Rational arc length

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Finding an expression for the length of a curve is one of the simpler geometric applications of the integral. If f is a function with a continuous derivative, then the expression $\int_a^b \sqrt{1 + (f'(x))^2}\,dx$ gives the length of the curve $y = f(x)$ on an interval $[a, b]$. However, after writing out the integrand for familiar functions such as $y = x^2$ and $y = \sin x$, it quickly becomes apparent that, in general, finding an antiderivative is a challenge. Of course, a computer can give accurate approximations for the value of the integral for the length of a curve, but it would be nice to find the exact length rather than a decimal approximation. In his work on geometry (from 1637), Descartes stated that he believed it was not possible to determine the exact lengths of curves. However, just twenty years later, William Neile was able to find the length of arcs of semicubical parabolas (see Katz [1]). These curves have the form $y = kx^{3/2}$ and are usually the first examples or exercises given to students since the resulting integral is very easy to compute. In this paper, we are going to examine this curve and other related curves and consider problems such as the following: find rational numbers a and b so that the length of the curve over the interval $[a, b]$ is an integer. As we shall see, problems such as this provide a variety of opportunities for undergraduate students to explore some interesting mathematics arising from a few simple and accessible questions.

1. *Some simple cases for the curve* $y = x^{3/2}$

Consider the curve $y = x^{3/2}$. Given rational numbers a and b with $b > a \geqslant 0$, the length L of this curve on the interval $[a, b]$ is

$$L = \int_a^b \sqrt{1 + \frac{9}{4}x}\,dx = \left[\frac{8}{27}\left(1 + \frac{9}{4}x\right)^{3/2}\right]_a^b = \left(b + \frac{4}{9}\right)^{3/2} - \left(a + \frac{4}{9}\right)^{3/2} = t^3 - s^3,$$

where

$$s = \sqrt{a + \frac{4}{9}} \qquad \text{and} \qquad t = \sqrt{b + \frac{4}{9}}.$$

The number L is clearly rational when s and t are both rational. The converse is also true and is a consequence of the following theorem.

Theorem 1: Suppose that c and d are distinct positive rational numbers such that $\sqrt{d} - \sqrt{c}$ is a rational number. Then both $\sqrt{c}$ and $\sqrt{d}$ are rational numbers.

Proof: Noting that $\sqrt{d} - \sqrt{c}$ is a non-zero rational number and that $d - c = (\sqrt{d} - \sqrt{c})(\sqrt{d} + \sqrt{c})$ is a rational number, we find that $\sqrt{d} + \sqrt{c}$ is a rational number. Hence, the numbers

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$$\sqrt{c} = \frac{1}{2}\left(\left(\sqrt{d} + \sqrt{c}\right) - \left(\sqrt{d} - \sqrt{c}\right)\right) \quad \text{and} \quad \sqrt{d} = \frac{1}{2}\left(\left(\sqrt{d} + \sqrt{c}\right) + \left(\sqrt{d} - \sqrt{c}\right)\right)$$

are rational numbers.

Suppose that L is a rational number. Applying Theorem 1 to the rational numbers

$$c = \left(a + \frac{4}{9}\right)^3 \qquad \text{and} \qquad d = \left(b + \frac{4}{9}\right)^3,$$

we find that the numbers $\sqrt{c} = s^3$ and $\sqrt{d} = t^3$ are rational. It then follows that s and t are rational numbers since s^2 and t^2 are rational numbers.

We now focus our attention on finding intervals $[a, b]$ that generate integer values for L. By the above work, we need to express L in the form $t^3 - s^3$ for rational numbers s and t. For an initial computation, suppose that $s = \frac{2}{3}$ so that $a = 0$. Writing $t = \frac{p}{q}$ with $\gcd(p, q) = 1$, we find that the number

$$t^3 - s^3 = \left(\frac{p}{q}\right)^3 - \frac{8}{27} = \frac{27p^3 - 8q^3}{27q^3}$$

must be an integer. Since 27 divides the numerator, it follows that 3 divides q. With $q = 3Q$, the fraction reduces to

$$\frac{p^3 - 8Q^3}{27Q^3},$$

revealing that Q^3 divides p^3 and thus Q divides p. Since p and Q are relatively prime, we find that $Q = 1$ and $q = 3$. Hence we need to choose p so that

$$L = \left(\frac{p}{3}\right)^3 - \frac{8}{27} = \frac{p^3 - 8}{27}$$

is an integer. It is easy to verify that the only integers that provide suitable choices for p are those of the form $9k + 2$, where k is a positive integer. It then follows that

$$b = \left(\frac{p}{3}\right)^2 - \frac{4}{9} = \frac{(9k+2)^2 - 4}{9} = 9k^2 + 4k$$

and

$$L = \frac{(9k+2)^3 - 8}{27} = k\left(27k^2 + 18k + 4\right).$$

For example, using the values 1, 2 and 3 for k, we find that

the length of the curve $y = x^{3/2}$ is 49 on the interval [0,13];
the length of the curve $y = x^{3/2}$ is 296 on the interval [0,44];
the length of the curve $y = x^{3/2}$ is 903 on the interval [0,93].

By subtracting the first two of these lengths, it follows that

the length of the curve $y = x^{3/2}$ is 247 on the interval [13,44].

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Similarly, if $s = \frac{7}{3}$, then $a = 5$ and we would like integer values of p so that

$$L = \left(\frac{p}{3}\right)^3 - \frac{343}{27} = \frac{p^3 + 8}{27} - 13$$

is an integer. Hence, any integer of the form $9k - 2$ is a suitable choice for p and it follows that

$$b = \frac{(9k - 2)^2 - 4}{9} = 9k^2 - 4k$$

and

$$L = \frac{(9k - 2)^3 - 343}{27} = (k - 1)\left(27k^2 + 9k + 13\right).$$

In this case, using the values 2, 3 and 4 for k, we find that

the length of the curve $y = x^{3/2}$ is 139 on the interval [5,28];
the length of the curve $y = x^{3/2}$ is 566 on the interval [5,69];
the length of the curve $y = x^{3/2}$ is 1443 on the interval [5,128];
the length of the curve $y = x^{3/2}$ is 427 on the interval [28,69],

where the fourth example follows by subtraction of the first two lengths. Many more such values can be found, but the numbers do get large rather quickly.

In general, if a and $b > a$ are any two integers from the sequence $\{9k^2 + 4k\}$ or any two integers from the sequence $\{9k^2 - 4k\}$, then the length of the curve $y = x^{3/2}$ on the interval $[a, b]$ is an integer. These two sequences are the only sequences of integer values for the endpoints that give integer values for L. To see this, let $a \geqslant 0$ and $b > a$ be integers and write

$$L = \frac{1}{27}\left((9b + 4)^{3/2} - (9a + 4)^{3/2}\right).$$

Applying Theorem 1 once again, the numbers $(9a + 4)^{3/2}$ and $(9b + 4)^{3/2}$ must be rational and consequently the integers $9a + 4$ and $9b + 4$ must be squares. Denoting these squares m^2 and n^2, respectively, we find that

$$a = \frac{m^2 - 4}{9} \qquad \text{and} \qquad b = \frac{n^2 - 4}{9},$$

for integers m and n with $n > m \geqslant 2$. It follows that m and n must be of the form $9k \pm 2$ in order for a and b to be integers. We then need $n^3 - m^3$ to be a multiple of 27 to guarantee that L is an integer. This only occurs when m and n have the same form, either $9k + 2$ or $9k - 2$. These values for m and n generate the two sequences listed above.

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2. *A general result for the curve* $y = x^{3/2}$

Now consider the following question. For what positive integers N does there exist an interval $[a, b]$ with rational endpoints for which the length of the curve $y = x^{3/2}$ has the value N? Eliminating the denominators from the arc length equation $N = t^3 - s^3$ given at the beginning of Section 1, this question is equivalent to asking for values of N that generate solutions to the Diophantine equation $p^3 - q^3 = Nr^3$. Assuming that the integers p, q and r are positive solutions to this equation, we can let $t = \frac{p}{r}$ and $s = \frac{q}{r}$, then compute

$$a = s^2 - \frac{4}{9} = \frac{q^2}{r^2} - \frac{4}{9} \qquad \text{and} \qquad b = t^2 - \frac{4}{9} = \frac{p^2}{r^2} - \frac{4}{9}.$$

Since a must be non-negative, we see that the added condition $\frac{q}{r} \geqslant \frac{2}{3}$ is required. For example,

the equation $2^3 - 1^3 = 7 \times 1^3$ generates the values $t = 2$ and $s = 1$;
the equation $3^3 - 2^3 = 19 \times 1^3$ generates the values $t = 3$ and $s = 2$;
the equation $11^3 - 3^3 = 163 \times 2^3$ generates the values $t = \frac{11}{2}$ and $s = \frac{3}{2}$.

We thus find that

the length of the curve $y = x^{3/2}$ is 7 on the interval $\left[\frac{5}{9}, \frac{32}{9}\right]$;
the length of the curve $y = x^{3/2}$ is 19 on the interval $\left[\frac{32}{9}, \frac{77}{9}\right]$;
the length of the curve $y = x^{3/2}$ is 163 on the interval $\left[\frac{65}{36}, \frac{1073}{36}\right]$.

Of course, these examples are just a small sample of specific cases that can occur.

When does the Diophantine equation $p^3 - q^3 = Nr^3$ have a solution? It turns out that, depending on the value of N, the equation either has no solutions or an infinite number of distinct solutions. For example, the fact that the equation has no solutions when $N = 1$ (or any other perfect cube) is a simple case of Fermat's Last Theorem. Since any cubes that appear in N can be absorbed by r, we can assume that no cube divides N, that is, we may assume that N is cube-free. (An integer $N > 1$ is cube-free if it is not divisible by any cube greater than 1.) The following theorem is the first step in establishing that there are infinitely many solutions for a given value of N once it is known that the equation has at least one solution.

Theorem 2: Let $N > 1$ be a cube-free positive integer and suppose that x, y and z are integers such that $x^3 + y^3 = Nz^3$. Then

$$\left(x\left(x^3 + 2y^3\right)\right)^3 + \left(-y\left(2x^3 + y^3\right)\right)^3 = N\left(z\left(x^3 - y^3\right)\right)^3.$$

Proof: Performing some algebra, we find that

$$\begin{aligned}&\left(x\left(x^3 + 2y^3\right)\right)^3 - \left(y\left(2x^3 + y^3\right)\right)^3 \\ &= \left(x^{12} + 6x^9y^3 + 12x^6y^6 + 8x^3y^9\right) - \left(y^{12} + 6x^3y^9 + 12x^6y^6 + 8x^9y^3\right)\end{aligned}$$

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$$= \left(x^{12} - 2x^9y^3 + x^6y^6\right) - \left(x^6y^6 - 2x^3y^9 + y^{12}\right)$$

$$= \left(x^6 - y^6\right)\left(x^6 - 2x^3y^3 + y^6\right)$$

$$= \left(x^3 + y^3\right)\left(x^3 - y^3\right)\left(x^3 - y^3\right)^2$$

$$= N\left(z\left(x^3 - y^3\right)\right)^3,$$

as desired.

Using Theorem 2, it can be shown (see Sierpiński [2]) that if the equation $x^3 + y^3 = Nz^3$, where $N > 2$ is a cube-free integer, has a solution that satisfies $\gcd(x, y) = 1$ and $z \neq 0$, then it has infinitely many such solutions. However, there are infinitely many values of N for which there are no non-trivial solutions (see Selmer [3]). The first few examples are the integers 1, 2, 3, 4, 5, 8, 10 and 11. (The solution $1^3 + 1^3 = 2 \times 1^3$ is considered a trivial solution for the case $N = 2$.) Hence, there are no intervals with rational endpoints for which the length of the curve $y = x^{3/2}$ on the given interval has one of these values. Selmer proves (among other such results) that no non-trivial solutions exist when N is an odd prime of the form $9k + 2$ or $9k + 5$. He also provides a complete list of all of the positive integers $N \leqslant 500$ for which there is a solution, along with a basic solution for each such N. For example,

$$37^3 + 17^3 = 6 \times 21^3; \qquad 89^3 + 19^3 = 12 \times 39^3;$$
$$2^3 + (-1)^3 = 7 \times 1^3; \qquad 7^3 + 2^3 = 13 \times 3^3;$$
$$2^3 + 1^3 = 9 \times 1^3; \qquad 683^3 + 397^3 = 15 \times 294^3.$$

However, the existence of a non-trivial solution to $x^3 + y^3 = Nz^3$ does not guarantee that the integer N corresponds to an interval for which the length of the curve $y = x^{3/2}$ on that interval has the value N. The equation $x^3 + y^3 = Nz^3$ can always be represented so that x and z are positive and $x > |y|$. In order for such an equation to correspond to an appropriate interval for the arc length of our curve, we need $y < 0$ and $-\frac{y}{z} \geqslant \frac{2}{3}$. As is evident in Theorem 2, the formula generates solutions for which the sign of y often alternates, but it is not clear that eventually a solution appears for which $-\frac{y}{z} \geqslant \frac{2}{3}$; that this is guaranteed to happen is proved in Theorem 3. We first checked all of the N values in Selmer's table up to 100 and found that each corresponds to an appropriate interval. To illustrate a simple such computation, one application of Theorem 2 transforms the equation $2^3 + 1^3 = 9 \times 1^3$ into $20^3 - 17^3 = 9 \times 7^3$. Letting

$$a = \left(\frac{17}{7}\right)^2 - \frac{4}{9} = \frac{2405}{441} \qquad \text{and} \qquad b = \left(\frac{20}{7}\right)^2 - \frac{4}{9} = \frac{3404}{441},$$

we find that the length of the curve $y = x^{3/2}$ is 9 on the interval $[a, b]$. Using similar calculations for other values of N (and recording our previous work), we find that the length of the curve $y = x^{3/2}$

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has a value of 6 on the interval $\left[\frac{316730513681}{102515232400}, \frac{510815921681}{102515232400}\right]$;

has a value of 7 on the interval $\left[\frac{5}{9}, \frac{32}{9}\right]$;

has a value of 9 on the interval $\left[\frac{2405}{441}, \frac{3404}{441}\right]$;

has a value of 12 on the interval $\left[\frac{131728847038283}{247090289054700}, \frac{1253943314566283}{247090289054700}\right]$;

has a value of 13 on the interval $\left[\frac{492548}{336675}, \frac{1955423}{336675}\right]$.

A table of all possible N values for $N < 100$ and the corresponding interval on which the curve $y = x^{3/2}$ has an arc length of N can be found in the appendix. As indicated in these few examples, it should be pointed out that some of the appropriate fractions involve integers with hundreds or (for $N = 78$) thousands of digits; this is not a total surprise given the use of cubes in the expressions appearing in Theorem 2.

As the following theorem shows, this connection between integer arc length for the curve $y = x^{3/2}$ and the Diophantine equation $x^3 + y^3 = Nz^3$ is true in general. A few of the elementary but tedious computational details needed in the proof are left to the reader.

Theorem 3: There exist non-negative rational numbers a and b such that the arc length of the curve $y = x^{3/2}$ on the interval $[a, b]$ equals the positive integer N if, and only if, there exist non-zero integers p, q and r such that $p \neq q$ and $p^3 + q^3 = Nr^3$.

Proof: Our previous work shows that an arc length N, where N is a positive integer, corresponds to a non-trivial solution to $p^3 + (-q)^3 = Nr^3$. For the converse, suppose that there exist non-zero integers p, q and r such that $p \neq q$ and $p^3 + q^3 = Nr^3$. From Selmer's results, we know that $N > 5$. If p and r are positive and q is negative with $-\frac{q}{r} \geqslant \frac{2}{3}$, then (as discussed above) there is an appropriate interval $[a, b]$. If these conditions are not satisfied, there are several cases to consider.

Suppose first that p, q and r are positive with $p > q$. The formulas in Theorem 2 generate a solution to the equation $P^3 + Q^3 = NR^3$, where P and R are positive, Q is negative, and $P > |Q|$. If $\frac{-Q}{R} \geqslant \frac{2}{3}$, then we are finished. (It is not difficult to show that this inequality is guaranteed to hold when $\frac{q}{r} \geqslant \frac{1}{3}$.) Suppose that this is not the case and write the equation as $t^3 - s^3 = N$, where $t = \frac{P}{R}$ and $s = \frac{Q}{R}$ are positive rational numbers. Our assumption is that $0 < s < \frac{2}{3}$ and, assuming that $N \geqslant 8$ (we have already solved the cases $N = 6$ and $N = 7$ directly), we know that $t > 2$.

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Next, we let $\rho = \dfrac{t}{s}$ and note that $\rho > 3$. With $m = \dfrac{N}{s^3}$, the equation $t^3 - s^3 = N$ becomes

$$\rho^3 + (-1)^3 = m \times 1^3.$$

Letting

$$x = \frac{\rho(\rho^3 - 2)}{\rho^3 + 1} \qquad \text{and} \qquad y = \frac{2\rho^3 - 1}{\rho^3 + 1},$$

we find that x and y are positive rational numbers that satisfy $x^3 + y^3 = m$ (we are just using a fractional form of the equation in Theorem 2) and $x > y$. The inequality $x > y$ follows from the fact that

$$\rho(\rho^3 - 2) > 2\rho^3 - 1 \qquad \Leftrightarrow \qquad \rho^4 - 2\rho^3 - 2\rho + 1 > 0$$

is true for all $\rho > 2.3$. With

$$u = \frac{x(x^3 + 2y^3)}{x^3 - y^3} = \frac{\rho(\rho^3 - 2)}{\rho^3 + 1} \cdot \frac{\rho^3(\rho^3 - 2)^3 + 2(2\rho^3 - 1)^3}{\rho^3(\rho^3 - 2)^3 - (2\rho^3 - 1)^3},$$

$$v = \frac{y(2x^3 + y^3)}{x^3 - y^3} = \frac{2\rho^3 - 1}{\rho^3 + 1} \cdot \frac{2\rho^3(\rho^3 - 2)^3 + (2\rho^3 - 1)^3}{\rho^3(\rho^3 - 2)^3 - (2\rho^3 - 1)^3},$$

we have $u^3 - v^3 = m$ and $u > v$. We claim that $v > 4$. To see this, note that (as above) the denominator in the expression for v is positive for all $\rho > 2.3$. Performing some simple algebra, it follows that the inequality $v > 4$ is equivalent to

$$6\rho^3\left((2\rho^3 - 1)^3 - (\rho^3 - 2)^3\right) + 3(2\rho^3 - 1)^3 > 0,$$

which is clearly true for all $\rho > 3$. Multiplying the equation $u^3 - v^3 = m$ by s^3 yields

$$(us)^3 - (vs)^3 = N \qquad \text{with} \qquad vs > 4s.$$

If $0 < vs < \frac{2}{3}$, we repeat this process. After a finite number of steps, we obtain an equation of the form $\tau^3 - \sigma^3 = N$, where $\sigma \geqslant \frac{2}{3}$. These values for σ and τ then generate an interval $[a, b]$ with rational endpoints so that the length of the curve $y = x^{3/2}$ on $[a, b]$ is N.

Now suppose that p and r are positive, that q is negative, and that the ratio $-\frac{q}{r}$ is less than $\frac{2}{3}$. Since the quantities

$$p(p^3 + 2q^3) = p(Nr^3 + q^3) \qquad \text{and} \qquad -q(2p^3 + q^3) = -q(Nr^3 + p^3)$$

are positive, the formulas in Theorem 2 generate positive integers P, Q and R such that $P^3 + Q^3 = NR^3$. We may assume that $P > Q$ and then repeat the steps in the first part of the proof to generate an interval $[a, b]$ with rational endpoints so that the length of the curve $y = x^{3/2}$ on $[a, b]$ is N.

Putting the two parts of the proof together, along with Sierpiński's result concerning an infinite number of solutions, we find that each value of N

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either never occurs as an arc length over an interval with rational endpoints or occurs as an arc length on an infinite number of such intervals. (We do not claim that this technique finds all such intervals.) To emphasise the fact that this result is more theoretical than practical, the equation

$$5563^3 + 53^3 = 78 \times 1302^3$$

shows that the starting fraction (53/1302 in this case) can be very small. Five iterations of Theorem 2 are required to generate numbers for which $-\frac{y}{z} > \frac{2}{3}$ and the resulting fractions needed for the rational endpoints that generate an arc length of 78 involve integers with roughly 7670 digits.

We mention in passing the slightly different curve $y = \frac{2}{3}x^{3/2}$. If we repeat the above analysis, we find that the length L of this curve on an interval $[a, b]$ with rational endpoints is given by

$$L = \frac{2}{3}\left((1+b)^{3/2} - (1+a)^{3/2}\right) = \frac{2}{3}\left(\frac{p^3}{r^3} - \frac{q^3}{r^3}\right)$$

and thus $(12L)r^3 = (2p)^3 - (2q)^3$, where we have used the previously proved facts that $1 + b$ and $1 + a$ must be squares of rational numbers. Modifying the proof of Theorem 3, we can obtain the following result; the details are left to the reader.

Theorem 4: There exist non-negative rational numbers a and b such that the arc length of the curve $y = \frac{2}{3}x^{3/2}$ on the interval $[a, b]$ equals the positive integer L if, and only if, there exist non-zero integers p, q and r such that $p^3 + q^3 = (12L)r^3$.

To determine one such interval for this modified curve, one that gives an integer value that is not possible for the curve $y = x^{3/2}$, suppose we begin with the equation $2089^3 - 901^3 = (12 \times 11) \times 399^3$. With

$$a = \left(\frac{901}{2 \times 399}\right)^2 - 1 = \frac{174997}{636804} \quad \text{and} \quad b = \left(\frac{2089}{2 \times 399}\right)^2 - 1 = \frac{3727117}{636804},$$

we find that the length of the curve $y = \frac{2}{3}x^{3/2}$ on the interval $[a, b]$ is

$$\frac{2}{3}\left(\frac{2089^3}{8 \times 399^3} - \frac{901^3}{8 \times 399^3}\right) = \frac{1}{12}\left(\frac{2089^3 - 901^3}{399^3}\right) = 11.$$

The reader is encouraged to explore other curves of the form $y = kx^{3/2}$ for various values of k. To be specific, consider the curve $y = \frac{2}{3}\sqrt{\frac{p}{q}}\,x^{3/2}$, where p and q are relatively prime positive integers, and search for positive integers b for which

$$L = \int_0^b \sqrt{1 + \frac{p}{q}x}\,dx = \frac{2q}{3p}\left(\left(1 + \frac{p}{q}b\right)^{3/2} - 1\right)$$

is an integer. Due to the variety of cases that can occur (such as whether or not 3 divides pq and/or p is even and/or q is cube-free), this problem offers many different avenues to explore.

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3. *Extensions to other curves*

Consider the curve $y = \frac{4}{5}x^{5/4}$. The arc length of this curve on an interval $[0, b]$ is

$$L = \int_0^b \sqrt{1 + \sqrt{x}}\, dx.$$

Making the substitution $u = \sqrt{1 + \sqrt{x}}$ and letting $c = \sqrt{1 + \sqrt{b}}$ yields

$$L = \int_1^c 4u^2\left(u^2 - 1\right) du = \frac{4}{5}\left(c^5 - 1\right) - \frac{4}{3}\left(c^3 - 1\right).$$

We can guarantee that L is an integer if $c = 15k + 1$ and thus $b = (15k\,(15k + 2))^2$, where k is a positive integer. Using $k = 1$ as an example, the length of the curve $y = \frac{4}{5}x^{5/4}$ on the interval $[0, 65025]$ is

$$\int_0^{(15\times 17)^2} \sqrt{1 + \sqrt{x}}\, dx = \int_1^{16} 4u^2\left(u^2 - 1\right) du = 833400.$$

It is clear that the relevant integers are very large. Performing a similar computation for the curve $y = \frac{3}{4}x^{4/3}$ leads to the integral

$$L = \int_0^b \sqrt{1 + x^{2/3}}\, dx.$$

In this case, the substitution $u = \sqrt{1 + x^{2/3}}$ and thus $x = \left(u^2 - 1\right)^{3/2}$ gives

$$\int_1^c 3u^2\sqrt{u^2 - 1}\, du = \frac{3}{8}\left(c\left(2c^2 - 1\right)\sqrt{c^2 - 1} - \ln\left(c + \sqrt{c^2 - 1}\right)\right),$$

where $c = \sqrt{1 + b^{2/3}}$. It is rather hopeless to seek integer values of b that make this quantity an integer.

Given these two examples, we are left wondering which curves of the form $y = \frac{p}{q}x^{p/q}$, where p and $q < p$ are positive integers, allow us to find integer values for arc length. As we have seen in these last two examples (exponents $\frac{5}{4}$ and $\frac{4}{3}$), the appropriate substitution in the integral has the form

$$u = \sqrt{1 + x^{2(p-q)/q}} \qquad \text{and thus} \qquad x = \left(u^2 - 1\right)^{q/(2(p-q))}.$$

Unless the new integrand is a polynomial, we are unlikely to obtain rational arc lengths on intervals with rational endpoints. We thus need the exponent on the term $u^2 - 1$ to be an integer, that is,

$$\frac{q}{2\,(p - q)} = n \Leftrightarrow \frac{p - q}{q} = \frac{1}{2n} \Leftrightarrow \frac{p}{q} = 1 + \frac{1}{2n} = \frac{2n + 1}{2n}$$

for some positive integer n. Consequently, we can consider the arc length of curves of the form

$$y = \frac{2n}{2n + 1}\, x^{(2n+1)/(2n)},$$

where n is a positive integer, on the interval $[0, b]$ and seek integer values of b so that the length L of the curve is an integer. (See Moen [4] for the arc length of these curves on the specific interval $[0, 1]$.) For these curves, we

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have

$$L = \int_0^b \sqrt{1 + x^{1/n}}\, dx.$$

Making the substitution $u = \sqrt{1 + x^{1/n}}$ and letting $c = \sqrt{1 + b^{1/n}}$ yields

$$\begin{aligned} L &= \int_1^c 2nu^2 \left(u^2 - 1\right)^{n-1} du \\ &= 2n \int_1^c \sum_{k=0}^{n-1} \binom{n-1}{k} u^{2k+2} (-1)^{n-1-k} du \\ &= \sum_{k=0}^{n-1} (-1)^{n-1-k} 2n \binom{n-1}{k} \frac{c^{2k+3} - 1}{2k+3}. \end{aligned}$$

Assuming that c is an integer (see the discussion after Theorem 5), we need

$$n\binom{n-1}{k}\left(c^{2k+3} - 1\right) \equiv 0 \pmod{(2k+3)}$$

for $k = 0, 1, \ldots, n-1$. This congruence equation will certainly hold if either $n\binom{n-1}{k}$ or $c - 1$ is congruent to 0 modulo $2k + 3$. For example, when $n = 4$ and $n = 5$, we need

$$\begin{aligned} 4 \times 1 \times \left(c^3 - 1\right) &\equiv 0 \pmod 3; & 5 \times 1 \times \left(c^3 - 1\right) &\equiv 0 \pmod 3; \\ 4 \times 3 \times \left(c^5 - 1\right) &\equiv 0 \pmod 5; & 5 \times 4 \times \left(c^5 - 1\right) &\equiv 0 \pmod 5; \\ 4 \times 3 \times \left(c^7 - 1\right) &\equiv 0 \pmod 7; & 5 \times 6 \times \left(c^7 - 1\right) &\equiv 0 \pmod 7; \\ 4 \times 1 \times \left(c^9 - 1\right) &\equiv 0 \pmod 9; & 5 \times 4 \times \left(c^9 - 1\right) &\equiv 0 \pmod 9; \\ & & 5 \times 1 \times \left(c^{11} - 1\right) &\equiv 0 \pmod{11} \end{aligned}$$

respectively. It is not difficult to verify that $c = 105j + 1$ and $c = 231j + 1$, respectively, satisfy the appropriate conditions. Therefore, we find that

$$b = \left(c^2 - 1\right)^4 = (105j\,(105j + 2))^4$$

and

$$b = \left(c^2 - 1\right)^5 = (231j\,(231j + 2))^5,$$

where j is any positive integer, are the appropriate values of b that generate integer values for the arc length of the curves $y = \frac{8}{9}x^{9/8}$ and $y = \frac{10}{11}x^{11/10}$, respectively. If, for a given value of n, our only goal is to find a sequence for the b values rather than seeking all such values, then we can use the primorial function $(2n + 1)\#$, which is defined as the product of all of the

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primes that are less than or equal to $2n + 1$, to obtain a value for c. For example, we have

$$c = \tfrac{1}{2}(9\#)j + 1 = (3 \times 5 \times 7)j + 1 = 105j + 1 \text{ for } n = 4,$$

$$c = \tfrac{1}{2}(11\#)j + 1 = (3 \times 5 \times 7 \times 11)j + 1 = 1155j + 1 \text{ for } n = 5.$$

As noted above, we can eliminate the factor of 5 for the $n = 5$ case since the modulo 5 condition is automatically satisfied for any value of c. For a second example, leaving the details to the reader, when n is 14 (and thus the curve is $y = \frac{28}{29}x^{29/28}$), we can use the number

$$3 \times 5 \times 17 \times 19 \times 23 \times 29 + 1 = 3231616$$

for c rather than the much larger $\frac{1}{2}(29\#) + 1 = 3234846616$ and then find that the appropriate b values are $(ck(ck + 2))^{14}$ for each positive integer k. However, this number is huge even for $k = 1$; it is an integer with 183 digits. The fact that each prime only needs to appear once in the product is a consequence of the following result.

Theorem 5: If p is an odd prime and c is a positive integer such that $c \equiv 1 \pmod{p}$, then $c^{p^n} \equiv 1 \pmod{p^n}$ for each positive integer n.

Proof: We use the Principle of Mathematical Induction. The statement is trivial when $n = 1$. Although not necessary, we check the case in which $n = 2$. Note that

$$c^{p^2} - 1 = \left(c^p\right)^p - 1^p = \left(c^p - 1\right)\sum_{k=0}^{p-1}\left(c^p\right)^k.$$

Since p divides $c^p - 1$ and (working modulo p)

$$\sum_{k=0}^{p-1}\left(c^p\right)^k \equiv \sum_{k=0}^{p-1} 1 \equiv p \equiv 0,$$

we find that p^2 divides $c^{p^2} - 1$. Now suppose that $c^{p^n} \equiv 1 \pmod{p^n}$ for some positive integer n. Then

$$c^{p^{n+1}} - 1 = \left(c^{p^n}\right)^p - 1^p = \left(c^{p^n} - 1\right)\sum_{k=0}^{p-1}\left(c^{p^n}\right)^k.$$

By hypothesis, the number p^n divides $c^{p^n} - 1$ and since (working modulo p)

$$\sum_{k=0}^{p-1}\left(c^{p^n}\right)^k \equiv \sum_{k=0}^{p-1} 1 \equiv p \equiv 0,$$

we find that p divides $\sum_{k=0}^{p-1}\left(c^{p^n}\right)^k$. It follows that $c^{p^{n+1}} \equiv 1 \pmod{p^{n+1}}$. By the Principle of Mathematical Induction, we conclude that $c^{p^n} \equiv 1 \pmod{p^n}$ for each positive integer n. We note in passing that a careful reading of the proof reveals that the conclusion $c^{p^n} \equiv 1 \pmod{p^{n+1}}$ is also valid.

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To illustrate this result with a specific example, suppose that we seek a positive integer c so that

$$c^{3\times 5^2\times 7^3} \equiv 1 \pmod{3 \times 5^2 \times 7^3}.$$

Let c be a positive integer that satisfies the three conditions

$$c \equiv 1 \pmod 3, \qquad c \equiv 1 \pmod 5, \qquad c \equiv 1 \pmod 7.$$

The choice $c = 106$ provides a simple example of such an integer. Using Theorem 5, we find that

$$c^3 \equiv 1 \pmod 3, \qquad c^{5^2} \equiv 1 \pmod{5^2} \qquad \text{and} \qquad c^{7^3} \equiv 1 \pmod{7^3},$$

and thus

$$3 \text{ divides } c^{3\times 5^2\times 7^3} - 1 = \left(c^3\right)^{5^2\times 7^3} - 1;$$

$$5^2 \text{ divides } c^{3\times 5^2\times 7^3} - 1 = \left(c^{5^2}\right)^{3\times 7^3} - 1;$$

$$7^3 \text{ divides } c^{3\times 5^2\times 7^3} - 1 = \left(c^{7^3}\right)^{3\times 5^2} - 1.$$

It follows that $3 \times 5^2 \times 7^3$ divides $c^{3\times 5^2\times 7^3} - 1$. In other words, each prime factor is needed only once when we are seeking appropriate c values.

We complete this discussion by verifying that the number $c = \sqrt{1 + b^{1/n}}$ must be an integer. Consider the two polynomials

$$P(x) = \left(x^2 - 1\right)^n - b \qquad \text{and} \qquad Q(x) = \int_1^x 2nt^2\left(t^2 - 1\right)^{n-1} dt - L,$$

where b and L are positive integers. The polynomials P and Q have rational coefficients and the algebraic number c is a root of both polynomials. (See abstract algebra texts such as [5] for a discussion of algebraic numbers.) Given the form of c as a square root, we see that the degree of c (as an algebraic number) must either be 1 or an even integer. The polynomial Q has degree $2n + 1$ and $Q'(x) = 2nx^2\left(x^2 - 1\right)^{n-1}$. If n is odd, then $Q(x)$ is an increasing function with only one real root. If n is even, then Q has a relative maximum at $x = -1$. Since

$$Q(-1) = \int_{-1}^1 2nt^2\left(1 - t^2\right)^{n-1} dt - L = 4n\int_0^1 t^2\left(1 - t^2\right)^{n-1} dt - L$$

$$< 4n\int_0^1 t\left(1 - t^2\right)^{n-1} dt - L = 2 - L \leqslant 0$$

for all $L \geqslant 2$, we find that $Q(x)$ has only one real root in this case as well. (We can exclude $L = 1$ since the length of any of these curves is greater than 1 when $b = 1$.) Factorising $Q(x)$ into a product of irreducible polynomials with rational coefficients gives us something like

$$Q(x) = Q_1(x)Q_2(x)\ldots Q_k(x).$$

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Since $Q(x)$ has odd degree and only one real root, exactly one of the polynomials $Q_i(x)$ has odd degree. Since c must be a root of this polynomial, we find that the degree of c is odd, which means that the degree of c is 1. Hence the number c is rational. Since the number $b^{1/n}$ can only be rational when it is an integer, it follows that c is an integer. This discussion requires a slight modification in the case when b is the power of an integer that is a divisor of n, say $b = \beta^{\ell}$ with $n = \ell m$. We simply note that the degree of $c = \sqrt{1 + \beta^{1/m}}$ is either 1 or an even number and proceed as above.

Summarising our findings, we have shown that for each positive integer n there is a sequence of integer values $\{b_k^{(n)}\}$ so that the length of the curve

$$y = \frac{2n}{2n+1} x^{(2n+1)/(2n)}$$

on the interval $\left[0, b_k^{(n)}\right]$ is an integer for each value of k. Lest we take this result for granted, consider the curve $y = \frac{3}{2}x^{2/3}$. The arc length L of this curve on the interval $[0, b]$ is given by

$$L = \int_0^b \sqrt{1 + x^{-2/3}}\, dx = \left[\left(x^{2/3} + 1\right)^{3/2}\right]_0^b = \left(b^{2/3} + 1\right)^{3/2} - 1.$$

Suppose that $b = u^3/v^3$, where u and v are relatively prime positive integers. Then

$$L + 1 = \left(\frac{u^2 + v^2}{v^2}\right)^{3/2} = \frac{u^2 + v^2}{v^3}\sqrt{u^2 + v^2}.$$

The only way for L to be rational is for u and v to be part of a Pythagorean triple. Writing $u^2 + v^2 = w^2$, we then have

$$L + 1 = \frac{w^3}{v^3},$$

and this quantity can never be an integer. In other words, there is no positive rational number b (and thus certainly not an integer) for which the length of the curve $y = \frac{3}{2}x^{2/3}$ on the interval $[0, b]$ is an integer. However, to complete this proof, we need to establish that b has the form of the ratio of two cubes whenever L is rational. Suppose that $b = \frac{u}{v}$, where u and v are relatively prime, and that the quantity

$$\left(b^{2/3} + 1\right)^{3/2} = \left(\frac{u^{2/3}}{v^{2/3}} + 1\right)^{3/2} = \frac{\left(u^{2/3} + v^{2/3}\right)^{3/2}}{v}$$

is a rational number. Assigning the numerator the ratio $\frac{x}{y}$ for relatively prime positive integers x and y, we find that

$$\left(u^{2/3} + v^{2/3}\right)^{3/2} = \frac{x}{y} \Leftrightarrow (uy)^{2/3} + (vy)^{2/3} = x^{2/3} \Leftrightarrow \left(u^2v^2\right)^{1/3} + \left(v^2y^2\right)^{1/3} = \left(x^2\right)^{1/3}.$$

Referring to Morgan [6], it follows that

$$u^2y^2 = \gamma\alpha^3,\ v^2y^2 = \gamma\beta^3 \text{ and thus } \alpha^3v^2 = \beta^3u^2$$

for positive integers α, β and γ with $\gcd(\alpha, \beta) = 1$. Since $\gcd(u, v) = 1 =$

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$\gcd(\alpha, \beta)$, the last equation tells us that $u^2 = a^3$ and $v^2 = \beta^3$. It then follows that both u and v are cubes.

As an aside, we mention a further elementary problem. Consider again the function f defined by $f(x) = \frac{2}{3}x^{3/2}$ and search for positive numbers b so that (i) the point $(b, f(b))$ has rational coordinates, (ii) the area A under the curve $y = f(x)$ on the interval $[0, b]$ is a rational number, and (iii) the length L of the curve $y = f(x)$ on the interval $[0, b]$ is a rational number. It turns out that this problem, as with the curve $y = \frac{3}{2}x^{2/3}$ just considered, is another situation in which Pythagorean triples make an appearance.

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Appendix: integer values for lengths of the curve $y = x^{3/2}$

To obtain these results, we started with the data in Selmer's table and then used the computer program *Maple* to generate further solutions (using the equation in Theorem 2) until an appropriate equation of the form $x^3 - y^3 = Mz^3$ appeared, that is, when x, y and z are all positive and the ratio y/z is greater than $2/3$. The appropriate interval required to generate the designated value of N is $[a, b]$, where

$$a = \left(\frac{y}{z}\right)^2 - \frac{4}{9} \qquad \text{and} \qquad b = \left(\frac{x}{2}\right)^2 - \frac{4}{9}.$$

We have not included the a and b values because the numbers in the fractions are rather large for most values of N. In addition, we have omitted those cases in which the integers x, y and z are too large to be read clearly. Given the formulas in Theorem 2, there are multiple choices for the intervals giving a certain integer value for the arc length. For the few cases (such as 7 and 19) in which the numbers are reasonable, we have recorded two sets of x, y and z values. It seems likely that the corresponding intervals are the

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best possible (in the sense of the smallest denominators) for each value of N, but we have not verified this fact.

N	x	y	z
6	2237723	1805723	960540
7	2	1	1
7	1265	1256	183
9	20	17	7
12	63963143	26919143	27226290
13	2513	1388	1005
15	303083663039	277818514559	75276116916
17	121606000577709480360	26974999705067357761	47121226547163077743
19	3	2	1
19	78251704	52579329	25996285
20	the integer for x requires 75 digits		
22	684469533791312783	661146496267328783	112919729369578740
26	3	1	1
26	53972175	52609231	7643944
28	87	55	26
30	1105275779	1057859459	177025128
31	137	65	42
33	12319835066783	6729969554783	3619703493420
34	631	359	182
35	129	124	19
37	4	3	1
37	8467676481	2228010160	2525625557
42	the integer for x requires 43 digits		
43	805	229	228
48	2237723	1805723	480270
49	11	2	3
49	48106342365030785	32924229736357496	11557109112932721
50	23417	11267	6111
51	the integer for x requires 376 digits		
53	1872	1819	217

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56	4	2	1
56	2530	2512	183
58	28747	14653	7083
61	5	4	1
62	22187	21035	2964
63	4	1	1
65	191	146	39
67	839959056446753	372715154118128	200600312998995
68	2538163	472663	620505
69	15409	10441	3318
70	2803753	1715113	623844
71	197	126	43
72	40	34	7
75	17351	11951	3606
78	the integer for x requires 3836 digits		
79	13	4	3
79	1605750231462853762	1528070497148513440	193490529458191518
84	64334824343	63328588343	5270776170
85	2570129	2404889	330498
86	31811	22595	6216
87	1176498611	907929611	216266610
89	12836017	12398760	1328873
90	1241	431	273
91	6	5	1
91	472	465	37
92	the integer for x requires 71 digits		
94	15642626656646177	15616184186396177	590736058375050
96	63963143	26919143	13613145
97	14	5	3
97	2832713504497390136	2799894968113535105	200421477873478047
98	5	3	1
98	260171463471	228469300015	38711273968

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