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Lineability, continuity, and antiderivatives in the non-Archimedean setting

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Abstract. This work focuses on the ongoing research of lineability (the search for large linear structures within certain non-linear sets) in non-Archimedean frameworks. Among several other results, we show that there exist large linear structures inside each of the following sets: (i) functions with a fixed closed subset of continuity, (ii) all continuous functions that are not Darboux continuous (or vice versa), (iii) all functions whose Dieudonné integral does not behave as an antiderivative, and (iv) functions with finite range and having antiderivative.

1 Introduction and Preliminaries

In mathematical analysis, many examples of functions with some sort of pathological behavior or enjoying certain special properties have been studied. Just to cite some seminal works, for instance, we could refer the reader to the famous example of Weierstrass (also known as *Weierstrass' monster*, a continuous nowhere differentiable function in $\mathbb{R}^{\mathbb{R}}$). Although the first known example of this kind dates back to Bolzano (1781–1848) in 1822 (see, *e.g.,* [23]), the first published example of such an f appeared in the 1886 paper [36] by Weierstrass (1815–1897). The example, defined as

$$W(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \cos(13^n \pi x),$$

was first presented by Weierstrass to the Prussian Academy of Sciences in 1872.

Once a mathematical object enjoying some kind of special property is found, the natural question is how many of these special objects one can find. Of course this question has many different interpretations (depending on the context) such as cardinality, prevalence, residuality, genericity, etc. Here, we are interested in finding large algebraic structures (dense manifolds, Banach spaces, algebras, etc.) within non-linear sets. This direction of research has, for the past decade, been referred to as *lineability and spaceability*. One of the earliest results in this direction was proved by Gurariy in 1966 ([20], see also [21]), who showed that the set of Weierstrass' monsters contains an infinite dimensional vector space. However, in 1940, Levine and Milman [27] showed that the subset of $\mathcal{C}([0,1])$ of all functions of bounded variation does not contain a closed infinite dimensional subspace. Later, Hencl [22] (see also [30])

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Figure 1: Bernard Bolzano and Karl Theodor Wilhelm Weierstrass.

showed that any separable Banach space is isometrically isomorphic to a subspace of $\mathcal{C}[0,1]$ whose non-zero elements are nowhere approximately differentiable and nowhere Hölder.

These last examples show that both positive and negative results can be obtained within this trend; for more details, see [17]. The term *lineable* and some of its variants were coined by V. I. Gurariy himself in the early 2000s ([2, 33]) and has, since attracted the attention of many mathematicians in different areas (see, *e.g.*, [1, 2, 3, 6, 8–10, 12, 13, 16, 29], and the references therein, for a current state of the art on this topic).

Before going further, let us recall some definitions, which, nowadays, have become standard terminology. We say that a subset A of a vector space V over a field $\mathbb{K}$ is:

- *lineable* in V if there is an infinite dimensional vector space M such that $M\backslash\{0\} \subseteq A$;
- κ-*lineable* in V (where κ is a cardinal number) if there exists a vector space M of dimension κ and $M\backslash\{0\} \subseteq A$. Thus, $\aleph_0$-lineability means lineability, where $\aleph_0$ stands for the cardinality of the set of natural numbers.

If, in addition, V is a topological vector space over $\mathbb{K}$, then A is said to be:

- *spaceable* in V if there exists a closed infinite dimensional vector space M such that $M\backslash\{0\} \subseteq A$.

Finally, and following [3, 7], if V is a topological vector space contained in a (not necessarily unital) algebra and if κ and β are any (finite or infinite) cardinal numbers, then A is called:

- *algebrable* if there is an algebra M that is not finitely generated with $M\backslash\{0\} \subseteq A$;
- κ-*algebrable* if there exists an algebra M such that $M\backslash\{0\} \subseteq A$ and M is a κ-dimensional vector space. Moreover, A is (κ,β)-*algebrable* if there is an algebra

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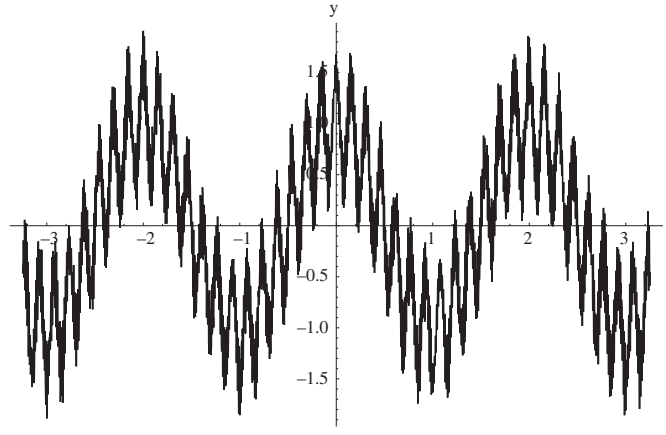


Figure 2: Sketch of the original Weiertrass' monster.

M such that $M\backslash\{0\} \subseteq A, \dim(M) = \kappa$ and there exists a minimal subset S of generators for the algebra M such that $\text{card}(S) = \beta$. Here, by $S = \{s_\alpha : \alpha \in I\}$ is a minimal system of generators of M, we mean that M is the algebra generated by S, and for every $\alpha_0 \in I$, s_{α_0} does not belong to the algebra generated by $S\backslash\{s_{\alpha_0}\}$.

In this work, we continue our ongoing study, started in [25, 26] (see also [14]), investigating this line of research for the spaces of functions over non-Archimedean fields.

In order to make this paper self-contained, we will include a brief selection of some standard concepts and notation concerning non-Archimedean functional analysis right before Section 2 begins. In Section 2, we will show, among other results, that: (i) for a fixed closed subset F, the set of functions $\mathbb{Q}_p \to \mathbb{Q}_p$ that are continuous only at the points of F is $2^{\mathfrak{c}}$-lineable; (ii) the set of Darboux continuous functions $\mathbb{Z}_p \to \mathbb{Z}_p$ that are not continuous is $\mathfrak{c}$-lineable; (iii) the set of everywhere discontinuous functions $\mathbb{N}_0 \to \mathbb{Q}_p$ such that their Dieudonné integral are not an antiderivative is $\mathfrak{c}$-lineable; (iv) the set of discontinuous functions $\mathbb{Q}_p \to \mathbb{Q}_p$ with finite range that have an antiderivative is $(\mathfrak{c}, \mathfrak{c})$-algebrable. It should be noted that, although many results in the classical setting have a non-Archimedean counterpart, their proofs are essentially different and may require an entirely new approach.

Let us then, as announced earlier, begin with a summary of some basic definitions from p-adic analysis (for a more profound treatment of this topic, we refer the interested reader to [19, 24, 31, 35]).

We will use standard set-theoretical notation. As usual, $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ denote the sets of all natural numbers, natural numbers including zero, integers, rational, real, and complex numbers, respectively. Also, as mentioned earlier, $\aleph_0$ and $\mathfrak{c}$ denote the cardinalities of $\mathbb{N}$ and $\mathbb{R}$, respectively. We identify each cardinal number with the first ordinal of the same cardinality (thus, a cardinal κ is equal to the set of all ordinals less than κ, denoted $\kappa = \{\alpha : \alpha < \kappa\}$).

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Figure 3: Vladimir Ilyich Gurariy was born in 1935 in Kharkov, Ukraine. In 1991 he moved to the USA and worked in Kent State University (Ohio) until his passing in 2005.

Let us also recall that a non-Archimedean field is a field $\mathbb{K}$ equipped with a function (valuation)

$$|\cdot| : \mathbb{K} \longrightarrow [0, \infty)$$

such that:

- $|x| = 0$ if and only if $x = 0$;
- $|xy| = |x||y|$;
- $|x + y| \leq \max\{|x|, |y|\}$,

for all $x, y \in \mathbb{K}$. Clearly, $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. An immediate consequence of so-called *strong triangle inequality* is that $|x| \neq |y|$ implies $|x + y| = \max\{|x|, |y|\}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except 0 into 1 and $|0| = 0$. This valuation is referred to as the "trivial" one. By the Big Ostrowski Theorem (see [35, Theorem 1.2]), any complete valued field $\mathbb{K}$ that is not topologically isomorphic to $\mathbb{R}$ or $\mathbb{C}$ is non-Archimedean.

Let us fix a prime number p throughout this work. For any non-zero integer $n \neq 0$, let $\mathrm{ord}_p(n)$ be the highest power of p that divides n. Then we define $|n|_p = p^{-\mathrm{ord}_p(n)}$, $|0|_p = 0$, and $|\frac{n}{m}|_p = p^{-\mathrm{ord}_p(n)+\mathrm{ord}_p(m)}$. The completion of the field of rationals, $\mathbb{Q}$, with respect to the p-adic metric $d(x, y) = |x - y|_p$ is called the *field of p-adic numbers* $\mathbb{Q}_p$. The metric d satisfies the so-called strong triangle inequality $|x \pm y|_p \leq \max\{|x|_p, |y|_p\}$. Ostrowski's Theorem states that every non-trivial absolute value on $\mathbb{Q}$ is equivalent (*i.e.*, defines the same topology) to an absolute value $|\cdot|_p$, where p is a prime number, or the usual absolute value (see [19]). It asserts that we have at our disposal all the absolute values on $\mathbb{Q}$.

Let $a \in \mathbb{Q}_p$ and let r be a positive number. The set

$$B_r(a) = \left\{x \in \mathbb{Q}_p : |x - a|_p < r\right\}$$

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is called the *open ball of radius r with center a*, and

$$\tilde{B}_r(a) = \{x \in \mathbb{Q}_p : |x - a|_p \leq r\}$$

is called the closed ball of radius r with center a. The ring of integers in $\mathbb{Q}_p$ is denoted by $\mathbb{Z}_p$, *i.e.*,

$$\mathbb{Z}_p = \left\{x \in \mathbb{Q}_p : |x|_p \leq 1\right\}.$$

Note that every $x \in \mathbb{Z}_p$ can be expanded in canonical form as

$$x = a_0 + a_1 p + \cdots + a_k p^k + \cdots,$$

with $a_k \in \{0, 1, \ldots, p-1\}$, $k \geq 0$. We know that $\mathbb{Z}_p$ is a compact set [19].

Let us remark that the definition of derivative of a p-adic function and the definition of an analytic function in p-adic setting are similar to the classical real functions (for more details, see [35]).

Let V be a linear space over a field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : V \to [0, \infty)$ is said to be a non-Archimedean norm if the following conditions hold:

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|\lambda x\| = |\lambda|\|x\|$, for all $\lambda \in \mathbb{K}$ and $x \in V$;
(iii) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$, for all $x, y \in V$.

Then $(V, \|\cdot\|)$ is called a *non-Archimedean normed space*. Throughout this article we will consider all vector spaces and algebras taken over the field $\mathbb{K} = \mathbb{Q}_p$ (unless stated otherwise).

Let us also recall the set-theoretical notion of almost disjoint sets that shall be frequently used in our proofs. Two sets A_1 and A_2 are said to be almost disjoint if $\text{card}(A_1 \cap A_2) < \text{card}(A_i)$ for $i = 1, 2$. A well-known theorem of W. Sierpiński is that an infinite set of the cardinality κ contains more than κ subsets of the cardinality κ that are pairwise almost disjoint (see [34]). We fix a family $\{A_\alpha : \alpha \in I\}$ of infinite pairwise almost disjoint subsets of $\mathbb{N}$ and write $A_\alpha = \{\alpha_n : n \in \mathbb{N}\}$ where (α_n) is an increasing sequence (notice that the cardinality of I can be assumed to be equal to $\mathfrak{c}$).

We need a theorem due to Fichtenholz and Kantorovich [15] (see also [4]). For any set X, there exists a family $\mathcal{B} \subseteq \mathcal{P}(X)$ (called a family of *independent sets*) of cardinality $2^{\text{card}(X)}$ such that for any finite sequences $B_1, \ldots, B_n \in \mathcal{B}$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{0, 1\}$, we have $B_1^{\varepsilon_1} \cap \cdots \cap B_n^{\varepsilon_n} \neq \varnothing$, where $B^1 = B$ and $B^0 = X \backslash B$. Moreover, all the sets in $\mathcal{B}$ can be chosen with cardinality $\text{card}(X)$. Here, $\mathcal{P}(X)$ denotes the power set of X.

2 The Main Results

We begin with a proposition that improves a result from the classical case (presented in [18, Theorem 5.1]) whose proof needs of a totally different approach and, also, the non-Archimedean counterpart of [5, Theorem 14] (proved adapting the idea of independent sets appearing in [5] and [4]).

Proposition 2.1 *Let F be a fixed closed subset of $\mathbb{Q}_p$. The set of functions $\mathbb{Q}_p \to \mathbb{Q}_p$ that are continuous only at the points belonging to F is $2^{\mathfrak{c}}$-lineable.*

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Proof For any $x, y \in \mathbb{Q}_p$, let us define the relation $\sim$ given by

$$x \sim y \quad \text{if and only if} \quad x - y \in \mathbb{Q},$$

and let $\{B_\alpha : \alpha < \mathfrak{c}\}$ be the partition of $\mathbb{Q}_p$ resulting from this equivalence relation. Notice that every B_α is dense in $\mathbb{Q}_p$. Also, let $\{N_\zeta : \zeta < 2^\mathfrak{c}\}$ be an independent family of subsets of $\mathfrak{c}$ and observe that $\operatorname{card}(N_\zeta) = \mathfrak{c}$.

For any $\zeta < 2^\mathfrak{c}$, put $B^{(\zeta)} = \cup_{\alpha \in N_\zeta} B_\alpha$. Notice that for every $\zeta_1 < \cdots < \zeta_n < 2^\mathfrak{c}$ and $\varepsilon_i \in \{0,1\}$ for $1 \le i \le n$, the set

$$(B^{(\zeta_1)})^{\varepsilon_1} \cap \cdots \cap (B^{(\zeta_n)})^{\varepsilon_n} = \bigcup_{\alpha \in N_{\zeta_1}^{\varepsilon_1} \cap \cdots \cap N_{\zeta_n}^{\varepsilon_n}} B_\alpha$$

is a dense proper subset of $\mathbb{Q}_p$; that is, the family $\{B^{(\zeta)} : \zeta < 2^\mathfrak{c}\}$ is an independent family of cardinality $2^\mathfrak{c}$.

For a fixed closed subset F, set $d(x,F) = \min\{|x-c|_p : c \in F\} \in \{0\} \cup \{p^t : t \in \mathbb{Z}\}$, and define $g : \mathbb{Q}_p \to \mathbb{Q}_p$ by $g(x) = |d(x,F)|_p$ and $g_\alpha : \mathbb{Q}_p \to \mathbb{Q}_p$, for any $\alpha < \mathfrak{c}$ by $g_\alpha(x) = g(x) \cdot \chi_{B_\alpha}(x)$.

Now, let us define, for any $\zeta < 2^\mathfrak{c}$,

$$f_\zeta(x) = \begin{cases} g_\alpha(x) & \text{if } x \in B_\alpha \text{ and } \alpha \in N_\zeta, \\ 0 & \text{otherwise.} \end{cases}$$

Next, consider the vector space obtained by spanning the functions $\{f_\zeta : \zeta < 2^\mathfrak{c}\}$.

To prove that the set $\{f_\zeta : \zeta < 2^\mathfrak{c}\}$ is linearly independent, let $H = \lambda_1 f_{\zeta_1} + \cdots + \lambda_n f_{\zeta_n} = 0$ for some $n \in \mathbb{N}, \lambda_i \in \mathbb{Q}_p \backslash \{0\}$ and $\zeta_i < 2^\mathfrak{c}$. Fix $1 \le i \le n$ and take $\varepsilon_i = 1$ and $\varepsilon_j = 0$ for $j \ne i$. We can find $\alpha \in N_{\zeta_i}$ such that $\alpha \notin N_{\zeta_j}$ for every $1 \le j \le n, j \ne i$. By definition, for any $x \in B_\alpha \backslash F$, we have $H(x) = \lambda_i |d(x,F)|_p = 0$, that is, $\lambda_i = 0$.

We now turn to studying the continuity of the linear combinations of f_α's. Let $H = \lambda_1 f_{\zeta_1} + \cdots + \lambda_n f_{\zeta_n}$ for some $n \in \mathbb{N}, \lambda_i \in \mathbb{Q}_p \backslash \{0\}$ and $\zeta_i < 2^\mathfrak{c}$. Let $x_0 \in F$ and (x_n) be a sequence converging to x_0. Since $0 \le |g_\alpha(x_n)|_p \le |g(x_n)|_p$ and $|g(x_n)|_p \to |g(x_0)|_p = 0$, it follows that H is continuous at x_0.

Finally, we show that H is not continuous at any point $x_0 \notin F$. Take $\ell \in \mathbb{Z}$ be such that $d(x_0, F) = p^{-\ell}$, so that $g(x_0) = p^\ell$. Since there exists θ such that $x_0 \in B_\theta$ and $x_0 \notin B_\alpha$ for all $\alpha \ne \theta$, we have $g_\theta(x_0) = p^\ell$ and $g_\alpha(x_0) = 0$ for all $\alpha \ne \theta$. But, on one hand, that there exists $\beta \ne \theta$ with $\beta \in N_{\zeta_1}^1 \cap N_{\zeta_2}^0 \cdots \cap N_{\zeta_n}^0$ together with the fact that B_β is dense in $\mathbb{Q}_p$ imply that there is a sequence (a_n) in B_β such that $a_n \to x_0$. Then, by definition of f_{ζ_i}'s, we must have $H(a_n) \to \lambda_1 p^\ell$. On the other hand, there is a $\gamma \ne \theta$ with $\gamma \in N_{\zeta_1}^0 \cap N_{\zeta_2}^0 \cdots \cap N_{\zeta_n}^0$ and a sequence (b_n) in B_γ such that $b_\gamma \to x_0$. Thus, we must have $H(b_n) \to 0$. This shows that H cannot be continuous at any point outside of F. ∎

Let us recall that, following [31, 32], a subset C of a valued field $\mathbb{K}$ is called *convex* if $\lambda x + (1-\lambda) y \in C$ for every $x, y \in C$, and $\lambda \in \mathbb{K}$ with $|\lambda| \le 1$. It can be seen that the only convex subsets of $\mathbb{K}$ are $\varnothing, \mathbb{K}$, the singleton sets and balls. A function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is called *Darboux continuous* if for each convex set $C \subseteq \mathbb{Z}_p$, the image $f(C)$ is convex in $\mathbb{Z}_p$.

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For any $\lambda, \mu \in \mathbb{Q}_p$ and subsets $A, B \subseteq \mathbb{Q}_p$, we set

$$\lambda A + \mu B = \{\lambda a + \mu b : a \in A, b \in B\}.$$

We need the following straightforward observation.

Lemma 2.2 *For any $\lambda, \mu \in \mathbb{Q}_p$ with $|\lambda|_p \geq |\mu|_p$, $\lambda\mathbb{Z}_p + \mu\mathbb{Z}_p = \lambda\mathbb{Z}_p$.*

Now we present our next result.

Theorem 2.3 *The set of Darboux continuous functions $\mathbb{Z}_p \to \mathbb{Z}_p$ that are not continuous is $\mathfrak{c}$-lineable.*

Proof Let $\{A_\alpha : \alpha \in I\}$ be a family of almost disjoint subsets of $\mathbb{N}$. For any $\alpha \in I$, let $A_\alpha = \{\alpha_n : n \in \mathbb{N}\}$ and

$$D_\alpha = \left\{ x = \sum_{i=1}^{\infty} a_i p^i \in \mathbb{Z}_p : (\exists n)(\forall i > \alpha_n)(i \notin A_\alpha \to a_i = 0) \right\}.$$

For any $x = \sum_{i=1}^{\infty} a_i p^i \in D_\alpha$, define $n_x = \min\{n \in \mathbb{N} : (\forall i > \alpha_n)(i \notin A_\alpha \to a_i = 0)\}$. Next, define $f_\alpha : \mathbb{Z}_p \to \mathbb{Z}_p$ by

$$f_\alpha(x) = \begin{cases} a_{\alpha_{n_x}} + a_{\alpha_{n_x+1}} p + a_{\alpha_{n_x+2}} p^2 + \cdots & \text{if } x \in D_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Let us first observe that f_α maps any ball onto $\mathbb{Z}_p$. In order to show this, let us take any $b \in \mathbb{Z}_p$ and $t \in \mathbb{Z}$. For every $x = \sum_{i=1}^{\infty} a_i p^i$, we find a point $y \in B_{p^{-t}}(b)$ such that $f_\alpha(y) = x$. Next, write $b = \sum_{i=1}^{\infty} b_i p^i$ and set $\ell = \min\{m \in \mathbb{N} : m > t, m \notin A_\alpha\}$ and $\alpha_{\bar{\ell}} = \min\{\alpha_k \in A_\alpha : \alpha_k > \ell\}$. Now define $y = b_0 + b_1 p + \cdots + b_t p^t + p^\ell + \sum_{i=0}^{\infty} a_i p^{\alpha_{\bar{\ell}+i}}$ and note that $y \in B_{p^{-t}}(b)$ and $f_\alpha(y) = x$.

We show that any linear combination of the functions f_α is a Darboux continuous function. To prove this, let $F = \lambda_1 f_{\alpha_1} + \cdots + \lambda_m f_{\alpha_m}$ for some $m \in \mathbb{N}$, $\lambda_i \in \mathbb{Q}_p$ and $\alpha_i \in I$. We show that for any ball $B_{p^{-t}}(b)$, $F(B_{p^{-t}}(b)) = \lambda_{\bar{n}}\mathbb{Z}_p$ where $1 \leq \bar{n} \leq m$ and $|\lambda_{\bar{n}}|_p = \max\{|\lambda_i|_p : 1 \leq i \leq m\}$. That $F(B_{p^{-t}}(b)) \subseteq \lambda_{\bar{n}}\mathbb{Z}_p$ is clear from the previous paragraph and Lemma 2.2. To show the reverse relation, let $b = \sum_{i=1}^{\infty} b_i p^i$ and let $x = \sum_{i=1}^{\infty} a_i p^i$ be an arbitrary element in $\mathbb{Z}_p$. We find an element y in the ball $B_{p^{-t}}(b)$ such that for all $j \neq \bar{n}$, $f_{\alpha_j}(y) = 0$ and $f_{\alpha_{\bar{n}}}(y) = x$. In view of the fact that A_α are almost disjoint, we can find an $\bar{m} \in \mathbb{N}$ such that $A^o_{\alpha_{\bar{n}}} := \{\alpha_{\bar{m}}, \alpha_{\bar{m}+1}, \dots\}$ and $A^o_{\alpha_{\bar{n}}} \cap (\cup_{\substack{j=1 \\ j\neq\bar{n}}}^{m} A_{\alpha_j}) = \varnothing$. Write

$$\mathbb{N} \setminus \bigcup_{j=1}^{m} A_{\alpha_j} := \{\gamma_1, \gamma_2, \dots\},$$

$$\{\gamma_1, \gamma_2, \dots\} \setminus \{1, 2, \dots, \alpha_{\bar{m}}\} := \{\delta_1, \delta_2, \dots\}.$$

Notice that there exists i such that $\delta_i > t$. Now, set $\widetilde{m} = \min\{i : \delta_i > t\}$ and $\widehat{m} = \min\{i : \alpha_i > \delta_{\tilde{m}}\}$. Define $y = b_0 + b_1 p + \cdots + b_t p^t + p^{\delta_{\tilde{m}}} + a_0 p^{\alpha_{\hat{m}}} + a_1 p^{\alpha_{\hat{m}+1}} + a_2 p^{\alpha_{\hat{m}+2}} + \cdots$. It is clear that $F(y) = \lambda_{\bar{n}} f_{\alpha_{\bar{n}}}(y) = \lambda_{\bar{n}} x$. This proves our claim. We conclude that F is Darboux continuous and nowhere continuous.

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Finally, let us show that this set of functions is linearly independent. In order to do that, let $F = \lambda_1 f_{\alpha_1} + \cdots + \lambda_m f_{\alpha_m} = 0$ for some $m \in \mathbb{N}$, $\lambda_i \in \mathbb{Q}_p \backslash \{0\}$ and $\alpha_i \in I$. For any $1 \leq j \leq m$, define $x_j = \sum_{i=1}^{\infty} a_i p^i \in \mathbb{Z}_p$ where

$$a_i = \begin{cases} p-1 & \text{if } i \in A_{\alpha_j}, \\ 0 & \text{otherwise.} \end{cases}$$

Observing that $x_j \in D_{\alpha_j}$ and $x_j \notin D_{\alpha_s}$ for any $s \in \{1, 2, \ldots, m\} \backslash \{j\}$, we see that $F(x_j) = \lambda_j f_{\alpha_j}(x_j) = -\lambda_j = 0$, which means that $\lambda_j = 0$, finishing the proof. ∎

Remark 2.4 Let us recall that the proof of the previous proposition shows that there is a $\mathfrak{c}$-linearly independent set of functions on $\mathbb{Z}_p$ mapping any non-empty ball onto $\mathbb{Z}_p$. Such functions are called *everywhere surjective* in [2] (see also [11]).

Proposition 2.5 *The set of continuous functions $\mathbb{Z}_p \to \mathbb{Z}_p$ that are not Darboux continuous is $(\mathfrak{c}, \mathfrak{c})$-algebrable.*

Proof For any $n \in \mathbb{N}_0$, let

$$B_n = \left\{ x \in \mathbb{Q}_p : |x|_p = p^{-n} \right\}.$$

Let $\{A_\alpha : \alpha \in I\}$ be a family of almost disjoint subsets of $\mathbb{N}$. For any $\alpha \in I$, define $f_\alpha : \mathbb{Z}_p \to \mathbb{Z}_p$ by

$$f_\alpha(x) = \begin{cases} 1 & x \in B_0, \\ p^n & x \in B_n,\ n \in A_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $f_\alpha(\mathbb{Z}_p) \subseteq \{p^0, p^1, p^2, \ldots\} \cup \{0\}$ and f_α is continuous. That f_α is not Darboux continuous is clear from the fact that $\{0, 1\} \subseteq f_\alpha(\mathbb{Z}_p) \neq \mathbb{Z}_p$.

Observing that

$$(f_\alpha \cdot f_\beta)(x) = \begin{cases} 1 & x \in B_0, \\ p^{2n} & x \in B_n,\ n \in A_\alpha \cap A_\beta, \\ 0 & \text{otherwise,} \end{cases}$$

and recalling that the convex sets in $\mathbb{Z}_p$ other than singletons and empty set are infinite, we infer that the algebra generated by the set $S = \{f_\alpha : \alpha \in I\}$ has the required properties. We show that S is a minimal system of generators for this algebra.

In order to achieve this, we shall show that, for any $\gamma \in I$, $f_\gamma \notin \mathcal{A}(S \backslash \{f_\gamma\})$. Take any polynomial without free term

$$F = P(f_{\alpha_1}, \ldots, f_{\alpha_k}) = \sum_{i=1}^{m} \lambda_i \prod_{j=1}^{k} f_{\alpha_j}^{n_{j,i}},$$

where $\lambda_i \neq 0$, $k, m \in \mathbb{N}$, and $n_{j,i} \in \mathbb{N}_0$. That $f_\gamma \notin \{f_{\alpha_1}, \ldots, f_{\alpha_k}\}$ together with the almost disjointness of the family A_α's implies that there is $m_\gamma \in A_\gamma$ such that $m_\gamma \notin A_{\alpha_i}$

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for all $i \leq k$. Thus, $F(p^{m_\gamma}) = 0$ and $f_\gamma(p^{m_\gamma}) = p^{m_\gamma}$, and hence $F \neq f_\gamma$. This finishes the proof. ∎

The problems of antiderivative and integration in the ultrametric setting are much subtler than in the classical real case. As a matter of fact, we do not know whether an integral exists for which the usual Fundamental Theorem of Calculus holds true (see [31, p. 93]). In 1944, Dieudonné defined an integral for an arbitrary function $g : \mathbb{N}_0 \to \mathbb{Q}_p$. Note that this integral coincides with the van der Put integral; for more details, see [28]. Let $x = \sum_{n=0}^{\infty} x_n p^n$ be the standard representation of $x \in \mathbb{N}_0$. Put

$$x^{(n)} = x_0 + x_i p + \cdots + x_{n-1} p^{n-1}$$

for $n \geq 1$. Thus, $x^{(n+1)} - x^{(n)} = x_n p^n$ if $n \geq r(x)$ where the suffix $r(x) \geq 0$ is such that $x_n = 0$ for all $n \geq r(x)$. Let us now define the Dieudonné integral $f(x)$ of $g(x)$ as the series

$$f(x) = \sum_{n=1}^{\infty} (x^{(n+1)} - x^{(n)}) g(x^{(n)}).$$

Proposition 2.6 *The set of everywhere discontinuous functions $\mathbb{N}_0 \to \mathbb{Q}_p$ whose Dieudonné integral is not an antiderivative is $\mathfrak{c}$-lineable.*

Proof Let $\{A_\alpha : \alpha \in I\}$ be a family of almost disjoint subsets of $\mathbb{N}$. For any $\alpha \in I$, define $g_\alpha : \mathbb{N}_0 \to \mathbb{Q}_p$ as

$$g_\alpha(x) = \begin{cases} p^{p^{\alpha_x}} & x \in \mathbb{N}, \\ 1 & x = 0. \end{cases}$$

Clearly, g_α is nowhere continuous, and its Dieudonné integral is given by

$$f_\alpha(x) := \sum_{n=1}^{\infty} (x^{(n+1)} - x^{(n)}) g_\alpha(x^{(n)}) = \sum_{n=1}^{\infty} (x_n p^n)(p^{p^{\alpha_{x^{(n)}}}}),$$

where $x = \sum_{n=0}^{\infty} x_n p^n$.

Next, let us observe that $f'_\alpha = 0 \neq g_\alpha$. Indeed, let $x, y \in \mathbb{N}_0$ be distinct and $|x - y|_p = p^{-\ell}$ for some $\ell \in \mathbb{N}$. Then $x_n = y_n$ for all $n = 0, 1, \ldots, \ell - 1$, and we can write

$$f_\alpha(x) - f_\alpha(y) = (x_\ell - y_\ell) p^{\ell + p^{\alpha_T}} + \textit{ terms with order greater than } p,$$

where $T = x_0 + x_1 p + \cdots + x_{\ell-1} p^{\ell-1}$. It follows that

$$\left| \frac{f_\alpha(x) - f_\alpha(y)}{x - y} \right|_p \leq p^{-p^{\alpha_T}},$$

and therefore,

$$\lim_{x \to y} \frac{f_\alpha(x) - f_\alpha(y)}{x - y} = f'_\alpha(y) = 0$$

Now, let us consider the vector space generated by the set $\{g_\alpha : \alpha \in I\}$. It is clear that any linear combination of g_α has the properties required in the proposition. It remains to show that the set $\{g_\alpha : \alpha \in I\}$ is linearly independent.

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To this end, let

$$F = \lambda_1 g_{\alpha_1} + \cdots + \lambda_m g_{\alpha_m} = 0,$$

where $m \in \mathbb{N}$, $\lambda_i \in \mathbb{Q}_p \backslash \{0\}$, and $\alpha_i \in I$. By definition of A_α's, there is a natural number ℓ such that for all natural numbers $k > \ell$, we have

$$F(k) = \lambda_1 p^{p^{\alpha_{1,k}}} + \cdots + \lambda_m p^{p^{\alpha_{m,k}}} = 0,$$

where $\alpha_{i,k} \in A_{\alpha_i}$ are distinct natural numbers. Evaluating F at m distinct natural numbers $n > \ell$, we obtain

$$\begin{pmatrix} p^{p^{\alpha_1,n_1}} & p^{p^{\alpha_2,n_1}} & \cdots & p^{p^{\alpha_m,n_1}} \\ p^{p^{\alpha_1,n_2}} & p^{p^{\alpha_2,n_2}} & \cdots & p^{p^{\alpha_m,n_2}} \\ \vdots & \vdots & \ddots & \vdots \\ p^{p^{\alpha_1,n_m}} & p^{p^{\alpha_2,n_m}} & \cdots & p^{p^{\alpha_m,n_m}} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{pmatrix} = 0,$$

where n_j are natural numbers greater than ℓ. Let $A = \left[p^{p^{\alpha_{i,n_j}}}\right]_{i,j}$, and notice that

$$\det(A) = \sum_{\sigma \in S_m} \varepsilon_\sigma \left(p^{p^{\alpha_{1\sigma(n_1)}} + \cdots + p^{\alpha_{m\sigma(n_m)}}}\right),$$

where $\varepsilon = \pm 1$ and S_m is the symmetric group on m elements. It is easily verified that if $\varepsilon_i = \pm 1$ and β_i are distinct natural numbers, then it is impossible to have that $\varepsilon_1 p^{\beta_1} + \cdots + \varepsilon_n p^{\beta_n} = 0$. Therefore, we infer that $\det(A) \neq 0$, and $\lambda_i = 0$ for $1 \leq i \leq m$. This completes the proof. ∎

Theorem 2.7 *The set of discontinuous functions $\mathbb{Q}_p \to \mathbb{Q}_p$ with finite range that have an antiderivative is $(\mathfrak{c}, \mathfrak{c})$-algebrable.*

Proof For any $n \in \mathbb{N}_0$, let

$$B_0 = \mathbb{Z}_p, \qquad B_n = \{x \in \mathbb{Q}_p : |x|_p = p^n\} \quad (n \geq 1).$$

Let $\{A_\alpha : \alpha \in I\}$ be a family of almost disjoint subsets of $\mathbb{N}$. For every $\alpha \in I$, define $f_\alpha : \mathbb{Q}_p \to \mathbb{Q}_p$ by

$$f_\alpha(x) = \begin{cases} 0 & x \in B_n,\ n \in A_\alpha, \\ 0 & x \in B_0 \backslash \{0\}, \\ 1 & \text{otherwise.} \end{cases}$$

Note that f_α is discontinuous at 0 and has an antiderivative g_α given by

$$g_\alpha(x) = \begin{cases} 0 & x \in B_n, n \in A_\alpha, \\ a_n p^n + \cdots + a_{2n} p^{2n} & x \in B_0 \backslash \{0\}, x = \sum_{j=0}^{\infty} a_j p^j, |x|_p = p^{-n}, \\ x & \text{otherwise.} \end{cases}$$

Let us consider the algebra generated by the set $\{f_\alpha : \alpha \in I\}$. It can be easily seen that the elements of this algebra enjoy the required properties.

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It remains to show that the set $S = \{f_\alpha : \alpha \in I\}$ is a minimal system of generators for the algebra.To this end, we show that for any $\theta \in I$, $f_\theta \notin \mathcal{A}(S\backslash\{f_\theta\})$, the algebra generated by $S\backslash\{f_\theta\}$. Take any polynomial without free term $P(f_{\alpha_1}, \ldots, f_{\alpha_k}) = \sum_{i=1}^{m} \lambda_i \prod_{j=1}^{k} f_{\alpha_j}^{n_{j,i}}$ where $\lambda_i \neq 0$, $k, m \in \mathbb{N}$, and $n_{j,i} \in \mathbb{N}_0$. Noting that ${f_\alpha}^n = f_\alpha$ for all $\alpha \in I$ and natural number n, we find that

$$F = P(f_{\alpha_1}, \ldots, f_{\alpha_k}) = \sum_{i=1}^{m} \lambda_i \prod_{j=1}^{k} f_{\alpha_j}^{\varepsilon_{i,j}},$$

where $\varepsilon_{i,j} \in \{0, 1\}$. That $f_\theta \notin \{f_{\alpha_1}, \ldots, f_{\alpha_k}\}$ together with almost disjointness of the family A_α's implies that there is $m_\theta \in A_\theta$ such that $m_\theta \notin A_{\alpha_i}$ for all $i \leq k$. Thus, $F(p^{-m_\theta}) = 1$ and $f_\theta(p^{-m_\theta}) = 0$, and hence $F \neq f_\theta$. This completes the proof. ∎

We conclude with the following result. Recall that by an analytic function on an open ball containing 0, we mean a function f defined from the ball into $\mathbb{Q}_p$ that can be expressed as a power series, *i.e.*, $f(x) = \sum_{i=0}^{\infty} a_i x^i$ for all x belongs to the ball, where $a_i \in \mathbb{Q}_p$ are coefficients. For the convenience of the reader, let us recall the statement of Strassman's Theorem (see,*e.g.*, [19]), which is needed in the proof of our last result. Let $f(x) = \sum_{i=0}^{\infty} a_i x^i$ be a non-zero power series with coefficients in $\mathbb{Q}_p$, and suppose that $f(x)$ converges for all $x \in \mathbb{Z}_p$. Let N be the integer defined by the two conditions

$$|a_N|_p = \max_i |a_i|_p \quad \text{and} \quad |a_i|_p < |a_N|_p \quad \text{for } i > N.$$

Then the function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ has, at most, N zeros.

Proposition 2.8 *The set of analytic functions $\mathbb{Z}_p \to \mathbb{Z}_p$ that do not have an analytic antiderivative is* $\mathfrak{c}$*-lineable.*

Proof For any $\alpha \in I$, define $f_\alpha : \mathbb{Z}_p \to \mathbb{Z}_p$ by

$$f_\alpha(x) = \sum_{i=1}^{\infty} p^{\alpha_i} x^{p^{\alpha_i}-1},$$

where $\{A_\alpha : \alpha \in I\}$ is a family of almost disjoint subsets of $\mathbb{N}$. Note that f_α is analytic on $\mathbb{Z}_p$ because that $p^{\alpha_i} \to 0$ and an antiderivative of f_α is given by $g_\alpha(x) = \sum_{i=1}^{\infty} x^{p^{\alpha_i}}$ that is not analytic since the series is divergent at $x = 1$. The vector space spanned by the set f_α's is the required linear space. Indeed,for any $\lambda, \mu \in \mathbb{Q}_p$, and $\alpha, \beta \in I$, let $F(x) = \lambda f_\alpha(x) + \mu f_\beta(x)$, and observe that $F(x) = \sum_{i=1}^{\infty} \Gamma_i p^{\gamma_i} x^{p^{\gamma_i}-1}$, where $\gamma_i \in A_\alpha \cup A_\beta$ and $\Gamma_i \in \{\lambda, \mu, \lambda + \mu\}$. Now it is clear that F is analytic, and its antiderivative $\sum_{i=1}^{\infty} \Gamma_i x^{p^{\gamma_i}}$ is not convergent at $x = 1$. On the other hand, by Strassman's Theorem, we find that if $F(x) = 0$ on $\mathbb{Z}_p$, then $\lambda = \mu = 0$, proving the result. ∎

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