Conics and a generalised conical pendulum

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1 *Introduction*

There is a long tradition of using geometry to solve problems from mechanics. Unfortunately this tradition is not practised much in schools and university any more.

With this exposition we would like to demonstrate how elementary properties of ellipses can be used to solve a problem related to the conical pendulum. The problem of the conical pendulum is to consider a mass attached to one end of a light inextensible string of length a with the other end attached at the top of a vertical rod. The mass is moving about the rod in uniform circular motion in a horizontal plane. Given the angular velocity of the mass, the question is to determine the angle the string makes with the rod.

One could conduct such an experiment by threading a bead onto a light string of length $2a$ and attaching the two ends to the top of a rod. Imagine now that we keep one end of the string attached to the top of the rod and fix the other end further down the rod. We assume that the bead can move freely (without friction) along the string. Given an angular velocity, the question is to find the equilibrium height where the bead revolves in uniform circular motion in a horizontal plane about the rod. The classical conical pendulum and our particular generalisation are shown in Figure 1.

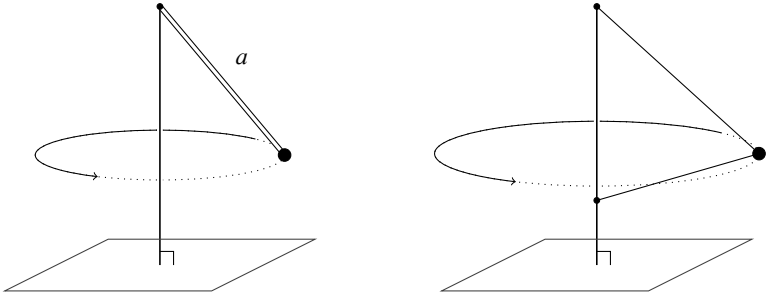


FIGURE 1: Conical pendulum and its generalisation

An ellipse can be characterised as the locus of points so that the sum of the distances to two given points is constant. In the problem under consideration, the point on the ellipse corresponds to the bead, and the sum of the distances is given by the length $2a$ of the string. The focal points of the ellipse are the points where the string is attached to the rod. Hence, at any time, the bead lies on an ellipse with the major axis given by the rod. It seems natural to solve the problem under consideration by exploiting the geometry of the ellipse.

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According to Newton's Second Law, the resultant force on the bead produces an acceleration. To keep the bead moving with angular velocity ω on a circular path of radius r, this acceleration must have magnitude $r\omega^2$ and be directed horizontally towards the centre of motion. The force is commonly known as the centripetal force and has magnitude $mr\omega^2$.

Suppose that the bead is located at the point P and has mass m. We assume that the string of length $2a$ is attached to the point S_2 on top of the vertical rod and to the point S_1 further down the rod. As the bead can move freely along the string, the tension in the string is the same at any point on the string. The forces acting on the bead P are the gravitational force mg acting downwards and the tension forces of magnitude T acting along the string towards the points S_1 and S_2. The constant g is the gravitational constant. The tension force T is determined by the gravitational force and the centripetal force $mr\omega^2$ keeping the bead on the circular path of radius with angular velocity ω. We show a cross-section in Figure 2, and after establishing some little-known properties of ellipses we provide a geometric solution of the problem of circular motion.

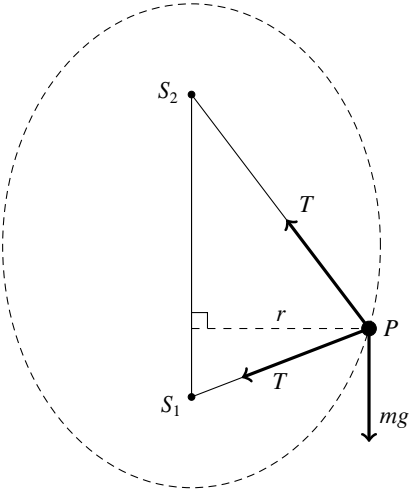


FIGURE 2: The forces acting on the bead P

We then proceed to discuss a problem that is mathematically equivalent to the above, but makes sense for an arbitrary conic. We consider a small ball rolling frictionless in uniform circular motion along the inside of a bowl whose cross section through the central axis is an arbitrary conic. The interesting fact in case of a parabola is that there is precisely one frequency ω for which uniform circular motion is possible. Before we solve the problem of circular motion we collect some geometric properties of ellipses required for the solution.

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2 *Interlude: Properties of ellipses*

In this section we consider some geometric properties of ellipses useful for solving our problem of circular motion. One relates the angles the tangent and the focal chords make with the major axis of the ellipse.

Let S_1 and S_2 be the focal points of an ellipse with semi-major axis of length a, eccentricity $e \in [0, 1)$ and centre O. Let P denote a point on the ellipse. We will make use of the following standard facts on ellipses found in every textbook on conics such as for instance [1, 2] or some advanced school textbooks such as [3]:

(i) The focus-directrix definition: An ellipse is the locus of points P such that the ratio of the distances of P from the focus S_1 (or S_2) and from a line ℓ is the constant e. The line ℓ is called a directrix. An ellipse has two focal points S_1, S_2 and two directrices. We also use the fact that the focal points and directrices have distance ae and a/e respectively from O.

(ii) The sum of the lengths of the focal chords is constant ('string property'):

$$|S_1P| + |S_2P| = 2a. \tag{1}$$

(iii) The angles the two focal chords make with the tangent are equal ('reflection property'). We denote that common angle by α. Some new proofs are presented in [4, 5].

We further denote by β the angle between the major axis and the tangent to the ellipse at P. The configuration is shown in Figure 3. Key to the solution of our problem of circular motion is the following relationship between α and β. It is not easily found in the standard literature.

Lemma 1: Assume that $\cos\beta \neq 0$. With the above definitions we have

$$e = \frac{\cos\alpha}{\cos\beta}, \tag{2}$$

where e is the eccentricity of the ellipse.

Proof: To prove (2) denote by T the point of intersection of the line S_1P and the perpendicular to the tangent at P through S_2. Further denote by M the midpoint between T and S_2. Finally, let R be the point of intersection of TS_2 and the parallel to the major axis through P as shown in Figure 3.

Due to the reflection property of the ellipse, M lies on the tangent, ΔS_2TP is isosceles with $|TP| = |S_2P|$ and ΔS_2PM is right-angled at M. Looking at the right angled triangles RPM and S_2PM we see that

$$\cos\beta = \frac{|MP|}{|RP|} \qquad \text{and} \qquad \cos\alpha = \frac{|MP|}{|S_2P|}.$$

Hence, eliminating $|MP|$ from the two equations, we obtain

$$|RP| = \frac{|MP|}{\cos\beta} = \frac{|S_2P|\cos\alpha}{\cos\beta}. \tag{3}$$

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As ΔS_2TP is isosceles we have $|S_1T| = |S_1P| + |TP| = |S_1P| + |S_2P| = 2a$ by (1). We also have $|S_1S_2| = 2ae$. As RP and S_1S_2 are parallel, ΔS_2S_1T and ΔRPT are similar. Considering ratios of corresponding sides we obtain

$$\frac{|RP|}{|PT|} = \frac{|S_1S_2|}{|S_1T|} = \frac{2ae}{2a} = e. \tag{4}$$

Substituting (3) into (4) and using that $|PT| = |S_2P|$ yields (2).

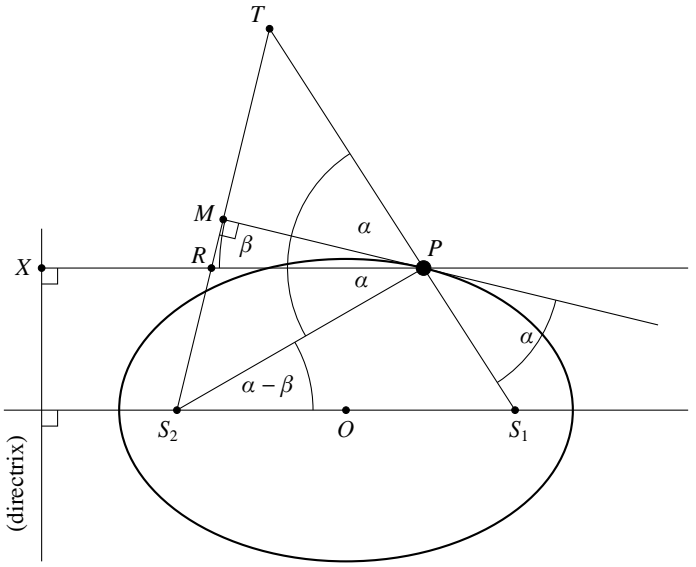


FIGURE 3: Ellipse with tangent, focal chords and directrix

We finally compute the length $|S_2P|$ in terms of α and β by using the focus-directrix definition of the ellipse. Let X denote the intersection of the parallel to the major axis through P with the directrix as shown in Figure 3. The focal point S_2 and the directrix have distance ae and a/e from O respectively. Using angles on parallel lines we have $\angle S_1S_2P = \alpha - \beta$. Hence

$$|XP| = \frac{a}{e} - ae + |S_2P| \cos(\alpha - \beta).$$

By the focus-directrix definition of the ellipse we therefore have

$$e = \frac{|S_2P|}{|XP|} = \frac{|S_2P|}{\frac{a}{e} - ae + |S_2P| \cos(\alpha - \beta)}. \tag{5}$$

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Solving (5) for $|S_2P|$ we obtain

$$|S_2P| = \frac{a(1 - e^2)}{1 - e\cos(\alpha - \beta)}. \tag{6}$$

Up to a different choice of angle, this is the polar representation of the ellipse from the focus S_2 as often used in the Kepler problem, where S_2 represents the sun and P a planet.

3 *Balancing the forces*

There are many ways to compute the balance of forces as discussed above. The most common way is to resolve all forces in the horizontal and vertical directions. This can be done, but it seems more natural to resolve forces in the tangent and normal directions of the ellipse. Before we do so we determine the dimensions of the ellipse on which P lies.

First note that the semi-major axis of the ellipse is given by half the length of the string, namely a. The distance between S_1 and S_2 needs to be smaller than the length of the string. Hence we can uniquely determine a positive constant $e \in [0, 1)$ so that the distance between S_1 and S_2 is given by $2ae$. The constant e is the eccentricity of the ellipse. The special case $e = 0$ corresponds to the usual conical pendulum, where S_1 and S_2 coincide with the centre O of the ellipse.

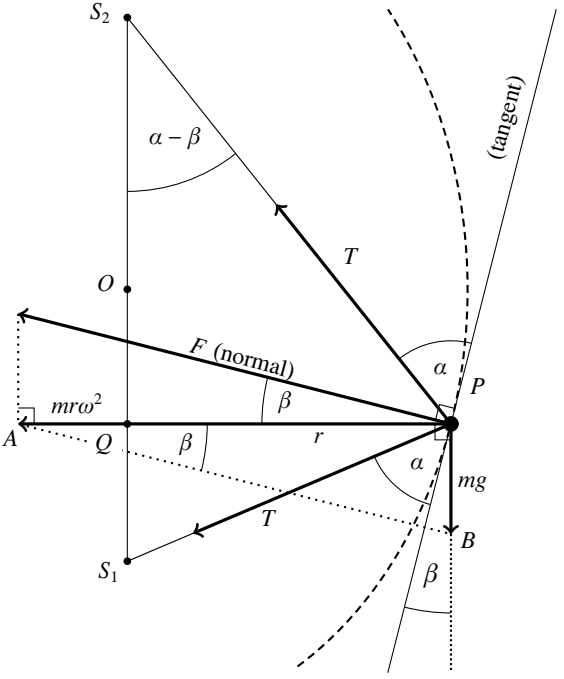


FIGURE 4: Resolution of forces in tangent and normal directions to the ellipse

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The bead is moving on a fixed horizontal plane and therefore it does not slip along the string. This means that the total force in the direction of the tangent to the ellipse is zero. By the reflection property of the ellipse, the angles the string at P makes with the tangent are equal. The angles are marked with α in Figure 4. Because the tension forces have the same magnitude, as indicated in Figure 2, their components in the direction of the tangent are therefore equal as well, pointing in opposite directions. Hence their vector sum F points in the direction of the normal to the ellipse at P. Since the total force on the bead in the direction of the tangent is zero, the vector sum of F and the gravitational force mg equals the centripetal force $mr\omega^2$. The configuration is shown in Figure 4.

Let β denote the angle the vector F makes with the horizontal. Then looking at the right-angled triangle ABP in Figure 4 we see that

$$\tan\beta = \frac{mg}{mr\omega^2} = \frac{g}{r\omega^2}. \tag{7}$$

Recall that F points in the normal direction to the tangent. Thus the angle the tangent makes with the vertical is β as well. It is our aim to show, using only the geometry of the ellipse, the following theorem.

Theorem 2: Consider the scenario from before. Let Q be the point on the vertical axis of the ellipse at the same height as P, and let $\alpha - \beta$ be the angle the string makes with the vertical at the upper end. Then

$$\cos(\alpha - \beta) = \frac{g + a\omega^2 e(1 - e^2)}{eg + a\omega^2(1 - e^2)} \tag{8}$$

and

$$|OQ| = \frac{g}{\omega^2(1 - e^2)} \tag{9}$$

if

$$\omega^2 > \frac{g}{a(1 - e^2)}. \tag{10}$$

The relationships (8) and (9) are both useful. The first gives a relationship between the frequency ω and the angle the string makes at the top end with the vertical. The second provides a relationship between ω and the distance of the bead below the horizontal plane through O. In both cases we recover the classical conical pendulum if $e = 0$. Then $\alpha = \frac{1}{2}\pi$ and $\varphi = \frac{1}{2}\pi - \beta$ is the angle the string makes with the vertical. Hence

$$\cos\varphi = \frac{g}{a\omega^2}. \tag{11}$$

In the case of (9) we used that $|OQ| = a\cos\varphi$ for the conical pendulum, where S_1, S_2 and coincide.

Proof: From the balance of forces we have an expression for $\tan\beta$ in (7). To complete the proof of Theorem 2 we have to relate $\tan\beta$ to the geometry of

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the ellipse. The key to do this is the formula (2) in Lemma 1. We can rewrite (2) in the form

$$e = \frac{\cos\alpha}{\cos\beta} = \frac{\cos((\alpha-\beta)+\beta)}{\cos\beta} = \cos(\alpha-\beta) - \sin(\alpha-\beta)\tan\beta. \qquad (12)$$

Note that $\alpha - \beta$ is the angle that the string makes with the vertical at the upper end where the string is attached. Solving (12) for $\tan\beta$ we obtain

$$\tan\beta = \frac{\cos(\alpha - \beta) - e}{\sin(\alpha - \beta)}. \qquad (13)$$

If we equate (7) from the resolution of forces and (13) we see that

$$\frac{g}{r\omega^2} = \tan\beta = \frac{\cos(\alpha - \beta) - e}{\sin(\alpha - \beta)}. \qquad (14)$$

Using (6) we see that

$$r = |PQ| = |S_2P| \sin(\alpha - \beta) = \frac{a(1 - e^2)\sin(\alpha - \beta)}{1 - e\cos(\alpha - \beta)}$$

and therefore by substitution into (14)

$$\frac{1 - e\cos(\alpha - \beta)}{a(1 - e^2)\sin(\alpha - \beta)}\frac{g}{\omega^2} = \frac{\cos(\alpha - \beta) - e}{\sin(\alpha - \beta)}.$$

Rearranging, we see that

$$\frac{g}{\omega^2(1 - e^2)} = a\frac{\cos(\alpha - \beta) - e}{1 - e\cos(\alpha - \beta)}. \qquad (15)$$

Solving for $\cos(\alpha - \beta)$ we find (8).

We next interpret the right-hand side of (15) and show that it is the distance of the bead from the horizontal plane through O. Using (6) we have

$$|OQ| = |S_2P|\cos(\alpha - \beta) - ae = \frac{a(1 - e^2)\cos(\alpha - \beta)}{1 - e\cos(\alpha - \beta)} - ae$$

$$= \frac{a(1-e^2)\cos(\alpha-\beta) - ae + ae^2\cos(\alpha-\beta)}{1 - e\cos(\alpha-\beta)} = a\frac{\cos(\alpha-\beta) - e}{1 - e\cos(\alpha-\beta)}.$$

Combining this with (15) we obtain (9). The restriction (10) comes from (9) and the fact that the length of the semi-major axis is a, so $|OQ| < a$.

4 *Related problems on an arbitrary conic*

Here we briefly discuss a related problem on uniform circular motion allowing us to generalise the problem to an arbitrary conic section: ellipse, parabola and hyperbola. We consider a bowl whose horizontal cross section is a circle, and whose vertical cross section through the central axis is a conic section. We let a small ball of mass m roll frictionless along the inside of the bowl in uniform circular motion of frequency ω and radius r in a

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horizontal plane as shown in Figure 5. We also take the ball sufficiently small so that any rotational energy is insignificant.

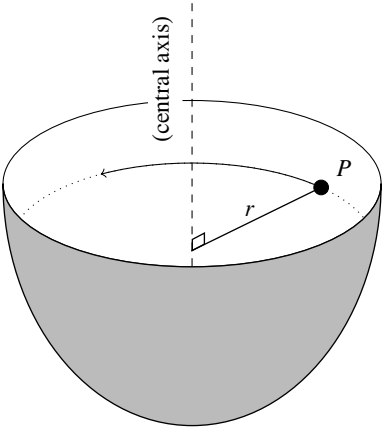


FIGURE 5: Ball rolling in uniform circular motion along the inside of a bowl

The forces acting on the ball are the centripetal force $mr\omega^2$ in the radial direction towards the central axis and the gravitational force mg in the vertical direction. As the motion is assumed to be frictionless, the force in the direction of the tangent to any vertical cross section through the axis vanishes, and hence the total force acts in the normal direction.

If β is the angle the tangent at P makes with the vertical, then the fact that the force in the direction of the tangent vanishes leads to condition (7) as before. The main difficulty is to express $\tan\beta$ using the geometry of the conic, leading to a solution of the problem on circular motion. We have solved the problem for the ellipse already in Theorem 2. We sketch the solutions for the parabola and the hyperbola. We only consider the case where the major axis of the conic is vertical. However, for the ellipse and the hyperbola we could also consider the case where the major axis is horizontal.

In the case of the ellipse we had a solution for *large* frequencies ω, more precisely for satisfying (10). It turns out that in the case of a hyperbola there is a solution for *small* frequencies. The case of the parabola is interesting because it turns out that there is precisely one frequency ω for which there is uniform circular motion.

Bowl whose vertical cross section is a parabola

Consider a bowl whose vertical cross section through the central axis is a parabola. The parabola is such that $2a$ is the distance from the focus S to the directrix. Let P be a point on the parabola. By the focus-directrix definition of the parabola $|XP| = |SP|$, where X is the projection of P onto the directrix as shown in Figure 6.

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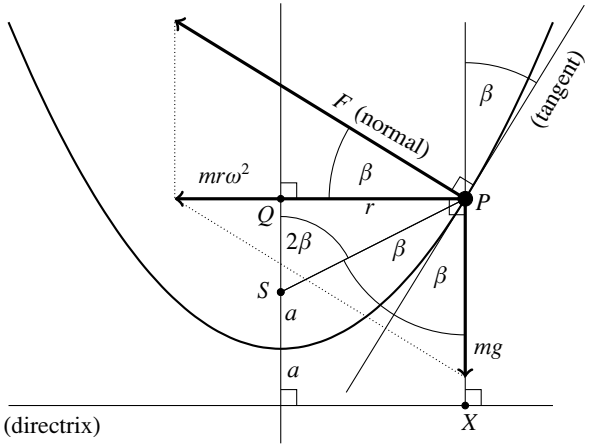


FIGURE 6: Cross section of paraboloid

Let β be the angle of inclination of the normal force F. By the reflection property β is also the angle the tangent at P makes with the focal chord SP and the vertical. Because the force in the tangential direction vanishes the vector sum of the gravitational force mg and F must equal the centripetal force $mr\omega^2$. Looking at Figure 6 we see that (7) is valid exactly as in case of the ellipse in the original problem of the generalised conical pendulum. We next use the geometry of the parabola to compute $r = |PQ|$. From Figure 6 we see that $|XP| = 2a + |SP| \cos 2\beta$ and hence

$$|SP| = |XP| = 2a + |SP| \cos 2\beta.$$

Therefore

$$|SP| = \frac{2a}{1 - \cos 2\beta}$$

and so

$$r = |PQ| = |SP| \sin 2\beta = \frac{2a \sin 2\beta}{1 - \cos 2\beta}. \tag{16}$$

Using the standard tangent half-angle formulae

$$\cos 2\beta = \frac{1 - \tan^2\beta}{1 + \tan^2\beta} \qquad \text{and} \qquad \sin 2\beta = \frac{2 \tan\beta}{1 + \tan^2\beta}$$

in (16) we obtain $\tan\beta = \frac{2a}{r}$. Equating $\tan\beta = \frac{2a}{r}$ with (7) we deduce that

$$\omega^2 = \frac{g}{2a}. \tag{17}$$

The result (17) is interesting since the frequency ω is uniquely determined by a and g. It means that there is exactly one frequency for which the ball undergoes uniform circular motion in a horizontal plane. For all others there cannot be an equilibrium of the forces, that is, the ball does not stay in a horizontal plane.

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Bowl whose vertical cross section is a hyperbola

We finally sketch a solution for a bowl whose cross section through the central axis is the upper branch of a hyperbola with eccentricity $e > 1$ and distance $2a$ between the two vertices. Let S_1 and S_2 be the focal points, O the centre and P a point on the hyperbola as shown in Figure 7.

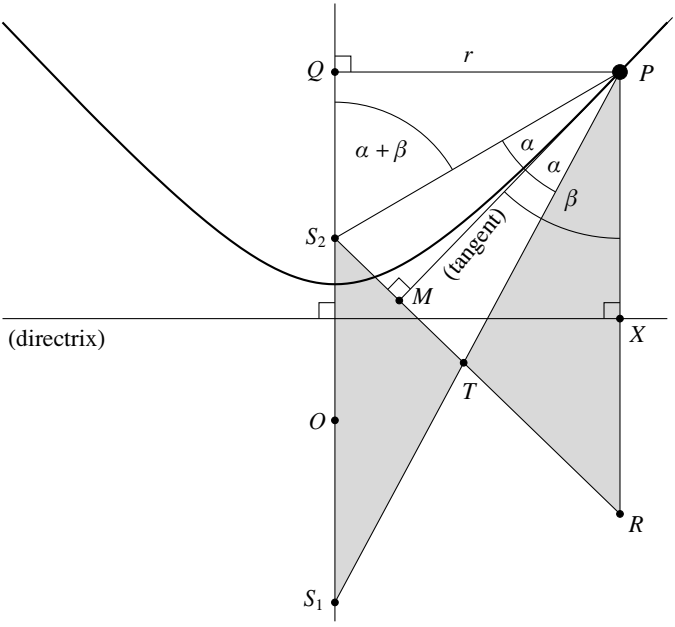


Figure 7: Hyperbola with focal points and directrix

Let β be the angle between the tangent at P and the major axis of the hyperbola. Then, as for the ellipse and the parabola, we obtain (7).

By the reflection property the angles between the tangent at P and the focal chords have a common value α. The key to solve the circular motion problem was the identity (2) relating the angles α and β. Proceeding similarly as in case of the ellipse we show that (2) is valid for the hyperbola. We define the points T, M and R as in the case of the ellipse: T is the point of intersection of S_1P and the perpendicular to the tangent at P through S_2M, is the midpoint between T and S_2, and R is the point of intersection of TS_1 and the parallel to the major axis through P as shown in Figure 7. Due to the reflection property of the hyperbola M lies on the tangent, ΔS_2TP is isosceles with $|TP| = |S_2P|$ and ΔS_2PM is right-angled at M. Clearly ΔS_1TS_2 and ΔPTR are similar. Hence, by looking at ratios of corresponding sides,

$$\frac{|S_1S_2|}{|S_1T|} = \frac{|PR|}{|TP|}. \tag{18}$$

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Looking at Figure 7 we see that ΔS_2MP and ΔPMR are right-angled at M and hence

$$|PR| = \frac{|PM|}{\cos\beta} = \frac{|S_2P|\cos\alpha}{\cos\beta}. \tag{19}$$

For any hyperbola $|S_1T| = |S_1P| - |S_2P| = |S_1P| - |TP| = 2a$ and $|S_1S_2| = 2ae$. Combining (18), (19) and taking into account that $|TP| = |S_2P|$ we deduce (2).

Now we compute $|S_2P|$ using the focus-directrix definition

$$e = \frac{|S_2P|}{|XP|} \tag{20}$$

of the hyperbola, where X is the projection of P onto the directrix. Clearly $\angle S_2PX = \alpha + \beta$. By looking at angles on parallel lines S_2P makes an angle of $\alpha + \beta$ with the vertical. The distance from the directrix to O is a/e. Hence from Figure 7 we see that

$$|XP| = ae - \frac{a}{e} + |S_2P|\cos(\alpha + \beta).$$

Using (20) we deduce that

$$e = \frac{|S_2P|}{ae - \frac{a}{e} + |S_2P|\cos(\alpha + \beta)}.$$

Solving for $|S_2P|$ we obtain

$$|S_2P| = \frac{a(e^2 - 1)}{1 - e\cos(\alpha + \beta)}, \tag{21}$$

corresponding to (6) in case of the ellipse. Using (2) we have

$$e = \frac{\cos a}{\cos\beta} = \frac{\cos((\alpha + \beta) - \beta)}{\cos\beta} = \cos(\alpha + \beta) + \sin(\alpha + \beta)\tan\beta$$

and hence

$$\tan\beta = \frac{e - \cos(\alpha + \beta)}{\sin(\alpha + \beta)}. \tag{22}$$

Equating (7) and (22) leads to

$$\frac{g}{\omega^2(e^2 - 1)} = a\frac{e - \cos(\alpha + \beta)}{1 - e\cos(\alpha + \beta)} \tag{23}$$

similar to (15) the ellipse. As in the case of the ellipse we want to interpret the right-hand side of (23) in terms of the length $|OQ|$. As $|OS| = ae$ we have, using (21),

$$\begin{aligned} |OQ| &= ae + |S_2P|\cos(\alpha + \beta) \\ &= ae + \frac{a(e^2 - 1)\cos(\alpha + \beta)}{1 - e\cos(\alpha + \beta)} = a\frac{e - \cos(\alpha + \beta)}{1 - e\cos(\alpha + \beta)}. \end{aligned} \tag{24}$$

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Combining (23) and (24) we obtain

$$|OQ| = \frac{g}{\omega^2(e^2 - 1)}, \tag{25}$$

an expression corresponding to (9) in the case of the ellipse. We could also solve (23) for $\cos(\alpha + \beta)$ and obtain an expression corresponding to (11). For the generalised conical pendulum there was a physical interpretation for the angle $\alpha - \beta$ in terms of the string. In the case of a small ball rolling inside a bowl this is not natural, so we only consider (25).

There is again a restriction on $|OQ|$. As the lowest point on the hyperbola is the vertex we have $|OQ| \geqslant a$. From (25) we therefore need to assume that

$$0 < \omega^2 < \frac{g}{a(e^2 - 1)}. \tag{26}$$

It seems counter-intuitive that $|OQ| \to \infty$ as $\omega \to 0$. Note however that, as $|OQ| \to \infty$, also $r \to \infty$. Hence the centripetal force $mr\omega^2$ turns out to be large enough. This shows that there is a solution for large frequencies satisfying (10) in the case of the ellipse and for small frequencies satisfying (26) in case of the hyperbola. The parabola is the limit case between the ellipse and the hyperbola and hence an extreme case of both inequalities applies. This explains why there is precisely one valid frequency in case of the parabola.

5 *Concluding remarks*

We have chosen a purely geometric approach to solve the problem of the generalised conical pendulum. In this section we mention some alternative approaches.

The first step was the derivation of (7). We then used the result from Lemma 1 to relate $\tan\beta$ to the geometry of the ellipse and the hyperbola. An alternative is to use calculus to compute $\tan\beta$. This requires the introduction of a coordinate system. The most convenient coordinate system is taking the x-axis to be the major axis of the ellipse pointing downwards, and the y-axis the minor axis pointing to the right in Figure 4. Then $\tan\beta$ is the slope of the tangent at P which can be computed by using any parametrisation of the ellipse, for instance $x = a\cos\theta$, $y = a\sqrt{1 - e^2}\sin\theta$. Doing so we can easily deduce (9) in the form

$$\cos\theta = \frac{g}{a\omega^2(1 - e^2)}$$

because $|OQ| = a\cos\theta$ if $\theta \in (0, \frac{1}{2}\pi)$.

A second alternative is a resolution of forces in the vertical and horizontal directions as done in a question from 2013 Mathematics Extension 2 Examination for the Higher School Certificate in New South Wales, Australia. This is the standard approach often taught in school or university courses. It is an approach that always works, but often requires

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numerous computations rather than clever geometric arguments. In the example of the generalised conical pendulum one would need to determine the sines and cosines of $\angle S_1PQ$ and $\angle S_2PQ$ in Figure 4, and then eliminate T from the equations. To do that one has to compute the lengths of the focal chords S_1P and S_2P which is most easily done by using the focus-directrix definition of the ellipse.

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