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Generalized Köthe function spaces. I

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1. *Introduction.* The idea of constructing a space of functions taking values in a locally convex space E from a linear space $\mathscr{F}$ of scalar valued functions is well known. We can, for example, define a space $\mathscr{F}(E)$ consisting of all E-valued functions $\boldsymbol{\phi}(t)$ such that $\langle \boldsymbol{\phi}(t), e' \rangle \in \mathscr{F}$ for all elements e' of the dual E' of E. Besides this construction there are others which arise in special cases. This idea has been used to obtain integrals of vector-valued functions (compare (2), Chapter III, § 4). Schwartz has also used it in his paper on differentiable vector-valued functions (9) whose main result is the famous kernel theorem, as well as in introducing vector-valued distributions. It is natural to expect that the space of vector-valued functions obtained will inherit some properties of the function space $\mathscr{F}$ and the vector space E. Therefore one usually starts from some function space $\mathscr{F}$ which has interesting properties.

For a given perfect sequence space λ, Pietsch (7), (8) has made an extensive study of the spaces $\lambda(E)$ and $\lambda[E]$ defined as follows. $\lambda(E)$ is the space of all E-valued sequences $\mathbf{e} = (e_i)$ such that, for each $e' \in E'$, the sequence $(\langle e_i, e' \rangle)$ belongs to λ; and $\lambda[E]$ is the space of all E-valued sequences $\mathbf{e} = (e_i)$ such that the series $\Sigma \alpha_i e_i$ is convergent in E for each element (α_i) of the associate λ^* of λ. On the other hand, more than fifteen years ago, Dieudonné (5) has generalized perfect sequence spaces to obtain perfect (or Köthe) function spaces. He has shown that the perfect function spaces have most of the interesting properties of the perfect sequence spaces.

Let a perfect function space Λ with its associate Λ^* be given. This paper is devoted to the study of the space $\Lambda[E]$ of E-valued functions $\boldsymbol{\phi}(t)$ such that, for each $\lambda^*(t) \in \Lambda^*$, the function $\lambda^*(t)\boldsymbol{\phi}(t)$ is integrable in the Bochner sense.

In a second paper, we will investigate the space $\Lambda(E)$ obtained, roughly speaking, in the same way as $\Lambda[E]$ but the Bochner integrability is replaced by the weak integrability.

2. *The space* $\Lambda[E]$ *associated with a Banach space* E. Let T be a locally compact, Hausdorff space which is countable at infinity† and μ be a positive Radon measure on T. We can write $T = \bigcup_{n=1}^{n=\infty} K_n$, where each K_n is compact and $K_n \subset K_{n+1}$. In the sequel we shall always identify a function with the equivalence class to which it belongs. We denote by Ω the space of all locally integrable real-valued functions. Provided with the topology determined by the countable system of semi-norms

$$N_n(f) = \int_{K_n} |f(t)|\, d\mu \quad (f \in \Omega;\ n = 1, 2, \ldots),$$

† This assumption is not really necessary, except for Propositions 3 and 4 and 10.

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Ω is a Fréchet space. The dual of Ω is the space Φ of all measurable, p.p. bounded functions of compact supports.

Let Γ be an arbitrary subset of Ω. The set Λ of all functions $f \in \Omega$ such that fg is integrable for all $g \in \Gamma$ is a linear subspace of Ω. We say that Λ is a perfect (or Köthe) function space and that Γ is its defining set. The perfect space Λ^* having Λ as defining set is called the associate of Λ. Λ and Λ^* are put into weak duality by the canonical bilinear form

$$\langle f, g \rangle = \int fg \, d\mu \quad (f \in \Lambda, \, g \in \Lambda^*).$$

Now let a Banach space E with its dual E' and a Köthe space Λ with its associate Λ^* be given. Consider the space $\Omega[E]$ of all E-valued functions $\mathbf{f}(t)$ which are locally integrable in the Bochner sense ((2), Chapters I–IV). A function $\mathbf{f}(t)$ belongs to $\Omega[E]$ if and only if it is measurable (2) and the real function $\|\mathbf{f}(t)\|$ is locally integrable. It is not difficult to see that equipped with the topology determined by the countable system of semi-norms

$$N_n(\mathbf{f}) = \int_{K_n} \|\mathbf{f}(t)\| \, d\mu \quad (\mathbf{f} \in \Omega[E]; \, n = 1, 2, \ldots),$$

$\Omega[E]$ is a Fréchet space. By $\Lambda[E]$ we denote the space of all E-valued functions $\boldsymbol{\phi}(t)$ such that for every $\lambda^*(t) \in \Lambda^*$ the function $\lambda^*(t)\boldsymbol{\phi}(t)$ is integrable in the Bochner sense.

An E-valued function $\boldsymbol{\phi}(t)$ belongs to $\Lambda[E]$ if and only if for each $\lambda^*(t) \in \Lambda^*$ the function $\lambda^*(t)\boldsymbol{\phi}(t)$ is measurable and the function $\|\lambda^*(t)\boldsymbol{\phi}(t)\|$ is integrable. Since the characteristic function of any compact subset of T belongs to Λ^* we see that $\boldsymbol{\phi}(t)$ belongs to $\Lambda[E]$ if and only if it is measurable and the real function $\|\boldsymbol{\phi}(t)\|$ belongs to Λ. This remark enables us to topologize the space $\Lambda[E]$ as follows. Let a system of semi-norms $\{p_\alpha\}_{\alpha \in \mathfrak{A}}$ defining a locally convex topology $\mathfrak{T}$ on Λ be given. We require the p_α's be 'monotonic', i.e. satisfy the condition (M). If $0 \leqslant f \leqslant g$ then $p_\alpha(f) \leqslant p_\alpha(g)$. By putting

$$p_\alpha(\boldsymbol{\phi}) = p_\alpha(\|\boldsymbol{\phi}\|) \quad (\alpha \in \mathfrak{A}, \boldsymbol{\phi} \in \Lambda[E]),$$

we obtain a system of semi-norms on $\Lambda[E]$ which defines a locally convex topology. We call this topology the topology associated with the topology $\mathfrak{T}$ of Λ and use the notation $\mathscr{T}$ to denote it. In fact, for $\boldsymbol{\phi}_1 \in \Lambda[E], \boldsymbol{\phi}_2 \in \Lambda[E]$, we have

$$\begin{aligned} p_\alpha(\boldsymbol{\phi}_1 + \boldsymbol{\phi}_2) &= p_\alpha(\|\boldsymbol{\phi}_1 + \boldsymbol{\phi}_2\|) \leqslant p_\alpha(\|\boldsymbol{\phi}_1\| + \|\boldsymbol{\phi}_2\|) \\ &\leqslant p_\alpha(\|\boldsymbol{\phi}_1\|) + p_\alpha(\|\boldsymbol{\phi}_2\|) = p_\alpha(\boldsymbol{\phi}_1) + p_\alpha(\boldsymbol{\phi}_2). \end{aligned}$$

Since the normal hull of a weakly bounded subset of Λ^* is weakly bounded and that of a weakly relatively compact subset is weakly relatively compact, it is not difficult to see that the topologies $\mathfrak{T}_s$, $\mathfrak{T}_n$, $\mathfrak{T}_k$ and $\mathfrak{T}_b$ of Λ may be defined by monotonic semi-norms. Thus the associated topologies $\mathscr{T}_s$, $\mathscr{T}_n$, $\mathscr{T}_k$ and $\mathscr{T}_b$ on $\Lambda[E]$ are well defined.

PROPOSITION 1. *$\Lambda \otimes E$ is dense in $\Lambda[E]$ provided with $\mathscr{T}_k$, the topology associated with the Mackey topology of Λ.*

Proof. For the proof we need the following result, which is not difficult to establish (compare, for example, (10)).

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LEMMA 1. *If Λ is a Köthe space and the sequence (f_n) of elements of Λ decreases pointwise to 0 p.p. then $f_n \to 0$ for the Mackey topology $\mathfrak{T}_k(\Lambda, \Lambda^*)$.*

Now let p be a monotonic semi-norm continuous for $\mathfrak{T}_k(\Lambda, \Lambda^*)$ and $\epsilon > 0$ and $\boldsymbol{\phi} \in \Lambda[E]$. Let G_n be an increasing sequence of compact sets of our locally compact space T such that $\boldsymbol{\phi}$ is continuous on each G_n and $\mu\left(T \Big/ \bigcup_1^\infty G_n\right) = 0$ ((2), Chapter IV, p. 169). Then $\|\boldsymbol{\phi} - \boldsymbol{\phi}\chi_{G_n}\| = \|\boldsymbol{\phi}\|\,\chi_{T \setminus G_n}$, where χ_{G_n} is the characteristic function of G_n, decreases pointwise to 0 p.p in Λ. Therefore $\|\boldsymbol{\phi} - \boldsymbol{\phi}\chi_{G_n}\| \to 0$ for $\mathfrak{T}_k(\Lambda, \Lambda^*)$ and there exists N such that

$$p(\boldsymbol{\phi} - \boldsymbol{\phi}\chi_{G_N}) \leqslant \tfrac{1}{2}\epsilon.$$

Because $\boldsymbol{\phi}$ is continuous on the compact set G_N, the set $\{\boldsymbol{\phi}(t) | t \in G_N\}$ is compact. So it is precompact and there is a finite set $e_1, \ldots, e_M$ of members of E such that for every $t \in G_N$ there exists i with

$$\|\boldsymbol{\phi}(t) - e_i\| \leqslant \frac{\epsilon}{2p(\chi_{G_N})}.$$

Put

$$H_i = \left\{t \middle| t \in G_N, \|\boldsymbol{\phi}(t) - e_i\| \leqslant \frac{\epsilon}{2p(\chi_{G_N})}\right\} \Big\setminus \bigcup_{j=1}^{i-1} H_j.$$

Then each H_i is measurable, being the difference of two closed sets; it is contained in G_N and so is relatively compact; the H_i's are mutually disjoint and $\bigcup_1^M H_i = G_N$. So $\sum_1^M \chi_{H_i} \otimes e_i \in \Lambda \otimes E$ and

$$p\left(\boldsymbol{\phi} - \sum_1^M \chi_{H_i} \otimes e_i\right) \leqslant p(\boldsymbol{\phi} - \boldsymbol{\phi}\chi_{G_N}) + p\left(\phi\chi_{G_N} - \sum_1^M \chi_{H_i} \otimes e_i\right)$$

$$\leqslant \tfrac{1}{2}\epsilon + p\left(\frac{\epsilon}{2p(\chi_{G_N})}\chi_{G_N}\right) = \epsilon.$$

(Of course, if $p(\chi_{G_N}) = 0$ then $p(\phi\chi_{G_N}) = 0$ and this case presents no difficulty.)

We recall (see (5), p. 101) that a subset H of Ω is called normal if the relations $f \in H$, $g \in \Omega$ and $|g| \leqslant |f|$ imply $g \in H$. We have

PROPOSITION 2. *Let $\mathfrak{S}$ be a family of normal, weakly bounded subsets of Λ^* such that Λ^* is the union of the members of $\mathfrak{S}$ and on Λ let $\mathfrak{T}$ be the topology of uniform convergence on the sets of $\mathfrak{S}$. Then $\Lambda[E]$ is complete when equipped with $\mathscr{T}$ the topology associated with the topology $\mathfrak{T}$.*

Proof. Since the characteristic function of a compact subset of T belongs to Λ^*, the topology induced on $\Lambda[E]$ by the strong topology of $\Omega[E]$ is coarser than $\mathscr{T}$. Therefore a $\mathscr{T}$-Cauchy directed system $\{\boldsymbol{\phi}_\alpha\}_{\alpha \in \mathfrak{A}}$ in $\Lambda[E]$ is also a strong Cauchy directed system in $\Omega[E]$ and hence converges to $\boldsymbol{\phi}_0(t)$ in $\Omega[E]$. On the other hand, for arbitrary α, $\beta \in \mathfrak{A}$; $\lambda^*(t) \in \Lambda^*$,

$$\int |\lambda^*(t)| \,.\, \|\boldsymbol{\phi}_\alpha(t) - \boldsymbol{\phi}_\beta(t)\|\, d\mu \geqslant \int |\lambda^*(t)|\, |\, \|\boldsymbol{\phi}_\alpha(t)\| - \|\boldsymbol{\phi}_\beta(t)\|\, |\, d\mu.$$

Therefore $\{\|\boldsymbol{\phi}_\alpha(t)\|\}_{\alpha \in \mathfrak{A}}$ is a $\mathfrak{T}$-Cauchy directed system in Λ and we know ((5),

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Theorem 5) that it converges to a function $\lambda_0(t)$ of Λ. Thus, for arbitrarily given $\epsilon > 0$ and for any compact subset K_n of T there exist $\alpha_1, \alpha_2 \in A$ such that

$$\int_{K_n} |\,\|\boldsymbol{\phi}_\alpha(t)\| - \|\boldsymbol{\phi}_0(t)\|\,|\,d\mu \leqslant \int_{K_n} \|\boldsymbol{\phi}_\alpha(t) - \boldsymbol{\phi}_0(t)\|\,d\mu \leqslant \tfrac{1}{2}\epsilon \quad \text{if} \quad \alpha \geqslant \alpha_1,$$

$$\int_{K_n} |\,\|\boldsymbol{\phi}_\alpha(t)\| - \lambda_0(t)|\,d\mu \leqslant \tfrac{1}{2}\epsilon \quad \text{if} \quad \alpha \geqslant \alpha_2.$$

Therefore

$$\int_{K_n} |\,\|\boldsymbol{\phi}_0(t)\| - \lambda_0(t)|\,d\mu \leqslant \epsilon.$$

Hence $\|\phi_0(t)\| = \lambda_0(t)$ p.p. and $\|\boldsymbol{\phi}_0(t)\| \in \Lambda$. Since $\boldsymbol{\phi}_0(t) \in \Omega[E]$, it is measurable and we see that $\boldsymbol{\phi}_0(t) \in \Lambda[E]$.

Now since $\{\boldsymbol{\phi}_\alpha(t)\}_{\alpha \in \mathfrak{A}}$ is a $\mathscr{T}$-Cauchy directed system, for every subset $S \in \mathfrak{S}$ and every $\epsilon > 0$, we can find an α_1 such that

$$\sup_{\lambda^* \in S} \int |\lambda^*(t)|\,\|\boldsymbol{\phi}_\alpha(t) - \boldsymbol{\phi}_\beta(t)\|\,d\mu \leqslant \tfrac{1}{2}\epsilon \quad \text{if} \quad \alpha, \beta \geqslant \alpha_1. \tag{1}$$

Let $\lambda_1^*(t)$ be an arbitrary element of S. We put $d\nu_1 = |\lambda_1^*(t)|\,d\mu$. $\{\boldsymbol{\phi}_\alpha(t)\}_{\alpha \in \mathfrak{A}}$ is a Cauchy directed system in $L_E^1(T, \nu_1)$ and there exists $\boldsymbol{\chi}_0(t)$ such that

$$\lim_\alpha \int |\lambda_1^*(t)|\,\|\boldsymbol{\phi}_\alpha(t) - \boldsymbol{\chi}_0(t)\|\,d\mu = 0.$$

Then

$$\lim_\alpha \int |\lambda_1^*(t)|\,|\,\|\boldsymbol{\phi}_\alpha(t)\| - \|\boldsymbol{\chi}_0(t)\|\,|\,d\mu = 0$$

so that $\|\boldsymbol{\chi}_0(t)\| = \lambda_0(t) = \|\boldsymbol{\phi}_0(t)\|$ p.p. with respect to $d\nu_1$. Therefore we can find an α_2 such that

$$\int |\lambda_1^*(t)|\,\|\boldsymbol{\phi}_\alpha(t) - \boldsymbol{\phi}_0(t)\|\,d\mu \leqslant \tfrac{1}{2}\epsilon \quad \text{if} \quad \alpha \geqslant \alpha_2. \tag{2}$$

From (1) and (2) we deduce that

$$\lim_\alpha \sup_{\lambda^* \in S} \int |\lambda^*(t)|\,\|\boldsymbol{\phi}_\alpha(t) - \boldsymbol{\phi}_0(t)\|\,d\mu = 0,$$

i.e. that $\boldsymbol{\phi}_0(t)$ is the $\mathscr{T}$-limit in $\Lambda[E]$ of $\{\boldsymbol{\phi}_\alpha(t)\}_{\alpha \in \mathfrak{A}}$. The proof is complete.

We now turn our attention to compact subsets of $\Lambda[E]$.

PROPOSITION 3. *Let the Köthe space Λ be provided with the topology $\mathfrak{T}$ of uniform convergence on a family $\mathfrak{S}$ of normal, weakly bounded subsets of Λ^* whose union is Λ^* and let $\Lambda[E]$ be provided with $\mathscr{T}$, the topology associated with the topology $\mathfrak{T}$. Then the three following properties of a subset H of $\Lambda[E]$ are equivalent:*

(i) *H is relatively compact;*

(ii) *H is relatively sequentially compact;*

(iii) *H is relatively countably compact.*

Proof. It is sufficient to show that (i) implies (ii) and (iii) implies (i).

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Proof that (i) *implies* (ii): The topology $\mathscr{T}$ is stronger than the topology induced on $\Lambda[E]$ by the metrizable topology of $\Omega[E]$, hence, if H is relatively compact in $\mathscr{T}$, these topologies coincide on $\bar{H}$, the closure of H. On the other hand, when the topology is metrizable, a subset which is compact is sequentially compact.

Proof that (iii) *implies* (i). Suppose now that H is relatively countably compact with respect to the topology $\mathscr{T}$. H is then precompact ((6), p. 39). Since, by Proposition 2, $\Lambda[E]$ is complete, H is relatively compact.

PROPOSITION 4. *Let $\Lambda[E]$ be topologized as in Proposition* 3. *A subset H of $\Lambda[E]$ is relatively compact if and only if it is relatively compact in the Fréchet space $\Omega[E]$ and every sequence of H which converges in $\Omega[E]$ also does so in $\Lambda[E]$.*

Proof. This Proposition is an immediate consequence of the last one.

3. *The dual system* $\langle\Lambda[E], \Lambda^*[E']\rangle$. By $\Lambda^*[E']$ we denote the space of all E'-valued functions $\psi(t)$ defined on T such that for each $\lambda(t)\in\Lambda$ the function $\lambda(t)\,\psi(t)$ is integrable in the Bochner sense. An E'-valued function $\psi(t)$ belongs to $\Lambda^*[E']$ if and only if it is measurable and $\|\psi(t)\|$ belongs to Λ^*.

PROPOSITION 5. *If $\boldsymbol{\phi}(t)\in\Lambda[E]$ and $\psi(t)\in\Lambda^*[E']$ then $\langle\boldsymbol{\phi}(t),\psi(t)\rangle$ is integrable.*

Proof. Since $\boldsymbol{\phi}(t)$ and $\psi(t)$ are both measurable, $\langle\boldsymbol{\phi}(t),\psi(t)\rangle$ is measurable ((2), Chapter IV, § 5, no. 3). On the other hand, the upper integral of $\langle\boldsymbol{\phi}(t),\psi(t)\rangle$ is finite because $|\langle\boldsymbol{\phi}(t),\psi(t)\rangle| \leqslant \|\boldsymbol{\phi}(t)\|\,.\,\|\psi(t)\|$ and the right-hand side is integrable.

The following proposition strengthens the last one:

PROPOSITION 6. *If $\boldsymbol{\phi}\in\Lambda[E]$, $g\in\Lambda^*$ and $g\geqslant 0$ then*

$$\sup\left\{|\langle\boldsymbol{\phi},\psi\rangle| = \left|\int\langle\boldsymbol{\phi}(t),\psi(t)\rangle\,d\mu\right| : \psi\in\Lambda^*[E'],\quad \|\psi\|\leqslant g\right\} = \int\|\boldsymbol{\phi}\|\,g\,d\mu.$$

Proof. Clearly $\sup\{|\langle\boldsymbol{\phi},\psi\rangle|\}\leqslant\int\|\boldsymbol{\phi}\|\,g\,d\mu$.

Conversely, suppose $c\geqslant 0$, $\epsilon>0$ are such that $c+\epsilon<\int\|\boldsymbol{\phi}\|\,g\,d\mu$. Then there is a function g_1 with $0\leqslant g_1\leqslant g$ of the form $g_1=\sum_1^N\xi_i\chi_{H_i}$, where the H_i are mutually disjoint, compact sets such that $\int\|\boldsymbol{\phi}\|\,g_1\,d\mu>c+\epsilon$. The existence of such a g_1 can be seen for example from (2), Chapter IV, § 4, no. 10, Proposition 19 and its corollaries. Using the arguments of Proposition 1, for each i there is a finite sequence $\{G_i^j\}_{1\leqslant j\leqslant M(i)}$ of mutually disjoint integrable subsets of H_i and a finite sequence $\{e_i^j\}_{1\leqslant j\leqslant M(i)}$ of elements of E such that

$$\int_{H_i}\left\|\boldsymbol{\phi}-\sum_{j=1}^{M(i)}\chi_{G_i^j}\otimes e_i^j\right\|d\mu\leqslant\frac{\epsilon}{2N\xi_i}.$$

Let $f_i^j\in E'$ be such that $\|f_i^j\|=1$ and $\langle f_i^j,e_i^j\rangle=\|e_i^j\|$. Set

$$\psi=\sum_{i=1}^N\xi_i\sum_{j=1}^{M(i)}\chi_{G_i^j}\otimes f_i^j.$$

Then $\psi\in\Lambda^*[E']$ and

$$\|\psi\|=\sum_{i=1}^N\xi_i\sum_{j=1}^{M(i)}\chi_{G_i^j}\leqslant\sum_{i=1}^N\xi_i\chi_{H_i}=g_1\leqslant g.$$

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Moreover
$$|\langle \phi, \psi \rangle| = \left| \int \langle \phi(t), \psi(t) \rangle \, d\mu \right| = \left| \sum_{i=i}^{N} \int_{H_i} \langle \phi(t), \psi(t) \rangle \, d\mu \right|$$
$$\geqslant \left| \sum_{i=1}^{N} \int_{H_i} \sum_{j=1}^{M(i)} \chi_{G_i^j}(t) \langle e_i^j, \psi(t) \rangle \, d\mu \right|$$
$$- \left| \sum_{i=1}^{N} \int_{H_i} \left\| \phi - \sum_{j=1}^{M(i)} \chi_{G_i^j}(t) \otimes e_i^j \right\| . \|\psi(t)\| \, d\mu \right.$$
$$\geqslant \sum_{i=1}^{N} \sum_{j=1}^{M(i)} \int_{G_i^j} \xi_i \|e_i^j\| \, d\mu - \sum_{i=1}^{N} \xi_i \frac{\epsilon}{2N\xi_i}$$
$$\geqslant \left| \sum_{i=1}^{N} \xi_i \int_{H_i} \left\| \sum_{j=1}^{M(i)} \chi_{G_i^j} \otimes e_i^j \right\| d\mu \right| - \tfrac{1}{2}\epsilon$$
$$\geqslant \sum_{i=1}^{N} \xi_i \int_{H_i} \|\phi(t)\| \, d\mu - \sum_{i=1}^{N} \xi_i \frac{\epsilon}{2N\xi_i} - \tfrac{1}{2}\epsilon$$
$$= \int \|\phi(t)\| \, g_1 \, d\mu - \epsilon > c.$$

Since c is arbitrary subject to $c < \int \|\phi(t)\| \, g \, d\mu$,

$$\sup \{ |\langle \phi, \psi \rangle| \} = \int \|\phi(t)\| \, g \, d\mu.$$

PROPOSITION 7. *The duality between $\Lambda[E]$ and $\Lambda^*[E']$ given by the canonical bilinear form*

$$\langle \phi, \psi \rangle = \int \langle \phi(t), \psi(t) \rangle \, d\mu \, (\phi \in \Lambda[E], \psi \in \Lambda^*[E'])$$

is separating.

Proof. This is a direct consequence of Proposition 6.

We define the normal hull H of a subset H of $\Lambda[E]$ to be the set of all functions of the form $\alpha(t)\phi(t)$ where $\phi(t) \in H$ and $\alpha(t)$ is a measurable function with $|\alpha(t)| \leqslant 1$ p.p.

PROPOSITION 8. *The normal hull $\tilde{H}$ of a $\mathfrak{T}_s(\Lambda[E], \Lambda^*[E'])$-bounded subset H of $\Lambda[E]$ is bounded in the same topology.*

Proof. By hypothesis, for every $\psi \in \Lambda^*[E']$, we have

$$\sup_{\phi \in H} \left| \int \langle \phi(t), \psi(t) \rangle \, d\mu \right| < \infty.$$

If $g(t)$ is an arbitrary function of $L^\infty(T, \mu)$ then $g(t)\psi(t) \in \Lambda^*[E']$ because $g(t)\psi(t)$ is measurable and $|g(t)| . \|\psi(t)\|$ belongs to Λ^*. Therefore the set

$$\{ f(t) \,|\, f(t) = \langle \phi(t), \psi(t) \rangle, \phi \in H \}$$

is a bounded subset of $L^1(T, \mu)$, that is

$$\sup_{\phi \in H} \int |\langle \phi(t), \psi(t) \rangle| \, d\mu < \infty.$$

Then
$$\sup_{\phi \in \tilde{H}} \int |\langle \phi(t), \psi(t) \rangle| \, d\mu < \infty.$$

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We call the topology of $\Lambda[E]$ determined by the system of semi-norms

$$p_\psi(\boldsymbol{\phi}) = \int |\langle \boldsymbol{\phi}(t), \psi(t)\rangle|\, d\mu (\boldsymbol{\phi} \in \Lambda[E], \psi \in \Lambda^*[E'])$$

the normal topology and use the symbol $\mathfrak{T}_n(\Lambda[E], \Lambda^*[E'])$ (or simply $\mathfrak{T}_n$) to denote it. The proof of the last result shows that in $\Lambda[E]$ the normal hull of a weakly bounded subset is even bounded in the normal topology. Actually a subset of $\Lambda[E]$ is weakly bounded if and only if it is bounded in the normal topology. We prove

PROPOSITION 9. *The normal topology* $\mathfrak{T}_n(\Lambda[E], \Lambda^*[E'])$ *of* $\Lambda[E]$ *is admissible for the dual system* $\langle \Lambda[E], \Lambda^*[E']\rangle$.

Proof. It is sufficient to show that the normal hull of any element ψ of $\Lambda^*[E']$, that is the set $\{\alpha(t)\psi(t) | \alpha(t) \in L^\infty, \|\alpha\|_\infty \leqslant 1\}$ is weakly compact. Let $\mathscr{F} = \{F_\beta\}$ be a filter on the normal hull of ψ. Let $\mathscr{G} = \{G_\beta\}$ where $G_\beta = \{\alpha(t): \alpha \in L^\infty, \alpha\psi = F_\beta\}$. Then $\{G_\beta\}$ is a filter basis on the unit ball of L^∞ and hence has an adherent point $\alpha_0(t)$. $\alpha_0(t)\psi(t)$ is an adherent point of $\{F_\beta\}$. In fact, since $\langle \boldsymbol{\phi}(t), \psi(t)\rangle \in L^1$ for every $\boldsymbol{\phi}(t) \in \Lambda[E]$, for $\epsilon > 0$ arbitrarily given, we can find in each G_β an $\alpha(t)$ such that

$$\left| \int (\alpha(t) - \alpha_0(t)) \langle \boldsymbol{\phi}(t), \psi(t)\rangle\, d\mu \right| < \epsilon,$$

$$\left| \int \langle \alpha(t)\psi(t) - \alpha_0(t)\psi(t), \boldsymbol{\phi}(t)\rangle\, d\mu \right| < \epsilon.$$

PROPOSITION 10. *If* E' *is separable then the topology* $\mathscr{T}_n$ *of* $\Lambda[E]$ *is admissible for the dual system* $\langle \Lambda[E], \Lambda^*[E']\rangle$.

Proof. If L is a linear functional on $\Lambda[E]$ continuous for $\mathscr{T}_n$ then there exists a $g \in \Lambda^*$, $g \geqslant 0$ such that $|L(\boldsymbol{\phi})| \leqslant \int \|\boldsymbol{\phi}(t)\|\, g(t)\, d\mu$ for every $\boldsymbol{\phi} \in \Lambda[E]$. Let F be the linear subspace of $L^1[E]$ consisting of elements of the form $\boldsymbol{\phi} g$, where $\boldsymbol{\phi} \in \Lambda[E]$ and on F define the linear functional L_1 by $L_1(\boldsymbol{\phi} g) = L(\boldsymbol{\phi})$. Then L_1 is well defined and on F has norm $\leqslant 1$, so, by the Hahn–Banach theorem, can be extended to a linear functional on $L^1[E]$ of norm $\leqslant 1$. Since, when E' is separable, the dual of $L^1[E]$ is $L^\infty[E']$, there is a $\psi \in L^\infty[E']$ such that $L_1(\boldsymbol{\phi} g) = \int \langle \boldsymbol{\phi}(t)\, g(t), \psi(t)\rangle\, d\mu$ for every $\boldsymbol{\phi} \in \Lambda[E]$. So

$$L(\boldsymbol{\phi}) = \int \langle \boldsymbol{\phi}(t), g(t)\, \psi(t)\rangle\, d\mu$$

and L is given by $g\psi \in \Lambda^*[E']$. It is obvious that the topology $\mathscr{T}_n$ is finer than the weak topology $\mathfrak{T}_s(\Lambda[E], \Lambda[E'])$.

COROLLARY. *If* E' *is separable then the space* $\Lambda[E]$ *is complete under the Mackey topology* $\mathfrak{T}_k(\Lambda[E], \Lambda^*[E'])$ *and the strong topology* $\mathfrak{T}_b(\Lambda[E], \Lambda^*[E'])$.

If the Banach space E' is separable, then $L^\infty[E']$ is the dual of $L^1[E]$ ((2), Chapter VI). In the sequel we shall assume that E is both separable and reflexive, so that the similarity between $\Lambda[E]$ and $\Lambda^*[E']$ is more complete.

PROPOSITION 11. *If* E *is separable and reflexive then* $L^\infty[E']$ *can be characterized as the space of all* E'*-valued functions* $\mathbf{g}(t)$ *such that for every* $\mathbf{f}(t) \in L^1[E]$, *the function* $\langle \mathbf{f}(t), \mathbf{g}(t)\rangle$ *is integrable.*

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Proof. The proof is essentially the same as the proof of a similar result in (1), p. 85. We prove that if $\mathbf{g}(t)$ is such that $\langle \mathbf{f}(t), \mathbf{g}(t)\rangle$ is integrable for every $\mathbf{f}(t) \in L^1[E]$ then $g(t) \in L^\infty[E']$. Put

$$\mathbf{g}_n(t) = \begin{cases} \mathbf{g}(t) & \text{if} \quad \|\mathbf{g}(t)\| \leqslant n, \\ n \dfrac{\mathbf{g}(t)}{\|\mathbf{g}(t)\|} & \text{if} \quad \|\mathbf{g}(t)\| \geqslant n. \end{cases}$$

Then $$|\langle \mathbf{f}(t), \mathbf{g}_n(t)\rangle \, d\mu| \leqslant n \int \|\mathbf{f}(t)\| \, d\mu \; (\mathbf{f} \in L^1[E]).$$

Thus $\langle \mathbf{f}(t), \mathbf{g}_n(t)\rangle$ tends to

$$\langle \mathbf{f}(t), \mathbf{g}(t)\rangle \quad \text{and} \quad |\langle \mathbf{f}(t), \mathbf{g}_n(t)\rangle| \leqslant |\langle \mathbf{f}(t), \mathbf{g}(t)\rangle|$$

which is integrable, by Lebesgue's theorem $\int \langle \mathbf{f}(t), \mathbf{g}_n(t)\rangle \, d\mu$ tends to $\int \langle \mathbf{f}(t), \mathbf{g}(t)\rangle \, d\mu$. Therefore $\int \langle \mathbf{f}(t), \mathbf{g}(t)\rangle \, d\mu$ is a continuous linear form on $L^1[E]$ and there exists a function $\mathbf{g}^*(t) \in L^\infty[E']$ such that

$$\int \langle \mathbf{f}(t), \mathbf{g}(t)\rangle \, d\mu = \int \langle \mathbf{f}(t), \mathbf{g}^*(t)\rangle \, d\mu (\mathbf{f}(t) \in L^1[E]).$$

Taking $\mathbf{f}$ to be of the form $\xi_{K_n}(t)\mathbf{e}$ where ξ_{K_n} is the characteristic function of the compact set K_n, letting $\mathbf{e}$ run through a countable dense subset of E we see that $\mathbf{g}(t) = \mathbf{g}^*(t)$ p.p. on K_n. Hence $\mathbf{g}(t) = \mathbf{g}^*(t)$ p.p. and $\mathbf{g}(t) \in L^\infty[E']$.

Similarly, if E' is separable then $\Phi[E']$ is the dual of $\Omega[E]$ and $\Phi[E']$ can be characterized as the space of all E'-valued functions $\mathbf{g}(t)$ such that, for each $\mathbf{f}(t) \in \Omega[E]$, the function $\langle \mathbf{f}(t), \mathbf{g}(t)\rangle$ is integrable.

PROPOSITION 12. *If E is separable and reflexive then $\Lambda^*[E']$ can be characterized as the space $(\Lambda[E])^\times$ of all E'-valued functions $\psi(t)$ such that for each $\boldsymbol{\phi}(t) \in \Lambda[E]$ the function $\langle \boldsymbol{\phi}(t), \psi(t)\rangle$ is integrable.*

Proof. We shall prove that $\psi \in L^1[E']$ if and only if $\langle \boldsymbol{\phi}(t), \psi(t)\rangle$ is integrable for each $\boldsymbol{\phi} \in L^\infty[E]$. The proof for the pair $\Lambda[E]$, $\Lambda^*[E']$ is similar. The 'only if' part is obvious. So suppose that $\langle \boldsymbol{\phi}(t), \psi(t)\rangle$ is integrable for each $\boldsymbol{\phi} \in L^\infty[E]$. Then for each $e \in E$, the function $\langle e, \psi(t)\rangle$ is integrable and hence measurable. Consequently, $\psi(t)$ is measurable in the norm topology of E'. We first show that ψ generates a continuous functional on $L^\infty[E]$. We recall that $T = \bigcup_{n=1}^{n=\infty} K_n$ where each K_n is compact and the sequence K_n is increasing. Put

$$\psi^n(t) = \psi(t) \quad \text{if} \quad t \in K_n \quad \text{and} \quad \|\psi(t)\| < n$$

$$= n \frac{\psi(t)}{\|\psi(t)\|} \quad \text{if} \quad t \in K_n \quad \text{and} \quad \|\psi(t)\| \geqslant n$$

$$= 0 \quad \text{if} \quad t \notin K_n.$$

Then $$\left| \int \langle \mathbf{f}(t), \psi^n(t)\rangle \, d\mu \right| \leqslant \|\mathbf{f}\|_\infty \int \|\psi^n(t)\| \, d\mu, \quad \mathbf{f} \in L^\infty[E].$$

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So that ψ^n generates a continuous functional on $L^\infty[E]$. But $\psi^n(t) \to \psi(t)$ everywhere and $|\langle \mathbf{f}(t), \psi^n(t)\rangle| \leqslant |\langle \mathbf{f}(t), \psi(t)\rangle|$, therefore by the dominated convergence theorem

$$\int \langle \mathbf{f}(t), \psi^n(t)\rangle \, d\mu \to \int \langle \mathbf{f}(t), \psi(t)\rangle \, d\mu$$

and by the Banach–Steinhaus theorem ((6), p. 173), $\mathbf{f} \to \int \langle \mathbf{f}(t), \psi(t)\rangle \, d\mu$ is a continuous linear functional L_ψ on $L^\infty[E]$. Next, let $e_1, e_2, \ldots, e_n, \ldots$ be a sequence dense in the unit ball of E. Put

$$h_n(t) = \sup_{1 \leqslant i \leqslant n} |\langle \psi(t), e_i\rangle|,$$

$$M_1 = \{t: h_n(t) = |\langle \psi(t), e_1\rangle|\},$$

$$M_2 = \{t: h_n(t) = |\langle \psi(t), e_2\rangle|, t \notin M_1\},$$

$$M_i = \left\{t: h_n(t) = |\langle \psi(t), e_i\rangle|, t \notin \bigcup_{j=1}^{i-1} M_j\right\}$$

and

$$\mathbf{g}^n(t) = \sum_{i=1}^{n} \chi_{M_i} e_i \operatorname{sign} \langle \psi(t), e_i\rangle\dagger.$$

Then $\|\mathbf{g}^n(t)\|_\infty = 1$ and

$$\int \langle \mathbf{g}^n(t), \psi(t)\rangle \, d\mu = \int h_n(t) \, d\mu \leqslant \|L_\psi\| \cdot \|\mathbf{g}^n\|_\infty = \|L_\psi\|.$$

Then by Fatou's lemma, $\liminf h_n(t) = \lim h_n(t) = \|\psi(t)\|$ is integrable.

Thus we see that if E is separable and reflexive, then the formal similarity between the pairs $\langle \Lambda[E], \Lambda^*[E']\rangle$ and $\langle \Lambda, \Lambda^*\rangle$ is complete. Therefore it is natural to ask oneself whether the first pair has the same interesting topological properties as the second one. Among other things, we want to know whether the normal hull of a $\mathfrak{T}_s(\Lambda[E], \Lambda^*[E'])$-relatively compact subset of $\Lambda[E]$ is relatively compact in the same topology. We feel that this is true but we have not been able to prove it. We note that the results of (3) cannot be applied to the pair $\langle \Lambda[E], \Lambda^*[E']\rangle$ to give a positive answer to our question because the example given at the bottom of p. 313, (3) is not correct if the spaces E_i are of infinite dimensions. On the other hand, it seems to us that (4) deals with a more restricted class of spaces because of condition C 3 (ii), p. 357. We prove a few results which might help in proving the conjecture mentioned above.

PROPOSITION 13. *If E' is separable then in $\Lambda[E]$ the notions of relatively weak compactness, relatively countably weak compactness and relatively sequentially weak compactness are equivalent.*

Proof. Since $\mathscr{T}_n$ is admissible (Proposition 10), the topology induced on $\Lambda[E]$ by the strong topology of $\Omega[E]$ is weaker than the Mackey topology of $\Lambda[E]$. Therefore relatively sequentially weak compactness and relatively countably weak compactness are equivalent (Smulian's theorem (6), p. 314). On the other hand, since $\Lambda[E]$ is complete under its Mackey topology (Corollary of Proposition 10), relatively countably weak compactness and relatively weak compactness are equivalent (Eberlein's theorem (6), p. 316).

† Sign $\langle \psi(t), e_i\rangle$ is defined to be $\exp(-i\theta)$ with θ = argument $\langle \psi(t), e_i\rangle$.

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PROPOSITION 14. *Let E' be separable and $h: E \to \Lambda^*$ be a linear mapping such that for every e in the unit ball of E, $|h(e)| \leqslant g_0$ where $0 \leqslant g_0 \in \Lambda^*$. Then there is a $\psi \in \Lambda^*[E']$ such that for each $e \in E$, $h(e)(t) = \langle e, \psi(t) \rangle$ p.p.*

Proof. Let $\{e_n\}_{1 \leqslant n}$ be a dense subset of E closed under addition and under multiplication by rationals. Let $\{g_n(t)\}$ be actual realizations of $h(e_n)$ such that $|g_n(t)| \leqslant \|e_n\| g_0(t)$ for every n and every t and let T_0 be the set of t for which there exist rationals α, β and integers n, m with $\alpha e_n + \beta e_m = 0$ but $\alpha g_n(t) + \beta g_m(t) \neq 0$. Then T_0 is the union of countably many negligible sets and is therefore negligible itself. Then for $t \notin T_0$ the mapping $e_n \to g_n(t)$ extends to a linear mapping on E which is continuous, since $|g_n(t)| \leqslant \|e_n\| g_0(t)$. Therefore there exists $\psi(t) \in E'$ such that $g_n(t) = \langle e_n, \psi(t) \rangle$ for all n. Observe that $\|\psi(t)\| \leqslant g_0(t)$. If for $t \in T_0$ we set $\psi(t) = 0$, then $g_n(t) = \langle e_n, \psi(t) \rangle$ for all n and almost every t.

Now, ψ is measurable, because for every $e \in E$ there exists a sequence of integers (n_k) with $e_{n_k} \to e$ in E, so

$$\langle e, \psi(t) \rangle = \lim_k \langle e_{n_k}, \psi(t) \rangle = \lim_k g_{n_k}(t) \quad (t \notin T_0)$$

which is measurable since each g_{n_k} is measurable. So, because E' is separable ψ is measurable. So $\psi \in \Lambda^*[E']$. Also

$$\begin{aligned} h(e)(t) &= \lim_k h(e_{n_k})(t) \text{ p.p. } (t) \\ &= \lim_k g_{n_k}(t) = \lim_k \langle e_{n_k}, \psi(t) \rangle \text{ p.p. } (t) \\ &= \langle e, \psi(t) \rangle \text{ p.p. } (t). \end{aligned}$$

So $h(e)(t) = \langle e, \psi(t) \rangle$ as required.

PROPOSITION 15. *If E' is separable a set C of $\Lambda[E]$ is bounded for $\mathfrak{T}_s(\Lambda[E], \Lambda^*[E'])$ if and only if the set $A = \{\|\phi\|, \phi \in C\}$ is bounded for $\mathfrak{T}_s(\Lambda, \Lambda^*)$.*

Proof. If E' is separable the dual of $\Lambda[E]$ equipped with $\mathscr{T}_n$ is $\Lambda^*[E']$, so C is $\mathfrak{T}_s(\Lambda[E], \Lambda^*[E'])$-bounded if and only if it is bounded for $\mathscr{T}_n$; i.e. if and only if A is bounded for $\mathfrak{T}_n(\Lambda, \Lambda^*)$. But $\mathfrak{T}_n(\Lambda, \Lambda^*)$ is admissible for the dual pair $\langle \Lambda, \Lambda^* \rangle$ so A is bounded for this topology if and only if it is bounded for $\mathfrak{T}_s(\Lambda, \Lambda^*)$.

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REFERENCES

(1) BANACH, S. *Théorie des opérations linéaires*. Monografje Matematyczne, Tome 1 (Warsaw, 1932).

(2) BOURBAKI, N. *Eléments de Mathématique*. Livre IV. *Intégration*. Actualités Sci. Ind. 1175 (Paris, 1952 or 1965; Chapters I–IV), 1244 (Paris, 1956; Chapter V), 1281 (Paris, 1958; Chapter VI).

(3) COOPER, J. L. B. Coordinated linear spaces. *Proc. London Math. Soc.* (3), **3** (1953), 305–327.

(4) COOPER, J. L. B. On a generalization of the Köthe coordinated spaces. *Math. Ann.* **162** (1966), 351–363.

https://doi.org/10.1017/S030500410000339X Published online by Cambridge University Press

(5) Dieudonné, J. Sur les espaces de Köthe. *J. Analyse Math.* **1** (1951), 81–115.
(6) Köthe, G. *Topologische lineare Räume* 1. Springer-Verlag (Berlin, 1960).
(7) Pietsch, A. *Verallgemeinerte vollkommene Folgenräume.* Sch. Forschungsinst. Math., Heft 12 (Berlin, 1962).
(8) Pietsch, A. Verallgemeinerte vollkommene Räume. *Studia Math.* **1** (1963), 89–91.
(9) Schwartz, L. Espaces de fonctions différentiables à valeurs vectorielles. *J. Analyse Math.* **4** (1955), 88–148.
(10) Welland, R. On Köthe spaces. *Trans. Amer. Math. Soc.* **112** (1964), 267–277.
(11) Nguyen Phuong Các. On Dieudonné's paper Sur les espaces de Köthe. *Proc. Cambridge Philos. Soc.* **62** (1966), 29–32.

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