Adv. Appl. Prob. **50,** 1217–1226 (2018)
*doi:*10.1017/apr.2018.57
© *Applied Probability Trust* 2018

REGENERATIVE PROCESSES FOR POISSON ZERO POLYTOPES

SERVET MARTÍNEZ,* *Universidad de Chile*
WERNER NAGEL,** *Friedrich-Schiller-Universität Jena*

Abstract

Let $(M_t : t > 0)$ be a Markov process of tessellations of $\mathbb{R}^\ell$, and let $(\mathcal{C}_t : t > 0)$ be the process of their zero cells (zero polytopes), which has the same distribution as the corresponding process for Poisson hyperplane tessellations. In the present paper we describe the stationary zero cell process $(a^t \mathcal{C}_{a^t} : t \in \mathbb{R})$, $a > 1$, in terms of some regenerative structure and we show that it is a Bernoulli flow. An important application is to STIT tessellation processes.

Keywords: Stochastic geometry; random tessellation; Poisson hyperplane tessellation; STIT tessellation; zero polytope; Bernoulli flow; regenerative process

2010 Mathematics Subject Classification: Primary 60D05; 60J25; 60J75; 37A25; 37A35

1. Introduction

Let us consider a process of Poisson hyperplane tessellations of the Euclidean space $\mathbb{R}^\ell$ for some $\ell \geq 1$. It is generated by a spatio-temporal Poisson process of hyperplanes marked with birth times. Let $\mathcal{H}$ be the space of hyperplanes in $\mathbb{R}^\ell$, endowed with the Borel σ-field associated to the Fell topology.

For a locally finite and translation invariant measure Λ on $\mathcal{H}$, consider the Poisson process $\hat{X}$ on $\mathcal{H} \times [0, \infty)$ with intensity measure $\Lambda \otimes \lambda_+$, where λ_+ denotes the Lebesgue measure on $[0, \infty)$. Then we define the process $(\hat{X}_t : t > 0)$ with $\hat{X}_t = \{(h, s) \in \hat{X} : s \leq t\}$ and the process $(X_t : t > 0)$, where X_t is the Poisson hyperplane tessellation (PHT) generated by $\{h : (h, s) \in \hat{X},\ s \leq t\}$.

The renormalized tessellation valued processes $(a^t X_{a^t} : t \in \mathbb{R})$, $a > 1$, is a time stationary Markov process. The main object of our study is its zero cell process. The essential motivation for investigating the zero cell process comes from stable with respect to iteration (STIT) tessellations. A STIT tessellation process $Y = (Y_t : t > 0)$ is a Markov process taking values in the space of tessellations of $\mathbb{R}^\ell$, and was first defined by Nagel and Weiss [10], who also showed that the zero cell processes of PHTs and STIT tessellations (with the same measure Λ) are identically distributed. Also, for STIT tessellations, the renormalized process $\mathcal{Z} = (\mathcal{Z}_t := a^t Y_{a^t} : t \in \mathbb{R})$ is time stationary.

Denote by $\mathcal{C}_t$ the zero cell (or zero polytope or Crofton polytope) which is the polytope in X_t or Y_t, containing the origin. The process $\mathcal{C} = (\mathcal{C}_t : t > 0)$ is well defined almost surely. So, we will study the process $\Gamma = (\Gamma_t := a^t \mathcal{C}_{a^t} : t \in \mathbb{R})$ which is a factor of $\mathcal{Z}$.

Received 31 August 2017; revision received 23 August 2018.
* Postal address: Departamento Ingeniería Matemática and Centro Modelamiento Matemático, Universidad de Chile, UMI 2807 CNRS, Casilla 170-3, Correo 3, Santiago, Chile. Email address: smartine@dim.uchile.cl
** Postal address: Fakultät für Mathematik und Informatik, Friedrich-Schiller-Universität Jena, 07737 Jena, Germany. Email address: werner.nagel@uni-jena.de

https://doi.org/10.1017/apr.2018.57 Published online by Cambridge University Press

In the present paper we mainly rely on results and methods developed for STIT tessellations. But note that, regarding zero cell processes, many formulations can be easily translated for PHTs by replacing the iteration of tessellations operation '$\boxplus$' by the superposition operation '$\sqcup$'. For example, (4) appears as

$$X_{t+s} \sim X_t \sqcup X'_s \quad \text{for all } t, s > 0,$$

where $X_t \sqcup X'_s$ is the PHT generated by $\hat{X}_t \cup \hat{X}'_s$ for independent Poisson hyperplane (marked with birth times) processes $\hat{X}_t$ and $\hat{X}'_s$.

We note that the distribution of the zero cell C^* of a random tessellation is determined by

$$(\mathbb{P}(C^* \supset K)\colon K \subset \mathbb{R}^\ell, K \text{ compact and convex});$$

cf. [8, Theorem 7.8], where a corresponding proposition is shown for the so-called containment functional. In the present paper, for fixed compact and convex K, we consider the 0–1 stationary process $(\mathbf{1}_{\{\Gamma_n \supset K\}}\colon n \in \mathbb{Z})$ associated to the zero cell of $\mathcal{Z}$. We prove that it is a regenerative process in the state 1 and study some of its properties. A main result of this paper is Proposition 2, where we construct the stationary process $(\mathbf{1}_{\{\Gamma_n \supset a\,K\}}\colon n \in \mathbb{Z})$, starting from $(\mathbf{1}_{\{\Gamma_n \supset K\}}\colon n \in \mathbb{Z})$. So, by recurrence, we can construct the family of regenerative processes $(\mathbf{1}_{\{\Gamma_n \supset a^i\,K\}}\colon n \in \mathbb{Z})$ with $i \geq 1$, starting from $(\mathbf{1}_{\{\Gamma_n \supset K\}}\colon n \in \mathbb{Z})$.

On the other hand, in [5] and [6] it was shown that $\mathcal{Z}$ is isomorphic to a time-continuous Bernoulli flow with infinite entropy (a flow is a stationary time-continuous process and its entropy is the mean gain of information in one unit of time). Because Γ is a factor of $\mathcal{Z}$, it is also Bernoulli. The unique thing we need to show is that the entropy is infinite, which follows from the fact that the law of Γ_1 is nonatomic.

The STIT tessellation can be used as a reference model for fracture patterns, and also for their development in time. Although the PHT process as well as the STIT process are Markov processes, obviously not all the processes derived from them, are Markov too. Therefore, it is of interest to also study regenerative properties of such processes.

The zero cell of a spatially stationary (homogeneous) tessellation can also be interpreted as the volume-weighted typical cell (in the sense of the Palm distribution) of a spatially stationary (homogeneous) tessellation. Thus, the results for the process of zero cells also provide information for the corresponding process of typical cells.

Even if the PHT, as a random tessellation of the space (but not a tessellation-valued process in time), is one of the classical and extensively studied models in stochastic geometry, there are still many open problems concerning the zero cell or the typical cell, e.g. the distribution of metric entities like volume, surface area, and diameter. Even if these problems are not yet solved, the spatio-*temporal* approach, which is intrinsic for the STIT model, can also be fruitful for the PHT. This motivates our investigation of the containment functional process.

Let us introduce some notation. Let $\mathbb{Z}$ denote the set of integers, and let $\mathbb{Z}_+ = \{n \in \mathbb{Z}\colon n \geq 0\}$ and $\mathbb{N} = \{n \in \mathbb{Z}\colon n > 0\}$. For a finite set I, we denote by $|I|$ the number of its elements. For $C \subseteq \mathbb{R}^\ell$, we denote by $\operatorname{int} C$ its interior, by $\operatorname{cl} C$ its closure and by $\partial C = \operatorname{cl} C \setminus \operatorname{int} C$ its boundary. For random elements, we use '$\sim$' to mean 'identically distributed as'.

1.1. The space of tessellations

A polytope is the convex hull of a finite point set. By definition, a *tessellation* T of $\mathbb{R}^\ell$ is a countable family of polytopes with a nonempty interior called the cells of T; we set

https://doi.org/10.1017/apr.2018.57 Published online by Cambridge University Press

$T = \{C(T)^l : l \in \mathbb{N}\}$, which satisfies

(i) $\mathbb{R}^\ell = \bigcup_{l\in\mathbb{N}} C(T)^l$ (covering),

(ii) $\operatorname{int}C(T)^l \cap \operatorname{int}C(T)^m = \varnothing$ if $l \neq m$ (disjoint interiors),

(iii) $|\{l \in \mathbb{N} : C(T)^l \cap K \neq \varnothing\}| < \infty$ for all compact $K \subset \mathbb{R}^\ell$ (locally finite).

The boundary of a tessellation is $\partial T = \bigcup_{l\in\mathbb{N}} \partial C(T)^l$. Note that T is determined by ∂T.

Let $\mathbb{T}$ be the space of tessellations of $\mathbb{R}^\ell$. For $b \neq 0$ and $A \subset \mathbb{R}^\ell$, we set $bA = \{bx : x \in A\}$. Then, for $T \in \mathbb{T}$ and $B \subset \mathbb{T}$, we define $bT = \{bC : C \in T\}$ and $bB = \{bT : T \in B\}$.

We fix a polytope with nonempty interior $W \subset \mathbb{R}^\ell$, and call it a *window*. A tessellation in W is a locally finite countable covering of W by polytopes with disjoint interiors. Let $\mathbb{T}_W$ be the space of tessellations of W. By compactness, each $R \in \mathbb{T}_W$ has a finite number of cells $|R|$. The trivial tessellation is $R = \{W\}$ in $\mathbb{T}_W$, and it has the boundary ∂W.

Let $T \in \mathbb{T}$ and $U \subseteq \mathbb{R}^\ell$ be a nonempty set such that $U = \operatorname{cl}(\operatorname{int}U)$. We define the restriction of T to U by

$$T \wedge U = \{C \cap U : C \in T,\ \operatorname{int}(C \cap U) \neq \varnothing\}.$$

When $\vec{T} = (T_l : l \in L)$ is a family of tessellations, we write $\vec{T} \wedge U = (T_l \wedge U : l \in L)$. For a window W and $T \in \mathbb{T}$, we have $T \wedge W \in \mathbb{T}_W$.

The spaces $\mathbb{T}$ and $\mathbb{T}_W$ are endowed with the Borel σ-fields, denoted by $\mathcal{B}(\mathbb{T})$ and $\mathcal{B}(\mathbb{T}_W)$, respectively, associated to the Fell topology; see [13, Chapter 12]. The measurability issues which play a role in the present paper were dealt with in detail in [6].

1.2. The STIT tessellation process

Let us recall the construction of the STIT tessellation process $Y^W = (Y_t^W : t \geq 0)$ in a bounded window W. Its law is determined by a (nonzero) locally finite and translation invariant measure Λ on the space of hyperplanes $\mathcal{H}$ in $\mathbb{R}^\ell$. We use the notation $[B] = \{H \in \mathcal{H} : H \cap B \neq \varnothing\}$ for a bounded Borel set $B \subset \mathbb{R}^\ell$, and if $0 < \Lambda([B]) < \infty$, we define a probability measure on $[B]$ by $\Lambda^B(\cdot) = \Lambda([B])^{-1}\Lambda(\cdot \cap [B])$.

The initial state of Y^W is $Y_0^W = \{W\}$. The window W has a random lifetime that is exponentially distributed with parameter $\Lambda([W])$. At the end of its lifetime, W is divided into two new cells, C_1 and C_2 say, by a random hyperplane with law Λ^W. Thus, the extant cells are divided consecutively: any cell C_i has a lifetime, exponentially distributed with parameter $\Lambda([C_i])$, and at the end of its lifetime it is divided by a random hyperplane into two new cells. The lifetime as well as the dividing hyperplane are conditionally (given the cell C) independent from all the other random variables used before. For $t \geq 0$, the tessellation Y_t^W is the set of all cells existing at this time.

A more precise and detailed description of the process is given in several earlier papers; see, e.g. in [6] or [10]. It was also shown that this process has the following spatial consistency property: for all pairs of windows $W' \subset W$, $Y^W \wedge W' \sim Y^{W'}$ holds. This property guarantees the existence of a process Y of STIT tessellations of the whole space $\mathbb{R}^\ell$ such that $Y \wedge W \sim Y^W$ for all windows W. A global construction (in the whole space $\mathbb{R}^\ell$) of Y is given in [7].

The restricted STIT process $Y^W = (Y_t^W : t \geq 0)$ is a pure-jump Markov process, and the STIT process $Y = (Y_t : t \geq 0)$ is a Markov process.

Moreover, it was shown in [9] that the distribution of the zero cell of the STIT tessellation Y_1 is identical to the distribution of the zero cell of a PHT with intensity measure Λ. This implies that, for all compact convex $K \subset \mathbb{R}^\ell$,

$$\mathbb{P}(\partial Y_1 \cap K = \varnothing) = \mathrm{e}^{-\Lambda([K])}. \tag{1}$$

https://doi.org/10.1017/apr.2018.57 Published online by Cambridge University Press

The following scaling property was shown in [10, Lemma 5]:

$$tY_t \sim Y_1 \quad \text{for all } t > 0. \tag{2}$$

1.3. Independent increments relation for STIT processes

Let $T = \{C(T)^k : k \in \mathbb{N}\} \in \mathbb{T}$ be a tessellation, and let $\vec{R} = (R_k : k \in \mathbb{N}) \in \mathbb{T}^{\mathbb{N}}$ be a sequence of tessellations with $R_k = \{C(R_k)^l : l \in \mathbb{N}\}$. We define the tessellation $T \boxplus \vec{R}$ (also referred to as iteration or nesting) by the set of cells resulting from the restrictions of the tessellations R_k to the cells $C(T)^k$:

$$T \boxplus \vec{R} = \{C(T)^k \cap C(R_k)^l : k \in \mathbb{N},\ l \in \mathbb{N},\ \operatorname{int}(C(T)^k \cap C(R_k)^l) \neq \varnothing\}. \tag{3}$$

For $s > 0$, let $\vec{Y}'_s = (Y'^m_s : m \in \mathbb{N})$ be a sequence of independent copies of the STIT tessellation Y_s, that is, $Y'^m_s \sim Y_s$, and also independent of Y_s. From the construction of Y we have the following relation, first stated in Lemma 2 of [10]:

$$Y_{t+s} \sim Y_t \boxplus \vec{Y}'_s \quad \text{for all } t, s > 0. \tag{4}$$

Furthermore, for all $0 < s_1 < \cdots < s_j$ and all $t > 0$, we have

$$(Y_t, Y_{t+s_1}, \ldots, Y_{t+s_j}) \sim (Y_t, Y_t \boxplus \vec{Y}'^{(1)}_{s_1}, \ldots, (((Y_t \boxplus \vec{Y}'^{(1)}_{s_1}) \boxplus \ldots) \boxplus \vec{Y}'^{(j)}_{s_j - s_{j-1}})). \tag{5}$$

2. The renormalized stationary process

2.1. Properties of the renormalized process

Fix $a > 1$ and define the renormalized process $\mathcal{Z} = (\mathcal{Z}_s : s \in \mathbb{R})$ by $\mathcal{Z}_s = a^s Y_{a^s}$ for $s \in \mathbb{R}$. Note that $\mathcal{Z}_0 = Y_1$. Since Y is a Markov process, so is $\mathcal{Z}$. From (2), all one-dimensional distributions of Z are identical. In Theorem 1.1. of [6] it was shown that $\mathcal{Z}$ is a time-stationary Markov process. The process $\mathcal{Z}$ inherits càdlàg trajectories (i.e. those that are right continuous with left limits) from Y, so it takes values in the space $D_{\mathbb{T}}(\mathbb{R})$ of càdlàg trajectories with values in $\mathbb{T}$ and time $\mathbb{R}$. (The space $D_{\mathbb{T}}(\mathbb{R})$ endowed with the Skorokhod topology is metrizable (see [2]), where the usual metric is given.) Let $\mu^{\mathcal{Z}}$ be the law of $\mathcal{Z}$ on $D_{\mathbb{T}}(\mathbb{R})$, and let $\sigma^t_{\mathbb{T}}$ denote shifts. Then $(D_{\mathbb{T}}(\mathbb{R}), \mu^{\mathcal{Z}}, (\sigma^t_{\mathbb{T}}))$ is a shift flow. The discrete process $\mathcal{Z}^d = (\mathcal{Z}_n : n \in \mathbb{Z})$ is also Markov and its law on $\mathbb{T}^{\mathbb{Z}}$ is denoted by $\mu^{\mathcal{Z}^d}$.

Let $\mathcal{Z} \wedge W = (\mathcal{Z}_s \wedge W : s \in \mathbb{R})$ and $\mathcal{Z}^d \wedge W = (\mathcal{Z}_n \wedge W : n \in \mathbb{Z})$ be the continuous and discrete processes, respectively, restricted to the window W. Their laws are respectively denoted by $\mu^{\mathcal{Z}}_W$ and $\mu^{\mathcal{Z}^d}_W$. The mappings $\mathcal{Z} \to \mathcal{Z} \wedge W$ and $\mathcal{Z}^d \to \mathcal{Z}^d \wedge W$ are factor maps. In Theorem 1.2 of [6] it was stated that $\mathcal{Z} \wedge W$, and so $\mathcal{Z}^d \wedge W$ also, is a mixing stationary Markov process. But the Markov property of $\mathcal{Z} \wedge W$ (which is not a simple consequence of $\mathcal{Z}$ being Markovian) was not shown in detail. Owing to the central role it plays in our main result, Lemma 2 below, and for completeness, we give a proof of it here.

Lemma 1. *The processes $\mathcal{Z} \wedge W$ and $\mathcal{Z}^d$ are Markov processes.*

Proof. It suffices to show that $\mathcal{Z} \wedge W$ is Markov. Fix $B \in \mathcal{B}(\mathbb{T}_W)$. For $b > 0$, we have $b^{-1}B \in \mathcal{B}(\mathbb{T}_{b^{-1}W})$ and, for all $T \in \mathbb{T}$, the relation $bT \wedge W \in B$ holds if and only if $T \wedge b^{-1}W \in b^{-1}B$. Thus, for $h > 0$,

$$\begin{aligned}
&\mathbb{P}(\mathcal{Z}_{t+h} \wedge W \in B \mid Z_s \wedge W,\ s \le t) \\
&\quad = \mathbb{P}(Y_{a^{t+h}} \wedge a^{-(t+h)}W \in a^{-(t+h)}B \mid Y_{a^s} \wedge a^{-s}W,\ s \le t) \\
&\quad = \mathbb{P}(Y_{a^{t+h}} \wedge a^{-(t+h)}W \in a^{-(t+h)}B \mid Y_{a^s} \wedge a^{-(t+h)}W,\ s \le t)
\end{aligned}$$

https://doi.org/10.1017/apr.2018.57 Published online by Cambridge University Press

$$= \mathbb{P}(Y_{a^{t+h}} \wedge a^{-(t+h)}W \in a^{-(t+h)}B \mid Y_{a^t} \wedge a^{-t}W))$$
$$= \mathbb{P}(\mathcal{Z}_{t+h} \wedge W \in B \mid \mathcal{Z}_t \wedge W),$$

where we used the fact that $(Y_u \wedge a^{-(t+h)}W : u > 0)$ is a Markov process and, due to spatial consistency of STIT tessellations, for all $s \le t$, the conditional distribution of $Y_{a^{t+h}} \wedge a^{-(t+h)}W$ given $Y_{a^s} \wedge a^{-s}W$ coincides with the conditional distribution of $Y_{a^{t+h}} \wedge a^{-(t+h)}W$ given $Y_{a^s} \wedge a^{-(t+h)}W$. □

Let $\vec{Y}' = (Y'^m : m \in \mathbb{N})$ be a sequence of independent copies of the STIT process Y, and independent of Y. From (4), it follows that

$$\mathcal{Z}_{n+1} \sim a\mathcal{Z}_n \boxplus a^{n+1}\vec{Y}'_{a^{n+1}-a^n}.$$

Since $a^{n+1}\vec{Y}'_{a^{n+1}-a^n} = (a/(a-1))(a^n(a-1)\vec{Y}'_{a^n(a-1)})$, we obtain, from (2),

$$(\mathcal{Z}_n, \mathcal{Z}_{n+1}) \sim \left(\mathcal{Z}_n, a\mathcal{Z}_n \boxplus \frac{a}{a-1}\vec{Y}'_1\right). \tag{6}$$

Let $(\vec{Y}_1^{'(i)} : i \ge 0)$ be independent copies of $\vec{Y}'_1$. A simple reiteration of (6) and using (5) yields the following formula for the finite-dimensional distributions of $\mathcal{Z}^d$:

$$(\mathcal{Z}_{n+i} : 0 \le i \le k) \sim \left(a^i\mathcal{Z}_n \boxplus_{j=1}^{i} \frac{a^{i+1-j}}{a-1}\vec{Y}_1^{'(j)} : 0 \le i \le k\right) \tag{7}$$

for $n \in \mathbb{Z}$ and $k \ge 0$. Here $M \boxplus_{i=1}^{k} \vec{M}^{'(i)}$ is an abbreviation for $(\ldots(M \boxplus \vec{M}^{'(1)}) \boxplus \ldots) \boxplus \vec{M}^{'(k)}$, where M is a tessellation and $\vec{M}^{'(i)}$ is a sequence of tessellations.

Let us now consider the joint distribution of the zero cell process of $\mathcal{Z}^d$, denoted by $\Gamma^d = (\Gamma_n : n \in \mathbb{Z})$. Let $\mathcal{C}_1^{'(i)}$ denote the zero cell of the first element of the sequence $\vec{Y}_1^{'(i)}$. Thus, $(\mathcal{C}_1^{'(i)} : i \ge 0)$ is a sequence of independent and identically distributed zero cells. Then (3) and (7) yield, for $n \in \mathbb{Z}$ and $k \ge 0$,

$$(\Gamma_{n+i} : 0 \le i \le k) \sim \left((a^i\Gamma_n) \cap \bigcap_{j=1}^{i} \frac{a^{i+1-j}}{a-1}\mathcal{C}_1^{'(j)} : 0 \le i \le k\right). \tag{8}$$

3. Regenerative structure of the stationary zero cell process

3.1. Stationary renewal sequences

Let $(V_n : n \in \mathbb{Z})$ be a stationary 0–1-valued sequence. We define the vector of transition probabilities $\vec{q} = (q_n : n \in \mathbb{N})$ by

$$q_n = \mathbb{P}(V_n = 1 \mid V_0 = 1) = \mathbb{P}(V_{n+i} = 1 \mid V_i = 1) \quad \text{for all } n \in \mathbb{N}. \tag{9}$$

By definition (see [12, Chapter 3, Section 3.7]), this stationary process *regenerates* at the 1-values if

$$\mathbb{P}(V_k = a_k,\ k = 1, \ldots, r \mid V_0 = 1,\ V_{-j_1} = 1, \ldots, V_{-j_n} = 1)$$
$$= \mathbb{P}(V_k = a_k,\ k = 1, \ldots, r \mid V_0 = 1) \tag{10}$$

for all $0 < j_1 < \cdots < j_n$. Then, from this regeneration property and (9),

$$\mathbb{P}(V_{i_0} = 1,\ V_{i_1} = 1, \ldots, V_{i_{n-1}} = 1,\ V_{i_n} = 1) = \left(\prod_{k=0}^{n-1} q_{i_{k+1}-i_k}\right)\mathbb{P}(V_{i_0} = 1). \tag{11}$$

For a deeper treatment of regenerative processes, see [1, Chapter VI].

https://doi.org/10.1017/apr.2018.57 Published online by Cambridge University Press

Let us consider the random set

$$\mathcal{V}^* = \{n \in \mathbb{Z} \colon V_n = 1\},$$

which is stationary too, i.e. for all $m \in \mathbb{Z}$, we have $\mathcal{V}^* \sim \mathcal{V}^* + m$.

The interarrival distribution $\vec{p} = (p_n \colon n \in \mathbb{N})$ is given by $p_n = \mathbb{P}(V_{n+k} = 1,\ V_{l+k} = 0,\ 0 < l < n \mid V_k = 1)$ for all $k \in \mathbb{Z}$. The stationarity of $\mathcal{V}^*$ implies that the mean recurrence time is finite, that is, $\rho = \sum_{n\in\mathbb{N}} n p_n < \infty$, and we have

$$\mathbb{P}(m \in \mathcal{V}^*) = \mathbb{P}(0 \in \mathcal{V}^*) = \mathbb{P}(V_0 = 1) = \rho^{-1} \quad \text{for all } m \in \mathbb{Z}. \tag{12}$$

We enumerate the elements of $\mathcal{V}^*$, i.e. we set $\mathcal{V}^* = \{V_i^* \colon i \in \mathbb{Z}\}$ by imposing the condition that $V_i^* < V_{i+1}^*$ for all $i \in \mathbb{Z}$ and letting $V_0^* = \inf\{n \in \mathbb{N} \colon n \in \mathcal{V}^*\}$. Then

$$p_n = \mathbb{P}(V_{i+1}^* - V_i^* = n) \quad \text{for all } i \geq 0,\ n > 0,$$

and

$$\mathbb{P}(V_0^* = k) = \rho^{-1} \sum_{m>k} p_m \quad \text{for all } k \geq 0.$$

For all these results on stationary renewal sets, see [4, Chapter II].

Note that the reverse process $-\mathcal{V}^*$ is also a stationary renewal sequence with the same interarrival law as $\mathcal{V}^*$.

Let us describe the interarrival distribution $\vec{p}$ of the renewal set $\mathcal{V}^*$ in terms of $\vec{q}$.

Proposition 1. *Let $n \in \mathbb{N}$ and $I \subseteq \{1, \dots, n-1\}$. Let $|I|$ be the number of elements of I and denote its elements by $i_1^I < \cdots < i_{|I|}^I$. Furthermore, set $i_0^I = 0$ and $i_{|I|+1}^I = n$. Then*

$$p_n = \mathbb{P}(V_0 = 1,\ V_k = 0,\ 0 < k < n \mid V_n = 1) = \rho \sum_{I\subseteq\{1,\dots,n-1\}} (-1)^{|I|} \Bigg(\prod_{k=0}^{|I|} q_{i_{k+1}^I - i_k^I} \Bigg),$$

where the sum includes the summand for $I = \varnothing$.

Proof. By (12) we obtain $\rho^{-1} p_n = \mathbb{P}(V_0 = 1,\ V_k = 0,\ 0 < k < n, V_n = 1)$. For $k = 0, \dots, n$, define the event $A_k = \{V_k = 1\}$. Then

$$\begin{aligned}
\rho^{-1} p_n &= \mathbb{P}\Bigg(A_0 \cap \bigcap_{k=1}^{n-1} A_k^{\mathrm{c}} \cap A_n \Bigg) \\
&= \mathbb{P}\Bigg(A_0 \cap \Bigg(\bigcup_{k=1}^{n-1} A_k \Bigg)^{\mathrm{c}} \cap A_n \Bigg) \\
&= \mathbb{P}\Big(A_0 \cap A_n \Big) - \mathbb{P}\Bigg(A_0 \cap \Bigg(\bigcup_{k=1}^{n-1} A_k \Bigg) \cap A_n \Bigg).
\end{aligned}$$

Now using the inclusion–exclusion principle and (11) yields the assertion. □

3.2. Regenerative properties of the stationary zero cell process

As already mentioned in the introduction, denote by $\mathcal{C}_t$ the zero cell of the PHT X_t or the STIT tessellation Y_t, respectively. The process $\mathcal{C} = (\mathcal{C}_t \colon t > 0)$ is well defined almost everywhere. The process $\Gamma = (\Gamma_t := a^t \mathcal{C}_{a^t} \colon t \in \mathbb{R})$ is the zero cell process of $\mathcal{Z}$ and $\Gamma^d = (\Gamma_n \colon n \in \mathbb{Z})$ is the zero cell sequence of $\mathcal{Z}^d$. Both processes Γ and Γ^d are stationary.

https://doi.org/10.1017/apr.2018.57 Published online by Cambridge University Press

Let us construct a regenerative sequence for the discrete stationary zero cell sequence. Let K be a compact and convex set containing 0 in its interior, and consider the random sequence $\mathcal{V}^K = (V_n^K : n \in \mathbb{Z})$ of 0–1-valued random variables with

$$V_n^K = \mathbf{1}_{\{\Gamma_n \supset K\}}, \qquad n \in \mathbb{Z}.$$

The random sequence $\mathcal{V}^K = (V_n^K : n \in \mathbb{Z})$ inherits stationarity from $\mathcal{Z}^d \wedge K$. From (1) we have

$$\mathbb{P}(V_n^K = 1) = \mathbb{P}(\Gamma_n \supset K) = \mathrm{e}^{-\Lambda([K])} \quad \text{for all } n \in \mathbb{Z}. \tag{13}$$

Lemma 2. *The* 0–1 *stationary sequence* $\mathcal{V}^K = (V_n^K : n \in \mathbb{Z})$ *satisfies the regenerative property (10).*

Proof. For $i_0 < i_1 < \cdots < i_n$ and two disjoint finite subsets I and J of $\mathbb{N}$,

$$\begin{aligned}
&\mathbb{P}(V_{i_n+k}^K = 1,\ k \in I,\ V_{i_n+k}^K = 0,\ k \in J \mid V_{i_n}^K = 1,\ V_{i_{n-1}}^K = 1, \dots, V_{i_0}^K = 1)\\
&\quad = \mathbb{P}(\mathcal{Z}_{i_n+k} \wedge K = \{K\},\ k \in I,\ \mathcal{Z}_{i_n+k} \wedge K \neq \{K\},\ k \in J \mid \mathcal{Z}_{i_n} \wedge K = \{K\},\\
&\qquad\quad \mathcal{Z}_{i_{n-1}} \wedge K = \{K\}, \dots, \mathcal{Z}_{i_0} \wedge K = \{K\})\\
&\quad = \mathbb{P}(\mathcal{Z}_{i_n+k} \wedge K = \{K\},\ k \in I,\ \mathcal{Z}_{i_n+k} \wedge K \neq \{K\},\ k \in J \mid \mathcal{Z}_{i_n} \wedge K = \{K\})\\
&\quad = \mathbb{P}(V_{i_n+k}^K = 1,\ k \in I,\ V_{i_n+k}^K = 0,\ k \in J \mid V_{i_n}^K = 1).
\end{aligned}$$

In the second equality we used the fact that $\mathcal{Z}^d \wedge K$ is a Markov process; see Lemma 1. □

Note that the conditioning on $\{V_{i_{n-j}}^K = 1\} = \{\mathcal{Z}_{i_{n-1}} \wedge K = \{K\}\}$, $j = 0, \dots, i_n$, is the conditioning on a *particular* value of the process $\mathcal{Z}^d \wedge K$. On the other hand, conditioning on $\{V_{i_{n-j}}^K = 0\} = \{\mathcal{Z}_{i_{n-1}} \wedge K \neq \{K\}\}$ would be a conditioning on a *set* of values, and, therefore, the Markov property of $\mathcal{Z}^d \wedge K$ does not imply that of $\mathcal{V}^K$.

Let us compute the transition probability vector $\vec{q}^{\,K} = (q_n^K : n \in \mathbb{N})$ of $\mathcal{V}^K$, which is given by (9).

Lemma 3. *For a compact and convex set* K *containing* 0 *in its interior, we have*

$$q_n^K = \mathrm{e}^{-(1-a^{-n})\Lambda([K])} \quad \textit{for all } n \in \mathbb{N}.$$

Proof. From (8) and from $\{\Gamma_0 \supset K, \Gamma_0 \supset a^{-n}K\} = \{\Gamma_0 \supset K\}$, we deduce that

$$\mathbb{P}(V_n^K = 1,\ V_0^K = 1) = \mathbb{P}(\Gamma_0 \supset K,\ \mathcal{C}_1^{'(j)} \supset (a-1)a^{-j}K,\ j = 1, \dots, n). \tag{14}$$

The $n+1$ random variables Γ_0, $\{\mathcal{C}_1^{'(j)} : j = 1, \dots, n\}$, are mutually independent; hence,

$$\begin{aligned}
&\mathbb{P}(V_n^K = 1, V_0^K = 1)\\
&\quad = \mathbb{P}(\Gamma_0 \supset K \mid \Gamma_0 \supset a^{-n}K)\mathbb{P}(\Gamma_0 \supset a^{-n}K, \mathcal{C}_1^{'(j)} \supset (a-1)a^{-j}K,\ j = 1, \dots, n).
\end{aligned}$$

Using the fact that the translation invariance of Λ implies that $\Lambda([a^{-n}K]) = a^{-n}\Lambda([K])$, we obtain $\mathbb{P}(\Gamma_0 \supset K \mid \Gamma_0 \supset a^{-n}K) = \mathrm{e}^{-(1-a^{-n})\Lambda([K])}$. Since

$$\mathbb{P}(\Gamma_0 \supset a^{-n}K, \mathcal{C}_1^{'(j)} \supset (a-1)a^{-j}K),\ j = 1, \dots, n) = \mathbb{P}(\Gamma_n \supset K) = \mathbb{P}(V_0^K = 1),$$

this yields $\mathbb{P}(V_n^K = 1 \mid V_0^K = 1) = \mathrm{e}^{-(1-a^{-n})\Lambda([K])}$. □

https://doi.org/10.1017/apr.2018.57 Published online by Cambridge University Press

Since the process $\mathcal{V}^K$ satisfies the regenerative property, we can associate to it the stationary regenerative set $\mathcal{V}^{K*}$. Let us study the conditional joint distribution $\mathbb{P}(\mathcal{V}^{aK*} \mid V^{K*})$.

If $K \subset K'$ are two compact convex sets then $\Gamma_i \supset K'$ implies that $\Gamma_i \supset K$ and, hence, $\mathcal{V}^{K'*} \subseteq V^{K*}$. In particular, for $a > 1$ and $0 \in K$, we have $\mathcal{V}^{aK*} \subseteq V^{K*}$. Furthermore, from (13) we have

$$\mathbb{P}(0 \in \mathcal{V}^{aK*}) = \mathbb{P}(V_0^{aK} = 1) = \mathrm{e}^{-a\Lambda([K])},$$

and from (12) for the mean recurrence time we have

$$\rho_{aK} = \mathrm{e}^{a\Lambda([K])}.$$

A straightforward calculation yields

$$\mathbb{P}(V_n^{aK} = 1 \mid V_n^K = 1) = \frac{\mathbb{P}(V_n^{aK} = 1)}{\mathbb{P}(V_n^K = 1)} = \mathrm{e}^{-(a-1)\Lambda([K])} \tag{15}$$

and

$$\rho_{aK}^{-1} = \mathrm{e}^{-(a-1)\Lambda([K])}\rho_K^{-1}.$$

Moreover, from (14) we obtain, for all $n \in \mathbb{Z}$,

$$\begin{aligned}\mathbb{P}(V_{n-1}^K = 1,\ V_n^K = 1) &= \mathbb{P}(\Gamma_{n-1} \supset K,\ \mathcal{C}_1'^{(1)} \supset (a-1)a^{-1}K)\\ &= \mathbb{P}(\Gamma_{n-1} \supset K)\mathrm{e}^{-(1-a^{-1})\Lambda([K])}.\end{aligned}$$

Also, it is easy to see that $\{V_n^{aK} = 1\} \subset \{V_n^K = 1,\ V_{n-1}^K = 1\}$, which implies that

$$\mathbb{P}(V_n^{aK} = 1 \mid V_n^K = 1,\ V_{n-1}^K = 1) = \mathrm{e}^{-(a+a^{-1}-2)\Lambda([K])} \tag{16}$$

and $\mathbb{P}(V_n^{aK} = 1 \mid V_n^K = 1,\ V_{n-1}^K = 0) = 0$. Hence, the values of $\mathbb{P}(V_n^{aK} = 1 \mid V_n^K = 1,\ V_{n-1}^K = 1)$ given in (16) are the key for the conditional joint distribution $\mathbb{P}(\mathcal{V}^{aK*} \mid V^{K*})$.

For $I \subset \mathbb{Z}$, the notation $V_I^K = 1$ means that $V_n^K = 1$ for all $n \in I$, and, similarly, $V_I^K = 0$ expresses that $V_n^K = 0$ for all $n \in I$.

Proposition 2. *Let $I^0 = [\alpha, \beta]$ be a finite interval in $\mathbb{Z}$ containing a finite family of disjoint intervals $I_t = [\alpha_t, \beta_t]$, $t = 1, \dots, l$, in $\mathbb{Z}$ that satisfies*

$$\alpha < \alpha_1, \qquad \beta_l < \beta, \qquad \alpha_t < \beta_t < \alpha_{t+1} - 1 \quad \text{for all } t = 1, \dots, l-1.$$

Let $\widehat{I}_t = [\alpha_t + 1, \beta_t]$ and $J_t \subseteq \widehat{I}_t$ for $t = 1, \dots, l$, and let $I = \bigcup_{t=1}^l I_t$, $\widehat{I} = \bigcup_{t=1}^l \widehat{I}_t$, and $J = \bigcup_{t=1}^l J_t$. Then

$$\begin{aligned}&\mathbb{P}(V_J^{aK} = 1,\ V_{\widehat{I}\setminus J}^{aK} = 0 \mid V_I^K = 1,\ V_{I^0\setminus I}^K = 0)\\ &\quad = \mathrm{e}^{-(a+a^{-1}-2)|J|\Lambda([K])}(1 - \mathrm{e}^{-(a+a^{-1}-2)\Lambda([K])})^{|\widehat{I}\setminus J|}.\end{aligned}$$

Proof. For all $i \in J$, we obtain

$$\begin{aligned}&\mathbb{P}(V_i^{aK} = 1, V_j^{aK} = 1,\ j \in J, j < i, V_j^{aK} = 0,\ j \in \widehat{I} \setminus J, j < i \mid V_I^K = 1,\ V_{I^0\setminus I}^K = 0)\\ &\quad = \mathbb{P}(V_i^{aK} = 1 \mid V_j^{aK} = 1,\ j \in J, j < i, V_j^{aK} = 0,\ j \in \widehat{I} \setminus J, j < i, V_I^K = 1,\ V_{I^0\setminus I}^K = 0)\\ &\qquad \times \mathbb{P}(V_j^{aK} = 1,\ j \in J, j < i, V_j^{aK} = 0,\ j \in \widehat{I} \setminus J, j < i \mid V_I^K = 1,\ V_{I^0\setminus I}^K = 0).\end{aligned} \tag{17}$$

https://doi.org/10.1017/apr.2018.57 Published online by Cambridge University Press

Let us consider the first factor in the expression above. Note that the condition $V_I^K = 1$ implies that $V_i^K = 1$ and $V_{i-1}^K = 1$ for all $i \in \widehat{I}$. An application of (8) yields

$$\begin{aligned}
&\mathbb{P}(V_i^{aK} = 1 \mid V_j^{a\,K} = 1,\ j \in J, j < i, V_j^{aK} = 0,\ j \in \widehat{I} \setminus J, j < i, V_I^K = 1,\ V_{I^0 \setminus I}^K = 0) \\
&\quad = \mathbb{P}\Bigg(a\Gamma_{i-1} \cap \frac{a}{a-1}\mathcal{C}_1^{'(1)} \supset a\,K \mid a\Gamma_{i-1} \cap \frac{a}{a-1}\mathcal{C}_1^{'(1)} \supset K, \Gamma_{i-1} \supset K, \\
&\qquad\qquad V_j^{aK} = 1,\ j \in J, j < i, V_j^{aK} = 0,\ j \in \widehat{I} \setminus J, j < i, V_I^K = 1,\ V_{I^0 \setminus I}^K = 0\Bigg) \\
&\quad = \mathbb{P}\Bigg(\Gamma_{i-1} \supset K, \mathcal{C}_1^{'(1)} \supset (a-1)\,K \mid \Gamma_{i-1} \supset K, \mathcal{C}_1^{'(1)} \supset \frac{a-1}{a}K, \\
&\qquad\qquad V_j^{aK} = 1,\ j \in J, j < i, V_j^{aK} = 0,\ j \in \widehat{I} \setminus J, j < i, V_I^K = 1,\ V_{I^0 \setminus I}^K = 0\Bigg) \\
&\quad = \mathbb{P}\Bigg(\mathcal{C}_1^{'(1)} \supset (a-1)K \mid \mathcal{C}_1^{'(1)} \supset \frac{a-1}{a}K,\Bigg) \\
&\quad = \mathrm{e}^{-(a+a^{-1}-2)\Lambda([K])}.
\end{aligned}$$

The third equality follows from the fact that $\mathcal{C}_1^{'(1)}$ is independent from Γ_{i-1} as well as from $\{V_j^{a\,K} = 1,\ j \in J, j < i, V_j^{a\,K} = 0,\ j \in \widehat{I} \setminus J, j < i, V_I^K = 1,\ V_{I^0 \setminus I}^K = 0\}$. Now, a recursive application of (17) yields the proposition. □

Remark 1. The results show that $\mathcal{V}^{a\,K*}$ is not constructed by an independent thinning of V^{K*}, where each 1 of V^{K*} would be kept in $\mathcal{V}^{aK*}$ with probability $\alpha \in (0, 1)$ given by (15) (or deleted with a probability $1 - \alpha$) independent of all the other 1s in V^*. For thinning in renewal processes, see [3, Chapter 10, Problem 10.5.16].

Now we show that the zero cell process is a Bernoulli flow. Let $Y = (Y_t : t > 0)$ be the STIT process, and let $\mathcal{C} = (\mathcal{C}_t : t > 0)$ be the process of its zero cells. Denote by $\mathcal{K}'$ the family of all nonempty compact convex sets in $\mathbb{R}^\ell$. Then the mapping $\mathbb{T} \to \mathcal{K}'$, $Y_t \mapsto \mathcal{C}_t$, is a measurable mapping with respect to the Borel σ-fields in $\mathbb{T}$ and $\mathcal{K}'$. The proof of the measurability is completely similar to that of Theorem 10.3.2. of [13]. Moreover, the process $\mathcal{C} = (\mathcal{C}_t : t > 0)$ inherits from Y the property of having càdlàg trajectories, i.e. $\mathcal{C}$ takes values in $D_{\mathcal{K}'}(\mathbb{R}_+)$ and the mapping $D_{\mathbb{T}}(\mathbb{R}_+) \to D_{\mathcal{K}'}(\mathbb{R}_+)$, $(Y_t : t \in \mathbb{R}) \mapsto (\mathcal{C}_t : t \in \mathbb{R})$ is measurable.

The process $\Gamma = (\Gamma_s = a^s \mathcal{C}_{a^s} : s \in \mathbb{R})$ is the zero cell process of $\mathcal{Z}$, it takes values in $D_{\mathcal{K}'}(\mathbb{R})$, and its law μ^Γ of Γ on $D_{\mathbb{T}}(\mathbb{R})$ is stationary. The mapping

$$\Theta\colon D_{\mathbb{T}}(\mathbb{R}) \to D_{\mathcal{K}'}(\mathbb{R}), \qquad (Z_t : t \in \mathbb{R}) \mapsto (\Gamma_t : t \in \mathbb{R})$$

is a factor map, that is, $\mu^\Gamma = \mu^{\mathcal{Z}} \circ \Theta^{-1}$. Let $\Gamma^d = (\Gamma_n : n \in \mathbb{Z})$ be the zero cell sequence of $\mathcal{Z}^d$, its law on $\mathcal{K}'^{\mathbb{Z}}$ is denoted by μ^{Γ^d}. The mapping

$$\Theta^d : \mathbb{T}^{\mathbb{Z}} \to \mathcal{K}'^{\mathbb{Z}}, \qquad (\mathcal{Z}_n : t \in \mathbb{R}) \mapsto (\Gamma_n : n \in \mathbb{Z})$$

is also a factor map.

Proposition 3. *The flow* $(D_{\mathcal{K}'}(\mathbb{R}), (\Gamma_t : t \in \mathbb{R}), \mu^\Gamma)$ *is a Bernoulli flow of infinite entropy.*

In [5] and [6] it was shown that $\mathcal{Z}$ is a Bernoulli flow, so, since Θ is a factor map sending $\mathcal{Z}$ on Γ, the Bernoulli property of the zero cell process Γ follows; see [11, Theorem 6, p 54]. It remains to show that it is of infinite entropy. From the regeneration property and the proof in [6], we need only to show that the distribution of Γ_1 has a nonatomic part. However, we go a step further and show that this distribution is nonatomic.

https://doi.org/10.1017/apr.2018.57 Published online by Cambridge University Press

Lemma 4. *The random variable Γ_1 is nonatomic.*

Proof. Let $\mathcal{F}$ denote the set of all closed subsets of $\mathbb{R}^\ell$, and, for $A \subset \mathbb{R}^\ell$, $\mathcal{F}^A = \{F \in \mathcal{F} : A \cap F = \varnothing\}$ and $\mathcal{F}_A = \{F \in \mathcal{F} : A \cap F \neq \varnothing\}$. Furthermore, $B(x, n^{-1})$ denotes the ball with center x and radius n^{-1}. If P is a polytope and $F_0(P)$ the set of its vertices then

$$\{P\} = \mathcal{K}' \cap \mathcal{F}^{P^c} \cap \bigcap_{n\in\mathbb{N}} \bigcap_{x\in F_0(P)} \mathcal{F}_{B(x,n^{-1})},$$

which implies that the singleton $\{P\} \in \mathcal{B}(\mathcal{K}')$ (see [13, Section 2.1]). Hence, if we assume that Γ_1 is atomic then, because $\Gamma_1 \sim \Gamma_0 = \mathcal{C}_1$, there exists a polytope P such that $\mathbb{P}(\mathcal{C}_1 = P) > 0$. Regarding the construction of STIT tessellations, this implies that, for all $(\ell-1)$-dimensional faces f of P and their 'carrying' hyperplanes $h(f) \in \mathcal{H}$ with $h(f) \supset f$, $\Lambda(\{h(f)\}) > 0$. But this contradicts the property that $\Lambda(\{h\}) = 0$ for any hyperplane $h \in \mathcal{H}$, which is a consequence of the translation invariance of the measure Λ. □

Note that the ergodicity of $\mathcal{Z}^d$ implies that we can apply the Birkhoff ergodic theorem, which yields the following corollary.

Corollary 1. *For all compact convex sets $K \subset \mathbb{R}^\ell$, we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n-1} \mathbf{1}_{\Gamma_l \supset K} = \mathrm{e}^{-\Lambda([K])} = \lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n-1} \mathbf{1}_{\mathcal{C}_l \supset l^{-1}K}.$$

Acknowledgements

The authors are thankful for the support of the Program Basal CMM from CONICYT (Chile) and the DAAD (Germany). The helpful comments from the reviewers are gratefully acknowledged.

References

[1] Asmussen, S. (2003). *Applied Probability and Queues*, 2nd edn. Springer, New York.
[2] Ethier, S. N. and Kurtz, T. G. (1986). *Markov Processes*. John Wiley, New York.
[3] Grimmett, G. R. and Stirzaker, D. R. (1992). *Probability and Random Processes: Problems and Solutions*. Oxford University Press.
[4] Lindvall, T. (1992). *Lectures on the Coupling Method*. John Wiley, New York.
[5] Martínez, S. (2014). STIT tessellations are Bernoulli and standard. *Ergodic Theory Dynam. Systems* **34,** 876–892.
[6] Martínez, S. and Nagel, W. (2012). Ergodic Description of STIT tessellations. *Stochastics* **84,** 113–134.
[7] Mecke, J., Nagel, W. and Weiss, V. (2008). A global construction of homogeneous random planar tessellations that are stable under iteration. *Stochastics* **80,** 51–67.
[8] Molchanov, I. (2005). *Theory of Random Sets*. Springer, London.
[9] Nagel, W. and Weiss, V. (2003). Limits of sequences of stationary planar tessellations. *Adv. Appl. Prob.* **35,** 123–138.
[10] Nagel, W. and Weiss, V. (2005). Crack STIT tessellations: characterization of stationary random tessellations stable with respect to iteration. *Adv. Appl. Prob.* **37,** 859–883.
[11] Ornstein, D. S. (1974). *Ergodic Theory, Randomness and Dynamical Systems*. Yale University Press.
[12] Ross, S. M. (1983). *Stochastic Processes*. John Wiley, New York.
[13] Schneider, R. and Weil, W. (2008). *Stochastic and Integral Geometry*. Springer, Berlin.

https://doi.org/10.1017/apr.2018.57 Published online by Cambridge University Press