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Magnetohydrodynamic equations under anisotropic conditions

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Scaling analysis is used to derive approximations of magnetohydrodynamics with self-consistent leading-order dynamics under general conditions of anisotropy. Both incompressible and weakly compressible limits are considered. The horizontal magnetic and velocity fields obey dynamics given by a reduced closed set of equations, but the vertical components have different decoupled dynamics in the different regimes. Conservation laws are also discussed. It is shown that the reduced equations are not self-consistent unless either the ratio of vertical to horizontal length scales is large or the fluid is gravitationally stratified and the ratio of length scales is small. New equations are derived for a rotating stratified plasma, which are an extension of the quasigeostrophic equations of neutral fluids.

1. Introduction

Reduced dynamical equations sets are often of great use in studying physical systems. Because they separate physical processes from one another by construction, they are well suited to the study of dynamics on a particular time scale without extraneous detail to confuse the analysis. In addition, they are more efficiently solvable by numerical computation. Two reduced equation sets that have been greatly successful in fluid dynamics are quasigeostrophy (Pedlosky 1979) and reduced magnetohydrodynamics (Strauss 1976, 1977; Montgomery 1982), sometimes called the Strauss equations. We refer to these as QG and RMHD respectively. QG is a reduction of the Navier–Stokes equations based on the assumption of anisotropy due to strong rotation about and stable density stratification along a vertical axis. RMHD is derived from magnetohydrodynamics (MHD) based on the assumption of anisotropy due to a dominant vertical magnetic field. Both are three-dimensional (3D) systems, but have horizontal motions that are nearly non-divergent, as in two-dimensional (2D) systems. In the QG case vertical length scales are assumed to be much smaller than horizontal length scales, whereas in RMHD the vertical length scales are assumed to be much longer.

In this paper we derive reduced equations from MHD under general assumptions of anisotropy. We compare the scalings obtained with those of RMHD and QG and arrive at a set of equations for a fluid plasma in a rotating, stratified environment. These equations are applicable to low-frequency motions (i.e. those evolving at advective or shear-Alfvén rates) in systems

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such as the stably stratified radiative interior of the Sun or the upper altitudes of the Earth's ionosphere/magnetosphere, where a background magnetic field, fluid rotation and stratification are all present. In the process, we clarify the relationship between the different scaling regimes, and in particular address the seemingly contradictory assumptions of vertical length scale traditionally associated with QG and RMHD.

We begin with the MHD equations in a frame rotating with angular frequency $\Omega\hat{\mathbf{z}}$ and with a gravitational force $-g\hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ is the unit vertical vector. Anticipating the anisotropic nature of our final equations, vector field quantities are assumed to be horizontally two-dimensional. Thus the velocity field is $(\mathbf{v}, w)$ and the magnetic field is $(\mathbf{B}, B_* + b)$, with B_* a constant. Partial derivatives with respect to coordinates are indicated by subscripts. The total density is $\rho^{\mathrm{T}} = \rho_0 + \bar{\rho}(z) + \rho$, and the total pressure is $p^{\mathrm{T}} = p_0 - gz\rho_0 + \bar{p}(z) + p$, with constants ρ_0 and p_0. No generality is lost in assuming a hydrostatic, stably stratified, motionless reference state $g\bar{\rho} = \bar{p}_z$, because $\bar{\rho}$ and $\bar{p}$ may always be considered to be zero.

In order to perform a scaling analysis, we rewrite all variables in terms of their dimensionless counterparts:

$$\rho^{\mathrm{T}} \to \rho_0 \rho^{\mathrm{T}}, \qquad \bar{\rho} \to \bar{\rho}_0 \bar{\rho}, \qquad \rho \to D\rho, \tag{1}$$

$$p^{\mathrm{T}} \to p_0 p^{\mathrm{T}}, \qquad \bar{p} \to \bar{p}_0 \bar{p}, \qquad p \to Pp, \tag{2}$$

$$\mathbf{v} \to U\mathbf{v}, \qquad w \to Ww, \qquad \mathbf{B} \to U(4\pi\rho_0)^{1/2}\mathbf{B}, \qquad b \to W(4\pi\rho_0)^{1/2} b, \tag{3}$$

$$\boldsymbol{\nabla} \to \frac{1}{L}\boldsymbol{\nabla}, \qquad \partial_z \to \frac{1}{H}\partial_z, \qquad \partial_t \to \frac{U}{L}\partial_t. \tag{4}$$

Here we have assumed that the velocity field $(\mathbf{v}, w)$ has magnitude comparable to the Alfvén speed $(\mathbf{B}, b)/(4\pi\rho_0)^{1/2}$. This assumption of equipartition is sometimes called the 'Alfvén effect' (Kraichnan 1965), and is based on the facts that Alfvén waves propagating along a large-scale magnetic field obey this scaling and that MHD turbulent cascades often lead to it as well. Such equipartition has long been observed both in the interplanetary magnetic field (see e.g. Belcher and Davis 1971; Roberts *et al.* 1987) and in turbulent numerical solutions (see e.g. Fyfe *et al.* 1977; Politano *et al.* 1989; Biskamp and Welter 1989; Kinney *et al.* 1995). All dimensionless fields are now assumed to be of order 1, with the relative magnitudes of the physical fields determined by the dimensionless parameters

$$\frac{U}{2\Omega L} \equiv R, \qquad \frac{gHD}{\rho_0 U^2} \equiv \frac{1}{\sigma^2}, \tag{5}$$

$$\frac{H}{L} \equiv \alpha, \qquad \frac{P}{\rho_0 U^2} \equiv \frac{1}{\mu^2}, \tag{6}$$

$$\frac{W/H}{U/L} \equiv \epsilon, \qquad \frac{B_*/(4\pi\rho_0)^{1/2} H}{U/L} \equiv \lambda. \tag{7}$$

The plasma β may be specified in terms of these parameters, but it must be

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measured relative to the strongest contribution to the magnetic pressure, i.e. the largest of U^2, $B_* b$ and b^2. Thus the plasma β is

$$\beta = \mathrm{Min}\left[\frac{1}{\mu^2}, \frac{1}{\mu^2\alpha^2\epsilon\lambda}, \frac{1}{\mu^2\alpha^2\epsilon^2}\right]. \tag{8}$$

The dimensionless MHD equations are

$$\begin{aligned}\rho^{\mathrm{T}}(\mathbf{v}_t + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} + \epsilon w\mathbf{v}_z) = {} & \mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{B} + (\lambda+\epsilon b)\,\mathbf{B}_z \\ & -\tfrac{1}{2}\boldsymbol{\nabla}(\mathbf{B}\cdot\mathbf{B}) - \alpha^2\epsilon\lambda\boldsymbol{\nabla}b - \alpha^2\epsilon^2\tfrac{1}{2}\boldsymbol{\nabla}b^2 \\ & +\frac{1}{R}\rho^{\mathrm{T}}\hat{\mathbf{z}}\times\mathbf{v} - \frac{1}{\mu^2}\boldsymbol{\nabla}p,\end{aligned} \tag{9}$$

$$\mathbf{B}_t + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{B} + \epsilon w\mathbf{B}_z = \mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{v} + (\lambda+\epsilon b)\,\mathbf{v}_z - \mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{v} + \epsilon w_z) \tag{10}$$

$$\alpha^2\epsilon\rho^{\mathrm{T}}(w_t + \mathbf{v}\cdot\boldsymbol{\nabla}w + \epsilon w w_z) = \alpha^2\epsilon\mathbf{B}\cdot\boldsymbol{\nabla}b - \frac{1}{2}(\mathbf{B}\cdot\mathbf{B})_z - \frac{1}{\sigma^2}\rho - \frac{1}{\mu^2}p_z, \tag{11}$$

$$b_t + \mathbf{v}\cdot\boldsymbol{\nabla}b + \epsilon w b_z = \mathbf{B}\cdot\boldsymbol{\nabla}w - \left(b + \frac{\lambda}{\epsilon}\right)\boldsymbol{\nabla}\cdot\mathbf{v}, \tag{12}$$

$$\frac{D}{\rho_0}(\rho_t + \mathbf{v}\cdot\boldsymbol{\nabla}\rho + \epsilon w\rho_z) + \epsilon\frac{\bar{\rho}_0}{\rho_0}w\bar{\rho}_z = -\rho^{\mathrm{T}}(\boldsymbol{\nabla}\cdot\mathbf{v} + \epsilon w_z), \tag{13}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} + \epsilon b_z = 0. \tag{14}$$

Here ρ^{T} refers to the combination $1 + (\bar{\rho}_0/\rho_0)\,\rho + (D/\rho_0)\,\rho$. To close this system, one must specify an equation of state and an internal energy equation.

We seek to reduce the temporal order of this system by making assumptions about the size of certain dimensionless parameters. We are seeking solutions in which the fields are anisotropic with respect to their vertical and horizontal components, i.e.

$$\epsilon \ll 1. \tag{15}$$

We also assume

$$\lambda \sim 1, \tag{16}$$

which is the same as assuming $B_* \gg b$ (if $B_* \sim b$ then B_* may simply be absorbed into b). The assumption $\epsilon \ll 1$ does not restrict the relative magnitude of $\{w, b\}$ compared with $\{\mathbf{v}, \mathbf{B}\}$ *per se*, nor does it explicitly refer to the aspect ratio α. Only the combination ϵ, the ratio of advective time scales for vertical and horizontal motions of the fluid, is specified. In practice, a number of different physical environments can result in small ϵ. Our approach is a traditional asymptotic expansion in ϵ, such that $\mathbf{v} = \mathbf{v}^0 + \epsilon\mathbf{v}^1 + \epsilon^2\mathbf{v}^2 + \dots$, and similarly for all fields. In so doing, we seek only the choices of parameters in (5) that give rise to a self-consistent ordering of all of the dynamical balances. Therefore we can only state that our derived equations represent a formally consistent subdynamics of 3D MHD. Although intuition suggests that under the appropriate conditions this dynamics will be achieved, we cannot claim to have proved so. A formal multiple-time-scale analysis such as has been used to derive RMHD (Zank and Mattheus 1992) and nearly incompressible MHD (Zank and Mattheus 1993) is a stronger indication of the realizability of such subdynamics, but an actual proof must be constructive, by explicit calculation of solutions.

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Equation (14) allows us to write

$$\mathbf{B}^0 = \hat{\mathbf{z}} \times \boldsymbol{\nabla} A. \tag{17}$$

Density fluctuations are assumed to be much smaller than the background, i.e.

$$D \ll \rho_0, \qquad \bar{\rho}_0 \ll \rho_0, \tag{18}$$

though the relative sizes of D and $\bar{\rho}_0$ are left open for the time being. This assumption and (13) imply that $\boldsymbol{\nabla} \cdot \mathbf{v}^0 = 0$, so we write

$$\mathbf{v}^0 = \hat{\mathbf{z}} \times \boldsymbol{\nabla} \Psi. \tag{19}$$

By applying $\hat{\mathbf{z}} \cdot \boldsymbol{\nabla} \times$ to (9) and (10) and spatially integrating the latter, we obtain

$$D_t \nabla^2 \Psi - [A, \nabla^2 A] = \lambda \nabla^2 A_z + \frac{\epsilon}{R} \boldsymbol{\nabla} \cdot \mathbf{v}^1, \tag{20}$$

$$D_t A = \lambda \Psi_z, \tag{21}$$

where terms of order ϵ have been dropped, brackets denote the Jacobian in horizontal coordinates, and $D_t \equiv \partial_t + \mathbf{v} \cdot \boldsymbol{\nabla} = \partial_t + [\Psi, \cdot\,]$.

Equations (20) and (21) will determine the evolution of A and Ψ in the several regimes examined below. The only assumption made in deriving them is that ϵ is small. In particular, no restriction is made on the aspect ratio α, which may be large or small. However, in order to constitute a valid physical model, (11)–(13) must also be considered in conjunction with (20) and (21), and the entire set of equations must be shown to have self-consistent leading-order balances. The physical interpretation and evolution of the 'slave' fields (i.e. those whose evolution does not affect A and Ψ) b, w, p, and ρ will be different in different parameter regimes. Our goal is to derive the scalings under which RMHD or QG are self-consistent and to reconcile the apparent disparity in aspect ratio mentioned earlier between these two anisotropic reduced equation sets. We will also explore the difference between the incompressible and weakly compressible limits.

2. Weakly rotating, non-gravitational, incompressible motions

In a weakly rotating fluid, $R \approx 1$ or larger. Non-gravitational means

$$1/\sigma^2 = \bar{\rho}_0 = \bar{p}_0 = 0.$$

By taking the incompressible limit, we assume $D/\rho_0 \leqslant \epsilon^2$, so that (13) implies

$$\boldsymbol{\nabla} \cdot \mathbf{v}^1 = -w_z^0. \tag{22}$$

We wish to explore the consistency of RMHD with respect to the aspect ratio, and look first at $\alpha \sim \epsilon^{-1/2}$, so that $\beta = 1/\mu^2$. In that case (9) gives

$$p = -b - \tfrac{1}{2}\mathbf{B} \cdot \mathbf{B} + \pi \tag{23}$$

for $\beta \sim 1$. No generality is lost by taking $\beta \sim 1$, because if $\beta \gtrsim \epsilon^{-1}$ then (9) gives $p^0 = 0$ and p^1 satisfies (23). The field π satisfies

$$\nabla^2 \pi = \hat{\mathbf{z}} \cdot (\boldsymbol{\nabla} \times [\Psi, \boldsymbol{\nabla} \Psi] - \boldsymbol{\nabla} \times [A, \boldsymbol{\nabla} A]) + \frac{1}{R} \nabla^2 \Psi, \tag{24}$$

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whose solution is unique up to an arbitrary function $F(z,t)$. Thus π is a diagnostic component of the pressure field that is directly forced by the Ψ and A fields.

The leading-order equations for the velocity and magnetic fields in dimensional form are

$$D_t \nabla^2 \Psi = [A, \nabla^2 A] + B_* \nabla^2 A_z, \tag{25a}$$

$$D_t A = B_* \Psi_z, \tag{25b}$$

$$D_t b = [A, w] + B_* w_z, \tag{25c}$$

$$D_t w = [A, b] + B_* b_z - \pi_z. \tag{25d}$$

In the dimensional equations we have rescaled A, b and B_* by $(4\pi\rho_0)^{1/2}$ and π by ρ_0. The structure of (24) and (25) is that Ψ and A have an independent dynamics, the Strauss equations, while π is a diagnostic field forced by Ψ and A, which in turn forces the coupled equations for b and w (ρ is irrelevant in this non-gravitational, incompressible regime). Thus, p, π, b and ω are slave fields to the independently evolving Ψ and A. Equations (25) support linear waves of the form $w \propto e^{ik(B_* t \pm z)}$, which are absent in the full 3D MHD equations. However, by construction, we have chosen an ordering in which all terms are of the same order of magnitude, so the propagation of these modes will always be interfered with by the $[A, w]$ and $[A, b]$ terms, except for those modes with no dependence on the horizontal coordinates. Such plane waves can always be removed by adding the appropriate $F(z,t)$ to π (see above). Note that the system (25) conserves the energy

$$H = \int (\tfrac{1}{2}|\boldsymbol{\nabla}\Psi|^2 + \tfrac{1}{2}|\boldsymbol{\nabla} A|^2). \tag{26}$$

Alternatively, if the aspect ratio is large ($\alpha \sim \epsilon^{-1}$), then $\beta \sim 1/\mu^2\alpha^2\epsilon\lambda$, and (9) gives

$$\boldsymbol{\nabla}(b + \beta p) = 0. \tag{27}$$

This ordering regime divides into a 'high-beta' and a 'low-beta' case. In 'low-beta' RMHD (Strauss 1976), $\beta \sim \epsilon$. Then (27) gives $b = 0$, since any constant can be absorbed into B_*, and the vertical velocity satisfies $D_t w = 0$. The next-order components, b^1 and w^2, obey the same equations as in (25). Thus large-aspect-ratio, 'low-beta' RMHD is equivalent to (25).

In 'high-beta' RMHD (Strauss 1977) $\beta \sim 1$, so that $p = -b + F(z,t)$ and one obtains

$$D_t \nabla^2 \Psi = [A, \nabla^2 A] + B_* \nabla^2 A_z, \tag{28a}$$

$$D_t A = B_* \Psi_z, \tag{28b}$$

$$D_t b = [A, w] + w_z, \tag{28c}$$

$$D_t w = [A, b] + b_z, \tag{28d}$$

which differ from (25) only in the absence of the π term. That is, the slave fields b and w now have an unforced free evolution that still does not influence the Ψ, A dynamics. At the next order in ϵ, however, π as given in (24) will force the

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equations for b^1 and w^1. This system separately conserves both H as given in (26) and

$$h = \int (\tfrac{1}{2}w^2 + \tfrac{1}{2}b^2). \tag{29}$$

Finally, if the aspect ratio is not large ($\alpha \sim 1$) then (23) is replaced by

$$p = -\tfrac{1}{2}\mathbf{B}\cdot\mathbf{B} - \pi, \tag{30}$$

where $\beta \sim 1$ and π is again defined as in (24). In addition, (11) gives

$$p_z + \tfrac{1}{2}(\mathbf{B}\cdot\mathbf{B})_z = 0, \tag{31}$$

which implies that

$$\pi_z = 0. \tag{32}$$

Even given the ability to add an arbitrary $F(z,t)$ to π, (32) is not generally preserved under evolution by (20), (21) and (24) unless Ψ and A are independent of z, leading simply to 2D MHD. Thus for RMHD to be self-consistent in three dimensions, one must have a large aspect ratio $\alpha \gtrsim \epsilon^{-1/2}$.

3. Weakly rotating, non-gravitational, weakly compressible motions

For weakly compressible motions, we take $D/\rho_0 \sim \epsilon$, so that the first-order velocity field has finite divergence:

$$\Delta \equiv \boldsymbol{\nabla}\cdot\mathbf{v}^1 + w_z = D_t\rho. \tag{33}$$

We also adopt an isentropic equation of state $p^{\mathrm{T}} = p^{\mathrm{T}}(\rho^{\mathrm{T}})$ and define the sound speed $c^2 \equiv dp^{\mathrm{T}}/d\rho^{\mathrm{T}}$. This gives

$$D_t p = -\kappa\Delta, \tag{34}$$

with $\kappa \equiv \epsilon\mu^2c^2/U^2$, which introduces the Mach number U/c as a parameter. The weakly compressible limit is defined as taking $U^2/c^2 \sim \epsilon$. For moderate aspect ratio ($\alpha \sim \epsilon^{-1/2}, \beta \sim 1$) therefore, $\kappa \sim 1$. Equations (34), (23) and (12) then combine to give

$$(1+\kappa)\,\Delta = [A,w] + w_z + D_t\pi - \tfrac{1}{2}D_t(\mathbf{B}\cdot\mathbf{B}), \tag{35}$$

with π again defined by (24). The final, dimensional equations are

$$D_t\nabla^2\Psi = [A,\nabla^2 A] + B_*\nabla^2 A_z, \tag{26a}$$

$$D_t A = B_*\Psi_z, \tag{36b}$$

$$D_t\left\{b + \frac{B_*}{B_*^2+c^2}(\tfrac{1}{2}\mathbf{B}\cdot\mathbf{B} - \pi)\right\} = \frac{c^2}{B_*^2+c^2}([A,w] + B_* w_z), \tag{36c}$$

$$D_t w = [A,b] + B_* b_z - \pi_z, \tag{36d}$$

which conserve H as defined above.

In the large-aspect-ratio, 'high-beta' limit ($\alpha \sim \epsilon^{-1}, \beta \sim 1$), Δ is given by

$$(1+\kappa)\,\Delta = [A,w] + w_z, \tag{37}$$

giving

$$D_t\nabla^2\Psi = [A,\nabla^2 A] + B_*\nabla^2 A_z, \tag{38a}$$

$$D_t A = B_*\Psi_z, \tag{38b}$$

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$$D_t b = \frac{c^2}{B_*^2 + c^2}([A, w] + B_* w_z), \tag{38c}$$

$$D_t w = [A, b] + B_* b_z, \tag{38d}$$

which conserves H and

$$h = \int \left(\frac{c^2}{c^2 + B_*^2} \frac{1}{2} w^2 + \frac{1}{2} b^2 \right) \tag{39}$$

separately.

Note that one may take c^2/B_*^2 to be any size without interfering with the ordering of the equations. Taking $c^2 \gg B_*^2$ (very small Mach number) transforms (36) and (38) into (25) and (28) respectively. Taking $B_*^2 \gg c^2$ in (38) recovers the equations of the original 'high-beta' RMHD derivation of Strauss (1977).

As in the incompressible case, Ψ and A satisfy the Strauss equations in (36) and (38), and thus are decoupled from the forced or free slave dynamics of p, π, b and w. Here the density fluctuations can be diagnosed from p and the equation of state. As in the incompressible case, the aspect ratio α is restricted to be $\gtrsim \epsilon^{-1/2}$.

4. Gravitationally stratified, weakly rotating motions

We have shown that for RMHD to be valid, the aspect ratio α must be bounded from below. However, the restriction on α may be removed if $1/\sigma^2$ is finite. In essence, the addition of another term to (31) introduces a degree of freedom that relaxes (32). Now we choose $\bar{\rho}_0$ and $\bar{p}_0$ finite and treat only the incompressible limit, i.e. $\kappa \gg 1$, which implies that $\Delta = 0$. Equation (13) gives

$$\frac{D}{\bar{\rho}_0} D_t \rho + \epsilon w \bar{\rho}_z = 0. \tag{40}$$

We choose $D/\bar{\rho}_0 \sim \epsilon$ without loss of generality, since a smaller $D/\bar{\rho}_0$ simply shifts the ordering of w. Likewise, we may take $\sigma^2 \sim 1$ and $\mu^2 \sim 1$. We wish to use (40) as a diagnostic equation for w, but, in order to do so, we must have $\alpha^2 \epsilon \ll 1$ so that the inertial terms are absent from (11), which becomes

$$\rho = -p_z - \tfrac{1}{2}(\mathbf{B} \cdot \mathbf{B})_z, \tag{41}$$

and can appropriately be called magnetohydrostatic balance. The plasma β is of order 1. If $\alpha^2 \epsilon \ll 1$ is not satisfied then no reduction of the temporal order of the system is possible, and one simply obtains incompressible 3D MHD. Note that in the stratified description the aspect ratio is bounded from above, rather than from below as in the non-gravitational scalings. The horizontal pressure balance from (9) is

$$p = -\tfrac{1}{2}\mathbf{B} \cdot \mathbf{B} + \pi, \tag{42}$$

where π again satisfies (24). The final equations in dimensional form are

$$D_t \nabla^2 \Psi = [\mathrm{A}, \nabla^2 A] + B_* \nabla^2 A_z, \tag{43a}$$

$$D_t A = B_* \Psi_z, \tag{43b}$$

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$$D_t\left\{b-B_*\left(\frac{\pi_z}{N^2}\right)_z-\frac{1}{N^2}[A,\pi_z]\right\}=0, \tag{43c}$$

$$w=\frac{1}{N^2}D_t\pi_z, \tag{43d}$$

where $N^2=g\bar{\rho}_z/\rho_0$ is the Brunt–Väisälä frequency, which is generally a function of z. In obtaining the b equation, we have used the identities

$$(D_tf)_z=D_tf_z+[\Psi_z,f],\qquad [A,D_tf]=D_t[A,f]-[\Psi_z,f].$$

Here, as in the non-gravitational scalings, the independent dynamics for Ψ and A are the Strauss equations, but the slave dynamics for p, b and w have quite different forms. In particular, the relation for w is wholly diagnostic, as is the relation for b (and hence p) in a Lagrangian reference frame. We have derived a new fluid parcel invariant for MHD in this regime:

$$\chi\equiv b-B_*\left(\frac{\pi_z}{N^2}\right)_z-\frac{1}{N^2}[A,\pi_z]. \tag{44}$$

Equations (43) now conserve H as well as integrals of arbitrary functions of χ.

5. Gravitationally stratified, strongly rotating motions

In a strongly rotating fluid $R\sim\epsilon$, and in order to maintain the leading-order geostrophic and hydrostatic pressure balances in the momentum equations, one must have $\mu^2\sim\sigma^2\sim\epsilon$. Therefore

$$\left.\begin{aligned}p&=\pi=\Psi,\\ \rho&=-p_z.\end{aligned}\right\} \tag{45}$$

The $\mathbf{B}\cdot\mathbf{B}$ term and the other contributions to π in (24) are negligible, higher-order components of p. The plasma β is of order ϵ^{-1}. The horizontal divergence term makes a contribution to the vertical vorticity equation (20) in this scaling, since $\epsilon/R\sim1$. Thus the final, dimensional equations are

$$D_t\zeta-[A,J]=B_*J_z, \tag{46a}$$

$$D_tA=B_*\Psi_z, \tag{46b}$$

$$J=\nabla^2A, \tag{46c}$$

$$\zeta=\nabla^2\Psi+\left(\frac{2\Omega^2}{N^2}\Psi_z\right)_z, \tag{46d}$$

$$D_t\left\{b-\left(\frac{2B_*\Omega}{N^2}\Psi_z\right)_z-\frac{2\Omega}{N^2}[A,\Psi_z]\right\}=0, \tag{46e}$$

$$w=\frac{2\Omega}{N^2}D_t\Psi_z. \tag{46f}$$

The set of equations (46), which we call by QGMHD, is the principal new result of this paper. The equations for Ψ and A differ from RMHD because of the Ω^2/N^2 term. In the non-magnetic limit, where B_*, A, J and b all tend to 0, (46) are the quasigeostrophic equations, and ζ is the potential vorticity. In general,

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they are closely related to (43), with a diagnostic relation for w and a Lagrangian parcel conservation relation for b. This system conserves

$$H = \int\left(\frac{1}{2}|\boldsymbol{\nabla}\Psi|^2+\frac{\Omega^2}{N^2}(\Psi_z)^2+\frac{1}{2}|\boldsymbol{\nabla}A|^2\right), \tag{47}$$

as well as integrals of functions of χ, defined as in (44) except with $\pi = 2\Omega\Psi$.

The generalization of the independent Ψ, A dynamics in QGMHD has several important physical consequences. For instance, vertical Alfvén waves are dispersive, with the dispersion relation

$$\omega^2 = B_*^2\, k_z^2 \frac{1}{1+\Omega^2 k_z^2/N^2 k_\perp^2}. \tag{48}$$

In addition, the stability properties of magnetic stationary states are different (Kinney *et al*. 1996) from those of RMHD.

6. Conclusions

We have demonstrated the existence of a number of self-consistent dynamical subsystems of MHD. All subsystems have in common a decoupled, anisotropic evolution for the non-divergent horizontal components of the velocity and magnetic field, but they differ in the slave dynamics of the vertical components, the pressure and the density fields. The statement of anisotropy, which we assumed a priori, is a statement that the nonlinear rate of transfer of information in the vertical direction is much slower than in the horizontal. The physical reasons for this may vary. In RMHD transfer is slow simply because the vertical length scales are large, whereas in QGMHD vertical motions are restricted by rotation and the restoring forces of bouyancy and pressure.

In QGMHD the potential vorticity ζ, which is a parcel invariant in the absence of a magnetic field, has a source due to the Lorentz torque. However, related to the vertical magnetic field there is a new quantity χ, which is passively advected by the horizontal velocity components. In QGMHD the ratio Ω/N is a free parameter, which may be taken to zero and bring the horizontal motions in (46) back to the form of ordinary RMHD, but one must not assume that a smooth transition between regimes will be possible in a physical system. RMHD requires a large aspect ratio α, while QGMHD requires α to be small. In particular, the slave dynamics for non-gravitational systems (i.e. (25), (28), (36) and (38)) are non-overlapping with those of the stratified systems (i.e. (43) and (46)). Therefore any transition between the two regimes must pass through a moderate-α regime in which the dynamics are neither RMHD nor QGMHD.

It is remarkable that a single pair of coupled equations for Ψ and A (those of QGMHD, which contain the Strauss equations as a subset) provide a self-consistent, leading-order, reduced dynamics for a very wide range of anisotropic MHD regimes. They are therefore of potentially great interest as a nearly universal anisotropic reduction of general 3D MHD.

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