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Numerical Solution of Stochastic Ito-Volterra Integral Equations using Haar Wavelets

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Abstract. This paper presents a computational method for solving stochastic Ito-Volterra integral equations. First, Haar wavelets and their properties are employed to derive a general procedure for forming the stochastic operational matrix of Haar wavelets. Then, application of this stochastic operational matrix for solving stochastic Ito-Volterra integral equations is explained. The convergence and error analysis of the proposed method are investigated. Finally, the efficiency of the presented method is confirmed by some examples.

AMS subject classifications: 65T60; 60H20; 45L05

Key words: Ito integral, Stochastic Ito-Volterra integral equations, Stochastic operational matrix, Haar wavelets, Error analysis.

1. Introduction

Stochastic functional equations (SFEs) are becoming increasingly important due to their application for modelling stochastic phenomena in different fields, e.g. biology, chemistry, epidemiology, mechanics, microelectronics, economics, and finance. The behavior of dynamical systems in these fields are often dependent on a noise source and a Gaussian white noise, governed by certain probability laws, so that modeling such phenomena naturally requires the use of various stochastic differential equations or, in more complicated cases, stochastic Volterra integral equations and stochastic integro-differential equations [1–5].

Similar to the difficulty of the deterministic functional equations, we have difficulty in finding the solution of stochastic functional equations. As analytic solutions of these equations are not available in many cases, numerical approximation becomes a practical way to face this difficulty. In previous works various numerical methods have been used for approximate the solution of SFEs. Here we only mention Kloeden and Platen [1], Oksendal [2], Maleknejad *et al.* [3,4], Cortes *et al.* [5,6], Murge *et al.* [7], Khodabin *et al.* [8,9], Zhang [10,11], Jankovic [12] and Heydari *et al.* [13].

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Recently, different orthogonal basis functions, such as block pulse functions, Walsh functions, Fourier series, orthogonal polynomials and wavelets, were used to estimate solutions of functional equations. As a powerful tool, wavelets have been extensively used in signal processing, numerical analysis, and many other areas. Wavelets permit the accurate representation of a variety of functions and operators [14–17]. Haar wavelets have been widely applied in system analysis, system identification, optimal control and numerical solution of integral and differential equations [18, 19]. In this paper we consider the following stochastic Ito-Volterra integral equation

$$X(t) = f(t) + \int_0^t k_1(s,t)X(s)ds + \int_0^t k_2(s,t)X(s)dB(s), \quad t \in [0,T), \tag{1.1}$$

where $X(t)$, $f(t)$, $k_1(s,t)$ and $k_2(s,t)$, for $s,t \in [0,T)$, are the stochastic processes defined on the same probability space (Ω, F, P), and $X(t)$ is unknown. Also $B(t)$ is a Brownian motion process and $\int_0^t k_1(s,t)X(s)dB(s)$ is the Ito integral [2, 20]. In order to solving this stochastic Ito-Volterra integral equation we first derive the Haar wavelets stochastic integration operational matrix. Then the stochastic operational matrix for Haar wavelets along with Haar wavelets basis are used to derive a numerical solution. Convergence and error analysis of the proposed method are also investigated. Numerical results show efficiency of the proposed method.

This paper is organized as follows: In Section 2 some basic properties of the Haar wavelets are described. In Section 3 stochastic operational matrix for Haar wavelets and a general procedure for deriving this matrix are introduced. In Section 4 a computational method based on the stochastic operational matrix of Haar wavelets are proposed for solving stochastic Ito-Volterra integral equations. In Section 5 Convergence and error analysis of the proposed method are also investigated. Some numerical examples are presented in Section 6. Finally, a conclusion is given in Section 7.

2. Haar wavelets and Block pulse functions

In this section we describe some basic properties of the Haar wavelets. For this purpose we first introduce the Block pulse functions (BPFs), function approximation by BPFs and their operational matrices. Then the relations between Haar wavelets and BPFs are investigated. Finally, we derive some important formulas for Haar wavelets that are useful for the next sections.

2.1. Block pulse functions

BPFs have been studied by many authors and applied for solving different problems. In this section we recall definition and some properties of the BPFs [3, 4, 21, 22].

The m-set of BPFs are defined as

$$b_i(t) = \begin{cases} 1 & (i-1)h \le t < ih, \\ 0 & \text{otherwise}, \end{cases} \tag{2.1}$$

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in which $t \in [0,T)$, $i = 1,2,...,m$ and $h = \frac{T}{m}$. The set of BPFs are disjoint with each other in the interval $[0,T)$ and

$$b_i(t)b_j(t) = \delta_{ij}b_i(t), \quad i,j = 1,...,m, \tag{2.2}$$

where δ_{ij} is the Kronecker delta. The set of BPFs defined in the interval $[0,T)$ are orthogonal with each other, that is

$$\int_0^T b_i(t)b_j(t)dt = h\delta_{ij}, \quad i,j = 1,...,m. \tag{2.3}$$

If $m \to \infty$ the union of BPF sets $\{b_i(t)\}_{i=1}^{\infty}$ form a complete basis for $L^2[0,T)$, so an arbitrary real bounded function $f(t)$, which is square integrable in the interval $[0,T)$, can be expanded into a Block pulse series as

$$f(t) \simeq \sum_{i=1}^{m} f_i b_i(t), \tag{2.4}$$

where

$$f_i = \frac{1}{h}\int_0^T b_i(t)f(t)dt, \quad i = 1,...,m. \tag{2.5}$$

Rewriting Eq. (2.4) in the vector form we have

$$f(t) \simeq \sum_{i=1}^{m} f_i b_i(t) = F^T\Phi(t) = \Phi^T(t)F, \tag{2.6}$$

in which

$$\Phi(t) = [b_1(t), b_2(t), ..., b_m(t)]^T, \quad F = [f_1, f_2, ..., f_m]^T. \tag{2.7}$$

Morever, any two dimensional function $k(s,t) \in L^2([0,T_1] \times [0,T_2])$ can be expanded with respect to BPFs such as

$$k(s,t) = \Phi^T(t)K\Phi(t), \tag{2.8}$$

where $\Phi(t)$ is the m-dimensional BPFs vectors respectively, and K is the $m \times m$ BPFs coefficient matrix with (i,j)-th element

$$k_{ij} = \frac{1}{h_1h_2}\int_0^{T_1}\int_0^{T_2} k(s,t)b_i(t)b_j(s)dtds, \quad i,j = 1,...,m, \tag{2.9}$$

and $h_1 = T_1/m$ and $h_2 = T_2/m$. Let $\Phi(t)$ be the BPFs vector, then we have

$$\Phi^T(t)\Phi(t) = 1, \tag{2.10}$$

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and

$$\Phi(t)\Phi^T(t) = \begin{pmatrix} b_1(t) & 0 & \dots & 0 \\ 0 & b_2(t) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b_m(t) \end{pmatrix}_{m\times m}. \tag{2.11}$$

For an m-vector F we have

$$\Phi(t)\Phi^T(t)F = \tilde{F}\Phi(t), \tag{2.12}$$

where $\tilde{F}$ is an $m \times m$ matrix, and $\tilde{F} = diag(F)$. Also, it is easy to show that for an $m \times m$ matrix A

$$\Phi^T(t)A\Phi(t) = \tilde{A}^T\Phi(t), \tag{2.13}$$

where $\tilde{A} = diag(A)$ is a m-vector.

2.2. Haar wavelets

The orthogonal set of of Haar wavelets $h_n(t)$ consists a set of square waves deffined as follows [14, 18, 19]

$$h_n(t) = 2^{\frac{j}{2}}\psi\left(2^j t - k\right), \quad j \geq 0, \quad 0 \leq k < 2^j, \quad n = 2^j + k, \quad n, j, k \in \mathbb{Z}, \tag{2.14}$$

where

$$h_0(t) = 1, \quad 0 \leq t < 1, \quad \psi(t) = \begin{cases} 1 & 0 \leq t < \frac{1}{2}, \\ -1 & \frac{1}{2} \leq t < 1, \end{cases} \tag{2.15}$$

Each Haar wavelets $h_n(t)$ has the support $\left[\frac{k}{2^j}, \frac{k+1}{2^j}\right)$, so that it is zero elsewhere in the interval $[0, 1)$. The Haar wavelets $h_n(t)$ are pairwise orthonormal in the interval $[0, 1)$ and

$$\int_0^1 h_i(t)h_j(t)dt = \delta_{ij}, \tag{2.16}$$

where δ_{ij} is the Kronecker delta. Any square integrable function $f(t)$ in the interval $[0, 1)$ can be expanded in term of Haar wavelets as

$$f(t) = c_0 h_0(t) + \sum_{i=1}^{\infty} c_i h_i(t), \quad i = 2^j + k, \quad j \geq 0, \quad 0 \leq k < 2^j, \quad j, k \in \mathbb{N}, \tag{2.17}$$

where c_i is given by

$$c_i = \int_0^1 f(t)h_i(t)dt, \quad i = 0, 2^j + k, \quad j \geq 0, \quad 0 \leq k < 2^j, \quad j, k \in \mathbb{N}. \tag{2.18}$$

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The infinite series in Eq. (2.17) can be truncated after $m = 2^J$ terms (J is level of wavelet resolution), that is

$$f(t) \simeq c_0 h_0(t) + \sum_{i=1}^{m-1} c_i h_i(t), \quad i = 2^j + k, \quad 0 \leq j \leq J-1, \quad 0 \leq k < 2^j, \tag{2.19}$$

rewritting this equation in the vector form we have,

$$f(t) \simeq C^T H(t) = H(t)^T C, \tag{2.20}$$

in which C and $H(t)$ are Haar coefficients and wavelets vectors as

$$C = [c_0, c_1, ..., c_{m-1}]^T, \quad H(t) = [h_0(t), h_1(t), ..., h_{m-1}(t)]^T. \tag{2.21}$$

Any two dimensional function $k(s,t) \in L^2([0,1) \times [0,1))$ can be expanded with respect to Haar wavelets as

$$k(s,t) = H^T(t) K H(t), \tag{2.22}$$

where $H(t)$ is the Haar wavelets vector and K is the $m \times m$ Haar wavelets coefficients matrix with (i,l)-th element can be obtained as

$$k_{il} = \int_0^1 \int_0^1 k(s,t) H_i(t) H_l(s) dt ds, \quad i,l = 1, ..., m. \tag{2.23}$$

2.3. Relation between the BPFs and Haar wavelets

In this section we will derive the relation between the BPFs and Haar wavelets. It is worth mention that in this section we set $T = 1$ in definition of BPFs.

Theorem 2.1. *Let $H(x)$ and $\Phi(x)$ be the m-dimensional Haar wavelets and BPFs vector respectively, the vector $H(x)$ can be expanded by BPFs vector $\Phi(x)$ as*

$$H(t) = Q\Phi(t), \quad m = 2^J, \tag{2.24}$$

where Q is an $m \times m$ matrix and

$$Q_{il} = 2^{\frac{l}{2}} h_{i-1}\left(\frac{2l-1}{2m}\right), \quad i,l = 1, ...m, \quad i-1 = 2^j + k, \quad 0 \leq k < 2^j. \tag{2.25}$$

Proof. Let $H_i(t), i = 1, 2, ..., m$ be the i-th element of Haar wavelets vector. Expanding $H_i(t)$ into an m-term vector of BPFs, we have

$$H_i(t) = \sum_{l=1}^{m} Q_{il} b_l(t) = Q_i^T B(t), \quad i = 1, ..., m, \tag{2.26}$$

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where Q_i is the i-th row and Q_{il} is the (i,l)-th element of matrix Q. By using the orthogonality of BPFs we have

$$Q_{il} = \frac{1}{h}\int_0^1 H_i(t)b_l(t)dt = \frac{1}{h}\int_{\frac{l-1}{m}}^{\frac{l}{m}} H_i(t)dt = 2^{\frac{j}{2}}m\int_{\frac{l-1}{m}}^{\frac{l}{m}} h_{i-1}(t)dt, \tag{2.27}$$

by using mean value theorem for integrals in the last equation we can write

$$Q_{ij} = 2^{\frac{j}{2}}m\left(\frac{l}{m}-\frac{l-1}{m}\right)h_{i-1}(\eta_l) = 2^{\frac{j}{2}}h_{i-1}(\eta_l), \quad \eta_l \in \left(\frac{l-1}{m},\frac{l}{m}\right), \tag{2.28}$$

As $h_{i-1}(t)$ is a constant function on the interval $(\frac{l-1}{m},\frac{l}{m})$, we can choose $\eta_l = \frac{2l-1}{2m}$ so we have

$$Q_{il} = 2^{\frac{j}{2}}h_{i-1}\left(\frac{2l-1}{2m}\right), \quad i,l = 1,...m. \tag{2.29}$$

and this prove the desired result. □

For an example the matrix $Q_{8\times 8}$ has the following form

$$Q_{8\times 8} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix}. \tag{2.30}$$

Remark 2.1. According to the definition of matrix Q in (2.24) it is easy to see that

$$Q^{-1} = \frac{1}{m}Q^T. \tag{2.31}$$

The following Remark is the consequence of relations (2.12), (2.13) and Theorem 2.1.

Remark 2.2. For an m-vector F we have

$$H(t)H^T(t)F = \tilde{F}H(t), \tag{2.32}$$

in which $\tilde{F}$ is an $m\times m$ matrix as

$$\tilde{F} = Q\bar{F}Q^{-1}, \tag{2.33}$$

where $\bar{F} = diag\left(Q^{-1}F\right)$. Moreover, it can be easy to show that for an $m\times m$ matrix A

$$H^T(t)AH(t) = \tilde{A}^TH(t), \tag{2.34}$$

where $\tilde{A}^T = UQ^{-1}$ and $U = diag(Q^TAQ)$ is an m-vector.

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3. Stochastic operational matrix of Haar wavelets

In this section we obtain the stochastic integration operational matrix for Haar wavelets. For this purpose we remind some useful results for BPFs [3, 4].

Lemma 3.1. ([3]) *Let $\Phi(t)$ be the BPFs vector defined in* (2.7). *Then integration of this vector can be derived as*

$$\int_0^t \Phi(s)ds \simeq P\Phi(t), \tag{3.1}$$

where $P_{m\times m}$ is called the operational matrix of integration for BPFs and is given by

$$P = \frac{h}{2}\begin{bmatrix} 1 & 2 & 2 & \dots & 2 \\ 0 & 1 & 2 & \dots & 2 \\ 0 & 0 & 1 & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 2 \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}_{m\times m}. \tag{3.2}$$

Lemma 3.2. ([3]) *Let $\Phi(t)$ be the BPFs vector defined in* (2.7). *Then the Ito integral of this vector can be derived as*

$$\int_0^t \Phi(s)dB(s) \simeq P_s\Phi(t), \tag{3.3}$$

where P_s is called the stochastic operational matrix of integration for BPFs and is given by

$$P_s = \begin{bmatrix} B\left(\frac{h}{2}\right) & B(h) & B(h) & \dots & B(h) \\ 0 & B\left(\frac{3h}{2}\right) - B(h) & B(2h) - B(h) & \dots & B(2h) - B(h) \\ 0 & 0 & B\left(\frac{5h}{2}\right) - B(2h) & \dots & B(3h) - B(2h) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & B\left(\frac{(2m-1)h}{2}\right) - B((m-1)h) \end{bmatrix}_{m\times m} \tag{3.4}$$

Now we are ready to derive a new operational matrix of stochastic integration for the Haar wavelets basis. For this end we use BPFs and the matrix Q introduced in (2.24).

Theorem 3.1. *Suppose $H(t)$ be the Haar wavelets vector defined in* (2.21)*, the integral of this vector can be derived as*

$$\int_0^t H(s)ds \simeq \frac{1}{m}QPQ^TH(t) = \Lambda H(t), \tag{3.5}$$

where Q is introduced in (2.24) *and P is the operational matrix of integration for BPFs derived in* (3.2).

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Proof. Let $H(t)$ be the Haar wavelets vector, by using Theorem 2.1 and Lemma 3.1 we have

$$\int_0^t H(s)ds \simeq \int_0^t Q\Phi(s)ds = Q\int_0^t \Phi(s)ds = QP\Phi(t), \tag{3.6}$$

now, Theorem 2.1 and Remark 2.1 give

$$\int_0^t H(s)ds \simeq QP\Phi(t) = \frac{1}{m}QPQ^T H(t) = \Lambda H(t), \tag{3.7}$$

and this complete the proof. □

Theorem 3.2. *Suppose $H(t)$ be the Haar wavelets vector defined in* (2.21)*, the Ito integral of this vector can be derived as*

$$\int_0^t H(s)dB(s) \simeq \frac{1}{m}QP_sQ^T H(t) = \Lambda_s H(t), \tag{3.8}$$

where Λ_s is called stochastic operational matrix for Haar wavelets, Q is introduced in (2.24) *and P_s is the stochastic operational matrix of integration for BPFs derived in* (3.4)*.*

Proof. Let $H(t)$ be the Haar wavelets vector, by using Theorem 2.1 and Lemma 3.2 we have

$$\int_0^t H(s)dB(s) \simeq \int_0^t Q\Phi(s)dB(s) = Q\int_0^t \Phi(s)dB(s) = QP_s\Phi(t), \tag{3.9}$$

now, Theorem 2.1 and Remark 2.1 result

$$\int_0^t H(s)dB(s) = QP_s\Phi(t) = \frac{1}{m}QP_sQ^T H(t) = \Lambda_s H(t), \tag{3.10}$$

and this complete the proof. □

4. Description of the proposed method

In this section, we apply the stochastic operational matrix of Haar wavelets for solving stochastic Ito-Volterra integral equations. Consider the following stochastic Ito-Volterra integral equations as

$$X(t) = f(t) + \int_0^t k_1(s,t)X(s)ds + \int_0^t k_1(s,t)X(s)dB(s), \quad t \in [0,T), \tag{4.1}$$

where $X(t)$, $f(t)$, $k_1(s,t)$ and $k_2(s,t)$, for $s,t \in [0,T)$, are the stochastic processes defined on the same probability space (Ω, F, P), and $X(t)$ is unknown. Also $B(t)$ is a Brownian motion process and $\int_0^t k_1(s,t)X(s)dB(s)$ is the Ito integral. For solving this

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problem by using the stochastic operational matrix of Haar wavelets, we approximate $X(t), f(t), k_1(s,t)$ and $k_2(s,t)$ in terms of Haar wavelets as follows

$$f(t) \simeq F^T H(t) = F H^T(t), \tag{4.2}$$
$$X(t) \simeq X^T H(t) = X H^T(t), \tag{4.3}$$
$$k_1(s,t) \simeq H^T(s) K_1 H(t) = H^T(t) K_1^T H(s), \tag{4.4}$$
$$k_2(s,t) \simeq H^T(s) K_2 H(t) = H^T(t) K_2^T H(s), \tag{4.5}$$

where X and F are Haar wavelets coefficients vector, K_1 and K_2 are Haar wavelets coefficients matrices defined in Eqs. (2.21) and (2.22). Substituting above approximations in Eq. (4.1), we have

$$\begin{aligned} X^T H(t) \simeq F^T H(t) + H^T(t) K_1 \left(\int_0^t H(s) H^T(s) X ds \right) \\ + H^T(t) K_2 \left(\int_0^t H(s) H^T(s) X dB(s) \right), \end{aligned} \tag{4.6}$$

using Remark 2.2 we get

$$X^T H(t) \simeq F^T H(t) + H^T(t) K_1 \left(\int_0^t \tilde{X} H(s) ds \right) + H^T(t) K_2 \left(\int_0^t \tilde{X} H(s) dB(s) \right),$$

where $\tilde{X}$ is a linear function of vector X. Now applying the operational matrices Λ and Λ_s for Haar wavelets derived in Eqs. (3.5) and (3.8) we have

$$X^T H(t) \simeq F^T H(t) + H^T(t) K_1 \tilde{X} \Lambda H(t) + H^T(t) K_2 \tilde{X} \Lambda_s H(t), \tag{4.7}$$

by setting $X_1 = K_1 \tilde{X} \Lambda, X_2 = K_2 \tilde{X} \Lambda_s$ and using Remark 2.2 we derive

$$X^T H(t) - \tilde{X}_1 H(t) - \tilde{X}_2 H(t) \simeq F^T H(t), \tag{4.8}$$

in which where $\tilde{X}_1$ and $\tilde{X}_2$ are linear function of vectors X_1 and X_2. This equation is hold for all $t \in [0,1)$, so we can write

$$X^T - \tilde{X}_1 - \tilde{X}_2 \simeq F^T, \tag{4.9}$$

since $\tilde{X}_1$ and $\tilde{X}_1$ is linear function of X, Eq. (4.9) is a linear system of equations for unknown vector X. After solving this linear system and determining X, we can approximate solution of stochastic Volterra integral equation (4.1) by substituting the obtained vector X in Eq. (4.3).

5. Error analysis

In this section we consider the convergence properties of the proposed wavelet method for solving the stochastic Ito-Voltera integral equations (4.1).

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Theorem 5.1. *Suppose that* $f(t) \in L^2[0,1)$ *with bounded first derivative, i.e.,* $|f'(t)| \leq M$. *Let* $e_m(t) = f(t) - \sum_{i=0}^{m-1} f_i h_i(t)$. *Then*

$$\|e_m(t)\|_2 \leq \frac{M}{\sqrt{3}m}, \tag{5.1}$$

that is, the Haar wavelets series is convergent.

Proof. By definition of error $e_m(t)$ we have

$$\|e_m(t)\|_2^2 = \int_0^1 \left(\sum_{i=m}^{\infty} f_i h_i(t) \right)^2 dt = \sum_{i=m}^{\infty} f_i^2, \tag{5.2}$$

where $i = 2^j + k,\ m = 2^J,\ J > 0$ and

$$f_i = \int_0^1 h_i(t) f(t) dt = 2^{\frac{j}{2}} \left(\int_{k2^{-j}}^{(k+\frac{1}{2})2^{-j}} f(t)dt - \int_{(k+\frac{1}{2})2^{-j}}^{(k+1)2^{-j}} f(t)dt \right),$$

by the mean value theorem for integrals there are $\eta_{j1} \in \left[k2^{-j}, \left(k+\frac{1}{2}\right)2^{-j}\right]$ and $\eta_{j2} \in \left[\left(k+\frac{1}{2}\right)2^{-j}, (k+1)2^{-j}\right]$ such that

$$\begin{aligned} f_i = \int_0^1 h_i(t) f(t) dt &= 2^{\frac{j}{2}} \left(f(\eta_{j1}) \int_{k2^{-j}}^{(k+\frac{1}{2})2^{-j}} dt - f(\eta_{j2}) \int_{(k+\frac{1}{2})2^{-j}}^{(k+1)2^{-j}} dt \right) \\ &= 2^{-\frac{j}{2}-1} \Big(f(\eta_{j1}) - f(\eta_{j2}) \Big) \\ &= 2^{-\frac{j}{2}-1} \left(\eta_{j1} - \eta_{j2} \right) f'(\eta_j), \quad \eta_{j2} < \eta_j < \eta_{j2}, \end{aligned} \tag{5.3}$$

which yields

$$\begin{aligned} \|e_m(t)\|_2^2 = \sum_{i=m}^{\infty} f_i^2 &= \sum_{i=m}^{\infty} 2^{-j-2} \left(\eta_{j1} - \eta_{j2} \right)^2 \Big(f'(\eta_j) \Big)^2 \\ &\leq \sum_{i=m}^{\infty} 2^{-j-2} 2^{-2j} M^2 = \sum_{i=m}^{\infty} 2^{-3j-2} M^2 \\ &= \sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} 2^{-3j-2} M^2 = \frac{M^2}{3m^2}. \end{aligned} \tag{5.4}$$

The desired estimate (5.1) follows directly from (5.4). □

Theorem 5.2. *Suppose that* $f(s,t) \in L^2([0,1) \times [0,1))$ *with bounded partial derivatives derivative,* $\left| \frac{\partial^2 f}{\partial s \partial t} \right| \leq M$, *and let*

$$e_m(s,t) = f(s,t) - \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} f_{ij} h_i(s) h_j(t).$$

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Then

$$\|e_m(s,t)\|_2 \leq \frac{M}{3m^2}. \tag{5.5}$$

Proof. By definition of error $e_m(s,t)$ we have

$$\|e_m(s,t)\|_2^2 = \int_0^1 \left(\sum_{i=m}^{\infty} \sum_{l=m}^{\infty} f_{il} h_i(s) h_l(t) \right)^2 dt = \sum_{i=m}^{\infty} \sum_{l=m}^{\infty} f_{il}^2, \tag{5.6}$$

where $i = 2^j + k,\ l = 2^{j'} + k,\ m = 2^J, J > 0$ and

$$f_{ij} = \int_0^1 \int_0^1 h_i(s) h_l(t) f(s,t) ds dt.$$

Similar to the proof of Theorem 5.1, there are $\eta_j, \eta_{j1},\ \eta_{j2}, \eta_{j'},\ \eta_{j_1'}$ and $\eta_{j_2'}$ such that

$$\begin{aligned} f_{ij} &= \int_0^1 h_i(s) \left(\int_0^1 h_l(t) f(s,t) dt \right) ds \\ &= \int_0^1 h_i(s) \left[2^{-\frac{j'}{2}-1} \left(\eta_{j_1'} - \eta_{j_2'} \right) \frac{\delta f(s,j')}{\delta t} \right] ds \\ &= 2^{-\frac{j}{2}-\frac{j'}{2}-2} \left(\eta_{j_1'} - \eta_{j_2'} \right) \left(\eta_{j_1} - \eta_{j_2} \right) \frac{\partial^2 f(\eta_j, \eta_{j'})}{\partial t \partial s}, \end{aligned} \tag{5.7}$$

Consequently, using (5.6) gives

$$\begin{aligned} \|e_m(s,t)\|_2^2 &= \sum_{i=m}^{\infty} \sum_{l=m}^{\infty} 2^{-j-j'-4} \left(\eta_{j_1'} - \eta_{j_2'} \right)^2 (\eta_{j_1} - \eta_{j_2})^2 \left| \frac{\partial^2 f\left(\eta_j, \eta_{j'}\right)}{\partial s \partial t} \right|^2 \\ &\leq \sum_{i=m}^{\infty} \sum_{l=m}^{\infty} M^2 2^{-3j-3j'-4} = M^2 \sum_{i=m}^{\infty} 2^{-3j-2} \sum_{l=m}^{\infty} 2^{-3j'-2} = \frac{M^2}{(3m^2)^2}, \end{aligned}$$

which yields (5.5). □

Theorem 5.3. *Suppose $X(t)$ is the exact solution of* (1.1) *and $X_m(t)$ is its Haar wavelets approximate solution whose coefficients are obtained by* (2.18). *Also assume that*

$$a)\ \|X(t)\| \leq \rho, \qquad t \in [0,1]\,; \tag{5.8}$$

$$b)\ \|k_1(s,t)\| \leq M_1, \qquad s,t \in [0,1] \times [0,1]\,; \tag{5.9}$$

$$c)\ \|k_2(s,t)\| \leq M_2, \qquad s,t \in [0,1] \times [0,1]\,; \tag{5.10}$$

$$d)\ (M_1 + \Gamma_{1m}) + |B(t)|\,(M_2 + \Gamma_{2m}) < 1. \tag{5.11}$$

Then

$$\|X(t) - X_m(t)\| \leq \frac{\Upsilon_m + \rho\Gamma_{1m} + |B(t)|\,\rho\Gamma_{2m}}{\left(1 - [(M_1 + \Gamma_{1m}) + |B(t)|\,(M_2 + \Gamma_{2m})]\right)}, \tag{5.12}$$

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where

$$\Upsilon_m = \sup_{t\in[0,1]} \frac{|f'(t)|}{\sqrt{3}\,m}, \qquad \Gamma_{im} = \frac{1}{3m^2} \sup_{s,t\in[0,1]} \left|\frac{\partial^2 k_i(s,t)}{\partial s \partial t}\right|, \quad i = 1, 2. \tag{5.13}$$

Proof. From (1.1) we have

$$X(t) - X_m(t) = f(t) - f_m(t) + \int_0^t (k_1(s,t)X(s) - k_{1m}(s,t)X_m(s))\, ds + \int_0^t (k_2(s,t)X(s) - k_{2m}(s,t)X_m(s))\, dB(s), \tag{5.14}$$

so, by the mean value theorem, we can write

$$\|X(t) - X_m(t)\| \le \|f(t) - f_m(t)\| + t\,\|(k_1(s,t)X(s) - k_{1m}(s,t)X_m(s))\| + |B(t)|\,\|(k_2(s,t)X(s) - k_{2m}(s,t)X_m(s))\|. \tag{5.15}$$

Now by using Theorems 5.1 and 5.2 we have

$$\begin{aligned} &\|(k_j(s,t)X(s) - k_{jm}(s,t)X_m(s))\| \\ &\le \|k_j(s,t)\|\,\|X(t) - X_m(t)\| + \|(k_j(s,t) - k_{jm}(s,t))\|\,\|X(t)\| \\ &\quad + \|(k_j(s,t) - k_{jm}(s,t))\|\,\|X(t) - X_m(t)\| \\ &\le (M_j + \Gamma_{jm})\,\|X(t) - X_m(t)\| + \rho\Gamma_{jm}, \quad j = 1, 2. \end{aligned} \tag{5.16}$$

Substituting (5.16) into (5.15) gives

$$\|X(t) - X_m(t)\| \le \Upsilon_m + t\Big((M_1 + \Gamma_{1m})\,\|X(t) - X_n(t)\| + \rho\Gamma_{1m}\Big) + |B(t)|\,\Big((M_2 + \Gamma_{2m})\,\|X(t) - X_m(t)\| + \rho\Gamma_{2m}\Big). \tag{5.17}$$

Using Assumption (d) yields (5.12). □

6. Numerical examples

In this section, we consider some numerical examples to illustrate the efficiency and reliability of the Haar wavelets operational matrices in solving stochastic Ito-Volterra integral equation.

Example 6.1. Consider the following stochastic Ito-Volterra integral equation

$$X(t) = 1 + \int_0^t s^2 X(s) ds + \int_0^t sX(s) dB(s), \quad s, t \in [0, 1],$$

where $X(t)$ is an unknown stochastic process defined on the probability space (Ω, F, P), and $B(t)$ is a Brownian motion process. The exact solution of this stochastic Ito-Volterra integral equation is

$$X(t) = \exp\left(\frac{t^3}{6} + \int_0^t sdB(s)\right).$$

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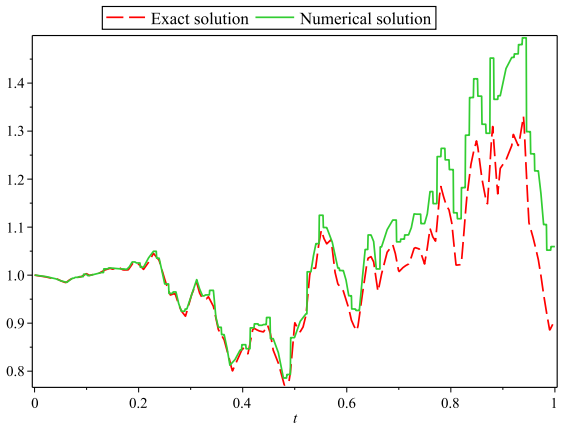


Figure 1: The exact and approximate solution for $m = 2^8$.

Table 1: The mean square error of numerical results for different values of m.

t	$m = 2^4$	$m = 2^5$	$m = 2^6$	$m = 2^7$
0.1	0.00584517	0.00291620	0.00548314	0.00017926
0.3	0.03623059	0.01266214	0.00064642	0.00437771
0.5	0.02501065	0.02793773	0.02173427	0.01950540
0.7	0.11121400	0.11338947	0.08232844	0.06034106
0.9	0.13708159	0.13834241	0.10531861	0.16808202

This stochastic Ito-Volterra integral equation is solved by using the Haar wavelets stochastic operational matrix and the proposed method in Section 4 for different values of $m = 2^J$. For $m = 2^8$ the approximate solution computed by the presented method and exact solution of are represented in Fig. 1. The mean square error of the numerical results are shown in the Table 1 for different values of m. The numerical results reveal the accuracy and efficiency of the proposed method.

Example 6.2. Let us consider the following stochastic Volterra integral equation

$$X(t) = \frac{1}{12} + \int_0^t \cos(s)X(s)ds + \int_0^t \sin(s)X(s)dB(s), \quad s,t \in [0,1],$$

where $X(t)$ is an unknown stochastic process defined on the probability space (Ω, F, P), and $B(t)$ is a Brownian motion process. The exact solution of this stochastic Volterra integral equation is

$$X(t) = \frac{1}{12}\exp\left(\frac{-t}{4} + \sin(t) + \frac{\sin(2t)}{8} + \int_0^t \sin(s)dB(s)\right).$$

This stochastic Volterra integral equation is solved by using the Haar wavelets stochastic operational matrix and the proposed method in Section 4 for different values of $m = 2^J$. In Fig. 2 the approximate solution computed by the presented method and

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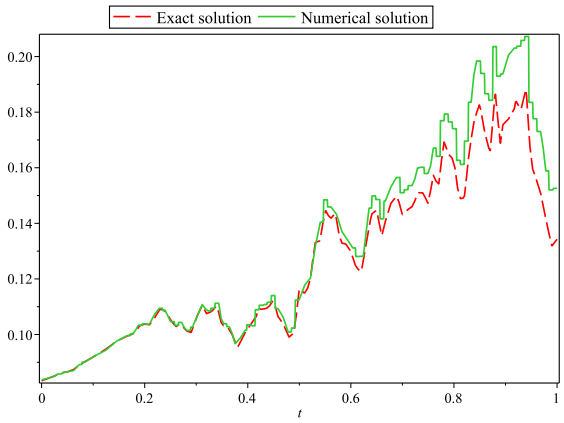


Figure 2: The exact and approximate solution for $m = 2^8$.

Table 2: The mean square error of numerical results for different values of m.

t	$m = 2^4$	$m = 2^5$	$m = 2^6$	$m = 2^7$
0.1	0.00053004	0.00027469	0.00050676	0.00001588
0.3	0.00395271	0.00138373	0.00008404	0.00047759
0.5	0.00338270	0.00351333	0.00272917	0.00240353
0.7	0.01475822	0.01485328	0.01068856	0.00776023
0.9	0.01655444	0.01681289	0.01260596	0.02018859

exact solution are shown for $m = 2^8$. The mean square error of the numerical results for different values of m are shown in the Table 2. The results of the proposed method show that the proposed method is efficient for solving this stochastic Ito-Volterra integral equation.

Example 6.3. Consider the following stochastic Ito-Volterra integral equation

$$X(t) = \frac{1}{3} + \int_0^t \ln(s+1)X(s)ds + \int_0^t \sqrt{\ln(s+1)}X(s)dB(s), \quad s,t \in [0,1],$$

where $X(t)$ is an unknown stochastic process defined on the probability space (Ω, F, P), and $B(t)$ is a Brownian motion process. The exact solution of this stochastic Volterra integral equation is

$$X(t) = \frac{1}{3}\exp\left(\frac{-t}{2} + \frac{1}{2}t\ln(t+1) + \frac{1}{2}\ln(t+1) + \int_0^t \sqrt{\ln(s+1)}dB(s)\right).$$

This stochastic Ito-Volterra integral equation is solved by using the Haar wavelets stochastic operational matrix and the proposed method in Section 4 for different values of $m = 2^J$. For $m = 2^8$ the approximate solution computed by the presented method and exact solution of are represented in Fig. 3. The mean square error of the numerical results are shown in the Table 3 for different values of m. The obtained results reveal the accuracy and efficiency of the proposed method.

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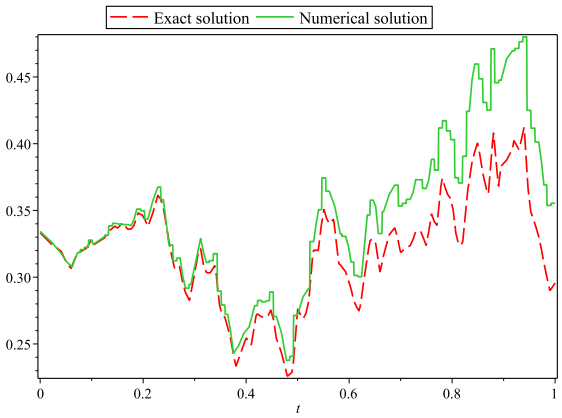


Figure 3: The exact and approximate solution for $m = 2^8$.

Table 3: The mean square error of numerical results for different values of m.

t	$m = 2^4$	$m = 2^5$	$m = 2^6$	$m = 2^7$
0.1	0.00102441	0.00004900	0.00536218	0.00027883
0.3	0.03197400	0.01382562	0.00394493	0.00584073
0.5	0.02523497	0.02250541	0.01612351	0.01509745
0.7	0.06074940	0.05837467	0.04217635	0.03371450
0.9	0.07133546	0.06910521	0.05259680	0.06922094

7. Conclusion

A new stochastic operational matrix for Haar wavelets is derived. The BPFs and their relations to Haar wavelets are employed to derive this stochastic operational matrix. By using this stochastic operational matrix an efficient computational method is proposed for solving stochastic Ito-Volterra integral equations. The convergence and error analysis of the proposed method are considerd. Efficiency of this method and good reasonable degree of accuracy is confirmed by some numerical examples.

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