} x_{\alpha_n})| - |\tau(y_{n-1} x_{\alpha_{n+1}})| - |\tau(y_n x_{\alpha_{n+1}})| \\ &> \tfrac{1}{4}\epsilon, \quad \text{for } n \geqslant 2, \end{aligned}$$

and, by this contradiction, the proof of the implication (i) $\Rightarrow$ (ii) is complete. The proof of the implication (i) $\Rightarrow$ (iii) is similar.

(*b*) The implication (i) $\Rightarrow$ (iv) is given in Proposition 2·6. (iv) $\Rightarrow$ (ii) follows from the assumption that $E \subseteq \mathscr{M}_0$ and $x_\alpha \downarrow 0 \subseteq E$ and so $x_\alpha \to 0$ for the measure topology.

The implication (v) $\Rightarrow$ (iii) is an immediate consequence of the inequality

$$\max(|\tau(ye_n x)|, |\tau(xe_n y)|) \leqslant \int_{[0,\infty)} \mu_t(y)\, \mu_t(xe_n)\, dt.$$

To establish the implication (i) $\Rightarrow$ (v) assume that (v) is not satisfied. There exist $\epsilon > 0$, a sequence $\{y_n\} \subseteq K$, a sequence $\{e_n\}$ of mutually orthogonal projections in $\mathscr{M}$ and $0 \leqslant x \in E$ such that

$$\int_{[0,\infty)} \mu_t(y_n)\, \mu_t(xe_n)\, dt > \epsilon, \quad n = 1, 2, \ldots.$$

From Proposition 1·1, there exists a sequence $\{x_n\} \subseteq E$ with

$$\mu(x_n) \prec\prec \mu(xe_n) \leqslant \mu(x)$$

and
$$|\tau(y_n x_n)| > \epsilon, \quad n = 1, 2, \ldots. \tag{A}$$

Now
$$\mu(e_n x) = \mu^{\frac{1}{2}}(x^* e_n x) \leqslant \mu^{\frac{1}{2}}(x^*(\sum_{m \geqslant n} e_m) x), \quad n = 1, 2, \ldots.$$

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Since $x \in \tilde{\mathcal{M}}_0$, it follows that

$$\mu^{\frac{1}{2}}(x^*(\sum_{m \geqslant n} e_m)\, x) \downarrow_n 0,$$

and consequently the inequality

$$\int_{[0,1)} \mu_t(x_n)\, dt \leqslant \int_{[0,1)} \mu_t(x e_n)\, dt$$

implies that $x_n \to 0$ in $G(\mathcal{M})$. Hence $\{x_n\}$ converges to zero for the measure topology and it follows that the inequalities in (A) together with Proposition 2·6 imply that K is not relatively $\sigma(E^\times, E)$-compact. ∎

COROLLARY 2·9. *Assume that E is fully symmetric, that $E \subseteq \tilde{\mathcal{M}}_0$ and that $E^\times \subseteq \tilde{\mathcal{M}}_0$. If $K \subseteq E^\times$ is relatively $\sigma(E^\times, E)$-compact, then so is the set*

$$\Omega(K) = \bigcup \{\Omega(y) : y \in K\}.$$

We let $\mathcal{S}_0 = \{f \in L^0(\mathbb{R}^+) : \mu_t(f) \to 0 \text{ as } t \to \infty\}$.

PROPOSITION 2·10. *Assume that $E(\mathbb{R}^+)$ is maximal and that $E(\mathbb{R}^+)$, $E(\mathbb{R}^+)^\times \subseteq \mathcal{S}_0$. If $K \subseteq E(\mathcal{M})$ is bounded, then the following statements are equivalent.*

(i) *K is relatively $\sigma(E(\mathcal{M}), E(\mathcal{M})^\times)$-compact.*

(ii) *$\{\mu(x) : x \in K\}$ is relatively $\sigma(E(\mathbb{R}^+), E(\mathbb{R}^+)^\times)$-compact.*

Proof. The implication (ii) ⇒ (i) follows from Proposition 2·8.

(i) ⇒ (ii). If $\{\mu(x) : x \in K\}$ is not relatively $\sigma(E(\mathbb{R}^+), E(\mathbb{R}^+)^\times)$-compact, then there exist $\epsilon > 0$, $0 \leqslant f \in E(\mathbb{R}^+)^\times$, a sequence $\{f_n\} \subseteq \Omega(f)$ which converges to zero for the measure topology, and a sequence $\{y_n\} \subseteq K$ such that

$$\int_{[0,\infty)} \mu_t(f_n)\, \mu_t(y_n)\, dt > \epsilon, \quad n = 1, 2, \ldots.$$

It follows again from Proposition 1·1 that there exists a sequence $\{z_n\} \subseteq E^\times(\mathcal{M})$ with

$$z_n \prec\prec f_n$$

and such that $$|\tau(z_n y_n)| > \epsilon, \quad n = 1, 2, \ldots.$$

It follows that $z_n \to 0$ for the measure topology. We set

$$\Lambda(f) = \{y \in \tilde{\mathcal{M}} : \mu(y) \prec\prec \mu(f)\}.$$

As in the proof of Proposition 2·4, $\Lambda(f)$ is $\sigma(E^\times(\mathcal{M}), E(\mathcal{M}))$-compact. An inspection of the proof of Proposition 2·6 now shows that

$$\sup\{|\tau(z_n y)| : y \in K\} \to 0 \quad \text{as } n \to \infty,$$

and this contradiction completes the proof of the Proposition. ∎

The preceding Proposition 2·10 was first proved for trace ideals by Garling [15], theorem 12 and in the general commutative setting by Fremlin [13], corollary 42. Even in the case of trace ideals, the present approach appears to yield a somewhat deeper structural insight than the methods of [15].

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3. *Topological properties of symmetric operator spaces*

The characterizations of weak compactness given in the preceding section are analogues of similar results that are well known in the theory of Banach lattices [1]. In this section we develop this analogy further by giving characterizations of various topological properties such as weak sequential completeness, perfectness and reflexivity which extend known results in the theory of Banach function spaces, and which in the case of trace ideals have been given in [9]. We point out that the machinery developed in [7] and in the present paper permits the use of ideas that are familiar in the setting of Banach lattices and so we will indicate only the essential lines of argument. It should be emphasized however, as is well known, that arguments based on lattice structure are not directly applicable in the non-commutative setting.

PROPOSITION 3·1. *Consider the following statements.*

(i) *E is perfect.*

(ii) *E is $\sigma(E, E^\times)$ sequentially complete.*

(iii) *$0 \leqslant x_n \uparrow_n \subseteq E$, $\sup_n \|x_n\|_E < \infty$ implies $\sup_n x_n$ exists in E.*

The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) *always hold. If $E^{\times\times} \subseteq \tilde{\mathcal{M}}_0$ and if the natural embedding of E into $E^{\times\times}$ is an isometry, then* (iii) $\Rightarrow$ (i).

Proof. (i) $\Rightarrow$ (ii). If $\{x_n\} \subseteq E$ is $\sigma(E, E^\times)$-Cauchy then the argument of [9], proposition 3·1 shows that

$$\lim_\alpha \sup_n |\tau(x_n y_\alpha)| = 0,$$

for all systems $\{y_\alpha\} \subseteq E^\times$ with $y_\alpha \downarrow_\alpha 0$ holding in $E^\times$. Consequently any weak*-accumulation point of $\{x_n\}$ in $E^{\times *}$ is normal and hence belongs to $E^{\times\times} = E$.

The implication (ii) $\Rightarrow$ (iii) follows from the fact that each increasing norm bounded sequence in the positive cone of E is $\sigma(E, E^\times)$-Cauchy since the positive cone of $E^\times$ is generating.

Assume now that $E^{\times\times} \subseteq \tilde{\mathcal{M}}_0$. If $0 \leqslant x \in E^{\times\times}$ has spectral representation

$$x = \int_{[0,\infty)} \lambda \, de_\lambda^x,$$

set

$$x_n = \int_{[1/n,\, n)} \lambda \, de_\lambda^x, \quad n = 1, 2, \ldots.$$

It follows that $\{x_n\} \subseteq E$ and that

$$x = \sup_n x_n$$

holds in $E^{\times\times}$. Condition (iii) now readily implies that $x \in E$ and the proof is complete. ∎

The norm on E is called *order continuous* if and only if $0 \leqslant x_\alpha \downarrow_\alpha 0$ in E implies $\|x_\alpha\|_E \downarrow_\alpha 0$. Since E is order complete, it is not difficult to see that the norm on E is order continuous if and only if $\|x_n\|_E \downarrow_n 0$ for every sequence $\{x_n\} \subseteq E$ for which $x_n \downarrow_n 0$ in E.

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PROPOSITION 3·2. *The following statements are equivalent.*

(i) *E is $\sigma(E, E^*)$ sequentially complete.*

(ii) *E is maximal and has order continuous norm.*

(iii) *Each norm bounded increasing sequence in E is convergent.*

(iv) *E contains no copy of c_0.*

Proof. (ii) $\Rightarrow$ (i). Since order continuity of the norm on E implies that via Proposition 1·2 $E^* = E^\times$, this implication follows immediately from the implication (i) $\Rightarrow$ (ii) of Proposition 3·1.

Before proceeding to the proof of the remaining implications, we make some simple remarks. If $\tau(1) = +\infty$ and if $1 \in E^{\times\times}$, then there exists a sequence $\{e_n\}$ of mutually orthogonal projections in $H(\mathscr{M})$ with

$$\tau(e_n) \geqslant \tfrac{1}{2}, \quad n = 1, 2, \ldots$$

so that

$$\|e_n\|_E \geqslant \|e_n\|_{E^{\times\times}} = \sup\left\{\int_{[0,\infty)} \mu_t(e_n)\,\mu_t(y)\,dt : y \in E^\times,\ \|y\|_{E^\times} \leqslant 1\right\}$$

$$\geqslant \sup\left\{\int_{[0,1/2)} \mu_t(y)\,dt : y \in E^\times,\ \|y\|_{E^\times} \leqslant 1\right\}, \quad \text{for} \quad n = 1, 2, \ldots.$$

Since

$$\sum_{j=1}^{n} e_j \leqslant 1, \quad n = 1, 2, \ldots$$

holds in $E^{\times\times}$, it follows that $\{e_n\}$ is $\sigma(E, E^\times)$ convergent to zero. If the embedding of E into $E^{\times\times}$ is an isometry then a standard argument now shows that the closure in E of $H(\mathscr{M})$ contains a subspace which is order isomorphic to c_0. Consequently, if the embedding of E into $E^{\times\times}$ is an isometry and if E contains no (positive) copy of c_0, a fortiori if E is $\sigma(E, E^*)$ sequentially complete, then $E^{\times\times} \subseteq \tilde{\mathscr{M}}_0$.

(i) $\Rightarrow$ (ii). To show that E has order continuous norm, suppose that $x_n \downarrow_n 0$ holds in E. By [3], theorem 1 the positive cone of E^* is generating; it follows that the sequence $\{x_n\}$ is $\sigma(E, E^*)$-Cauchy and so by (i) $\{x_n\}$ is $\sigma(E, E^*)$-convergent. Necessarily $\{x_n\}$ is weakly convergent to zero. By Mazur's theorem, there exists a sequence $\{y_n\}$ of convex combinations of the sequence $\{x_n\}$ such that $\inf_n \|y_n\|_E = 0$, and since the sequence $\{x_n\}$ is decreasing, it follows also that $\inf_n \|x_n\|_E = 0$. Consequently the norm on E is order continuous and from this it follows further that the embedding of E into $E^{\times\times}$ is an isometry. The assumption (i) and the preceding remarks imply that $E^{\times\times} \subseteq \tilde{\mathscr{M}}_0$. An appeal to Proposition 3·1 now completes the proof of the implication (i) $\Rightarrow$ (ii).

Essentially the same argument yields the implication (iii) $\Rightarrow$ (ii) and the implication (ii) $\Rightarrow$ (iii) is a simple consequence of [7], corollary 5·12.

Since the implication (i) $\Rightarrow$ (iv) is trivial the proof of the Proposition will be completed by establishing the implication (iv) $\Rightarrow$ (iii). To this end, if $\{x_n\} \subseteq E$ is a norm-bounded increasing sequence in E and if

$$y_n = x_{n+1} - x_n, \quad n = 1, 2, \ldots,$$

then

$$\sum_{n=1}^{\infty} |\phi(y_n)| < \infty$$

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holds for each $\phi \in E^*$ since the positive cone of E^* is generating. Consequently, if E contains no copy of c_0, then the series $\sum_{n=1}^{\infty} y_n$ is (unconditionally) convergent in E by the well-known criterion of Bessaga–Pelczynski ([**8**], theorem V·8). ∎

PROPOSITION 3·3. *If E has order continuous norm, and if $E^{\times\times} \subseteq \tilde{\mathcal{M}}_0$, then the following statements are equivalent.*

(i) *$E^{\times}$ has order continuous norm.*

(ii) *The unit ball of E is conditionally $\sigma(E, E^*)$ sequentially compact.*

(iii) *E contains no copy of l^1.*

Proof. The equivalence of statements (ii) and (iii) is a special case of Rosenthal's l^1-theorem ([**8**], chapter XI).

(ii) $\Rightarrow$ (i). We first show that the unit ball of E is relatively countably $\sigma(E^{\times\times}, E^{\times})$-compact. If $\{x_n\}$ is a sequence in the unit ball of E, then, by (ii), $\{x_n\}$ has a $\sigma(E, E^{\times})$-Cauchy subsequence $\{x_{n_k}\}$. Since $E^{\times\times}$ is maximal (see [**7**], proposition 5·7), Proposition 3·1 implies that $E^{\times\times}$ is $\sigma(E^{\times\times}, E^{\times})$ sequentially complete. Hence $\{x_{n_k}\}$ converges for $\sigma(E^{\times\times}, E^{\times})$ in $E^{\times\times}$, and so the sequence $\{x_n\}$ has a $\sigma(E^{\times\times}, E^{\times})$ cluster point in $E^{\times\times}$.

Now Proposition 2·2 implies that the unit ball of E is relatively $\sigma(E^{\times\times}, E^{\times})$-compact, and so Proposition 2·8 implies that

$$\sup\{|\tau(xy_\alpha)| : x \in E,\ \|x\|_E \leqslant 1\} \to_\alpha 0$$

for all systems $\{y_\alpha\} \subseteq E^{\times}$ with $y_\alpha \downarrow_\alpha 0 \subseteq E^{\times}$, that is, $E^{\times}$ has order continuous norm.

(i) $\Rightarrow$ (ii). Order continuity of the norm on E implies that $E^* = E^{\times}$ and order continuity of the norm on $E^{\times}$ implies that $E^{**} = E^{\times\times}$. If $\{x_n\}$ is any bounded sequence in E, it follows that $\{x_n\}$ is relatively $\alpha(E^{\times\times}, E^{\times})$-compact in $E^{\times\times}$. Let $0 \leqslant e \in E^{\times}$ be any projection with $0 < \tau(e) < \infty$. From Proposition 2·4, Šmulian's theorem and a diagonal argument, there exists a subsequence $\{y_n\} \subseteq \{x_n\}$ such that the restriction of $\{y_n\}$ to $\bigcup_{k=1}^{\infty} \Omega(ke)$ is pointwise Cauchy. Since $H(\mathcal{M}) \subseteq \bigcup_{k=1}^{\infty} \Omega(ke)$, it follows that $\{y_n\}$ is $\sigma(E, H(\mathcal{M}))$-Cauchy. Order continuity of the norm on $E^{\times}$ now implies that $\{y_n\}$ is $\sigma(E, E^{\times})$-Cauchy and the proof of the implication is complete. ∎

COROLLARY 3·4. *The following statements are equivalent.*

(i) *E is reflexive.*

(ii) *E contains no copy of c_0 or l^1.*

The preceding Corollary is a simple consequence of Propositions 3·2, 3·3 after noting, as in the proof of Proposition 3·2, that statement (ii) implies that $E^{\times\times} \subseteq \tilde{\mathcal{M}}_0$.

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