Some alternative derivations of Craig's formula

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Introduction

In the performance of digital communication systems over fading channels, the error analysis is typically modelled using a Gaussian probability distribution. One function central to the analysis is what engineers routinely refer to as the (Gaussian) *Q-function* and is defined by

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{u^2}{2}\right) du. \tag{1}$$

This is the canonical representation used for the function. In this paper a number of derivations of an important alternative representation for the Q-function known as Craig's formula will be given.

In statistics the Q-function corresponds to the tail probability of the standard normal distribution. To see this, replacing the argument x with $-x$ in (1), if a transformation of $u \to -u$ is made one obtains

$$Q(-x) = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp\left(-\frac{u^2}{2}\right) du. \tag{2}$$

Here $\Phi(x)$ is the cumulative distribution function for the standard normal distribution. Adding (1) to (2), from the property of the definite integral over a contiguous interval of integration, we can write

$$\begin{aligned} Q(-x) + Q(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp\left(-\frac{u^2}{2}\right) du + \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{u^2}{2}\right) du \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \exp\left(-\frac{u^2}{2}\right) du. \end{aligned}$$

or

$$Q(-x) + Q(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \exp\left(-\frac{u^2}{2}\right) du, \tag{3}$$

since the integrand is an even function between symmetric limits. Substituting $u = t\sqrt{2}$ into (3) and making use of the well-known result for the value of the probability (Gaussian) integral

$$\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}, \tag{4}$$

after rearranging one obtains

$$Q(x) = 1 - Q(-x) = 1 - \Phi(x). \tag{5}$$

Thus the Q-function corresponds to the tail probability of the standard normal distribution, as claimed.

Computationally the form of the Q-function given by (1) turns out to be

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less than ideal. This is the case since the interval of integration is neither fixed nor finite. Instead, a representation where the argument appears in the integrand rather than in the lower limit of integration and where the interval of integration is both fixed and finite, is preferred.

Remarkably it was not until 1991 when, more as a by-product of other work he was engaged in, Craig gave the following alternative expression for the Q-function [1]

$$Q(x) = \frac{1}{\pi}\int_0^{\pi/2} \exp\left(-\frac{x^2}{2\sin^2\theta}\right) d\theta, \qquad x > 0. \tag{6}$$

This surprisingly simple formula, while it had been anticipated by others [2, 3, 4], appears to have not been given before. In this alternative form, the Q-function is referred to as either *Craig's formula* or as *Craig's representation* for the Q-function. Notice that in this form Craig's formula is only valid for positive x, but used in conjunction with (5), can be extended to negative values.

Currently no simple substitution which takes the Q-function from its canonical form to the form given by (6) is known. Instead, as will be shown in this paper, methods akin to those used in the evaluation of the probability integral can be used to affect a transformation leading to such a representation. In addition to the standard derivation, which we give first, a number of other new derivations for Craig's formula are given. It should come as no surprise to see the various derivations presented bear a strong resemblance to those used in the evaluation of the probability integral, a good collection of which can be found in [5]. As the Q-function and the probability integral are almost the same, differing only in the value found in their lower limit of integration, our alternative derivations of Craig's formula provide an excellent example of how old ideas can be put to use in the solution of a new problem.

The error and complementary error functions

Before proceeding with the various alternative derivations of Craig's formula, as the ongoing discussion will make heavy use of the error and complementary error functions, we briefly remind the reader of these two important special functions. Their connection to the Q-function is also given. Substituting $t\sqrt{2}$ for u into the definition for $Q(x)$, one finds

$$Q(x) = \frac{1}{\sqrt{\pi}}\int_{x/\sqrt{2}}^{\infty} e^{-t^2} dt = \frac{1}{2}\operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right) = \frac{1}{2}\left[1 - \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right]. \tag{7}$$

Here the familiar integral definition for the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2} dt,$$

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and the result

$$\operatorname{erf}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt = 1 - \operatorname{erfc}(x),$$

which provides the relation between the error function and its complement, have been used.

Alternative derivations of Craig's formula

Method I: Conversion to polar coordinates

The 'standard' method, as commonly found in engineering texts [6, 7], proceeds in a manner similar to how one usually finds the probability integral evaluated. Here one introduces a double integral by squaring the function before making a transformation from Cartesian to polar coordinates. In our case, in order to arrive at Craig's formula, from the integral definition of the error and complementary functions their squares will be converted to polar coordinates before being evaluated. Starting with the error function, from its definition, for its square we write

$$\operatorname{erf}^2(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du \cdot \frac{2}{\sqrt{\pi}} \int_0^x e^{-v^2} dv = \frac{4}{\pi} \int_0^x \int_0^x e^{-(u^2+v^2)} du\, dv.$$

On converting to polar coordinates, using $u = r\cos\theta$ and $v = r\sin\theta$, the (finite) square region in the uv-plane needs to be divided into the following two non-overlapping regions:

$$\mathcal{R}_1 = \left\{(r, \theta) : 0 \leqslant r \leqslant \frac{x}{\cos\theta},\ 0 \leqslant \theta \leqslant \frac{\pi}{4}\right\},$$

and

$$\mathcal{R}_2 = \left\{(r, \theta) : 0 \leqslant r \leqslant \frac{x}{\sin\theta},\ \frac{\pi}{4} < \theta \leqslant \frac{\pi}{2}\right\}.$$

Writing $u^2 + v^2 = r^2$ and since $du\, dv = r\, dr\, d\theta$ as the Jacobian for the transformation is r, one has

$$\operatorname{erf}^2(x) = \frac{4}{\pi} \int_0^{\pi/4} \int_0^{x/\cos\theta} re^{-r^2} dr\, d\theta + \frac{4}{\pi} \int_{\pi/4}^{\pi/2} \int_0^{x/\sin\theta} re^{-r^2} dr\, d\theta.$$

The r-integration is readily found. The result is

$$\operatorname{erf}^2(x) = 1 - \frac{2}{\pi} \int_0^{\pi/4} \exp\left(-\frac{x^2}{\cos^2\theta}\right) d\theta - \frac{2}{\pi} \int_{\pi/4}^{\pi/2} \exp\left(-\frac{x^2}{\sin^2\theta}\right) d\theta.$$

In a similar fashion, for the complementary error function from its square we can write

$$\operatorname{erfc}^2(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du \cdot \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-v^2} dv = \frac{4}{\pi} \int_x^\infty \int_x^\infty e^{-(u^2+v^2)} du\, dv.$$

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Converting to polar coordinates, the infinite rectangular region is divided into the following two non-overlapping regions:

$$\mathcal{R}_1 = \left\{(r, \theta) : \frac{x}{\sin\theta} < r \leqslant \infty, \quad 0 < \theta \leqslant \frac{\pi}{4}\right\},$$

and

$$\mathcal{R}_2 = \left\{(r, \theta) : \frac{x}{\cos\theta} \leqslant r < \infty, \quad \frac{\pi}{4} < \theta \leqslant \frac{\pi}{2}\right\}.$$

Again the r-integration can be readily performed with the result

$$\operatorname{erfc}^2(x) = \frac{2}{\pi}\int_0^{\pi/4} \exp\left(-\frac{x^2}{\sin^2\theta}\right)d\theta + \frac{2}{\pi}\int_{\pi/4}^{\pi/2} \exp\left(-\frac{x^2}{\cos^2\theta}\right)d\theta.$$

We now relate the squares of the error and complementary error functions to erfc (x). By observing that

$$\operatorname{erfc}^2(x) + \left(1 - \operatorname{erf}^2(x)\right) = \operatorname{erfc}^2(x) + 1 - (1 - \operatorname{erfc}(x))^2 = 2\operatorname{erfc}(x),$$

we find

$$\operatorname{erfc}(x) = \frac{1}{2} - \frac{1}{2}\operatorname{erf}^2(x) + \frac{1}{2}\operatorname{erfc}^2(x). \tag{8}$$

Substituting the integrals found for $\operatorname{erf}^2(x)$ and $\operatorname{erfc}^2(x)$ into the above expression, one finds

$$\operatorname{erfc}(x) = \frac{1}{\pi}\int_0^{\pi/2} \exp\left(-\frac{x^2}{\sin^2\theta}\right)d\theta + \frac{1}{\pi}\int_0^{\pi/2} \exp\left(-\frac{x^2}{\cos^2\theta}\right)d\theta.$$

From (7), in terms of the Q-function this can be written as

$$Q(x) = \frac{1}{2\pi}\int_0^{\pi/2} \exp\left(-\frac{x^2}{2\sin^2\theta}\right)d\theta + \frac{1}{2\pi}\int_0^{\pi/2} \exp\left(-\frac{x^2}{2\cos^2\theta}\right)d\theta.$$

If, in the second of these integrals $\theta \to \frac{\pi}{2} - \theta$, Craig's formula follows. If however in the first of these integrals $\theta \to \frac{\pi}{2} - \theta$, the following alternative (cosine) form for Craig's formula results

$$Q(x) = \frac{1}{\pi}\int_0^{\pi/2} \exp\left(-\frac{x^2}{2\cos^2\theta}\right)d\theta, \qquad x > 0.$$

Method II: *Using an auxiliary function*

The idea here is to consider an auxiliary function, related to our function of interest, but which turns out to be a constant function for all x in its domain. While it is not immediately obvious what such a function with these properties ought to be, the fourth and fifth methods to be considered shortly should at least provide a clue to possible types of auxiliary functions one may care to consider.

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We start by considering the following auxiliary function

$$I(x) = \left(\int_0^{\pi} e^{-t^2} dt\right)^2 + \int_0^1 \frac{e^{-x^2(t^2+1)}}{1+t^2}\, dt, \qquad x > 0. \tag{9}$$

On differentiating the auxiliary function with respect to x we have

$$I'(x) = 2e^{-x^2} \int_0^{\pi} e^{-t^2} dt - 2xe^{-x^2} \int_0^1 e^{-x^2 t^2}\, dt.$$

In obtaining this result, Leibniz' rule for differentiating under the integral sign has been used. In the second integral, if a substitution of $u = xt$ is made, the result $I'(x) = 0$ quickly follows showing the auxiliary function is indeed constant for all $x > 0$. To find the value for this constant, letting $x \to 0^+$ gives

$$I(x) \to \int_0^1 \frac{dt}{1+t^2} = \frac{\pi}{4},$$

so that $I(x) = \pi/4$ for all $x > 0$. As the first of the integrals appearing in (9) can be written in terms of the error function we have

$$\int_0^1 \frac{e^{-x^2(t^2+1)}}{1+t^2} dt = \frac{\pi}{4}\left(1 - \operatorname{erf}^2(x)\right). \tag{10}$$

A similar thing can be done for the complementary error function. In this case we start by considering the following auxiliary function

$$J(x) = \left(\int_x^{\infty} e^{-t^2} dt\right)^2 - \int_1^{\infty} \frac{e^{-x^2(t^2+1)}}{1+t^2}\, dt, \qquad x > 0. \tag{11}$$

On differentiating with respect to x we have

$$J'(x) = -2e^{-x^2} \int_x^{\infty} e^{-t^2} dt + 2xe^{-x^2} \int_1^{\infty} e^{-x^2 t^2} dt.$$

A substitution of $u = xt$ in the second integral once again reduces the derivative of the auxiliary function to zero, showing $J(x)$ is constant. Letting $x \to \infty$ in (11) we see $J(x) \to 0$. Thus $J(x) = 0$ for all $x > 0$. Writing the first of the integrals in (11) in terms of the complementary error function, one finds

$$\int_1^{\infty} \frac{e^{-x^2(1+t^2)}}{1+t^2}\, dt = \frac{\pi}{4} \operatorname{erfc}^2(x). \tag{12}$$

Substituting (10) and (12) into (8) yields

$$\operatorname{erfc}(x) = \frac{2}{\pi} \int_0^{\infty} \frac{e^{-x^2(1+t^2)}}{1+t^2}\, dt. \tag{13}$$

Setting $t = \cot\theta$ in (13), as $dt = -\operatorname{cosec}^2\theta\, d\theta$ and the term $1 + t^2$ becomes $\operatorname{cosec}^2\theta$, one obtains

$$\operatorname{erfc}(x) = \frac{2}{\pi} \int_0^{\pi/2} \exp\left(-\frac{x^2}{\sin^2\theta}\right) d\theta.$$

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from which the sine form for Craig's formula follows. If on the other hand a substitution of $t = \tan\theta$ is made in (13), as $dt = \sec^2\theta\, d\theta$ and the term $1 + t^2$ becomes $\sec^2\theta$, one obtains

$$\operatorname{erfc}(x) = \frac{2}{\pi}\int_0^{\pi/2} \exp\left(-\frac{x^2}{\cos^2\theta}\right) d\theta,$$

from which the cosine form for Craig's formula

$$Q(x) = \frac{1}{\pi}\int_0^{\pi/2} \exp\left(-\frac{x^2}{2\cos^2\theta}\right) d\theta,$$

follows.

As an added bonus, notice that if a substitution of $t = \tan\theta$ is made in (12) one obtains

$$\operatorname{erfc}^2(x) = \frac{4}{\pi}\int_{\pi/4}^{\pi/2} \exp\left(-\frac{x^2}{\cos^2\theta}\right) d\theta, \tag{14}$$

leading to the following Craig-like expression for the square of the Q-function

$$Q^2(x) = \frac{1}{\pi}\int_{\pi/4}^{\pi/2} \exp\left(-\frac{x^2}{2\cos^2\theta}\right) d\theta, \qquad x > 0,$$

while if a substitution of $t = \cot\theta$ is made, the following alternative Craig-like form for the square of the Q-function

$$Q^2(x) = \frac{1}{\pi}\int_0^{\pi/4} \exp\left(-\frac{x^2}{2\sin^2\theta}\right) d\theta, \qquad x > 0,$$

results.

Method III: *Solution of a differential equation*

In the next method we set up a differential equation which has the complementary error function as its solution. Let

$$I(x) = \int_0^{\infty} \frac{e^{-x^2t^2}}{1 + t^2}\, dt, \qquad x > 0. \tag{15}$$

Differentiating with respect to x we find

$$I'(x) = -2x\int_0^{\infty} \frac{t^2}{1 + t^2}\, e^{-x^2t^2} dt.$$

Note here Leibniz' rule for differentiating under the integral sign has been used. Writing the rational function term appearing in the integrand in terms of t as

$$\frac{t^2}{1 + t^2} = \frac{(1 + t^2) - 1}{1 + t^2} = 1 - \frac{1}{1 + t^2},$$

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we find

$$I'(x) = -2x\int_0^\infty e^{-x^2t^2}dt + 2x\int_0^\infty \frac{1}{1+t^2}e^{-x^2t^2}dt$$
$$= -2x\int_0^\infty e^{-x^2t^2}dt + 2xI(x).$$

In the above integral, on setting $u = xt$ we obtain

$$I'(x) = -\sqrt{\pi}\cdot\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-u^2}du + 2xI(x) = -\sqrt{\pi} + 2xI(x),$$

where once more we have used the result for the value of the probability integral. Thus

$$I'(x) - 2xI(x) = -\sqrt{\pi}, \tag{16}$$

a first-order linear differential equation. An integrating factor for this equation is $R(x) = e^{-x^2}$. Multiplying each side of (16) by this integrating factor allows one to write the linear differential equation in the following separable form

$$\frac{d}{dx}\left(e^{-x^2}I(x)\right) = -\sqrt{\pi}e^{-x^2}.$$

On separating and integrating

$$e^{-x^2}I(x) = -\frac{\pi}{2}\cdot\frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du + c. \tag{17}$$

Here c is an arbitrary constant of integration. To find the value of this constant, let $x \to 0^+$ in (15). Thus

$$I(x) \to \int_0^\infty \frac{dt}{1+t^2} = \frac{\pi}{2}.$$

Now letting $x \to 0^+$ in (17), one finds $c = \frac{\pi}{2}$. So from (17) we have

$$e^{-x^2}I(x) = \frac{\pi}{2}\left[1 - \operatorname{erf}(x)\right] = \frac{\pi}{2}\operatorname{erfc}(x).$$

But as the expression for $I(x)$ is known – it is after all given by (15) – we can write

$$e^{-x^2}\int_0^\infty \frac{e^{-x^2t^2}}{1+t^2}dt = \frac{\pi}{2}\operatorname{erfc}(x),$$

or

$$\operatorname{erfc}(x) = \frac{2}{\pi}\int_0^\infty \frac{e^{-x^2(1+t^2)}}{1+t^2}dt.$$

The sine and cosine forms for Craig's formula then follow when a substitution of either $t = \cot\theta$ or $t = \tan\theta$ is used, as was done in *Method II*.

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Method IV: Differentiating under the integral sign (again)

From the integral definition for the error function, on squaring and rearranging we can write

$$A_1(x) = \frac{\pi}{4}\operatorname{erf}^2(x) = \left(\int_0^x e^{-u^2}du\right)^2, \qquad x > 0.$$

Differentiating $A_1(x)$ with respect to x, after making use of the Fundamental Theorem of Calculus we have

$$A_1'(x) = 2\int_0^x e^{-u^2}du \cdot e^{-x^2} = 2e^{-x^2}\int_0^x e^{-u^2}du.$$

Now if we let $u = xy$

$$A_1'(x) = \int_0^1 2xe^{-x^2(1+y^2)}dy.$$

The function appearing under the integral sign is readily antidifferentiated with respect to x. Noting that

$$\frac{\partial}{\partial x}\left[\frac{e^{-x^2(1+y^2)}}{1+y^2}\right] = -2xe^{-x^2(1+y^2)}, \tag{18}$$

we can write

$$A_1'(x) = -\int_0^1 \frac{\partial}{\partial x}\left[\frac{e^{-x^2(1+y^2)}}{1+y^2}\right]dy = \frac{d}{dx}\int_0^1 \frac{e^{-x^2(1+y^2)}}{1+y^2}\,dy.$$

Setting

$$B_1(x) = \int_0^1 \frac{e^{-x^2(1+y^2)}}{1+y^2}\,dy$$

we have $A_1'(x) = -B_1'(x)$ for all $x > 0$. Thus there exists a constant c_1 such that

$$A_1(x) = -B_1(x) = c_1. \tag{19}$$

To find the value of the constant, let $x \to 0^+$ in the expressions for $A_1(x)$ and $B_1(x)$. Thus $A_1(x) \to 0$ while

$$B_1(x) \to \int_0^1 \frac{dy}{1+y^2} = \frac{\pi}{4}.$$

Hence $c_1 = \frac{\pi}{4}$ and (19) becomes

$$\int_0^1 \frac{e^{-x^2(1+y^2)}}{1+y^2}\,dy = \frac{\pi}{4}\left[1 - \operatorname{erf}^2(x)\right],$$

which is the same expression for the square of the error function obtained in *Method II* (see (10)).

Similarly, if instead we were to start with the integral definition for the complementary error function, on squaring and rearranging one can write

$$A_2(x) = \frac{\pi}{4}\operatorname{erfc}^2(x) = \left(\int_x^\infty e^{-u^2}du\right)^2.$$

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For $A_2(x)$, differentiating once again with respect to x, after making use of the Fundamental Theorem of Calculus we have

$$A_2'(x) = -2\int_x^\infty e^{-u^2}du \cdot e^{-x^2} = -2e^{-x^2}\int_x^\infty e^{-u^2}du.$$

Setting $u = xy$ gives

$$A_2'(x) = \int_1^\infty -2xe^{-x^2(1+y^2)}dy,$$

which after applying (18) becomes

$$A_2'(x) = \int_1^\infty \frac{\partial}{\partial x}\left[\frac{e^{-x^2(1+y^2)}}{1+y^2}\right]dy = \frac{d}{dx}\int_0^\infty \frac{e^{-x^2(1+y^2)}}{1+y^2}\,dy.$$

Setting

$$B_2(x) = \int_1^\infty \frac{e^{-x^2(1+y^2)}}{1+y^2}\,dy$$

we have $A_2'(x) = B_2'(x)$ for all $x > 0$ so there exists a constant c_2 such that

$$A_2(x) = B_2(x) + c_2. \tag{20}$$

To find the value of the constant, let $x \to 0^+$ in the expressions for $A_2(x)$ and $B_2(x)$. So

$$A_2(x) \to \left(\int_0^\infty e^{-u^2}du\right) = \left(\frac{\sqrt{\pi}}{2}\right)^2 = \frac{\pi}{4},$$

and

$$B_2(x) \to \int_1^\infty \frac{dy}{1+y^2} = \frac{\pi}{4}.$$

So $c_2 = 0$ and (20) becomes

$$\int_1^\infty \frac{e^{-x^2(1+y^2)}}{1+y^2}\,dy = \frac{\pi}{4}\,\mathrm{erfc}^2(x),$$

and is the same as expression (12) previously found. Craig's formula then follows in the same way as in *Method II*.

Method V: Another change of variables

In this second change of variables method we start by finding expressions for the square of the error and complementary error functions. For the square of the error function

$$\mathrm{erf}^2(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du \cdot \frac{2}{\sqrt{\pi}}\int_0^x e^{-v^2}dv = \frac{4}{\pi}\int_0^x\int_0^x e^{-(u^2+v^2)}du\,dv.$$

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Applying the transformation $(u, v) \rightarrow (t, yt)$ to the double integral, the Jacobian for such a transformation is equal to t while the limits of integration become

$$\left\{(y, t) : 0 \leqslant t \leqslant x, \ 0 \leqslant y \leqslant \frac{x}{t}\right\}.$$

Under these changes of variable the double integral can be written as

$$\operatorname{erf}^2(x) = \frac{4}{\pi}\int_0^x \int_0^{x/t} te^{-t^2(1+y^2)} dy\, dt.$$

Changing the order of integration, the region of integration in the yt-plane needs to be divided into two separate regions. For the first region

$$\mathscr{R}_1 = \{(y, t) : 0 \leqslant t \leqslant x, \ 0 \leqslant y < 1\}$$

while for the second region

$$\mathscr{R}_2 = \left\{(y, t) : 0 \leqslant t \leqslant \frac{x}{y}, \ 1 \leqslant y < \infty\right\}.$$

After this change of order we have

$$\operatorname{erf}^2(x) = \frac{4}{\pi}\int_0^1 \int_0^x te^{-t^2(1+y^2)} dt\, dy + \frac{4}{\pi}\int_1^\infty \int_0^{x/y} te^{-t^2(1+y^2)} dt\, dy.$$

Evaluating the inner t-integrals, and simplifying, one obtains

$$\operatorname{erf}^2(x) = 1 - \frac{2}{\pi}\int_0^1 \frac{e^{-x^2(1+y^2)}}{1+y^2} dy - \frac{2}{\pi}\int_1^\infty \frac{e^{-x^2\left(\frac{1+y^2}{y^2}\right)}}{1+y^2} dy.$$

If in the first integral we let $y = \cot\theta$ so that $dy = -\operatorname{cosec}^2\theta\, d\theta$ and in the second we let $y = \tan\theta$ so that $dy = \sec^2\theta\, d\theta$, after an appropriate change in the respective limits of integration we finally arrive at

$$\operatorname{erf}^2(x) = 1 - \frac{4}{\pi}\int_{\pi/4}^{\pi/2} \exp\left(-\frac{x^2}{\sin^2\theta}\right) d\theta. \tag{21}$$

Observe that (21) is equivalent to (10) on setting $t = \operatorname{cosec}^2\theta$ in (10).

Repeating the procedure for the square of the complementary error function

$$\operatorname{erfc}^2(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2} du \cdot \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-v^2} dv = \frac{4}{\pi}\int_x^\infty \int_x^\infty e^{-(u^2+v^2)} du\, dv.$$

Under the same change of variables of $(u, v) \rightarrow (t, yt)$ the Jacobian is the same as before while for the limits of integration we have

$$\left\{(y, t) : \frac{x}{t} \leqslant y < \infty, x \leqslant t < \infty\right\}.$$

Thus

$$\operatorname{erfc}^2(x) = \frac{4}{\pi}\int_x^\infty \int_{x/t}^\infty te^{-t^2(1+y^2)} dy\, dt.$$

A change in the order of the limits of integration is again applied. In doing so the region of integration in the yt-plane once again needs to be divided up

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into two separate regions. For the first region

$$\mathcal{R}_1 = \{(y, t) : x \leqslant t < \infty, \ 1 \leqslant y < \infty\},$$

while for the second region

$$\mathcal{R}_2 = \left\{(y, t) : \frac{x}{y} \leqslant t < \infty, \ 0 \leqslant y < 1\right\}.$$

After this change in the order of integration the double integral becomes

$$\operatorname{erfc}^2(x) = \frac{4}{\pi}\int_1^\infty \int_x^\infty te^{-t^2(1+y^2)}dt\,dy + \frac{4}{\pi}\int_0^1 \int_{x/y}^\infty te^{-t^2(1+y^2)}dt\,dy.$$

Evaluating the inner t-integrals and simplifying gives

$$\operatorname{erfc}^2(x) = \frac{2}{\pi}\int_1^\infty \frac{e^{-x^2(1+y^2)}}{1+y^2}\,dy + \frac{2}{\pi}\int_0^1 \frac{e^{-x^2\left(\frac{1+y^2}{y^2}\right)}}{1+y^2}\,dy.$$

In the first integral setting $y = \cot\theta$ gives $dy = -\operatorname{cosec}^2\theta\,d\theta$ and in the second setting $y = \tan\theta$ gives $dy = \sec^2\theta\,d\theta$. After the appropriate changes in the limits of integration are made one obtains

$$\operatorname{erfc}^2(x) = \frac{4}{\pi}\int_0^{\pi/4} \exp\left(-\frac{x^2}{\sin^2\theta}\right)d\theta. \tag{22}$$

Note (22) is equivalent to (14) on setting $\theta \to \frac{\pi}{2} - \theta$. The sine form for Craig's formula now follows on substituting (21) and (22) into (8) and converting the resulting expression found for $\operatorname{erfc}(x)$ into the Q-function.

Some final remarks

In finding Craig's formula the second, third, and fourth methods all depended on reducing the problem to the following integral representation for the complementary error function

$$\operatorname{erfc}(x) = \frac{2}{\pi}\int_1^\infty \frac{e^{-x^2(1+t^2)}}{1+t^2}\,dt. \tag{23}$$

Craig's formula then lies just a simple substitution away.

Of course the above integral form for the complementary error function was known well before Craig stumbled across his result, via other means, in late 1991. For example, the integral is found in [8, Eq. 3.2.7], [9, Eq. 7.4.11], and [2, Ex. 6, p. 29], and other earlier references to it no doubt exist. However it seems no one foresaw nor appreciated how a simple substitution of either $t = \tan\theta$ or $t = \cot\theta$ in (23) would bring it into a form that has proved to be so useful to the engineer (see, for example, [10] where the virtues of using the Q-function in Craig's form are expounded). As an example, it shows how something very simple can often be easily missed. Other slightly more involved derivations of Craig's formula not given here can be found in [11, 12] . Finally, it is worth noting that all of the methods given can be modified to find the value of the probability integral (see (4)). Using a limiting process, in those formulae containing $\operatorname{erf}^2(x)$ one lets $x \to \infty$ while in those containing $\operatorname{erfc}^2(x)$ one lets $x \to 0^+$.

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Centenary of the birth of Bill Tutte

Readers may have seen recent media coverage commemorating the 100th anniversary of the birth of William Thomas Tutte who was born in Newmarket, Suffolk on 14th May 1917. He showed prodigious mathematical talent at a very early age and went on to be a leading member of the codebreaking team based at Bletchley Park. Readers wishing to know more of him could usefully view his Wikipedia entry https://en.wikipedia.org/wiki/W._T._Tutte or the BBC documentary http://www.bbc.co.uk/news/uk-england-suffolk-29064159 or search more widely on the internet.

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