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Continuity of packing measure functions of self-similar iterated function systems

HUA QIU

Department of Mathematics, Nanjing University, Nanjing, China
(e-mail: huatony@gmail.com)

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Abstract. In this paper, we focus on the packing measures of self-similar sets. Let K be a self-similar set whose Hausdorff dimension and packing dimension equal s. We state that if K satisfies the strong open set condition with an open set $\mathcal{O}$, then

$$\mathcal{P}^s(K \cap B(x, r)) \geq (2r)^s$$

for each open ball $B(x, r) \subset \mathcal{O}$ centred in K, where $\mathcal{P}^s$ denotes the s-dimensional packing measure. We use this inequality to obtain some precise density theorems for the packing measures of self-similar sets. These theorems can be used to compute the exact value of the s-dimensional packing measure $\mathcal{P}^s(K)$ of K. Moreover, by using the above results, we show the continuity of the packing measure function of the attractors on the space of self-similar iterated function systems satisfying the strong separation condition. This result gives a complete answer to a question posed by Olsen in [**15**].

1. *Introduction*

In this paper, we will analyse the behaviour of the packing measures of self-similar sets with open set conditions or strong separation conditions. Recall the definition of packing measure, introduced by Tricot [**18**], Taylor and Tricot [**17**], which requires two limiting procedures. For $E \subset \mathbb{R}^d$ and $\delta > 0$, a δ-packing of E is a countable family of disjoint open balls of radii at most δ and with centres in E. For $s \geq 0$, the s-dimensional *packing premeasure* of E is defined as

$$P^s(E) = \inf_{\delta > 0}\{P^s_\delta(E)\},$$

where $P^s_\delta(E) = \sup\{\sum_{i=1}^{\infty} \operatorname{diam}(B_i)^s\}$, with the supremum taken over all δ-packings of E. Here $\operatorname{diam}(B_i)$ denotes the diameter of B_i. The s-dimensional *packing measure* of E is defined as

$$\mathcal{P}^s(E) = \inf\left\{\sum_{i=1}^{\infty} P^s(E_i) \;\middle|\; E \subset \bigcup_{i=1}^{\infty} E_i\right\}.$$

The *packing dimension* of E is defined as

$$\dim_P(E) = \inf\{s \geq 0 \mid \mathcal{P}^s(E) = 0\} = \sup\{s \geq 0 \mid \mathcal{P}^s(E) = \infty\}.$$

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The packing measure and packing dimension play an important role in the study of fractal geometry in a manner dual to the Hausdorff measure and Hausdorff dimension (see [**3, 9**] for further properties of these measures and dimensions).

Let $N \geq 2$ be an integer. Let $\mathbf{f} = \{f_1, f_2, \ldots, f_N\}$ be an *iterated function system* (*IFS*) on $\mathbb{R}^d$ of contractive similitudes. The corresponding *self-similar set* for $\mathbf{f}$ is the unique non-empty compact set $K \subset \mathbb{R}^d$, which is invariant under the action of the elements of $\mathbf{f}$:

$$K = \bigcup_{i=1}^{N} f_i(K).$$

It is well known that if $\mathbf{f}$ satisfies the *open set condition* (*OSC*), i.e. there exists a non-empty bounded open set $\mathcal{O} \subset \mathbb{R}^d$ such that $f_i(\mathcal{O}) \subset \mathcal{O}$ for all $1 \leq i \leq N$ and $f_i(\mathcal{O}) \cap f_j(\mathcal{O}) = \emptyset$ for all $i \neq j$, then the Hausdorff dimension $\dim_H(K)$ and the packing dimension $\dim_P(K)$ of K coincide, and the common value $s = \dim_H(K) = \dim_P(K)$ is given by the following formula:

$$\sum_{i=1}^{N} r_i^s = 1, \tag{1}$$

where r_i denotes the contraction ratio of f_i for $1 \leq i \leq N$. Moreover, the Hausdorff measure and packing measure of K are finite and positive. This was proved by Moran [**12**] in 1946 and rediscovered by Hutchinson [**6**] in the 1980s. Since the intersection of $\mathcal{O}$ and K may be empty, the OSC is in general too weak to infer results. One can strengthen the definition as follows: *the strong open set condition (SOSC)* holds if and only if $\mathcal{O} \cap K \neq \emptyset$. Schief proved that SOSC is equivalent to OSC in the Euclidean case, see [**16**]. There is another separation condition called the *strong separation condition (SSC)*, which is satisfied if $f_i(K) \cap f_j(K) = \emptyset$ for all i, j with $i \neq j$. Obviously, SSC implies SOSC and the implication may not be inverted. In this paper, we will frequently assume these two conditions.

We will need some standard notations from symbolic dynamics. For each positive integer k, let

$$W_k = \{1, 2, \ldots, N\}^k = \{\mathbf{i} = i_1 i_2, \ldots, i_k : i_j \in \{1, 2, \ldots, N\}\}$$

denote the space of words of length k with symbols $\{1, 2, \ldots, N\}$. Also, for $k = 0$, we define $W_0 = \{\emptyset\}$ and call $\emptyset$ the empty word. Moreover, set $W = \bigcup_{k \geq 0} W_k$ and denote the length of $\mathbf{i} \in W$ by $|\mathbf{i}|$. Assume now that $\mathbf{f} = \{f_1, f_2, \ldots, f_N\}$ is an *IFS* with an invariant set K. Let $f_\emptyset = \mathrm{id}$, $r_\emptyset = 1$, $K_\emptyset = K$. For each non-empty finite word $\mathbf{i} = i_1 i_2 \cdots i_k$, let $f_\mathbf{i} = f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_k}$, $r_\mathbf{i} = r_{i_1} r_{i_2} \cdots r_{i_k}$ and $K_\mathbf{i} = f_\mathbf{i}(K)$. Then $K = \bigcup_{\mathbf{i} \in W_k} K_\mathbf{i}$ for each k.

For a Borel measure μ on $\mathbb{R}^d$ and a Borel set E, we let $\mu|_E$ denote the restriction of μ to E. Let λ denote the self-similar measure satisfying

$$\lambda = \sum_i r_i^s \lambda \circ f_i^{-1}.$$

It is well known that under the assumption of OSC, $\lambda = \mathcal{H}^s|_K / \mathcal{H}^s(K) = \mathcal{P}^s|_K / \mathcal{P}^s(K)$, and $\lambda(K_\mathbf{i}) = r_\mathbf{i}^s$ for each $\mathbf{i} \in W$. Hence the measure λ is the normalized s-dimensional Hausdorff measure restricted to K.

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We always assume that K is in a general position, i.e. not contained in a hyperplane. In [**10**], it is proved that, under this assumption and the OSC, the intersection of K with any l-dimensional C^1 submanifold of $\mathbb{R}^d$ with $0 < l < d$ is an $\mathcal{H}^s$-null set, and therefore a $\mathcal{P}^s$-null set.

Since the definitions of Hausdorff and packing measures are sometimes awkward to work with, there are very few non-trivial examples of sets in $E \subset \mathbb{R}^d$ for which the exact Hausdorff measure $\mathcal{H}^{\dim_H(E)}(E)$ or packing measure $\mathcal{P}^{\dim_P(E)}(E)$ of E is known. For examples, see [**1, 5**]. Zhou and Feng [**19**] give a recent review of relevant open questions in this field. In particular, there is no formula similar to (1) for the Hausdorff measure or packing measure of a self-similar set. In view of this, it is natural to ask if the Hausdorff measure and packing measure vary continuously with the IFS.

To make the above question precise, we introduce the following notations.

Let N be a positive integer with $N \geq 2$ and let $X \subset \mathbb{R}^d$ be a compact set. Let $\mathbf{f} = (f_1, f_2, \ldots, f_N)$ be an IFS on X satisfying the OSC. In order to emphasize the relation between the corresponding fractal characteristics with $\mathbf{f}$, we write $K(\mathbf{f})$ for the self-similar set associated with $\mathbf{f}$, and we write $s(\mathbf{f})$ for the common value of the Hausdorff dimension and the packing dimension of $K(\mathbf{f})$. Also, let $\lambda(\mathbf{f})$ denote the normalized $s(\mathbf{f})$-dimensional Hausdorff measure restricted to $K(\mathbf{f})$, and write the contraction ratio of f_i as $r_i(\mathbf{f})$ for each $1 \leq i \leq N$ and $r_*(\mathbf{f}) = \min_{1 \leq i \leq N} r_i(\mathbf{f})$.

Write

$$M_{\mathrm{OSC}} = \{\mathbf{f} \mid \mathbf{f} = (f_1, f_2, \ldots, f_N) \text{ is a self-similar IFS on } X \text{ satisfying the OSC}\},$$

$$M_{\mathrm{SSC}} = \{\mathbf{f} \mid \mathbf{f} = (f_1, f_2, \ldots, f_N) \text{ is a self-similar IFS on } X \text{ satisfying the SSC}\}.$$

It is obvious that $M_{\mathrm{SSC}} \subset M_{\mathrm{OSC}}$. We equip M_{OSC} and M_{SSC} with the metric induced by

$$D(\mathbf{f}, \mathbf{g}) = \max_{1 \leq i \leq N} \{\|f_i - g_i\|_\infty\}$$

for $\mathbf{f}, \mathbf{g} \in M_{\mathrm{OSC}}$. It is not difficult to see that M_{SSC} is an open subset of M_{OSC}. Below, $d(\cdot, \cdot)$ denotes the Euclidean metric between two points or two sets. For $\Delta > 0$, we write

$$M_\Delta = \{\mathbf{f} \mid \mathbf{f} \in M_{\mathrm{SSC}}, \text{ and } d(K_i(\mathbf{f}), K_j(\mathbf{f})) > \Delta \text{ for all } i \neq j\}.$$

Also, one can easily prove that $M_{\mathrm{SSC}} = \bigcup_{\Delta > 0} M_\Delta$ and each M_Δ is an open subset in M_{SSC}, see [**15**]. The metric spaces M_{OSC} and M_{SSC} provide a natural setting for investigating the extent to which the fractal characteristics of $K(\mathbf{f})$ vary continuously with $\mathbf{f}$. For example, let $\mathcal{K}$ denote the family of non-empty compact subsets of X equipped with the Hausdorff metric. Then the map

$$\mathbf{f} \to K(\mathbf{f})$$

from M_{OSC} into $\mathcal{K}$ is continuous, see [**2**]. It also follows immediately from (1) that the dimension map

$$\mathbf{f} \to s(\mathbf{f})$$

from M_{OSC} into $\mathbb{R}$ is continuous. Following this line of investigation, it is natural to ask if the measure maps

$$\mathbf{f} \to \mathcal{H}^{s(\mathbf{f})}(K(\mathbf{f})), \tag{2}$$

$$\mathbf{f} \to \mathcal{P}^{s(\mathbf{f})}(K(\mathbf{f})) \tag{3}$$

from M_{OSC} into $\mathbb{R}$ are continuous.

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For the Hausdorff measure map, Ayer and Strichartz [**1**] showed that in the special case of linear Cantor sets there is a point $\mathbf{f} \in M_{\mathrm{OSC}} \backslash M_{\mathrm{SSC}}$ at which the map (2) fails to be continuous. In [**15**], Olsen altered the space M_{OSC} to M_{SSC}, then proved the continuity, i.e. the map

$$\mathbf{f} \to \mathcal{H}^{s(\mathbf{f})}(K(\mathbf{f}))$$

from M_{SSC} into $\mathbb{R}$ is continuous.

Recall that in order to prove this continuity theorem, Olsen used a so-called explicit formula for the Hausdorff measure of self-similar sets, which was established in [**14**]. Indeed, he showed that the Hausdorff measure coincides with the infimum of the reciprocal densities. Let $\mathbf{f} \in M_{\mathrm{OSC}}$ and let $K(\mathbf{f})$ be the corresponding self-similar set. Let $s(\mathbf{f})$ be the Hausdorff dimension and let $\lambda(\mathbf{f})$ be the normalized $s(\mathbf{f})$-dimensional Hausdorff measure restricted to $K(\mathbf{f})$. Then

$$\mathcal{H}^{s(\mathbf{f})}(K(\mathbf{f})) = \inf\left\{ \frac{\operatorname{diam}(U)^{s(\mathbf{f})}}{\lambda(\mathbf{f})(U)} \;\middle|\; U \text{ is open and convex, } U \cap K(\mathbf{f}) \neq \emptyset \right\}.$$

Moreover, if $\mathbf{f}$ satisfies the SSC, i.e. $\Delta > 0$ exists such that $\mathbf{f} \in M_\Delta$, then

$$\mathcal{H}^{s(\mathbf{f})}(K(\mathbf{f})) = \inf\left\{ \frac{\operatorname{diam}(U)^{s(\mathbf{f})}}{\lambda(\mathbf{f})(U)} \;\middle|\; U \text{ is open and convex, } U \cap K(\mathbf{f}) \neq \emptyset, \operatorname{diam}(U) \geq \Delta \right\}. \tag{4}$$

As pointed out in [**14**], the above formulae are implicit in earlier work by Marion, Ayer and Strichartz, see [**1, 7, 8**]. They used these formulae to compute the exact value of the s-dimensional Hausdorff measure $\mathcal{H}^s(\mathcal{C})$ of certain linear Cantor subsets $\mathcal{C}$ of $\mathbb{R}$, where s denotes the Hausdorff dimension of $\mathcal{C}$.

Because $M_{\mathrm{SSC}} = \bigcup_{\Delta>0} M_\Delta$ and each M_Δ is an open subset in M_{SSC}, the proof of the Hausdorff measure continuity theorem can be simplified to prove that map (2) from M_Δ into $\mathbb{R}$ is continuous for each $\Delta > 0$. This is based on formula (4) for the Hausdorff measure $\mathcal{H}^{s(\mathbf{f})}(K(\mathbf{f}))$ of $K(\mathbf{f})$. It should be pointed out that the fact that the infimum in (4) is taken over sets U with $\operatorname{diam}(U) \geq \Delta$ plays a crucial role for the estimates involved in the proof. Actually, a detailed comparison between families from two adjacent IFSs on the normalized $s(\mathbf{f})$-dimensional Hausdorff measures of their respective elements is needed. The condition $\operatorname{diam}(U) \geq \Delta$ ensures that the comparison is successful.

However, for the packing measure map, it is not clear if the map (3) is continuous. Let $\mathcal{A}$ denote the family of analytic subsets of M_{SSC} and let $\sigma(\mathcal{A})$ denote the σ-algebra generated by $\mathcal{A}$. Olsen proved that the map (3) from M_{SSC} into $\mathbb{R}$ is $\sigma(\mathcal{A})$-measurable. However, this result is very weak compared to the continuity, see [**15**]. The proof also needs an explicit formula for the packing measure. A similar formula for the packing measure in the SSC case was also proved in [**14**], i.e. for each $\mathbf{f} \in M_\Delta$ with $\Delta > 0$, the following formula holds:

$$\mathcal{P}^{s(\mathbf{f})}(K(\mathbf{f})) = \sup\left\{ \frac{(2r)^{s(\mathbf{f})}}{\lambda(\mathbf{f})(B(x,r))}, x \in K(\mathbf{f}), 0 < r \leq \frac{1}{2}\Delta \right\}. \tag{5}$$

However, contrary to (4), the explicit formula (5) for the packing measure $\mathcal{P}^{s(\mathbf{f})}$ of $K(\mathbf{f})$ does not involve taking infima over sets whose diameters are bounded away from zero. Indeed, in (5) the infimum is taken over all balls with radii less than $\frac{1}{2}\Delta$. In particular,

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the infimum in (5) is taken over balls with arbitrarily small radii. For this reason, it is not possible to adapt the arguments in the proof of the Hausdorff measure continuity theorem (proving the continuity of $\mathbf{f} \to \mathcal{H}^{s(\mathbf{f})}(K(\mathbf{f}))$ using (4)) to prove the continuity of the map $\mathbf{f} \to \mathcal{P}^{s(\mathbf{f})}(K(\mathbf{f}))$ using (5).

On the other hand, results by Mattila and Mauldin [**11**] show that various somewhat related maps are discontinuous (in fact, not even Borel measurable), and it is therefore entirely plausible that the map $\mathbf{f} \to \mathcal{P}^{s(\mathbf{f})}(K(\mathbf{f}))$ is discontinuous.

Based on the above reasoning, Olsen posed the following open question in [**15**].

Question. Is the packing measure function in (3) from M_{SSC} into $\mathbb{R}$ continuous? If it is not continuous, is it of Baire class n for some positive integer n? If it is not of Baire class n for some positive integer n, is it Borel measurable?

In this paper we will show, somewhat surprisingly, that this map is continuous, which gives a complete answer to this question. This leads to our main result.

THEOREM 1.1. *The map* $\mathbf{f} \to \mathcal{P}^{s(\mathbf{f})}(K(\mathbf{f}))$ *from* M_{SSC} *into* $\mathbb{R}$ *is continuous.*

To prove Theorem 1.1, we first establish a new explicit formula for the packing measure $\mathcal{P}^{s(\mathbf{f})}(K(\mathbf{f}))$, where the infimum is taken over all balls with radii bounded away from zero. This formula is analogous to (4) (where the infimum is also taken over sets with diameters bounded away from zero) and replaces Olsen's formula (5). The formula is stated below. If $\mathbf{f} \in M_\Delta$, $\Delta > 0$, then

$$\mathcal{P}^{s(\mathbf{f})}(K(\mathbf{f})) = \sup\left\{\frac{(2r)^{s(\mathbf{f})}}{\lambda(\mathbf{f})(B(x,r))},\, x \in K(\mathbf{f}),\, \frac{1}{2}r_*(\mathbf{f})\Delta \le r \le \frac{1}{2}\Delta\right\}, \tag{6}$$

where $r_*(\mathbf{f}) = \min_{1\le i\le N} r_i(\mathbf{f})$. Next, using (6), we adapt the techniques for proving the Hausdorff measure continuity theorem to establish Theorem 1.1, showing that the map $\mathbf{f} \to \mathcal{P}^{s(\mathbf{f})}(K(\mathbf{f}))$ is continuous. Here we should point out that the key formula (6) is a direct corollary of the following theorem. This is another main result in this paper.

THEOREM 1.2. *Let* $K \subset \mathbb{R}^d$ *be the self-similar set associated with an IFS* $\mathbf{f} = \{f_1, f_2, \ldots, f_N\}$ *satisfying the SOSC for an open set* $\mathcal{O}$*, and with packing and Hausdorff dimensions* s*. Then*

$$\mathcal{P}^s(K \cap B(x,r)) \ge (2r)^s \tag{7}$$

for any ball $B(x,r) \subset \mathcal{O}$ *centred in* K*.*

A similar result in the SSC case was proved by Olsen in [**14**], and from which the explicit formula (5) was obtained.

It is not known whether the packing measure continuity theorem for the map $\mathbf{f} \to \mathcal{P}^{s(\mathbf{f})}(K(\mathbf{f}))$ still holds from M_{OSC} into $\mathbb{R}$. In the special setting of linear Cantor sets of the real line $\mathbb{R}$, Feng [**5**] discussed the exact value of the packing measure $\mathcal{P}^{\dim_P(\mathcal{C})}$ of self-similar Cantor sets $\mathcal{C}$ satisfying the OSC (where the open set is an interval). His result implies that $\mathcal{P}^{\dim_P(\mathcal{C})}$ depends continuously on the IFSs in M_{OSC}. In view of this, we suppose that Theorem 1.1 could be generalized to the following setting.

CONJECTURE. *The packing measure function in (3) from* M_{OSC} *into* $\mathbb{R}$ *is continuous.*

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However, we are not able to prove this.

This paper is organized as follows. In §2, we deal with the density theorem for the packing measure of self-similar sets with SOSC. First, we give the proof of Theorem 1.2, which plays an important role in giving the explicit formula of packing measures in the SOSC case. Secondly, we prove formula (6) using the so-called blow-up principle in the SSC case. Section 3 is devoted to the proof of Theorem 1.1 using the explicit formula (6).

2. *Density theorems for packing measures of self-similar sets*

We analyse the local behaviour of the packing measures of self-similar sets in this section. Let $N \geq 2$ be an integer. Let $\mathbf{f} = \{f_1, f_2, \ldots, f_N\}$ be an IFS on $\mathbb{R}^d$ of contractive similitudes. In this section, for the sake of simplicity, we always write K for the self-similar set of $\mathbf{f}$ and we write s for the common value of the Hausdorff dimension and the packing dimension of K. Also, let λ denote the normalized s-dimensional Hausdorff measure restricted to K; write r_i for the contraction ratio of f_i for each $1 \leq i \leq N$ and write $r_* = \min_{1\leq i\leq N} r_i$. Our main result in this section, i.e. Theorem 1.2, says that if K satisfies the SOSC, then

$$\mathcal{P}^s(K \cap B(x, r)) \geq (2r)^s$$

for all $B(x, r) \subset \mathcal{O}$ centred in K. This result has several applications to densities and can also be used to compute the exact value of the packing measure $\mathcal{P}^s(K)$ of K. Recall that in [**14**] Olsen also proved a density theorem for the packing measures of self-similar sets, which requires that the IFSs satisfy the SSC. In that setting, $r_0 > 0$ exists such that the above formula holds for all $x \in K$ and all $r \in (0, r_0]$. However, in the SOSC case, instead of finding a constant r_0, we require r to be small enough such that $B(x, r) \subset \mathcal{O}$. It is easy to check that our result is a natural generalization of the SSC case.

In order to prove Theorem 1.2, we will need the following lemma.

LEMMA 2.1. *Let $B(x, r) \subset \mathcal{O}$ be a ball centred in K and let k be a positive integer. Then*

$$\mathcal{P}^s\left(K \cap \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(\overline{B(x, r)})\right) = \mathcal{P}^s(K \cap \overline{B(x, r)}) > 0.$$

Proof. First, we prove that

$$K \cap \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(B(x, r)) = \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(K \cap B(x, r)). \tag{8}$$

Fix $y \in K \cap \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(B(x, r))$. Since $B(x, r) \subset \mathcal{O}$, $\mathbf{u} \in W_k$ exists such that $y \in f_{\mathbf{u}}(B(x, r)) \subset f_{\mathbf{u}}(\mathcal{O})$. We also have $y \in K = \bigcup_{\mathbf{i}\in W_k} K_{\mathbf{i}}$ and we therefore find $\mathbf{v} \in W_k$ such that $y \in f_{\mathbf{v}}(K) \subset f_{\mathbf{v}}(\overline{\mathcal{O}})$. Thus $y \in f_{\mathbf{u}}(\mathcal{O}) \cap f_{\mathbf{v}}(\overline{\mathcal{O}})$, and therefore $\mathbf{u} = \mathbf{v}$. Hence

$$y \in f_{\mathbf{u}}(B(x, r)) \cap f_{\mathbf{u}}(K) = f_{\mathbf{u}}(K \cap B(x, r)) \subset \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(K \cap B(x, r)).$$

The other direction is obvious. Hence formula (8) holds.

It follows from (8) that

$$\mathcal{P}^s\left(K \cap \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(B(x, r))\right) = \mathcal{P}^s\left(\bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(K \cap B(x, r))\right).$$

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However, since the SOSC is satisfied and $B(x, r) \subset \mathcal{O}$, the sets $f_{\mathbf{i}}(K \cap B(x, r))_{\mathbf{i}\in W_k}$ are pairwise disjoint. It therefore follows that

$$\begin{aligned}\mathcal{P}^s\left(K \cap \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(B(x, r))\right) &= \sum_{\mathbf{i}\in W_k} \mathcal{P}^s(f_{\mathbf{i}}(K \cap B(x, r))) \\ &= \sum_{\mathbf{i}\in W_k} r_{\mathbf{i}}^s \mathcal{P}^s(K \cap B(x, r)) \\ &= \mathcal{P}^s(K \cap B(x, r)).\end{aligned}$$

Since the intersection of K with any $(n-1)$-dimensional C^1 manifold is an $\mathcal{P}^s$-null set, we have

$$\mathcal{P}^s\left(K \cap \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(B(x, r))\right) = \mathcal{P}^s\left(K \cap \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(\overline{B(x, r)})\right),$$

and

$$\mathcal{P}^s(K \cap B(x, r)) = \mathcal{P}^s(K \cap \overline{B(x, r)}). \tag{9}$$

Using the above three equalities, we get

$$\mathcal{P}^s\left(K \cap \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(\overline{B(x, r)})\right) = \mathcal{P}^s(K \cap \overline{B(x, r)}).$$

Moreover, since $x \in K$, we deduce that $\mathcal{P}^s(K \cap \overline{B(x, r)}) > 0$. This completes the proof of Lemma 2.1. □

Proof of Theorem 1.2. In order to reach a contradiction, we assume that (7) is not satisfied, i.e. there exists a ball $B(x, r) \subset \mathcal{O}$, centred in K, such that

$$\mathcal{P}^s(K \cap B(x, r)) < (2r)^s.$$

From (9), we get

$$\mathcal{P}^s(K \cap \overline{B(x, r)}) < (2r)^s.$$

Thus we can find a number $0 < \kappa < 1$ with

$$(1 + \kappa)\mathcal{P}^s(K \cap \overline{B(x, r)}) < (2r)^s. \tag{10}$$

Next, fix $\delta > 0$ and choose a positive integer k such that

$$2r_{\mathbf{i}} r \le \delta$$

for all $\mathbf{i} \in W_k$. Let $\eta = \frac{1}{2}\kappa\mathcal{P}^s(K \cap \overline{B(x, r)})$. It follows from Lemma 2.1 that $\eta > 0$.

For a positive integer m, write

$$F_m = K \setminus \bigcup_{\mathbf{i}\in W_k} B\left(f_{\mathbf{i}}x, r_{\mathbf{i}}r + \frac{1}{m}\right),$$

and observe that

$$F_1 \subset F_2 \subset F_3 \subset \cdots$$

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and

$$\bigcup_m F_m = K \setminus \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(\overline{B(x,r)}).$$

If $K\setminus\bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(\overline{B(x,r)}) \neq \emptyset$, then there is a positive integer m_0 with $1/m_0 < \delta$ such that $F_{m_0} \neq \emptyset$, and

$$\mathcal{P}^s\left(K\setminus\bigcup_{\mathbf{i}\in W_k} B\left(f_{\mathbf{i}}x, r_{\mathbf{i}}r+\frac{1}{m_0}\right)\right) = \mathcal{P}^s(F_{m_0}) \geq \mathcal{P}^s\left(K\setminus\bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(\overline{B(x,r)})\right) - \frac{\eta}{2}. \tag{11}$$

We can also choose a $(1/m_0)$-packing $\{B(x_i,\rho_i)\}_i$ of $K\setminus\bigcup_{\mathbf{i}\in W_k} B(f_{\mathbf{i}}x, r_{\mathbf{i}}r+1/m_0)$ such that

$$\begin{aligned}\sum_i (2\rho_i)^s &\geq P^s_{1/m_0}\left(K\setminus\bigcup_{\mathbf{i}\in W_k} B\left(f_{\mathbf{i}}x, r_{\mathbf{i}}r+\frac{1}{m_0}\right)\right) - \frac{\eta}{2}\\ &\geq P^s\left(K\setminus\bigcup_{\mathbf{i}\in W_k} B\left(f_{\mathbf{i}}x, r_{\mathbf{i}}r+\frac{1}{m_0}\right)\right) - \frac{\eta}{2}\\ &\geq \mathcal{P}^s\left(K\setminus\bigcup_{\mathbf{i}\in W_k} B\left(f_{\mathbf{i}}x, r_{\mathbf{i}}r+\frac{1}{m_0}\right)\right) - \frac{\eta}{2}. \end{aligned}\tag{12}$$

Since $x\in K$ and $B(x,r)\subset\mathcal{O}$, $f_{\mathbf{i}}(B(x,r))\cap f_{\mathbf{j}}(B(x,r))=\emptyset$ for all $\mathbf{i}\neq\mathbf{j}$ in W_k, and for each $\mathbf{i}\in W_k$ we have $f_{\mathbf{i}}(x)\in K_{\mathbf{i}}\subset K$ and $2r_{\mathbf{i}}r\leq\delta$. Thus the family $\{f_{\mathbf{i}}(B(x,r))\}_{\mathbf{i}\in W_k}$ is a δ-packing of $K\cap\bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(\overline{B(x,r)})$.

Since $\{B(x_i,\rho_i)\}_i$ is also a $(1/m_0)$-packing of $K\setminus\bigcup_{\mathbf{i}\in W_k} B(f_{\mathbf{i}}x, r_{\mathbf{i}}r+1/m_0)$, we conclude that $\{f_{\mathbf{i}}(B(x,r))\}_{\mathbf{i}\in W_k}\bigcup\{B(x_i,\rho_i)\}_i$ is a δ-packing of K. Using this, we therefore conclude from (10), (11), (12), and Lemma 2.1 that

$$\begin{aligned}P^s_\delta(K) &\geq \sum_{\mathbf{i}\in W_k}(2r_{\mathbf{i}}r)^s + \sum_i (2\rho_i)^s\\ &\geq \sum_{\mathbf{i}\in W_k} r^s_{\mathbf{i}}(2r)^s + \mathcal{P}^s\left(K\setminus\bigcup_{\mathbf{i}\in W_k} B\left(f_{\mathbf{i}}x, r_{\mathbf{i}}r+\frac{1}{m_0}\right)\right) - \frac{\eta}{2}\\ &\geq (2r)^s + \mathcal{P}^s\left(K\setminus\bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(\overline{B(x,r)})\right) - \eta\\ &\geq (1+\kappa)\mathcal{P}^s(K\cap\overline{B(x,r)}) + \mathcal{P}^s\left(K\setminus\bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(\overline{B(x,r)})\right) - \eta\\ &= (1+\kappa)\mathcal{P}^s\left(K\cap\bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(\overline{B(x,r)})\right) + \mathcal{P}^s\left(K\setminus\bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(\overline{B(x,r)})\right) - \eta\\ &= \mathcal{P}^s(K) + \kappa\mathcal{P}^s\left(K\cap\bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(\overline{B(x,r)})\right) - \eta\\ &= \mathcal{P}^s(K) + 2\eta - \eta\\ &= \mathcal{P}^s(K) + \frac{1}{2}\kappa\mathcal{P}^s(K\cap\overline{B(x,r)}).\end{aligned}$$

Finally, letting $\delta\to 0$, we obtain

$$P^s(K) \geq \mathcal{P}^s(K) + \tfrac{1}{2}\kappa\mathcal{P}^s(K\cap\overline{B(x,r)}). \tag{13}$$

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In [**4**], it is proved that the packing premeasure P^s coincides with the packing measure $\mathcal{P}^s$ for compact subsets with a finite P^s-measure. Thus they coincide for K, and it follows from (13) that

$$\mathcal{P}^s(K) \geq \mathcal{P}^s(K) + \tfrac{1}{2}\kappa\mathcal{P}^s(K \cap \overline{B(x, r)}).$$

Since $\mathcal{P}^s(K)$ is positive and finite, and $\frac{1}{2}\kappa\mathcal{P}^s(K \cap \overline{B(x, r)}) > 0$, we obtain the contradiction.

On the other hand, if

$$K \setminus \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(\overline{B(x, r)}) = \emptyset,$$

i.e. $K \subset \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(\overline{B(x, r)})$, then the foregoing string of inequalities simplifies to

$$\begin{aligned} P^s_\delta(K) &\geq \sum_{\mathbf{i}\in W_k} (2r_{\mathbf{i}}r)^s \\ &= \sum_{\mathbf{i}\in W_k} r^s_{\mathbf{i}}(2r)^s \\ &= (2r)^s \\ &> (1+\kappa)\mathcal{P}^s(K \cap \overline{B(x, r)}) \\ &= (1+\kappa)\mathcal{P}^s\left(K \cap \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(\overline{B(x, r)})\right). \end{aligned}$$

Letting $\delta \to 0$ and using the fact that $\mathcal{P}^s(K) = P^s(K)$ gives

$$\begin{aligned} \mathcal{P}^s(K) &= P^s(K) \\ &\geq (1+\kappa)\mathcal{P}^s\left(K \cap \bigcup_{\mathbf{i}\in W_k} f_{\mathbf{i}}(\overline{B(x, r)})\right) \\ &= (1+\kappa)\mathcal{P}^s(K) > \mathcal{P}^s(K). \end{aligned}$$

This provides the desired contradiction.

The proof of Theorem 1.2 is completed. □

This result has applications to densities. For a given measure μ on $\mathbb{R}^d$ and $x \in \mathbb{R}^d$, the lower α-density of μ at x is defined by

$$\Theta^\alpha_*(\mu, x) = \liminf_{r\to 0} \frac{\mu(B(x, r))}{(2r)^\alpha}.$$

The upper α-density, $\Theta^{*\alpha}(\mu, x)$, is defined similarly by taking the upper limit. We have the following result. If $E \subset \mathbb{R}^d$ and $\alpha > 0$ with $0 < \mathcal{P}^\alpha(E) < \infty$, then

$$\Theta^\alpha_*(\mathcal{P}^\alpha|_E, x) = 1 \quad \text{for } \mathcal{P}^\alpha\text{-almost every } x \in E. \tag{14}$$

See the proof in [**9**]. We can then obtain the following corollary on the basis of (14) and Theorem 1.2.

COROLLARY 2.2. *Let $K \subset \mathbb{R}^d$ be the self-similar set satisfying the SOSC for an open set $\mathcal{O}$, with packing and Hausdorff dimensions s. Then*

$$\mathcal{P}^s(K) = \sup\left\{\frac{(2r)^s}{\lambda(B(x, r))} \,\middle|\, x \in K,\ B(x, r) \subset \mathcal{O}\right\}. \tag{15}$$

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Proof. Since $K \cap \mathcal{O} \neq \emptyset$, we can take a point $y \in K \cap \mathcal{O}$. Choose $\rho > 0$ such that the ball $B(y, \rho)$ is contained in $\mathcal{O}$ and $\mathcal{P}^s(K \cap B(y, \rho)) > 0$. Hence, from (14), there exists a point $z \in K \cap B(y, \rho)$ with $\Theta^s_*(\mathcal{P}^s|_K, z) = 1$. By the definition of $\Theta^s_*(\mathcal{P}^s|_K, z)$, there exists a sequence $\{r_n\}$ with each $r_n \leq \rho - d(z, y)$ and $r_n \to 0$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \frac{\mathcal{P}^s(K \cap B(z, r_n))}{(2r_n)^s} = 1.$$

Note that here all balls $B(z, r_n)$ are contained in $B(y, \rho) \subset \mathcal{O}$. Moreover, by Theorem 1.2, for each ball $B(x, r) \subset \mathcal{O}$ centred in K, we have $\mathcal{P}^s(K \cap B(x, r))/(2r)^s \geq 1$. Hence we get

$$\inf\left\{ \frac{\mathcal{P}^s(K \cap B(x, r))}{(2r)^s} \;\middle|\; x \in K,\ B(x, r) \subset \mathcal{O} \right\} = 1.$$

Since $\lambda = \mathcal{P}^s|_K / \mathcal{P}^s(K)$, (15) follows immediately from the above equation. □

After this work was completed, we learned that Morán [**13**] had proved, independently, the same result as Corollary 2.2. However, his proof is quite different from ours. In fact, in [**13**], the so-called self-similar tiling principle plays a central role in the proof. This principle says that any open subset U of K can be tiled by a countable set of similar copies of an arbitrarily given closed set with a positive Hausdorff or packing measure, where the tiling is exact in the sense that the part of U which cannot be covered by the tiles is of null measure. The continuity theorem is not studied in his paper.

The following lemma will be used in the following corollaries.

LEMMA 2.3. (Blow-up principle) *Let $B(x, r) \subset \mathcal{O}$ be centred in K. Then for any $1 \leq j \leq N$, $f_j(B(x, r))$ has the same reciprocal density as $B(x, r)$, i.e.*

$$\frac{(2r)^s}{\lambda(B(x, r))} = \frac{(2rr_j)^s}{\lambda(f_j(B(x, r)))}.$$

In other words, if $B(x, r) \subset f_j(\mathcal{O})$ is centred in K_j for some $1 \leq j \leq N$, then $f_j^{-1}(B(x, r))$ has the same reciprocal density as $B(x, r)$.

Proof. We only need to check that

$$\lambda(f_j(B(x, r))) = r_j^s \lambda(B(x, r)).$$

Actually,

$$\lambda(f_j(B(x, r))) = \lambda(f_j(B(x, r)) \cap K_j) + \sum_{i\neq j} \lambda(f_j(B(x, r)) \cap K_i).$$

Note that if $i \neq j$, then

$$f_j(B(x, r)) \cap K_i \subset f_j(\mathcal{O}) \cap f_i(K) \subset f_j(\mathcal{O}) \cap f_i(\overline{\mathcal{O}}) = \emptyset.$$

Hence

$$\lambda(f_j(B(x, r))) = \lambda(f_j(B(x, r) \cap K)) = r_j^s \lambda(B(x, r)).$$ □

Combining the above lemma and Corollary 2.2, we immediately obtain the following corollaries.

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COROLLARY 2.4. *Let $K \subset \mathbb{R}^d$ be the self-similar set satisfying the SOSC for an open set $\mathcal{O}$, with packing and Hausdorff dimensions s. Then*

$$\mathcal{P}^s(K) = \sup\left\{ \frac{(2r)^s}{\lambda(B(x,r))} \;\middle|\; x \in K,\ B(x,r) \subset \mathcal{O},\ B(x,r) \not\subseteq f_j(\mathcal{O}),\ 1 \le j \le N \right\}.$$

COROLLARY 2.5. *If $\Delta > 0$ and $d(K_i, K_j) > \Delta$ for all $i \neq j$, then*

$$\mathcal{P}^s(K) = \sup_{x \in K,\, \frac{1}{2} r_* \Delta \le r \le \frac{1}{2}\Delta} \frac{(2r)^s}{\lambda(B(x,r))},$$

where $r_* = \min_{1 \le i \le N} r_i$.

Proof. Let $\mathcal{O} = \bigcup_{x \in K} B(x, \frac{1}{2}\Delta)$. Then it is obvious that this open set $\mathcal{O}$ satisfies the SOSC, and therefore the previous results can be fully applied in the SSC case. Hence, from Corollary 2.2, we get

$$\mathcal{P}^s(K) = \sup_{x \in K,\, 0 < r \le \frac{1}{2}\Delta} \frac{(2r)^s}{\lambda(B(x,r))}. \tag{16}$$

By Lemma 2.3, we can limit $B(x,r)$ so that it is not contained in any $f_j(\mathcal{O})$. Fix $x \in K$, $0 < r \le \frac{1}{2}\Delta$. Then $B(x,r) \subset \mathcal{O}$ is a ball centred in K. Hence j exists such that $x \in K_j$. Obviously, $B(x,r) \cap K_j \neq \emptyset$, which gives $B(x,r) \cap f_j(\overline{\mathcal{O}}) \neq \emptyset$. Hence $B(x,r) \cap f_j(\mathcal{O}) \neq \emptyset$. But $B(x,r)$ cannot be contained in $f_j(\mathcal{O})$, i.e. $B(x,r) \not\subseteq \bigcup_{z \in K_j} B(z, \frac{1}{2} r_j \Delta)$. So $r \ge \frac{1}{2} r_j \Delta \ge \frac{1}{2} r_* \Delta$. Hence

$$\mathcal{P}^s(K) = \sup_{x \in K,\, \frac{1}{2} r_* \Delta \le r \le \frac{1}{2}\Delta} \frac{(2r)^s}{\lambda(B(x,r))}. \qquad \square$$

3. *Proof of Theorem 1.1*

In this section, we prove Theorem 1.1. In order to prove this theorem, we need some lemmas. Below, $D_H(\cdot,\cdot)$ denotes the Hausdorff metric on the family of all compact subsets of X.

LEMMA 3.1. *Let $\Delta > 0$, $\mathbf{f} \in M_\Delta$. Then $\beta > 0$ exists such that if $\mathbf{g} \in M_\Delta$ with $D(\mathbf{f}, \mathbf{g}) \le \beta$, then $r_*(\mathbf{g}) \ge \frac{1}{2} r_*(\mathbf{f})$.*

Proof. The proof is easy since the map $\mathbf{g} \to r_*(\mathbf{g})$ from M_Δ into $\mathbb{R}$ is continuous. $\square$

LEMMA 3.2. *Let $\Delta > 0$, $\mathbf{f} \in M_\Delta$, $0 < \rho < \frac{1}{4} r_*(\mathbf{f})\Delta$, $\gamma > 0$, and let β be the same as in Lemma 3.1. Then there exists $\delta_1 > 0$ with $\delta_1 \le \beta$ such that if $\mathbf{g} \in M_\Delta$ and $\mathbf{h} \in M_\Delta$ with $D(\mathbf{f}, \mathbf{g}) \le \delta_1$ and $D(\mathbf{f}, \mathbf{h}) \le \delta_1$, then for each ball $B(x,r)$ centred in $K(\mathbf{g})$ with radius $r \in [\frac{1}{2} r_*(\mathbf{g})\Delta, \frac{1}{2}\Delta]$, there exists a ball $B(y, r-\rho)$ centred in $K(\mathbf{h})$ such that*

$$\lambda(\mathbf{h})(B(y, r-\rho)) - \gamma \le \lambda(\mathbf{g})(B(x,r)).$$

Proof. Choose a positive integer k such that

$$\operatorname{diam}(f_{\mathbf{i}}(K(\mathbf{f}))) \le \frac{\rho}{8}$$

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for all $\mathbf{i} \in W_k$. By the continuity of the map $\mathbf{g} \to K(\mathbf{g})$ from M_Δ into $\mathcal{K}$ and the map $\mathbf{g} \to r_i(\mathbf{g})$ from M_Δ into $\mathbb{R}$, for each $1 \le i \le N$, we can choose $\delta_1 > 0$ with $\delta_1 \le \beta$ such that if $\mathbf{g} \in M_\Delta$ with $D(\mathbf{f}, \mathbf{g}) \le \delta_1$, then

$$\operatorname{diam}(g_{\mathbf{i}}(K(\mathbf{g}))) \le \frac{\rho}{4}, \tag{17}$$

$$D_H(f_{\mathbf{i}}(K(\mathbf{f})), g_{\mathbf{i}}(K(\mathbf{g}))) \le \frac{\rho}{8}, \tag{18}$$

$$|(r_{\mathbf{i}}(\mathbf{f}))^{s(\mathbf{f})} - (r_{\mathbf{i}}(\mathbf{g}))^{s(\mathbf{g})}| \le \frac{\gamma}{2N^k} \tag{19}$$

for all $\mathbf{i} \in W_k$.

Now fix $\mathbf{g}, \mathbf{h} \in M_\Delta$ with $D(\mathbf{f}, \mathbf{g}) \le \delta_1$, $D(\mathbf{f}, \mathbf{h}) \le \delta_1$, and also fix a ball $B(x, r)$ centred in $K(\mathbf{g})$ with radius $r \in [\frac{1}{2} r_*(\mathbf{g})\Delta, \frac{1}{2}\Delta]$. By Lemma 3.1, $r_*(\mathbf{g}) \ge \frac{1}{2} r_*(\mathbf{f})$, so $r - \rho > 0$.

Since $D_H(K(\mathbf{f}), K(\mathbf{g})) \le \rho/8$, $D_H(K(\mathbf{f}), K(\mathbf{h})) \le \rho/8$, we get $D_H(K(\mathbf{g}), K(\mathbf{h})) < \rho/2$. Hence there exists a point $y \in B(x, \rho/2)$ with $y \in K(\mathbf{h})$, which gives

$$B(y, r - \rho) \subset B\left(x, r - \frac{\rho}{2}\right). \tag{20}$$

If we denote

$$V_k = \left\{\mathbf{i} \mid \mathbf{i} \in W_k, h_{\mathbf{i}}(K(\mathbf{h})) \cap B\left(x, r - \frac{\rho}{2}\right) \ne \emptyset\right\},$$

then from (19) and (20) we obtain

$$\begin{aligned}
\lambda(\mathbf{h})(B(y, r - \rho)) - \gamma &\le \lambda(\mathbf{h})\left(B\left(x, r - \frac{\rho}{2}\right)\right) - \gamma \\
&\le \lambda(\mathbf{h})\left(\bigcup_{\mathbf{i} \in V_k} h_{\mathbf{i}}(K(\mathbf{h}))\right) - \gamma \\
&= \sum_{\mathbf{i} \in V_k} \lambda(\mathbf{h})(h_{\mathbf{i}}(K(\mathbf{h}))) - \gamma \\
&= \sum_{\mathbf{i} \in V_k} (r_{\mathbf{i}}(\mathbf{h}))^{s(\mathbf{h})} - \gamma \\
&\le \sum_{\mathbf{i} \in V_k} \left((r_{\mathbf{i}}(\mathbf{f}))^{s(\mathbf{f})} + \frac{\gamma}{2N^k}\right) - \gamma \\
&\le \sum_{\mathbf{i} \in V_k} \left((r_{\mathbf{i}}(\mathbf{g}))^{s(\mathbf{g})} + \frac{\gamma}{N^k}\right) - \gamma \\
&\le \sum_{\mathbf{i} \in V_k} (r_{\mathbf{i}}(\mathbf{g}))^{s(\mathbf{g})} \\
&= \sum_{\mathbf{i} \in V_k} \lambda(\mathbf{g})(g_{\mathbf{i}}(K(\mathbf{g}))).
\end{aligned}$$

Next, if we denote

$$U_k = \left\{\mathbf{i} \mid \mathbf{i} \in W_k, g_{\mathbf{i}}(K(\mathbf{g})) \cap B\left(x, r - \frac{\rho}{4}\right) \ne \emptyset\right\},$$

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then we must have $V_k \subset U_k$. In fact, if $\mathbf{i} \in V_k$, then $h_{\mathbf{i}}(K(\mathbf{h})) \cap B(x, r-\rho/2) \neq \emptyset$. Combining this with (18), we get $f_{\mathbf{i}}(K(\mathbf{f})) \cap B(x, r-3\rho/8) \neq \emptyset$, and using (18), once more we obtain $g_{\mathbf{i}}(K(\mathbf{g})) \cap B(x, r-\rho/4) \neq \emptyset$, which proves $V_k \subset U_k$. Hence

$$\lambda(\mathbf{h})(B(y, r-\rho)) - \gamma \leq \sum_{\mathbf{i}\in U_k} \lambda(\mathbf{g})(g_{\mathbf{i}}(K(\mathbf{g}))) = \lambda(\mathbf{g})\Big(\bigcup_{\mathbf{i}\in U_k} g_{\mathbf{i}}(K(\mathbf{g}))\Big).$$

Finally, note that if $\mathbf{i} \in W_k$ and $g_{\mathbf{i}}(K(\mathbf{g})) \cap B(x, r-\rho/4) \neq \emptyset$, then it follows from (17) that $g_{\mathbf{i}}(K(\mathbf{g})) \subset B(x, r)$, whence $\bigcup_{\mathbf{i}\in U_k} g_{\mathbf{i}}(K(\mathbf{g})) \subset B(x, r)$. Hence we have

$$\lambda(\mathbf{h})(B(y, r-\rho)) - \gamma \leq \lambda(\mathbf{g})(B(x, r)). \qquad \square$$

LEMMA 3.3. *Let $\Delta > 0$, $\mathbf{f} \in M_\Delta$, $\kappa > 0$, and let β be the same as in Lemma 3.1. Then $0 < \rho < \min\{\beta, \frac{1}{4} r_*(\mathbf{f})\Delta\}$ exists such that if $\mathbf{g}, \mathbf{h} \in M_\Delta$ with $D(\mathbf{f}, \mathbf{g}) \leq \rho$ and $D(\mathbf{f}, \mathbf{h}) \leq \rho$, and $r \in [\frac{1}{2} r_*(\mathbf{g})\Delta, \frac{1}{2}\Delta]$, then*

$$(2(r-\rho))^{s(\mathbf{h})} \geq (2r)^{s(\mathbf{g})} - \kappa.$$

Proof. Note that the map $(r, \mathbf{g}) \to (2r)^{s(\mathbf{g})}$ from $(0, \frac{1}{2}\Delta) \times M_\Delta$ into $\mathbb{R}$ is continuous. Moreover, for each fixed $\mathbf{g}$, the map is uniformly continuous on $r \in (0, \frac{1}{2}\Delta)$. Hence, for fixed $\mathbf{f} \in M_\Delta$, $\kappa > 0$, there exists $0 < \rho < \min\{\beta, \frac{1}{4} r_*(\mathbf{f})\Delta\}$ with the following property. If $\mathbf{g} \in M_\Delta$ with $D(\mathbf{f}, \mathbf{g}) \leq \rho$, and $r', r'' \in (0, \frac{1}{2}\Delta)$ with $|r' - r''| \leq \rho$, then

$$|(2r')^{s(\mathbf{g})} - (2r'')^{s(\mathbf{f})}| \leq \frac{\kappa}{2}. \tag{21}$$

Hence if we take $\mathbf{g} \in M_\Delta$ with $D(\mathbf{f}, \mathbf{g}) \leq \rho$, and $r' = r$, $r'' = r - \rho$ with $r \in [\frac{1}{2} r_*(\mathbf{g})\Delta, \frac{1}{2}\Delta]$, then, by (21), we get

$$|(2r)^{s(\mathbf{g})} - (2(r-\rho))^{s(\mathbf{f})}| \leq \frac{\kappa}{2}.$$

If we take $\mathbf{h} \in M_\Delta$ with $D(\mathbf{f}, \mathbf{h}) \leq \rho$, and $r' = r'' = r - \rho$ with $r \in [\frac{1}{4} r_*(\mathbf{f})\Delta, \frac{1}{2}\Delta]$, then, also by (21), we get

$$|(2(r-\rho))^{s(\mathbf{h})} - (2(r-\rho))^{s(\mathbf{f})}| \leq \frac{\kappa}{2}.$$

The above two inequalities and Lemma 3.1 give the desired result. $\square$

LEMMA 3.4. *Let c, C, $\kappa > 0$, $0 < \varepsilon < 1$ with $\kappa < c\varepsilon$ and $c < C$. Then $\gamma > 0$ exists such that*

$$\frac{x-\kappa}{y+\gamma} \geq \frac{x}{y} - \varepsilon$$

for all $x, y \in [c, C]$.

Proof. Take $\gamma = c(c\varepsilon - \kappa)/C$. Without losing generality, we may assume that $x/y - \varepsilon > 0$, i.e. $x - y\varepsilon > 0$. Then $\gamma \leq y(\varepsilon y - \kappa)/(x - y\varepsilon)$. Hence $(x - y\varepsilon)\gamma \leq y(\varepsilon y - \kappa)$, which yields $\kappa y + x\gamma \leq \varepsilon y(y + \gamma)$. Thus, dividing the above inequality by $y(y+\gamma) > 0$ gives $(\kappa y + x\gamma)/y(y+\gamma) \leq \varepsilon$, i.e. $x/y - (x-\kappa)/(y+\gamma) \leq \varepsilon$. $\square$

Proof of Theorem 1.1. Since $M_{\mathrm{SSC}} = \bigcup_{\Delta>0} M_\Delta$ and M_Δ are open subsets of M_{SSC} for all $\Delta > 0$, we only need to prove, for each $\Delta > 0$, that the map

$$\mathbf{f} \to \mathcal{P}^{s(\mathbf{f})}(K(\mathbf{f}))$$

from M_Δ into $\mathbb{R}$ is continuous.

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Fix $\Delta > 0$, $\mathbf{f} \in M_\Delta$ and let $0 < \varepsilon < 1$. We now find $\delta > 0$ such that if $\mathbf{g} \in M_\Delta$ with $D(\mathbf{f}, \mathbf{g}) \le \delta$, then

$$|\mathcal{P}^{s(\mathbf{f})}(K(\mathbf{f})) - \mathcal{P}^{s(\mathbf{g})}(K(\mathbf{g}))| \le \varepsilon.$$

By the continuity of the map $\mathbf{g} \to s(\mathbf{g})$ from M_Δ into $\mathbb{R}$, there exists $\delta_2 > 0$ with $\delta_2 \le \beta$ such that if $\mathbf{g} \in M_\Delta$ with $D(\mathbf{f}, \mathbf{g}) \le \delta_2$, then

$$\tfrac{1}{2}s(\mathbf{f}) \le s(\mathbf{g}) \le \tfrac{3}{2}s(\mathbf{f}),$$

where β is the same as in Lemma 3.1. Hence there exist $C_1, C_2 > 0$ such that if $\mathbf{g} \in M_\Delta$ with $D(\mathbf{f}, \mathbf{g}) \le \delta_2$, and $r \in [\frac{1}{2}r_*(\mathbf{g})\Delta, \frac{1}{2}\Delta]$, then

$$C_1 \le (2r)^{s(\mathbf{g})} \le C_2.$$

In [**15**, Lemma 3.1], the map $\mathbf{g} \to \lambda(\mathbf{g})$ from M_Δ into $\mathcal{M}$ is continuous, where $\mathcal{M}$ denotes the space consisting of all Borel regular probability measures equipped with weak topology. Hence there exist $\delta_3 > 0$, $C_3 > 0$ with $\delta_3 \le \beta$ such that if $\mathbf{g} \in M_\Delta$ with $D(\mathbf{f}, \mathbf{g}) \le \delta_3$, then

$$\lambda(\mathbf{g})(B(x, r)) \ge C_3$$

for all balls $B(x, r)$ with radius $r \in [\frac{1}{2}r_*(\mathbf{g})\Delta, \frac{1}{2}\Delta]$ centred in $K(\mathbf{g})$.

Put $c = \min\{C_1, C_3\}$, $C = C_2 + 1 + c$, $\kappa = c\varepsilon/4$, $\gamma > 0$ as in Lemma 3.4. Take $\delta = \min\{\rho, \delta_1, \delta_2, \delta_3\}$. We claim that if $\mathbf{g} \in M_\Delta$ with $D(\mathbf{f}, \mathbf{g}) \le \delta$, then

$$|\mathcal{P}^{s(\mathbf{f})}(K(\mathbf{f})) - \mathcal{P}^{s(\mathbf{g})}(K(\mathbf{g}))| \le \varepsilon.$$

To prove this, we show that if $\mathbf{g}, \mathbf{h} \in M_\Delta$ with $D(\mathbf{f}, \mathbf{g}) \le \delta$ and $D(\mathbf{f}, \mathbf{h}) \le \delta$, then

$$\mathcal{P}^{s(\mathbf{h})}(K(\mathbf{h})) \ge \mathcal{P}^{s(\mathbf{g})}(K(\mathbf{g})) - \varepsilon. \tag{22}$$

We therefore fix $\mathbf{g}, \mathbf{h} \in M_\Delta$ satisfying $D(\mathbf{f}, \mathbf{g}) \le \delta$ and $D(\mathbf{f}, \mathbf{h}) \le \delta$. It follows from Corollary 2.5 that there exists $B(x, r)$ centred in $K(\mathbf{g})$ with radius $r \in [\frac{1}{2}r_*(\mathbf{g})\Delta, \frac{1}{2}\Delta]$ such that

$$\frac{(2r)^{s(\mathbf{g})}}{\lambda(\mathbf{g})(B(x, r))} \ge \mathcal{P}^{s(\mathbf{g})}(K(\mathbf{g})) - \frac{\varepsilon}{2}. \tag{23}$$

Since $c \le (2r)^{s(\mathbf{g})} \le C$, $c \le \lambda(\mathbf{g})(B(x, r)) \le C$ and $\kappa = c\varepsilon/4 < c\varepsilon/2$, then, from Lemma 3.4,

$$\frac{(2r)^{s(\mathbf{g})} - \kappa}{\lambda(\mathbf{g})(B(x, r)) + \gamma} \ge \frac{(2r)^{s(\mathbf{g})}}{\lambda(\mathbf{g})(B(x, r))} - \frac{\varepsilon}{2}. \tag{24}$$

From Lemma 3.3, we have

$$(2(r - \rho))^{s(\mathbf{h})} \ge (2r)^{s(\mathbf{g})} - \kappa. \tag{25}$$

Then, from Lemma 3.2, there exists a ball $B(y, r - \rho)$ centred in $K(\mathbf{h})$ such that

$$\lambda(\mathbf{h})(B(y, r - \rho)) - \gamma \le \lambda(\mathbf{g})(B(x, r)). \tag{26}$$

Combining (16) and (23)–(26), we get

$$\begin{aligned}\mathcal{P}^{s(\mathbf{h})}(K(\mathbf{h})) &\ge \frac{2(r - \rho)^{s(\mathbf{h})}}{\lambda(\mathbf{h})(B(y, r - \rho))} \\ &\ge \frac{(2r)^{s(\mathbf{g})} - \kappa}{\lambda(\mathbf{g})(B(x, r)) + \gamma}\end{aligned}$$

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$$\geq \frac{(2r)^{s(\mathbf{g})}}{\lambda(\mathbf{g})(B(x,r))} - \frac{\varepsilon}{2}$$
$$\geq \mathcal{P}^{s(\mathbf{g})}(K(\mathbf{g})) - \varepsilon.$$

This proves (22), and hence the proof of Theorem 1.1 is completed. □

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