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On sign-representable matroids

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Abstract

A matroid M will be called sign-representable if, for every basis B of M, there is a $(0, 1, -1)$-matrix $[I_r | Y]$ representing M over $\mathbb{Q}$ in which the first r columns correspond to the members of B. The class of sign-representable matroids, which is closely related to the important class of regular matroids, is easily seen to be closed under both duality and the taking of minors. This paper proves several characterizations of the class, including a constructive one, and shows that the excluded minors for the class are $U_{2,5}$, $U_{3,5}$, the Fano matroid and its dual, and the rank-3 whirl.

1. *Introduction*

A regular matroid is one that is representable over all fields. Such matroids are among the most frequently studied (see, for example, [**1**, **4**, **6**, **7**, **10**, **12**, **13**]) and there are numerous alternative characterizations of these matroids, many of which are summarized in [**15**], chapter 3 and [**8**], chapter 13. In particular, a matroid is regular if and only if it is representable over both GF(2) and some field of characteristic other than two. Since the class of binary (GF(2)-representable) matroids is also well understood (see, for example, [**8**], chapter 9), this means that if $\mathscr{F}$ is a set of fields, then the class of matroids that are representable over every member of $\mathscr{F}$ has been characterized for all $\mathscr{F}$ containing GF(2). By contrast, we know of just one such characterization when GF(2) is not in $\mathscr{F}$, that of ternary (GF(3)-representable) matroids [**2**, **9**]. Thus one natural place to begin a search for other such results is among those sets $\mathscr{F}$ that contain GF(3). This search provides the underlying motivation for this paper.

One quickly sees that the case when GF(3) is the smallest member of $\mathscr{F}$ is not as well-behaved as the case when GF(2) is in $\mathscr{F}$. In the latter case, every choice of $\mathscr{F}$ yields one of only two classes of matroids: binary matroids if $\mathscr{F}$ contains only fields of characteristic two, and regular matroids otherwise. But, for example, the ternary affine plane, AG(2, 3), is $\mathbb{F}$-representable if and only if $\mathbb{F}$ contains a root of the equation $x^2 - x + 1 = 0$. Hence AG(2, 3) is GF(q)-representable if and only if q is a power of 3, q is a square, or $q \equiv 1 (\mathrm{mod}\ 3)$ (see [**5**], p. 100). Thus AG(2, 3) is both

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representable over infinitely many prime fields and non-representable over infinitely many prime fields.

Three natural classes of matroids to consider here arise when we make the following choices for $\mathscr{F}$: (i) all fields other than GF(2); (ii) all fields of characteristic other than two; and (iii) {GF(3), $\mathbb{Q}$}. The non-Fano matroid is in the second and third classes but not the first, so the first class is properly contained in both the second and third. But we know of no examples to show that the second and third classes are different. Moreover, a well-known conjecture of uncertain origin (see, for example, [**8**], conjecture $14 \cdot 1 \cdot 11$) is that the second class coincides with the class of matroids that are representable over all finite prime fields of odd order.

The class of sign-representable matroids that will be the focus of this paper arose in our attempts to characterize the third class noted above, the ternary matroids that are also representable over the rationals.

The matroid terminology used here will follow Oxley [**8**]. Let M be a matroid having ground set $\{e_1, e_2, \ldots, e_n\}$ and suppose that $\{e_1, e_2, \ldots, e_r\}$ is a basis B of M where $r \geqslant 1$. Let D be the $(0,1)$-matrix whose rows are labelled by $e_1, e_2, \ldots, e_r$, whose columns are labelled by $e_{r+1}, e_{r+2}, \ldots, e_n$, and whose (i,j)-entry is 1 if and only if $e_i \in C(e_{r+j}, B)$, the fundamental circuit of e_{r+j} with respect to B. The matrix D is called the B-fundamental-circuit incidence matrix of M, while $[I_r | D]$, the columns of which are labelled by $e_1, e_2, \ldots, e_n$, is called the partial representation [**11**] of M corresponding to the basis B. It is well known (see [**6**], proposition 2·2) that M is representable over a field $\mathbb{F}$ if and only if there is an assignment of non-zero elements of $\mathbb{F}$ to those of D to give a matrix D_1 such that $[I_r | D_1]$ represents M over $\mathbb{F}$. In particular, M is binary if and only if $[I_r | D]$ represents M over GF(2).

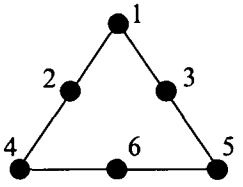
A signing of a $(0,1)$-matrix D is a $(0, 1, -1)$-matrix D' that is obtained from D by changing the signs of some (possibly empty) set of entries. Such a matrix D' is totally unimodular if, when viewed over $\mathbb{Q}$, every square submatrix of D' has determinant in $\{0, 1, -1\}$. It is well known that a matroid M of rank at least one is regular if and only if, for every partial representation $[I_r | X]$ of M, there is a totally unimodular signing X' of X such that $[I_r | X']$ represents M over $\mathbb{Q}$.

In this paper, we shall study a generalization of the class of regular matroids. In particular, we define a matroid M to be sign-representable if either M has rank zero, or M has rank at least one and, for every partial representation $[I_r | X]$ of M, there is a signing X' of X such that $[I_r | X']$ represents M over $\mathbb{Q}$.

It is well known (see, for example, [**8**], theorem 2·2·8) that if $[I_r | Y]$ represents an n-element matroid M over a field $\mathbb{F}$, then $[-Y^T | I_{n-r}]$ represents M^* over $\mathbb{F}$. Using this, one easily checks that the class of sign-representable matroids is closed under duality. Since the class is clearly closed under deletion, it follows that the class is also closed under contraction. Hence the class of sign-representable matroids is closed under the taking of minors. Moreover, this class clearly contains the class of regular matroids.

Regular matroids can also be characterized as those having some totally unimodular matrix representation. In view of this, the reader may wonder why we do not define a matroid M to be 'sign-representable' if there is some $(0, 1, -1)$-matrix $[I_r | Y]$ that represents M over $\mathbb{Q}$, or even if there is some $(0, 1, -1)$-matrix A, not necessarily in standard form, that represents M over $\mathbb{Q}$. The disadvantage of each of these alternative definitions is that the class of matroids satisfying the given

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(*a*)

	4	5	6
1	1	1	1
2	1	0	1
3	0	1	1

(*b*)

Fig. 1.

condition is not minor-closed. To see this, take a $(0, 1, -1)$-matrix $[I_3 | Y]$ that represents PG(2, 3) over GF(3) and let M be the 13-point rank-3 matroid for which this matrix is a $\mathbb{Q}$-representation. It is not difficult to check that M has a contraction that has $U_{2,5}$ as a restriction. But no simple rank-2 matroid that can be represented by a $(0, 1, -1)$-matrix has more than four points.

The next two theorems, the main results of this paper, are characterizations of the class of sign-representable matroids. The first of these results characterizes the class both constructively and by excluded minors. The constructive characterization relies on the fact that if $\mathscr{M}$ is a minor-closed class of matroids, then all the members of $\mathscr{M}$ can be obtained from the 3-connected members of the class by repeatedly using the operations of direct sum and 2-sum (see [**10**], (2·6)). The latter operation is closely related to the operation of series connection, which is discussed in detail in [**3**] and, for example, in [**8**], section 7·1. Let M_1 and M_2 be matroids with $E(M_1) \cap E(M_2) = \{p\}$. Then $S(M_1, M_2)$ denotes the series connection of M_1 and M_2 with respect to the basepoint p. If M_1 and M_2 both have at least three elements and neither M_1 nor M_2 has p as a loop or coloop, then $M_1 \oplus_2 M_2$, the 2-sum of M_1 and M_2, is $S(M_1, M_2)/p$.

The matroids F_7 and $\mathscr{W}^3$, which appear in the following theorem, are, respectively, the Fano matroid and the rank-3 whirl [**14**]. A geometric representation for the second of these matroids is shown in Figure 1 (*a*) above.

(1·1) THEOREM. *The following statements are equivalent for a matroid M.*

(i) *M is sign-representable.*

(ii) *M has no minor isomorphic to any of the matroids $U_{2,5}$, $U_{3,5}$, F_7, F_7^*, or $\mathscr{W}^3$.*

(iii) *M can be obtained from regular matroids and copies of $U_{2,4}$ by a sequence of direct sums and 2-sums.*

(iv) *M is ternary having no minor isomorphic to $\mathscr{W}^3$.*

The class of sign-representable matroids can also be characterized in terms of fields other than $\mathbb{Q}$.

(1·2) THEOREM. *Let $\mathbb{F}$ be a field of characteristic other than two or three. The following statements are equivalent for a matroid M.*

(i) *M is sign-representable.*

(ii) *M has rank zero or, for every partial representation $[I_r | X]$ of M, there is a signing X' of X such that $[I_r | X']$ is an $\mathbb{F}$-representation of M.*

(iii) *M has rank zero or, for every partial representation $[I_r | X]$ of M, there is a signing X' of X such that $[I_r | X']$ represents M over all fields of characteristic other than two.*

We note that if $\mathbb{F}$ has characteristic three, then all ternary matroids satisfy (ii) above but not (i). Moreover, if $\mathbb{F}$ has characteristic two, then, for example, $U_{2,4}$

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satisfies (i) but not (ii). Thus the condition that $\mathbb{F}$ does not have characteristic two or three is needed in the last result.

The proofs of these theorems will be given in Section 2. The following basic fact about series connection (see [3], proposition 4·10) will be important in these proofs.

(1·3) LEMMA. *Let p be an element of a connected matroid M. If $M \setminus p = N_1 \oplus N_2$ where both N_1 and N_2 are non-empty, then $M = S(M/E(N_1), M/E(N_2))$.*

A well-known matrix operation that we shall use in the next section is that of pivoting. Let $[I_r \mid Y]$ be a matrix over a field $\mathbb{F}$ where $Y = [y_{ij}]$. If y_{st} is a non-zero entry of Y, then a pivot on y_{st} in $[I_r \mid Y]$ first transforms column $r+t$ of $[I_r \mid Y]$ into the sth unit vector by (i) multiplying row s of $[I_r \mid Y]$ by $1/y_{st}$, and then (ii) adding appropriate multiples of row s to the other rows. Finally, to give a matrix in the form $[I_r \mid D]$, one interchanges columns s and $r+t$ of the matrix obtained from (ii).

2. *The proofs of the theorems*

We shall prove both of the theorems at the same time in the following:

Proof. The equivalence of (ii) and (iv) of (1·1) is an immediate consequence of the excluded-minor characterization of ternary matroids (see [**2**, **9**]). Therefore both theorems will be verified if we can establish that, in the following cyclic order, each of the four statements implies its successor: (1·2)(iii), (1·2)(ii), (1·1)(ii), (1·1)(iii). This is because proving these four implications will imply the equivalence of (1·1)(ii), (1·1)(iii), (1·2)(ii), and (1·2)(iii) for all allowable choices of the field $\mathbb{F}$ in (1·2)(ii). By taking $\mathbb{F}$ equal to $\mathbb{Q}$, (1·2)(ii) becomes (1·1)(i), and so (1·1)(i) is equivalent to the other four statements.

The fact that (1·2)(iii) implies (1·2)(ii) is immediate. Now suppose that M satisfies (1·2)(ii). Clearly the class of matroids satisfying (1·2)(ii) is closed under both duality and the taking of minors. Thus M has no F_7- or F_7^*-minor since each of F_7 and F_7^* is representable only over fields of characteristic two. Moreover, as noted earlier, there are at most four elements in a rank-two simple matroid that can be represented by a $(0, 1, -1)$-matrix. Thus M has no $U_{2,5}$-minor and, by duality, M has no $U_{3,5}$-minor. Finally, consider $\mathscr{W}^3$. With its elements labelled as in Figure 1 (*a*), $\mathscr{W}^3$ has the partial representation $[I_3 \mid X]$ where X is as shown in Figure 1 (*b*). If X has a signing X' so that $[I_3 \mid X']$ represents $\mathscr{W}^3$ over $\mathbb{F}$, then we may assume that

$$X' = \begin{bmatrix} 1 & 1 & 1 \\ a & 0 & 1 \\ 0 & b & 1 \end{bmatrix},$$

where $a, b \in \{1, -1\}$. But $\{4,5,6\}$ is a circuit of $\mathscr{W}^3$, and so $\det X' = 0$. Thus $a = b/(b-1)$. Since $b \in \{1, -1\}$, we must have $b = -1$. So $a = \frac{1}{2}$. Again, as $a \in \{1, -1\}$, we deduce that $-1 = \frac{1}{2}$, that is, $-2 = 1$. Hence $\mathbb{F}$ has characteristic three; a contradiction. We conclude that M has no $\mathscr{W}^3$-minor. Thus (1·2)(ii) implies (1·1)(ii).

Next suppose that M satisfies (1·1)(ii). We shall show that M satisfies (1·1)(iii). Suppose first that M is 3-connected. As M has no $U_{2,5}$-, $U_{3,5}$-, F_7-, or F_7^*-minor, M is ternary. If M is also binary, then M is regular. Hence we may assume that M is non-binary. Then M has a $U_{2,4}$-minor. But M has no $U_{2,5}$-, $U_{3,5}$-, or $\mathscr{W}^3$-minor. Thus, by Seymour's Splitter Theorem [**10**], M must be isomorphic to $U_{2,4}$. We conclude that the

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only 3-connected matroids satisfying (1·1)(ii) are regular matroids and $U_{2,4}$. But those matroids that satisfy (1·1)(ii) and are not 3-connected can be built up from 3-connected matroids satisfying (1·1)(ii) by direct sums and 2-sums. Hence all matroids satisfying (1·1)(ii) also satisfy (1·1)(iii).

Finally, we need to prove that (1·1)(iii) implies (1·2)(iii). The proof of this implication occupies the rest of the paper. Suppose that M satisfies (1·1)(iii). Then we may assume that M has rank at least one. If M is regular, then, for every partial representation $[I_r|X]$ of M, there is a totally unimodular signing X' of X. Then $[I_r|X']$ represents M over all fields and so (1·2)(iii) holds. If $M \cong U_{2,4}$, then $[I_2|\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}]$ is the only partial representation of M. Since $[I_2|\begin{smallmatrix}1 & 1\\ -1 & 1\end{smallmatrix}]$ represents $U_{2,4}$ over all fields of characteristic other than two, we again conclude that (1·2)(iii) holds.

Now let M_1 and M_2 be matroids on disjoint ground sets. Then, up to a permutation of the columns and a permutation of the rows, every partial representation of $M_1 \oplus M_2$ has the form

$$\begin{bmatrix} I_{r_1} & 0 & X_1 & 0 \\ 0 & I_{r_2} & 0 & X_2 \end{bmatrix},$$

where $[I_{r_1}|X_1]$ and $[I_{r_2}|X_2]$ are partial representations of M_1 and M_2, respectively. It follows without difficulty that if M_1 and M_2 satisfy (1·2)(iii), then so does $M_1 \oplus M_2$.

To complete the proof that (1·1)(iii) implies (1·2)(iii), we need only show that the 2-sum of two connected matroids satisfying (1·2)(iii) also satisfies (1·2)(iii). It is straightforward to show that the class of matroids satisfying (1·2)(iii) is closed under the operation of parallel extension. Using this and duality, we immediately obtain the following:

(2·1) LEMMA. *Suppose that N is a matroid satisfying* (1·2)(iii). *Then every parallel extension of N and every series extension of N satisfies* (1·2)(iii). ∎

Now let M_1 and M_2 be connected matroids each having at least three elements and suppose that $E(M_1) \cap E(M_2) = \{p\}$. Assume also that both M_1 and M_2 satisfy (1·2)(iii). Since $M_1 \oplus_2 M_2 = S(M_1, M_2)/p$, to show that $M_1 \oplus_2 M_2$ satisfies (1·2)(iii), it suffices to show that if $M = S(M_1, M_2)$, then M satisfies (1·2)(iii). The remainder of the proof will be devoted to showing this.

Let B be an arbitrary basis for M and $[I_r|X]$ be the corresponding partial representation. Now B can be written as a disjoint union of bases of M_1 and M_2. We shall distinguish two cases:

(I) $p \notin B$; and
(II) $p \in B$.

In case (I), $B = B_1 \dot{\cup} B_2$ where B_i is a basis of $M_i \setminus p$ and so, since $M \setminus p = (M_1 \setminus p) \oplus (M_2 \setminus p)$, the rows and columns of the partial representation $[I_r|X]$ of M can be reordered so that it takes the form

$$\begin{array}{c} \begin{matrix} B_1 & B_2 & A_1 & A_2 & p \end{matrix} \\ \begin{bmatrix} I_{r_1} & 0 & X_1 & 0 & P_1 \\ 0 & I_{r_2} & 0 & X_2 & P_2 \end{bmatrix} \end{array}$$

where $M|(B_i \cup A_i) = M_i \setminus p$ for $i = 1, 2$. Evidently $[I_{r_i}|X_i|P_i]$ is a partial representation for M_i. Thus there is a signing $[X_i'|P_i']$ of $[X_i|P_i]$ such that $[I_{r_i}|X_i'|P_i']$ represents M_i over all fields of characteristic other than two.

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Consider the matrix

$$\begin{array}{c} \begin{matrix} B_1 & B_2 & A_1 & A_2 & p \end{matrix} \\ \begin{bmatrix} I_{r_1} & 0 & X_1' & 0 & P_1' \\ 0 & I_{r_2} & 0 & X_2' & P_2' \end{bmatrix} \end{array}$$

and let $\mathbb{F}'$ be an arbitrary field of characteristic other than two. To complete the proof in case (I), we shall show that the matroid N for which the above matrix is an $\mathbb{F}'$-representation is equal to M. For $i = 1, 2$, we have $N|(B_i \cup A_i \cup p) \cong M_i$, and M_i is connected. Thus N must also be connected. Moreover,

$$N \setminus p = M \setminus p = (M_1 \setminus p) \oplus (M_2 \setminus p).$$

Thus, by Lemma 1·3, $N = S(N_1, N_2)$ where $N_1 = N/E(M_1 \setminus p)$ and $N_2 = N/E(M_2 \setminus p)$. But, for $i = 1, 2$, the set B_i spans $E(M_i \setminus p)$. Hence $N/E(M_i \setminus p) = N/B_i \setminus A_i$ and the last matroid is equal to M_j where $\{i, j\} = \{1, 2\}$. Therefore

$$N = S(M_2, M_1) = S(M_1, M_2) = M.$$

Next consider case (II). Without loss of generality, we may suppose that $B = B_1 \dot\cup B_2' \dot\cup p$ where B_1 is a basis of M_1 and $B_2' \dot\cup p$ is a basis of M_2. Let $A_1 = E(M_1 \setminus p) - B_1$ and $A_2 = E(M_2 \setminus p) - B_2'$. Now $M \setminus p = (M_1 \setminus p) \oplus (M_2 \setminus p)$. In $M \setminus p / B_1$, the elements of A_1 are loops. Thus, by suitably reordering the rows and columns, we may assume that $[I_r | X]$ has the form

$$\begin{array}{c} \begin{matrix} B_1 & B_2' & p & A_1 & A_2 \end{matrix} \\ \left[\begin{array}{ccccc} I_s & 0 & 0 & S_1 & \\ 0 & I_t & 0 & 0 & Y \\ 0 & 0 & 1 & 0 & \end{array}\right]. \end{array}$$

Consider the subset Z of A_2 consisting of those elements for which the last entry in the corresponding column of Y is a one. Then $p \cup Z$ is a cocircuit of M and, as M is connected, $|Z| \geqslant 1$. Clearly $p \cup A$ is a cocircuit of M/B_2'. But

$$M/B_2' = S(M_1, M_2)/B_2' = S(M_1, M_2/B_2'),$$

and M_2/B_2' has rank 1. Thus M_2/B_2' has $p \cup Z$ as a cocircuit and thus has $A_2 - Z$ as a set of loops. Therefore $A_2 - Z$ is a set of loops of M/B_2' and so we may assume that $[I_r | X]$ has the form

$$\begin{array}{c} \begin{matrix} B_1 & B_2' & p & A_1 & Z & A_2 - Z \end{matrix} \\ \begin{bmatrix} I_s & 0 & 0 & S_1 & S_2 & 0 \\ 0 & I_t & 0 & 0 & S_3 & S_4 \\ 0 & 0 & 1 & 0 & 1 \ldots 1 & 0 \ldots 0 \end{bmatrix}. \end{array}$$

Let M_1' be the matroid $M/B_2' \setminus (A_2 - Z)$. This matroid is the series connection of M_1 and a rank-one uniform matroid on $p \cup Z$. It therefore has $p \cup Z$ as a cocircuit and Z as a parallel class. Moreover, M_1' can be obtained from M_1 by adding an element z of Z in series with p and then adding all of the elements of $Z - z$ in parallel with z. Therefore, by Lemma 2·1, M_1' satisfies (1·2)(iii). The matrix

$$\begin{array}{c} \begin{matrix} B_1 & p & A_1 & Z \end{matrix} \\ \begin{bmatrix} I_s & 0 & S_1 & S_2 \\ 0 & 1 & 0 & 1 \ldots 1 \end{bmatrix} \end{array}$$

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is a partial representation for M_1'. Thus there is a signing of this partial representation that represents M_1' over all fields of characteristic other than two. Since all the elements of Z are parallel in M_1', we may assume that this signing is

$$\begin{array}{c} \begin{matrix} B_1 & p & A_1 & Z \end{matrix} \\ \begin{bmatrix} I_s & 0 & S_1' & S_2' \\ 0 & 1 & 0 & 1\ldots1 \end{bmatrix} \end{array}, \tag{1}$$

where all the columns of S_2' are identical, and S_2' contains no negative entries. Hence each row of S_2' consists either entirely of zeros or entirely of ones.

On contracting B_1 and deleting A_1 from M, we obtain the matroid M_2. This matroid has

$$\begin{array}{c} \begin{matrix} B_2' & p & Z & A_2-Z \end{matrix} \\ \begin{bmatrix} I_t & 0 & S_3 & S_4 \\ 0 & 1 & 1\ldots1 & 0\ldots0 \end{bmatrix} \end{array}$$

as a partial representation. This matrix has a signing of the form

$$\begin{array}{c} \begin{matrix} B_2' & p & Z & A_2-Z \end{matrix} \\ \begin{bmatrix} I_t & 0 & S_3' & S_4' \\ 0 & 1 & 1\ldots1 & 0\ldots0 \end{bmatrix} \end{array} \tag{2}$$

that represents M_2 over all fields of characteristic other than two.

Now let $\mathbb{F}'$ be an arbitrary field of characteristic other than two and let N be the matroid that is represented over $\mathbb{F}'$ by the matrix

$$\begin{array}{c} \begin{matrix} B_1 & B_2' & p & A_1 & Z & A_2-Z \end{matrix} \\ \begin{bmatrix} I_s & 0 & 0 & S_1' & S_2' & 0 \\ 0 & I_t & 0 & 0 & S_3' & S_4' \\ 0 & 0 & 1 & 0 & \boxed{1}\ldots1 & 0\ldots0 \end{bmatrix} \end{array}. \tag{3}$$

To complete the proof in case (II), we shall use Lemma 1·3 to show that $N = S(M_1, M_2) = M$. To do this, we shall show that

(*a*) $N \setminus p$ is disconnected having $E(M_1 \setminus p)$ and $E(M_2 \setminus p)$ as separators;

(*b*) $N/E(M_1 \setminus p) = M_2$; and

(*c*) $N/E(M_2 \setminus p) = M_1$.

From (*b*) and (*c*) it will follow, since M_1^* and M_2^* are connected restrictions of N^* having the element p in common, that N^* is connected and so N is connected.

Let the boxed entry of (3) be in the column labelled by z. Since each row of S_2' consists entirely of zeros or entirely of ones, when we pivot on this boxed entry, we obtain the matrix

$$\begin{array}{c} \begin{matrix} B_1 & B_2' & z & A_1 & p & Z-z & A_2-Z \end{matrix} \\ \begin{bmatrix} I_s & 0 & 0 & S_1' & S_5' & 0 & 0 \\ 0 & I_t & 0 & 0 & S_6' & S_7' & S_4' \\ 0 & 0 & 1 & 0 & 1 & 1\ldots1 & 0\ldots0 \end{bmatrix} \end{array}$$

for some S_5', S_6', and S_7'. Thus $N \setminus p$ has $B_1 \cup A_1$ and $B_2' \cup A_2$ as separators. Since

$$B_1 \cup A_1 = E(M_1 \setminus p) \tag{4}$$

and

$$B_2' \cup A_2 = E(M_2 \setminus p), \tag{5}$$

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we deduce that (a) holds. Moreover, from (2), (3), and (4), we conclude that (b) holds. To show (c), we consider (1) and (3). The matroid $N/B_2'/(A_2-Z)$ is M_1'. It has $p \cup Z$ as a cocircuit and Z as a parallel class. Moreover, $M_1 = M_1' \setminus (Z-z)/z = M_1'/z \setminus (Z-z)$. Since Z is a parallel class of M_1', we have $M_1 = M_1'/z \setminus (Z-z) = M_1'/Z$. Thus, as $N/B_2'/(A_2-Z) = M_1'$, we deduce that $N/(B_2' \cup A_2) = M_1$. We conclude, by (5), that (c) holds. ∎

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