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Intersection Theorem for DG-modules over non-positive DG-rings

Xiaoyan Yang
School of Science, Zhejiang University of Science and Technology, Hangzhou, China (yangxy@zust.edu.cn)

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By extending some basic results about cohomological dimension of tensor products to non-positive DG-rings, the Intersection Theorem for DG-modules is examined over commutative noetherian local DG-rings with bounded cohomology. Some applications are provided. The first is to improve the DG-setting of the amplitude inequality in [Forum Math. 22 (2010) 941–948]. The second is to show Minamoto's conjecture in [Israel J. Math. 242 (2021) 1–36]. The third is to obtain the DG-version of the Vasconcelos conjecture about Gorenstein rings.

Keywords: cohomological dimension; depth; Intersection Theorem; amplitude inequality; Cohen-Macaulay DG-ring; Gorenstein DG-ring

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1. Introduction

The *New Intersection Theorem* (NIT) of Peskine and Szpiro [15], Hochster [8] and Roberts [17] is a central result in the homological theory of commutative noetherian rings, which has a variety of applications (see Foxby's notes: Hyperhomological algebra and commutative algebra and [4, 18]), and was extended and polished by commutative algebra practitioners in several stages (see [1, 5, 6, 10, 11, 16]).

Let A be a commutative noetherian local ring and M, N two finitely generated A-modules with $\operatorname{projdim}_A M < \infty$. The NIT yields an inequality, i.e., *Intersection Theorem* (IT)

$$\dim_A N \leq \dim_A(M \otimes_A N) + \operatorname{projdim}_A M. \tag{$\dagger$}$$

By applying the inequality ($\dagger$) with $N = A$, one has $\dim A \leq \dim_A M + \operatorname{projdim}_A M$. By the Auslander–Buchsbaum formula, the inequality is equivalent to $\operatorname{cmd} A \leq \operatorname{cmd}_A M$, where $\operatorname{cmd}_A M = \dim_A M - \operatorname{depth}_A M$ is the *Cohen–Macaulay defect* of M which determines the failure of M to be Cohen–Macaulay. Furthermore, Yassemi [28] showed the inequalities

$$\dim_A N \leq \dim_A \mathrm{RHom}_A(M, N) \leq \dim_A(M \otimes_A N) + \operatorname{projdim}_A M. \tag{$\ddagger$}$$

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By [4, Theorem 16.2.24], $\mathrm{depth}_A\mathrm{RHom}_A(M,A) = \mathrm{depth}A$, it follows by (‡) that $\mathrm{cmd}A \leq \mathrm{cmd}_A\mathrm{RHom}_A(M,A) \leq \mathrm{cmd}_A M$. Also $\mathrm{grade}_A M + \dim_A M \leq \dim A$, so if M is *perfect* ($\mathrm{projdim}_A M = \mathrm{grade}_A M$) then $\mathrm{cmd}A = \mathrm{cmd}_A\mathrm{RHom}_A(M,A) = \mathrm{cmd}_A M$, where the *grade* of M is given by $\mathrm{grade}_A M = \min\{i \geq 0 | \mathrm{Ext}^i_A(M,A) \neq 0\}$.

On the other hand, Foxby generalized the NIT for complexes in his notes, for bounded complexes X, Y with finite cohomologies, showed the following IT for complexes

$$\dim_A Y \leq \dim_A(X \otimes^{\mathrm{L}}_A Y) + \mathrm{projdim}_A X$$

whenever $\mathrm{projdim}_A X < \infty$ (see [4, Theorem 18.5.3]). In 2005, Dibaei and Yassemi [5] continued to study the IT of complexes and proved the following inequalities

$$\dim_A Y - \sup X \leq \dim_A\mathrm{RHom}_A(X,Y) \leq \dim_A(X \otimes^{\mathrm{L}}_A Y) + \mathrm{projdim}_A X - \inf X,$$

which generalizes the inequalities (‡).

Despite the great success of the NIT and IT in classical commutative algebras, until now, it was completely missing from higher algebra. DG-rings (short for differential graded rings) allow us to use techniques of homological algebra of ordinary rings in a much wider context. The aim of this paper is to extend the IT to the setting of commutative noetherian local DG-rings.

An *associative DG-ring* A is a $\mathbb{Z}$-graded ring $A = \bigoplus_{i\in\mathbb{Z}} A^i$ with a diferential of degree 1, that satisfies the Leibniz rule. A DG-ring A is called *commutative* if $b \cdot a = (-1)^{i\cdot j} a \cdot b$ for $a \in A^i$ and $b \in A^j$, and $a^2 = 0$ if i is odd. A DG-ring A is called *non-positive* if $A^i = 0$ for all $i > 0$. A non-positive DG-ring A is called *noetherian* if the ring $\mathrm{H}^0(A)$ is noetherian and the $\mathrm{H}^0(A)$-module $\mathrm{H}^i(A)$ is finitely generated for all $i < 0$. If A is a noetherian DG-ring and $(\mathrm{H}^0(A), \bar{\mathfrak{m}}, \bar{\kappa})$ is a local ring, then we say that $(A, \bar{\mathfrak{m}}, \bar{\kappa})$ is *local noetherian*.

Unless stated to the contrary, we assume throughout this paper that A is a commutative and noetherian DG-ring.

The derived category of DG-modules over A will be denoted by $\mathrm{D}(A)$, and write $- \otimes^{\mathrm{L}}_A -$ for its monoidal product, $\mathrm{RHom}_A(-,-)$ for the internal hom. The full subcategory which consists of DG-modules whose cohomology is bounded above (resp. bounded below, bounded) is denoted by $\mathrm{D}^-(A)$ (resp. $\mathrm{D}^+(A)$, $\mathrm{D}^{\mathrm{b}}(A)$), denote by $\mathrm{D}_{\mathrm{f}}(A)$ the full subcategory of $\mathrm{D}(A)$ consisting of DG-modules with finite cohomology, and set $\mathrm{D}^-_{\mathrm{f}}(A) = \mathrm{D}^-(A) \cap \mathrm{D}_{\mathrm{f}}(A)$, $\mathrm{D}^+_{\mathrm{f}}(A) = \mathrm{D}^+(A) \cap \mathrm{D}_{\mathrm{f}}(A)$, $\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A) = \mathrm{D}^{\mathrm{b}}(A) \cap \mathrm{D}_{\mathrm{f}}(A)$. For a DG-module X, we set $\inf X := \inf\{n \in \mathbb{Z} | \mathrm{H}^n(X) \neq 0\}$, $\sup X := \sup\{n \in \mathbb{Z} | \mathrm{H}^n(X) \neq 0\}$ and $\mathrm{amp}X := \sup X - \inf X$. The n-th shift of X is denoted by $X[n]$ for $n \in \mathbb{Z}$.

Let $(A, \bar{\mathfrak{m}}, \bar{\kappa})$ be a local DG-ring. Shaul [22] extended the definitions of Krull dimensions and depth of complexes to the DG-setting. The *local cohomology Krull dimension* of a DG-module $X \in \mathrm{D}^-(A)$ is the number

$$\mathrm{lc.dim}_A X := \sup\{\dim_{\mathrm{H}^0(A)}\mathrm{H}^\ell(X) + \ell | \ell \in \mathbb{Z}\},$$

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where $\dim_{\mathrm{H}^0(A)}\mathrm{H}^\ell(X)$ is the usual Krull dimension of the $\mathrm{H}^0(A)$-module $\mathrm{H}^\ell(X)$. The *depth* of a DG-module $Y \in \mathrm{D}^+(A)$ is the number

$$\operatorname{depth}_A Y := \inf \mathrm{RHom}_A(\bar{\kappa}, Y).$$

The theory of classical Cohen–Macaulay rings, Cohen–Macaulay modules, and Gorenstein rings has also been extended to the DG-version in [22, 23]. This paper focuses on the local cohomological Krull dimension of the DG-modules $\mathrm{RHom}_A(X,Y)$ and $X \otimes^{\mathrm{L}}_A Y$ for appropriate X and Y in $\mathrm{D}(A)$. We wish to extend the IT to local DG-rings and obtain some characterizations about local Cohen–Macaulay and local Gorenstein DG-rings.

The paper is organized as follows.

In Section 2, we gather some preliminaries and results about the local cohomological Krull dimension of DG-modules. The preparatory work in Section 2 allows us to establish the IT over a local DG-ring A with bounded cohomology in Section 3 (see Theorem 3.1), a bit more generally, for $0 \not\simeq Y \in \mathrm{D}^-_{\mathrm{f}}(A)$ and $X \in \mathrm{D}^-(A)$ with $\operatorname{flatdim}_A X < \infty$ and $\operatorname{depth}_A X < \infty$, we show the following inequality:

$$\operatorname{lc.dim}_A Y \leq \operatorname{lc.dim}_A(X \otimes^{\mathrm{L}}_A Y) + \operatorname{flatdim}_A X. \tag{$\dagger'$}$$

Moreover, some upper and lower bounds of $\operatorname{lc.dim}_A(X \otimes^{\mathrm{L}}_A Y)$ and $\operatorname{lc.dim}_A \mathrm{RHom}_A(X,Y)$ are provided (see Theorem 3.5). When the DG-ring A is sequence-regular and $X, Y \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$, we improve the inequality ($\dagger'$) in Proposition 3.7, and prove the stronger inequalities:

$$\operatorname{lc.dim}_A X + \operatorname{lc.dim}_A Y \leq \operatorname{lc.dim} A + \operatorname{lc.dim}_A(X \otimes^{\mathrm{L}}_A Y).$$

In view of the inequalities ($\dagger'$), we extend in Section 4 some classical results about amplitude inequality, Cohen–Macaulay rings, and Gorenstein rings to local DG-rings with bounded cohomology. For $0 \not\simeq X, Y \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ with $\operatorname{projdim}_A X < \infty$, Jørgensen [12, Theorem 3.1] showed the amplitude inequality $\operatorname{amp}(X \otimes^{\mathrm{L}}_A Y) \geq \operatorname{amp} X$ when A is a sufficiently nice local DG-ring with $\operatorname{amp} A < \infty$. The first application of our results is to eliminate the hypothesis on A, and improve Jørgensen's result (see Proposition 4.1). Bass [2] raised the question of whether the existence of a finitely generated module of finite injective dimension would imply the Cohen–Macaulayness of the ring, the question that later became known as the Bass conjecture. For $0 \not\simeq X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ with $\operatorname{injdim}_A X < \infty$, Minamoto [13, Conjecture 2.36] conjectured that $\operatorname{amp} X \geq \operatorname{amp} A$. Shaul [22, Theorem 5.22(2)] showed the DG-version of Bass Conjecture and solved this conjecture under the noetherian model assumption, and conjectured in [22, Remark 5.25] that this assumption is redundant. As the second application, Minamoto's Conjecture or Shaul's Conjecture is proved (see Proposition 4.5). Vasconcelos [25] conjectured that rings of type one are Gorenstein. The last application is to get the DG-version of the Vasconcelos conjecture (see Proposition 4.7).

2. Preliminaries and some facts

This section is devoted to recalling some notions and basic facts.

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Projective, injective, and flat dimensions. For a DG-module $X \in \mathrm{D}(A)$, we recall the definition of the projective, injective, and flat dimensions of X introduced by Bird, Shaul, Sridhar and Williamson in [3], which are different from the definition, $\mathrm{pd}X$, $\mathrm{id}X$, and $\mathrm{fd}X$, introduced by Yekutieli [29] (also see [14]).

The *projective dimension* of X is defined by

$$\mathrm{projdim}_A X = \inf\{n \in \mathbb{Z} | \mathrm{H}^i(\mathrm{RHom}_A(X, Y)) = 0 \text{ for any } Y \in \mathrm{D}^{\mathrm{b}}(A) \text{ and } i > n + \sup Y\}.$$

The *injective dimension* of X is defined by

$$\mathrm{injdim}_A X = \inf\{n \in \mathbb{Z} | \mathrm{H}^i(\mathrm{RHom}_A(Y, X)) = 0 \text{ for any } Y \in \mathrm{D}^{\mathrm{b}}(A) \text{ and } i > n - \inf Y\}.$$

The *flat dimension* of X is defined by

$$\mathrm{flatdim}_A X = \inf\{n \in \mathbb{Z} | \mathrm{H}^{-i}(Y \otimes^{\mathrm{L}}_A X) = 0 \text{ for any } Y \in \mathrm{D}^{\mathrm{b}}(A) \text{ and } i > n - \inf Y\}.$$

We denote by $\mathcal{P} \subseteq \mathrm{D}(A)$ the full subcategory of direct summands of a direct sums of A. For $X \in \mathrm{D}^-(A)$ and $Y \in \mathrm{D}^+(A)$, it follows by [14, Theorems 2.22, 3.21, 4.13] that

$$\begin{aligned}
&\mathrm{projdim}_{\mathrm{H}^0(A)}(\mathrm{H}^0(A) \otimes^{\mathrm{L}}_A X) = \mathrm{projdim}_A X = \mathrm{pd}X - \sup X,\\
&\mathrm{injdim}_{\mathrm{H}^0(A)}\mathrm{RHom}_A(\mathrm{H}^0(A), Y) = \mathrm{injdim}_A Y = \mathrm{id}Y + \inf Y,\\
&\mathrm{flatdim}_{\mathrm{H}^0(A)}(\mathrm{H}^0(A) \otimes^{\mathrm{L}}_A X) = \mathrm{flatdim}_A X = \mathrm{fd}X - \sup X.
\end{aligned}$$

LEMMA 2.1. *Let A be a DG-ring with $\mathrm{amp}A < \infty$, $X \in D^-(A)$ and $Y \in D^+(A)$.*
(1) *If $\mathrm{projdim}_A X < \infty$, then $\inf X \geq \inf A - \mathrm{projdim}_A X$, i.e., $X \in D^b(A)$.*
(2) *If $\mathrm{injdim}_A Y < \infty$, then $\sup Y \leq -\inf A + \mathrm{injdim}_A Y$, i.e., $Y \in D^b(A)$.*
(3) *If $\mathrm{flatdim}_A X < \infty$, then $\inf X \geq \inf A - \mathrm{flatdim}_A X$, i.e., $X \in D^b(A)$.*

Proof. We just prove (1) since (2) follows dually.

(1) We use induction on $n = \mathrm{projdim}_A X + \sup X$. If $n = 0$, then $X \in \mathcal{P}[-\sup X]$ by [14, Lemma 2.14] and so $\inf X \geq \inf A - \mathrm{projdim}_A X$. Assume that $n \geq 1$. By [14, Lemma 2.13], there is an exact triangle $Z \to P \to X \rightsquigarrow$ so that $P \in \mathcal{P}[-\sup X]$ and $\mathrm{projdim}_A Z + \sup Z \leq \mathrm{projdim}_A Z + \sup X = n - 1$, it follows by the induction that $\inf X \geq \inf A - \mathrm{projdim}_A X$.

(3) For the $\mathrm{H}^0(A)$-module $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}^0(A), \mathbb{Q}/\mathbb{Z})$, it follows by [20, Theorem 5.7] that there is a DG-module E so that $\mathrm{H}^0(E) = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}^0(A), \mathbb{Q}/\mathbb{Z})$. Hence, [20, Theorem 4.10] and (2) imply that $\inf X = -\sup \mathrm{RHom}_A(X, E) \geq \inf A - \mathrm{injdim}_A \mathrm{RHom}_A(X, E) = \inf A - \mathrm{flatdim}_A X$ as $\mathrm{injdim}_A \mathrm{RHom}_A(X, E) = \mathrm{flatdim}_A X$ by [3, Lemma 7.12]. □

LEMMA 2.2. *([3]). Let A be a DG-ring with $\mathrm{amp}A < \infty$ and $X, Y, Z \in D(A)$.*
(1) *If either $X \in D^-_f(A)$, $Y \in D^+(A)$ and $\mathrm{injdim}_A Z < \infty$, or $X \in D^b_f(A)$ and $\mathrm{projdim}_A X < \infty$, then the natural map*

$$X \otimes^L_A RHom_A(Y, Z) \to RHom_A(RHom_A(X, Y), Z)$$

is an isomorphism in $D(A)$.

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(2) *If either* $X \in D_f^-(A)$, $Y \in D^+(A)$ *and* $flatdim_A Z < \infty$, *or* $X \in D_f^b(A)$ *and* $projdim_A X < \infty$, *then the natural map*

$$RHom_A(X, Y) \otimes_A^L Z \to RHom_A(X, Y \otimes_A^L Z)$$

is an isomorphism in $D(A)$.

Derived torsion. Let $\bar{\mathfrak{a}} \subseteq \mathrm{H}^0(A)$ be an ideal. An $\mathrm{H}^0(A)$-module $\bar{X}$ is called $\bar{\mathfrak{a}}$-*torsion* if for any $\bar{x} \in \bar{X}$ there is $n \in \mathbb{N}$ such that $\bar{\mathfrak{a}}^n \bar{x} = 0$. The category $\mathrm{D}_{\bar{\mathfrak{a}}\text{-tor}}(A)$ consisting of DG-modules X such that the $\mathrm{H}^0(A)$-module $\mathrm{H}^n(X)$ is $\bar{\mathfrak{a}}$-torsion for every n, is a triangulated subcategory of $\mathrm{D}(A)$. Following [21], the inclusion functor $\mathrm{D}_{\bar{\mathfrak{a}}\text{-tor}}(A) \hookrightarrow \mathrm{D}(A)$ has a right adjoint, and composing this right adjoint with the inclusion, one has a triangulated functor $\mathrm{R}\Gamma_{\bar{\mathfrak{a}}} : \mathrm{D}(A) \to \mathrm{D}(A)$, which is called the *derived* $\bar{\mathfrak{a}}$-*torsion functor*. The derived $\bar{\mathfrak{a}}$-adic completion $\mathrm{RHom}_A(\mathrm{R}\Gamma_{\bar{\mathfrak{a}}}(A), A)$, denoted by $\mathrm{L}\Lambda(A, \bar{\mathfrak{a}})$, is a commutative DG-ring.

Let $(A, \bar{\mathfrak{m}})$ be a local DG-ring. For $X \in \mathrm{D}_{\mathrm{f}}^-(A)$ and $Y \in \mathrm{D}^+(A)$, it follows by [22, Theorem 2.15] and [22, Proposition 3.3] that

$$\mathrm{lc.dim}_A X = \mathrm{sup}\mathrm{R}\Gamma_{\bar{\mathfrak{m}}}(X), \ \mathrm{depth}_A Y = \mathrm{inf}\mathrm{R}\Gamma_{\bar{\mathfrak{m}}}(Y).$$

Localization. Let $\pi_A : A \to \mathrm{H}^0(A)$ be the canonical surjection and $\pi_A^0 : A^0 \to \mathrm{H}^0(A)$ be its degree 0 component. Given a prime ideal $\bar{\mathfrak{p}} \in \mathrm{Spec}\mathrm{H}^0(A)$, let $\mathfrak{p} = (\pi_A^0)^{-1}(\bar{\mathfrak{p}}) \in \mathrm{Spec}A^0$, and define $A_{\bar{\mathfrak{p}}} := A \otimes_{A^0} A^0_{\mathfrak{p}}$. More generally, given $X \in \mathrm{D}(A)$, we define

$$X_{\bar{\mathfrak{p}}} := X \otimes_A A_{\bar{\mathfrak{p}}} = X \otimes_{A^0} A^0_{\mathfrak{p}} \in \mathrm{D}(A_{\bar{\mathfrak{p}}}).$$

A prime ideal $\bar{\mathfrak{p}} \in \mathrm{Spec}\mathrm{H}^0(A)$ is called an *associated prime* of X if $\mathrm{depth}_{A_{\bar{\mathfrak{p}}}} X_{\bar{\mathfrak{p}}} = \mathrm{inf} X_{\bar{\mathfrak{p}}}$. The set of associated primes of X is denoted by $\mathrm{Ass}_A X$. The *support* of X is the set

$$\mathrm{Supp}_A X := \{\bar{\mathfrak{p}} \in \mathrm{Spec}\mathrm{H}^0(A) | X_{\bar{\mathfrak{p}}} \not\simeq 0\}.$$

For $\bar{\mathfrak{p}} \in \mathrm{Spec}\mathrm{H}^0(A)$, denote by $E(A, \bar{\mathfrak{p}})$ the DG-module corresponding to the injective hull $E(\mathrm{H}^0(A), \bar{\mathfrak{p}})$ of the residue field $\mathrm{H}^0(A)_{\bar{\mathfrak{p}}}/\bar{\mathfrak{p}}\mathrm{H}^0(A)_{\bar{\mathfrak{p}}}$.

Next, we bring several necessary lemmas and their proofs.

LEMMA 2.3. (1) *For* $X \in D_f^-(A)$ *and* $Y \in D^+(A)$, *one has*

$$infRHom_A(X, Y) = inf\{infRHom_A(H^\ell(X), Y) - \ell | \ell \in \mathbb{Z}\}.$$

(2) *For* $X \in D_f^-(A)$ *and* $Z \in D^-(A)$, *one has*

$$sup(X \otimes_A^L Z) = sup\{sup(H^\ell(X) \otimes_A^L Z) + \ell | \ell \in \mathbb{Z}\}.$$

(3) *Let* $(A, \bar{\mathfrak{m}}, \bar{\kappa})$ *be a local DG-ring. For* $X, Y \in D_f^-(A)$, *one has*

$$sup(X \otimes_A^L Y) = supX + supY.$$

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Proof. (1) We first show that $\operatorname{infRHom}_A(X,Y) = \inf\{\operatorname{depth}_{A_{\bar{\mathfrak{p}}}} Y_{\bar{\mathfrak{p}}} - \sup X_{\bar{\mathfrak{p}}} | \bar{\mathfrak{p}} \in \operatorname{Spec}\mathrm{H}^0(A)\}$. For each $\bar{\mathfrak{p}} \in \operatorname{Spec}\mathrm{H}^0(A)$ one has the next (in)equalities

$$\begin{aligned} r := \operatorname{infRHom}_A(X,Y) &\leq \operatorname{infRHom}_{A_{\bar{\mathfrak{p}}}}(X_{\bar{\mathfrak{p}}}, Y_{\bar{\mathfrak{p}}}) \\ &\leq \operatorname{depth}_{A_{\bar{\mathfrak{p}}}} \operatorname{RHom}_{A_{\bar{\mathfrak{p}}}}(X_{\bar{\mathfrak{p}}}, Y_{\bar{\mathfrak{p}}}) \\ &= \operatorname{depth}_{A_{\bar{\mathfrak{p}}}} Y_{\bar{\mathfrak{p}}} - \sup X_{\bar{\mathfrak{p}}}, \end{aligned}$$

where the first inequality is by [13, Proposition 2.4], the second inequality is by [22, Proposition 3.3] and the equality is by [27, Proposition 4.9] and Nakayama's lemma. If $\bar{\mathfrak{p}} \in \operatorname{Ass}_{\mathrm{H}^0(A)} \mathrm{H}^r(\operatorname{RHom}_A(X,Y))$, then the claimed equality holds by [22, Proposition 3.3].

Set $i = \operatorname{infRHom}_A(X,Y)$ and $j = \inf\{\operatorname{infRHom}_A(\mathrm{H}^\ell(X), Y) - \ell | \ell \in \mathbb{Z}\}$. For $\bar{\mathfrak{p}} \in \operatorname{Spec}\mathrm{H}^0(A)$, set $s = \sup X_{\bar{\mathfrak{p}}}$. By the preceding proof, we have

$$\operatorname{depth}_{A_{\bar{\mathfrak{p}}}} Y_{\bar{\mathfrak{p}}} - \sup X_{\bar{\mathfrak{p}}} = \operatorname{depth}_{A_{\bar{\mathfrak{p}}}} Y_{\bar{\mathfrak{p}}} - s \geq \operatorname{infRHom}_A(\mathrm{H}^s(X), Y) - s \geq j,$$

and hence $i \geq j$. On the other hand, let $\ell \in \mathbb{Z}$ and $\mathrm{H}(\operatorname{RHom}_A(\mathrm{H}^\ell(X), Y)) \not\simeq 0$. Choose $\bar{\mathfrak{p}} \in \operatorname{Spec}\mathrm{H}^0(A)$ such that $\operatorname{infRHom}_A(\mathrm{H}^\ell(X), Y) = \operatorname{depth}_{A_{\bar{\mathfrak{p}}}} Y_{\bar{\mathfrak{p}}}$. Then

$$\operatorname{RHom}_A(\mathrm{H}^\ell(X), Y) - \ell \geq \operatorname{depth}_{A_{\bar{\mathfrak{p}}}} Y_{\bar{\mathfrak{p}}} - \sup X_{\bar{\mathfrak{p}}} \geq i.$$

Therefore, the equality holds.

(2) Set $E = \oplus_{\bar{\mathfrak{m}} \in \operatorname{Max}\mathrm{H}^0(A)} E(A, \bar{\mathfrak{m}})$. We have the following equalities

$$\begin{aligned} \sup(X \otimes^{\mathrm{L}}_A Z) &= -\operatorname{infRHom}_A(X, \operatorname{RHom}_A(Z,E)) \\ &= -\inf\{\operatorname{infRHom}_A(\mathrm{H}^\ell(X), \operatorname{RHom}_A(Z,E)) - \ell | \ell \in \mathbb{Z}\} \\ &= \sup\{\ell - \operatorname{infRHom}_A(\mathrm{H}^\ell(X) \otimes^{\mathrm{L}}_A Z, E) | \ell \in \mathbb{Z}\} \\ &= \sup\{\ell + \sup(\mathrm{H}^\ell(X) \otimes^{\mathrm{L}}_A Z) | \ell \in \mathbb{Z}\}, \end{aligned}$$

where the first and the last ones are by [20, Theorem 4.10], the second one is by (1).

(3) As $\mathrm{H}^{\sup X}(X)$ and $\mathrm{H}^{\sup Y}(Y)$ are finitely generated $\mathrm{H}^0(A)$-modules, one has

$$\mathrm{H}^{\sup X + \sup Y}(\bar{\kappa} \otimes^{\mathrm{L}}_A X \otimes^{\mathrm{L}}_A Y) \cong \bar{\kappa} \otimes_{\mathrm{H}^0(A)} \mathrm{H}^{\sup X}(X) \otimes_{\bar{\kappa}} \bar{\kappa} \otimes_{\mathrm{H}^0(A)} \mathrm{H}^{\sup Y}(Y) \neq 0,$$

which implies the desired equality. □

By the equality (2.14) of [22, Proposition 2.13], for $X \in \mathrm{D}^-(A)$, one has

$$\operatorname{lc.dim}_A X = \sup\{\dim \mathrm{H}^0(A)/\bar{\mathfrak{p}} + \sup X_{\bar{\mathfrak{p}}} | \bar{\mathfrak{p}} \in \operatorname{Spec}\mathrm{H}^0(A)\}. \qquad (*)$$

The following lemma is useful in the proof of the main results.

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LEMMA 2.4. *For $X \in D_f^-(A)$ and $Y \in D^-(A)$, there exists an equality*

$$lc.dim_A(X \otimes_A^L Y) = sup\{lc.dim_A(H^\ell(X) \otimes_A^L Y) + \ell | \ell \in \mathbb{Z}\}.$$

Proof. We have the following computations

$$\begin{aligned}
\mathrm{lc.dim}(X \otimes_A^{\mathrm{L}} Y) &= \sup\{\mathrm{dim}\mathrm{H}^0(A)/\bar{\mathfrak{p}} + \sup(X \otimes_A^{\mathrm{L}} Y)_{\bar{\mathfrak{p}}} | \bar{\mathfrak{p}} \in \mathrm{Spec}\mathrm{H}^0(A)\} \\
&= \sup\{\mathrm{dim}\mathrm{H}^0(A)/\bar{\mathfrak{p}} \\
&\quad + \sup\{\sup(\mathrm{H}^\ell(X_{\bar{\mathfrak{p}}}) \otimes_{A_{\bar{\mathfrak{p}}}}^{\mathrm{L}} Y_{\bar{\mathfrak{p}}}) + \ell | \ell \in \mathbb{Z}\} | \bar{\mathfrak{p}} \in \mathrm{Spec}\mathrm{H}^0(A)\} \\
&= \sup\{\mathrm{dim}\mathrm{H}^0(A)/\bar{\mathfrak{p}} + \sup(\mathrm{H}^\ell(X) \otimes_A^{\mathrm{L}} Y)_{\bar{\mathfrak{p}}} \\
&\quad + \ell | \bar{\mathfrak{p}} \in \mathrm{Spec}\mathrm{H}^0(A), \ell \in \mathbb{Z}\} \\
&= \sup\{\mathrm{lc.dim}_A(\mathrm{H}^\ell(X) \otimes_A^{\mathrm{L}} Y) + \ell | \ell \in \mathbb{Z}\},
\end{aligned}$$

where the second equality follows by Lemma 2.3(2), as desired. □

LEMMA 2.5. *Let $X \in D^+(A)$ and $Y \in D^-(A)$. If $flatdim_A Y < \infty$, then*

$$depth_A(X \otimes_A^L Y) = depth_A X + depth_A Y - depth A.$$

Proof. As $\mathrm{flatdim}_{\mathrm{H}^0(A)}(\mathrm{H}^0(A) \otimes_A^{\mathrm{L}} Y) < \infty$ and $\mathrm{RHom}_A(\mathrm{H}^0(A), X) \in \mathrm{D}^+(\mathrm{H}^0(A))$, $\mathrm{H}^0(A) \otimes_A^{\mathrm{L}} Y \in \mathrm{D}^{\mathrm{b}}(\mathrm{H}^0(A))$, one has the following equalities

$$\begin{aligned}
\mathrm{depth}_A(X \otimes_A^{\mathrm{L}} Y) &= \mathrm{depth}_{\mathrm{H}^0(A)}\mathrm{RHom}_A(\mathrm{H}^0(A), X \otimes_A^{\mathrm{L}} Y) \\
&= \mathrm{depth}_{\mathrm{H}^0(A)}(\mathrm{RHom}_A(\mathrm{H}^0(A), X) \otimes_A^{\mathrm{L}} Y) \\
&= \mathrm{depth}_{\mathrm{H}^0(A)}(\mathrm{RHom}_A(\mathrm{H}^0(A), X) \otimes_{\mathrm{H}^0(A)}^{\mathrm{L}} \mathrm{H}^0(A) \otimes_A^{\mathrm{L}} Y) \\
&= \mathrm{depth}_{\mathrm{H}^0(A)}\mathrm{RHom}_A(\mathrm{H}^0(A), X) + \inf(\bar{\kappa} \otimes_{\mathrm{H}^0(A)}^{\mathrm{L}} \mathrm{H}^0(A) \otimes_A^{\mathrm{L}} Y) \\
&= \mathrm{depth}_A X + \inf(\bar{\kappa} \otimes_A^{\mathrm{L}} Y),
\end{aligned}$$

where the first and the last ones are by [27, Remark 4.3(1)], the second one is by Lemma 2.2(2), the fourth one is by [4, Theorem 16.3.1]. Applied to $X = A$, we obtain the desired equality. □

3. Intersection Theorem for DG-modules

In this section, we examine the IT for DG-modules and study the local cohomology Krull dimension of the DG-modules $\mathrm{RHom}_A(X, Y)$ and $X \otimes_A^{\mathrm{L}} Y$, which extends some classical results proved by Christensen and Foxby and Holm in [4].

We begin with the following main theorem of this section, which is an extension of the IT in [5, Theorem 2.6] in the DG-setting.

THEOREM 3.1 *Let $(A, \bar{\mathfrak{m}}, \bar{\kappa})$ be a local DG-ring with $ampA < \infty$.*

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(1) *Let* $X \in D^-(A)$ *with* $\mathit{flatdim}_A X < \infty$ *and* $0 \not\simeq Y \in D_f^-(A)$. *If* $\mathit{depth}_A X < \infty$, *then*

$$\mathit{lc.dim}_A Y \leq \mathit{lc.dim}_A(X \otimes_A^L Y) + \mathit{flatdim}_A X.$$

(2) *For* $0 \not\simeq X, Y \in D_f^-(A)$ *with* $\mathit{projdim}_A X < \infty$, *one has an inequality*

$$\mathit{lc.dim}_A Y \leq \mathit{lc.dim}_A(X \otimes_A^L Y) + \mathit{projdim}_A X.$$

Proof. (1) As $\mathrm{depth}_A X < \infty$, $\mathrm{depth}_{\mathrm{H}^0(A)}(X \otimes_A^{\mathrm{L}} \mathrm{H}^0(A)) < \infty$. One has

$$\begin{aligned}
\mathrm{lc.dim}_A(X \otimes_A^{\mathrm{L}} Y) &= \sup\{\mathrm{lc.dim}_A(X \otimes_A^{\mathrm{L}} \mathrm{H}^\ell(Y)) + \ell | \ell \in \mathbb{Z}\} \\
&= \sup\{\mathrm{lc.dim}_{\mathrm{H}^0(A)}(X \otimes_A^{\mathrm{L}} \mathrm{H}^0(A) \otimes_{\mathrm{H}^0(A)}^{\mathrm{L}} \mathrm{H}^\ell(Y)) + \ell | \ell \in \mathbb{Z}\} \\
&\geq \sup\{\dim_{\mathrm{H}^0(A)} \mathrm{H}^\ell(Y)) + \ell | \ell \in \mathbb{Z}\} - \mathrm{flatdim}_{\mathrm{H}^0(A)}(X \otimes_A^{\mathrm{L}} \mathrm{H}^0(A)) \\
&= \mathrm{lc.dim}_A Y - \mathrm{flatdim}_A X,
\end{aligned}$$

where the first equality is by Lemma 2.4, the inequality follows by [4, Theorem 18.5.3] and local duality, the last equality follows by [3, Corollary 2.3].

(2) As $\mathrm{projdim}_A X < \infty$, it follows by [13, Theorem 2.16] that $\mathrm{depth}_A X < \infty$. Hence, (1) yields the desired inequality. □

Corollary 3.2. *Let* $(A, \bar{\mathfrak{m}}, \bar{\kappa})$ *be a local DG-ring with* $\mathit{ampA} < \infty$ *and* $Y \in D^-(A)$ *with* $\mathit{depth}_A Y < \infty$. *If either* $0 \not\simeq X \in D_f^b(A)$ *or* $\mathit{depth}_A X < \infty$, *then there is an inequality,*

$$\mathrm{flatdim}_A Y \geq \mathrm{depth}_A X - \mathrm{lc.dim}_A(Y \otimes_A^L X).$$

Proof. One can assume that $\mathrm{flatdim}_A Y < \infty$. If $0 \not\simeq X \in \mathrm{D}_{\mathrm{f}}^{\mathrm{b}}(A)$, then

$$\mathrm{flatdim}_A Y \geq \mathrm{lc.dim}_A X - \mathrm{lc.dim}_A(Y \otimes_A^{\mathrm{L}} X) \geq \mathrm{depth}_A X - \mathrm{lc.dim}_A(Y \otimes_A^{\mathrm{L}} X)$$

by Theorem 3.1(1) and [22, Proposition 3.5]. As $\mathrm{flatdim}_A Y < \infty$, $Y \otimes_A^{\mathrm{L}} X \in \mathrm{D}^{\mathrm{b}}(A)$. If $\mathrm{depth}_A X < \infty$, then $\mathrm{depth}_A(Y \otimes_A^{\mathrm{L}} X) < \infty$ and so $\mathrm{depth}_A(Y \otimes_A^{\mathrm{L}} X) = \mathrm{infR}\Gamma_{\bar{\mathfrak{m}}}(Y \otimes_A^{\mathrm{L}} X) \leq \mathrm{supR}\Gamma_{\bar{\mathfrak{m}}}(Y \otimes_A^{\mathrm{L}} X) \leq \mathrm{lc.dim}_A(Y \otimes_A^{\mathrm{L}} X)$ by [22, Theorem 2.15 and Proposition 3.3]. Thus, $\mathrm{flatdim}_A Y \geq -\mathrm{inf}(\bar{\kappa} \otimes_A^{\mathrm{L}} Y) = \mathrm{depth}_A X - \mathrm{depth}_A(Y \otimes_A^{\mathrm{L}} X)$ by the equality in the proof of Lemma 2.5, as desired. □

Corollary 3.3. *Let* $(A, \bar{\mathfrak{m}}, \bar{\kappa})$ *be a local DG-ring with* $\mathit{ampA} < \infty$, *and let* $X \in D_f^-(A)$ *with* $\mathit{flatdim}_A X < \infty$ *and* $0 \not\simeq Y \in D_f^-(A)$. *There exists an inequality,*

$$\mathrm{lc.dim}_A(X \otimes_A^L Y) - \mathrm{depth}_A(X \otimes_A^L Y) \geq \mathrm{lc.dim}_A Y - \mathrm{depth}_A Y.$$

In particular, $\mathrm{lc.dim}_A X - \mathrm{depth}_A X \geq \mathrm{lc.dim} A - \mathrm{depth} A$.

Proof. As $X \in \mathrm{D}_{\mathrm{f}}^-(A)$, $\mathrm{projdim}_A X = \mathrm{flatdim}_A X$ by [13, Proposition 2.2] and $X \in \mathrm{D}_{\mathrm{f}}^{\mathrm{b}}(A)$ by Lemma 2.1, it follows by [13, Corollary 2.31] and [3, Lemma 7.12]

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and Lemma 2.5 that

$$\operatorname{flatdim}_A X = -\inf(\bar{\kappa} \otimes^{\mathrm{L}}_A X) = \operatorname{depth}_A X - \operatorname{depth} A.$$

Hence, Theorem 3.1 implies that

$$\begin{aligned}\operatorname{lc.dim}_A(X \otimes^{\mathrm{L}}_A Y) - \operatorname{depth}_A(X \otimes^{\mathrm{L}}_A Y) \geq \operatorname{lc.dim}_A Y - \operatorname{flatdim}_A X \\ + \operatorname{depth} A - \operatorname{depth}_A X - \operatorname{depth}_A Y,\end{aligned}$$

we obtain the desired inequality. □

Following [22], a local DG-ring $(A, \bar{\mathfrak{m}})$ with $\operatorname{amp} A < \infty$ is called *local Cohen–Macaulay* if $\operatorname{amp}\mathrm{R}\Gamma_{\bar{\mathfrak{m}}}(A) = \operatorname{amp} A$. A DG-module $X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ is called *local Cohen–Macaulay* if

$$\operatorname{amp} X = \operatorname{amp} A = \operatorname{amp}\mathrm{R}\Gamma_{\bar{\mathfrak{m}}}(X).$$

A local DG-ring $(A, \bar{\mathfrak{m}})$ with $\operatorname{amp} A < \infty$ is called *Gorenstein* if $\operatorname{injdim}_A A < \infty$.

Remark 3.4. Let $(A, \bar{\mathfrak{m}}, \bar{\kappa})$ be a local DG-ring with $\operatorname{amp} A < \infty$.

(1) The next example shows that the condition $\operatorname{depth}_A X < \infty$ or $X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ cannot be dropped in Corollaries 3.2 and 3.3. Set $(A, \bar{\mathfrak{m}}) = (R, \mathfrak{m})$ be a local ring, $x = x_1, \cdots, x_n$ an R-regular sequence in $\mathfrak{m}$ and $\mathfrak{p}$ a prime ideal such that $x \notin \mathfrak{p}$. Set $X = E(R/\mathfrak{p})$ and $Y = R/x$. Then $X \notin \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(R)$ and $\operatorname{depth}_R X = \infty$. Thus, $\operatorname{flatdim}_R Y = n < \operatorname{depth}_R X - \operatorname{lc.dim}_R(Y \otimes^{\mathrm{L}}_R X)$ and $\operatorname{lc.dim}_R X - \operatorname{depth}_R X = -\infty < \operatorname{lc.dim} R - \operatorname{depth} R$.

(2) Let $X \in \mathrm{D}^{-}_{\mathrm{f}}(A)$. Then $\operatorname{depth}_A X < \infty$ by [22, Proposition 3.5]. One can assume that $\operatorname{flatdim}_A X < \infty$. Then $\operatorname{flatdim}_A X = \operatorname{depth} A - \operatorname{depth}_A X$, it follows by Corollary 3.3 that

$$\operatorname{flatdim}_A X \geq \operatorname{lc.dim} A - \operatorname{lc.dim}_A X.$$

(3) If there is a local Cohen–Macaulay DG-module $X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ with $\operatorname{projdim}_A X < \infty$, then $\operatorname{amp}\mathrm{R}\Gamma_{\bar{\mathfrak{m}}}(X) = \operatorname{lc.dim}_A X - \operatorname{depth}_A X \geq \operatorname{lc.dim} A - \operatorname{depth} A = \operatorname{amp}\mathrm{R}\Gamma_{\bar{\mathfrak{m}}}(A)$ by Corollary 3.3. As $\operatorname{amp}\mathrm{R}\Gamma_{\bar{\mathfrak{m}}}(A) \geq \operatorname{amp} A$ by [22, Theorem 4.1(1)], it implies that $\operatorname{amp}\mathrm{R}\Gamma_{\bar{\mathfrak{m}}}(A) = \operatorname{amp} A$ and A is local Cohen–Macaulay.

(4) Let $\bar{x}_1, \cdots, \bar{x}_n$ be a sequence in $\mathrm{H}^0(A)$ and $\bar{\mathfrak{p}}$ be minimal among the prime ideals containing $(\bar{x}_1, \cdots, \bar{x}_n)$. Set $X = A_{\bar{\mathfrak{p}}}//(\bar{x}_1/1, \cdots, \bar{x}_n/1)$. Then $\operatorname{projdim}_{A_{\bar{\mathfrak{p}}}} X \leq n$ by [14, Theorem 2.19]. Also $\operatorname{lc.dim}_{A_{\bar{\mathfrak{p}}}} X = 0$, so $\operatorname{lc.dim} A_{\bar{\mathfrak{p}}} \leq n$ by Theorem 3.1(2). Here, $Y//\bar{x}Y$ is the *cone* of the map $\bar{x} : Y \to Y$ in $\mathrm{D}(A)$ for $Y \in \mathrm{D}(A)$.

The next result is a generalization of [28, Theorem 3.7] and [5, Theorems 2.1 and 2.6].

Theorem 3.5 *Let $(A, \bar{\mathfrak{m}}, \bar{\kappa})$ be a local DG-ring with $\operatorname{amp} A < \infty$, and let $X, Y \in \mathrm{D}(A)$ such that $\mathrm{R}\mathit{Hom}_A(X, Y) \not\simeq 0$.*

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(*i*) *If* $X, Y \in D^b_f(A)$ *with either* $\mathrm{projdim}_A X < \infty$ *or* $\mathrm{injdim}_A Y < \infty$, *then*

$$\mathrm{lc.dim}_A \mathrm{RHom}_A(X, Y) \leq \mathrm{lc.dim} A - \mathrm{inf} X + \mathrm{sup} Y.$$

(*ii*) *If* $X \in D^-_f(A)$ *with* $\mathrm{projdim}_A X < \infty$ *and* $Y \in D^b(A)$, *then there is an inequality*

$$\mathrm{lc.dim}_A \mathrm{RHom}_A(X, Y) \leq \mathrm{lc.dim}_A Y + \mathrm{projdim}_A X.$$

(*iii*) *If* $X, Y \in D^b_f(A)$ *with* $\mathrm{projdim}_A X < \infty$, *then there are inequalities*

$$\mathrm{lc.dim}_A Y - \mathrm{sup} X \leq \mathrm{lc.dim}_A \mathrm{RHom}_A(X, Y) \leq \mathrm{lc.dim}_A(X \otimes^L_A Y) + \mathrm{projdim}_A X - \mathrm{inf} X.$$

Proof. (i) Fix a point $\bar{\mathfrak{p}} \in \mathrm{Spec}\mathrm{H}^0(A)$. Assume that $\mathrm{projdim}_A X < \infty$. Then

$$\begin{aligned}
&\mathrm{dim}\mathrm{H}^0(A)/\bar{\mathfrak{p}} + \mathrm{sup}\mathrm{RHom}_{A_{\bar{\mathfrak{p}}}}(X_{\bar{\mathfrak{p}}}, Y_{\bar{\mathfrak{p}}}) \\
&\quad \leq \mathrm{dim}\mathrm{H}^0(A)/\bar{\mathfrak{p}} + \mathrm{projdim}_{A_{\bar{\mathfrak{p}}}} X_{\bar{\mathfrak{p}}} + \mathrm{sup} Y_{\bar{\mathfrak{p}}} \\
&\quad = \mathrm{dim}\mathrm{H}^0(A)/\bar{\mathfrak{p}} + \mathrm{depth} A_{\bar{\mathfrak{p}}} - \mathrm{depth}_{A_{\bar{\mathfrak{p}}}} X_{\bar{\mathfrak{p}}} + \mathrm{sup} Y_{\bar{\mathfrak{p}}} \\
&\quad \leq \mathrm{dim}\mathrm{H}^0(A)/\bar{\mathfrak{p}} + \mathrm{dim}\mathrm{H}^0(A)_{\bar{\mathfrak{p}}} + \mathrm{inf} A_{\bar{\mathfrak{p}}} - \mathrm{depth}_{A_{\bar{\mathfrak{p}}}} X_{\bar{\mathfrak{p}}} + \mathrm{sup} Y_{\bar{\mathfrak{p}}} \\
&\quad \leq \mathrm{dim}\mathrm{H}^0(A) - \mathrm{inf} X_{\bar{\mathfrak{p}}} + \mathrm{sup} Y_{\bar{\mathfrak{p}}} \\
&\quad \leq \mathrm{lc.dim} A - \mathrm{inf} X + \mathrm{sup} Y,
\end{aligned}$$

where the equality is by [13, Theorem 2.16], the second inequality is by [22, Corollary 3.6] and the third one is by [22, Proposition 3.3]. Assume that $\mathrm{injdim}_A Y < \infty$. One has

$$\begin{aligned}
&\mathrm{dim}\mathrm{H}^0(A)/\bar{\mathfrak{p}} + \mathrm{sup}\mathrm{RHom}_{A_{\bar{\mathfrak{p}}}}(X_{\bar{\mathfrak{p}}}, Y_{\bar{\mathfrak{p}}}) \\
&\quad \leq \mathrm{dim}\mathrm{H}^0(A)/\bar{\mathfrak{p}} + \mathrm{injdim}_{A_{\bar{\mathfrak{p}}}} Y_{\bar{\mathfrak{p}}} - \mathrm{inf} X_{\bar{\mathfrak{p}}} \\
&\quad = \mathrm{dim}\mathrm{H}^0(A)/\bar{\mathfrak{p}} + \mathrm{depth} A_{\bar{\mathfrak{p}}} + \mathrm{sup} Y_{\bar{\mathfrak{p}}} - \mathrm{inf} X_{\bar{\mathfrak{p}}} \\
&\quad \leq \mathrm{dim}\mathrm{H}^0(A)/\bar{\mathfrak{p}} + \mathrm{dim}\mathrm{H}^0(A)_{\bar{\mathfrak{p}}} + \mathrm{inf} A_{\bar{\mathfrak{p}}} + \mathrm{sup} Y_{\bar{\mathfrak{p}}} - \mathrm{inf} X_{\bar{\mathfrak{p}}} \\
&\quad \leq \mathrm{dim}\mathrm{H}^0(A) + \mathrm{sup} Y_{\bar{\mathfrak{p}}} - \mathrm{inf} X_{\bar{\mathfrak{p}}} \\
&\quad \leq \mathrm{lc.dim} A + \mathrm{sup} Y - \mathrm{inf} X,
\end{aligned}$$

where the equality is by [13, Theorem 2.33]. Therefore, we obtain the desired inequality.

(ii) By [22, Remark 2.3], $\mathrm{lc.dim}_A \mathrm{RHom}_A(X, Y) \leq \mathrm{dim}\mathrm{H}^0(A) + \mathrm{sup}\mathrm{RHom}_A(X, Y) < \infty$. By the equality $(*)$ or [22, (2.14)], one can choose $\bar{\mathfrak{p}} \in \mathrm{Supp}_A X \cap \mathrm{Supp}_A Y$ so that

$$\mathrm{lc.dim}_A \mathrm{RHom}_A(X, Y) = \mathrm{dim}\mathrm{H}^0(A)/\bar{\mathfrak{p}} + \mathrm{sup}\mathrm{RHom}_A(X, Y)_{\bar{\mathfrak{p}}}.$$

By Lemmas 2.1(1) and 2.2(2), we have $\mathrm{RHom}_A(X, Y)_{\bar{\mathfrak{p}}} \simeq \mathrm{RHom}_{A_{\bar{\mathfrak{p}}}}(X_{\bar{\mathfrak{p}}}, A_{\bar{\mathfrak{p}}}) \otimes^{\mathrm{L}}_{A_{\bar{\mathfrak{p}}}} Y_{\bar{\mathfrak{p}}}$ and $\mathrm{sup}\mathrm{RHom}_{A_{\bar{\mathfrak{p}}}}(X_{\bar{\mathfrak{p}}}, A_{\bar{\mathfrak{p}}}) \leq \mathrm{sup}\mathrm{RHom}_A(X, A) = \mathrm{projdim}_A X$. Hence

$$\begin{aligned}
\mathrm{dim}\mathrm{H}^0(A)/\bar{\mathfrak{p}} + \mathrm{sup}\mathrm{RHom}_A(X, Y)_{\bar{\mathfrak{p}}} &\leq \mathrm{dim}\mathrm{H}^0(A)/\bar{\mathfrak{p}} + \mathrm{sup} Y_{\bar{\mathfrak{p}}} + \mathrm{sup}\mathrm{RHom}_{A_{\bar{\mathfrak{p}}}}(X_{\bar{\mathfrak{p}}}, A_{\bar{\mathfrak{p}}}) \\
&\leq \mathrm{lc.dim}_A Y + \mathrm{projdim}_A X.
\end{aligned}$$

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(iii) By the proof of (ii), one has $\mathrm{lc.dim}_A\mathrm{RHom}_A(X,Y) \leq \dim \mathrm{H}^0(A)/\bar{\mathfrak{p}} + \sup Y_{\bar{\mathfrak{p}}} + \mathrm{projdim}_A X$ for some $\bar{\mathfrak{p}} \in \mathrm{Supp}_A X \cap \mathrm{Supp}_A Y$. By Lemma 2.3(3), we have

$$\begin{aligned}\mathrm{lc.dim}_A\mathrm{RHom}_A(X,Y) &\leq \dim \mathrm{H}^0(A)/\bar{\mathfrak{p}} + \sup(X \otimes^{\mathrm{L}}_A Y)_{\bar{\mathfrak{p}}} - \sup X_{\bar{\mathfrak{p}}} + \mathrm{projdim}_A X \\ &\leq \mathrm{lc.dim}_A(X \otimes^{\mathrm{L}}_A Y) + \mathrm{projdim}_A X - \inf X.\end{aligned}$$

For the left-hand side of (iii), as $\mathrm{RHom}_A(X,A) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ and $\mathrm{projdim}_A\mathrm{RHom}_A(X,A) < \infty$, it follows by Theorem 3.1(2) that

$$\begin{aligned}\mathrm{lc.dim}_A Y &\leq \mathrm{lc.dim}_A(\mathrm{RHom}_A(X,A) \otimes^{\mathrm{L}}_A Y) + \mathrm{projdim}_A\mathrm{RHom}_A(X,A) \\ &= \mathrm{lc.dim}_A\mathrm{RHom}_A(X,Y) + \sup\mathrm{RHom}_A(\mathrm{RHom}_A(X,A),A) \\ &= \mathrm{lc.dim}_A\mathrm{RHom}_A(X,Y) + \sup X,\end{aligned}$$

where the first equality is by [27, Proposition 4.4(3)] and the second one follows from the isomorphism $X \simeq \mathrm{RHom}_A(\mathrm{RHom}_A(X,A),A)$, as claimed. □

The following characterization of Gorenstein DG-rings is a DG-version of Auslander–Bridger characterization of classical Gorenstein rings.

Corollary 3.6. *Let $(A,\bar{\mathfrak{m}},\bar{\kappa})$ be a local DG-ring with $\mathrm{amp}A < \infty$. The next are equivalent:*

(1) *A is local Gorenstein;*
(2) *$\mathrm{lc.dim}_A\mathrm{RHom}_A(X,A) < \infty$ for all $X \in D^b_f(A)$;*
(3) *$\mathrm{lc.dim}_A\mathrm{RHom}_A(T,A) < \infty$ for all finitely generated $\mathrm{H}^0(A)$-modules T;*
(4) *$\mathrm{lc.dim}_A\mathrm{RHom}_A(\bar{\kappa},A) < \infty$.*

Proof. (1) ⇒ (2) follows from Theorem 3.5(i) and (2) ⇒ (3) ⇒ (4) are trivial.

(4) ⇒ (1) One has the following equalities

$$\sup\mathrm{RHom}_A(\bar{\kappa},A) = \sup\mathrm{RHom}_A(\bar{\kappa},A)_{\bar{\mathfrak{m}}} = \mathrm{lc.dim}_A\mathrm{RHom}_A(\bar{\kappa},A) < \infty.$$

Hence, [13, Corollary 2.31] implies that $\mathrm{injdim}_A A < \infty$ and A is local Gorenstein. □

Let $(A,\bar{\mathfrak{m}},\bar{\kappa})$ be a local DG-ring and $X \in \mathrm{D}^+(A)$. An element $\bar{x} \in \bar{\mathfrak{m}}$ is called *X-regular* if it is $\mathrm{H}^{\mathrm{inf}X}(X)$-regular, that is, if the multiplication map

$$\bar{x} : \mathrm{H}^{\mathrm{inf}X}(X) \to \mathrm{H}^{\mathrm{inf}X}(X)$$

is injective. Inductively, we say that a sequence $\bar{x}_1,\cdots,\bar{x}_n \in \bar{\mathfrak{m}}$ is *X-regular* if $\bar{x}_1$ is X-regular and the sequence $\bar{x}_2,\cdots,\bar{x}_n$ is $X//\bar{x}_1X$-regular. Following [24], a local DG-ring $(A,\bar{\mathfrak{m}},\bar{\kappa})$ with $\mathrm{amp}A < \infty$ is called *sequence-regular* if $\bar{\mathfrak{m}}$ is generated by an A-regular sequence. If A is a regular local ring and M,N finitely generated A-modules, then there is an inequality

$$\dim_A M + \dim_A N \leq \dim A + \dim_A(M \otimes_A N),$$

which is due to Serre and is called *strong IT* (see Foxby's notes or [4, Lemma 20.1.32]). We next provide the DG-version of this theorem.

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PROPOSITION 3.7. *Let $(A, \bar{\mathfrak{m}}, \bar{\kappa})$ be a sequence-regular DG-ring and $X, Y \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$. One has*

$$\mathrm{lc.dim}_A X + \mathrm{lc.dim}_A Y \leq \mathrm{lc.dim} A + \mathrm{lc.dim}_A(X \otimes^L_A Y).$$

Proof. For $s, t \in \mathbb{Z}$ and $\bar{\mathfrak{p}} \in \mathrm{Spec}\mathrm{H}^0(A)$, as $\sup(\mathrm{H}^s(X) \otimes^{\mathrm{L}}_A \mathrm{H}^t(Y))_{\bar{\mathfrak{p}}} = \sup(\mathrm{H}^s(X) \otimes^{\mathrm{L}}_{\mathrm{H}^0(A)} \mathrm{H}^t(Y))_{\bar{\mathfrak{p}}}$ by Lemma 2.3(3), it follows by the equality $(*)$ and [4, Proposition 17.6.20] that $\mathrm{lc.dim}_A(\mathrm{H}^s(X) \otimes^{\mathrm{L}}_A \mathrm{H}^t(Y)) = \dim_{\mathrm{H}^0(A)}(\mathrm{H}^s(X) \otimes_{\mathrm{H}^0(A)} \mathrm{H}^t(Y))$. We have the next (in)equalities

$$\begin{aligned}
\mathrm{lc.dim}_A(X \otimes^{\mathrm{L}}_A Y) &= \sup_{s,t\in\mathbb{Z}}\{\mathrm{lc.dim}_A(\mathrm{H}^s(X) \otimes^{\mathrm{L}}_A \mathrm{H}^t(Y)) + s + t\} \\
&= \sup_{s,t\in\mathbb{Z}}\{\dim_{\mathrm{H}^0(A)}(\mathrm{H}^s(X) \otimes_{\mathrm{H}^0(A)} \mathrm{H}^t(Y)) + s + t\} \\
&\geq \sup_{s,t\in\mathbb{Z}}\{\dim_{\mathrm{H}^0(A)}\mathrm{H}^s(X) + s + \dim_{\mathrm{H}^0(A)}\mathrm{H}^t(Y) + t\} - \dim\mathrm{H}^0(A) \\
&= \mathrm{lc.dim}_A X + \mathrm{lc.dim}_A Y - \mathrm{lc.dim} A,
\end{aligned}$$

where the first equality is by Lemma 2.4, the inequality is by the above strong IT as the ring $\mathrm{H}^0(A)$ is regular by [24, Theorem 3.4]. □

Following [20], a *dualizing* DG-module R over A is a DG-module $R \in \mathrm{D}^+_{\mathrm{f}}(A)$ such that $\mathrm{injdim}_A R < \infty$ and the natural map $A \to \mathrm{RHom}_A(R, R)$ is an isomorphism in $\mathrm{D}(A)$. If A is local then dualizing DG-module over A is unique up to shift. We say that R is *normalized* if $\inf R = -\dim\mathrm{H}^0(A)$. The next corollary is an extension of the sharper amplitude inequality, see [9, Theorem 5.1] or [4, Theorem 20.1.36].

COROLLARY 3.8. *Let $(A, \bar{\mathfrak{m}})$ be a sequence-regular DG-ring with a normalized dualizing DG-module R and $X, Y \in D^b_f(A)$ with $\mathrm{projdim}_A X < \infty$. One has*

$$\mathrm{amp}(X \otimes^L_A Y) \geq \mathrm{amp}(R \otimes^L_A X) + \mathrm{amp} Y + \inf A.$$

Proof. By [24, Remark 7.4], A is local Cohen–Macaulay, so $-\inf A = \mathrm{amp} A = \mathrm{amp} R$ by [22, Proposition 4.4]. Write $L^* := \mathrm{RHom}_A(L, A)$ and $L^\dagger := \mathrm{RHom}_A(L, R)$ for $L \in \mathrm{D}(A)$. As $\mathrm{projdim}_A X < \infty$ and $\mathrm{projdim}_A X^* < \infty$, it follows by Lemma 2.2(1) that $X \simeq X^{**}$ and $(X \otimes^{\mathrm{L}}_A Y)^\dagger \simeq \mathrm{RHom}_A(X, Y^\dagger) \simeq \mathrm{RHom}_A(X^{**}, Y^\dagger) \simeq X^* \otimes^{\mathrm{L}}_A Y^\dagger$. By [20, Theorem 7.26] and [22, Theorem 2.15 and Proposition 3.3], one has $\mathrm{lc.dim}_A L^\dagger = \sup \mathrm{R}\Gamma_{\bar{\mathfrak{m}}}(L^\dagger) = -\inf L$, $\mathrm{depth}_A L^\dagger = \inf \mathrm{R}\Gamma_{\bar{\mathfrak{m}}}(L^\dagger) = -\sup L$ for $L \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$. As $\mathrm{projdim}_A X < \infty$, $X \otimes^{\mathrm{L}}_A Y \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$. Also $X^* \simeq \mathrm{RHom}_A(X, \mathrm{RHom}_A(R, R)) \simeq (R \otimes^{\mathrm{L}}_A X)^\dagger$ and $\sup(R \otimes^{\mathrm{L}}_A X) = \sup X + \sup R$ by

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Lemma 2.3(3), we have the next (in)equalities

$$\begin{aligned}\mathrm{amp}(X\otimes^{\mathrm{L}}_{A}Y) &= \mathrm{amp}\mathrm{R}\Gamma_{\bar{\mathfrak{m}}}(X\otimes^{\mathrm{L}}_{A}Y)^{\dagger}\\ &= \mathrm{lc.dim}_{A}(X\otimes^{\mathrm{L}}_{A}Y)^{\dagger}-\mathrm{depth}_{A}(X\otimes^{\mathrm{L}}_{A}Y)^{\dagger}\\ &= \mathrm{lc.dim}_{A}(X^{*}\otimes^{\mathrm{L}}_{A}Y^{\dagger})-\mathrm{depth}_{A}\mathrm{RHom}_{A}(X,Y^{\dagger})\\ &\geq \mathrm{lc.dim}_{A}X^{*}+\mathrm{lc.dim}_{A}Y^{\dagger}-\mathrm{lc.dim}A-\mathrm{depth}_{A}Y^{\dagger}+\sup X\\ &= \mathrm{lc.dim}_{A}(R\otimes^{\mathrm{L}}_{A}X)^{\dagger}+\mathrm{lc.dim}_{A}Y^{\dagger}-\mathrm{lc.dim}A-\mathrm{depth}_{A}Y^{\dagger}+\sup X\\ &= -\inf(R\otimes^{\mathrm{L}}_{A}X)-\inf Y-\mathrm{lc.dim}A+\sup Y+\sup X\\ &= -\inf(R\otimes^{\mathrm{L}}_{A}X)-\mathrm{lc.dim}A+\sup(R\otimes^{\mathrm{L}}_{A}X)-\sup R+\mathrm{amp}Y\\ &= \mathrm{amp}(R\otimes^{\mathrm{L}}_{A}X)+\mathrm{amp}Y+\inf A,\end{aligned}$$

where the inequality is by Proposition 3.7 and [27, Proposition 4.9], and the last equality is by $-\mathrm{lc.dim}A=-\mathrm{dim}\mathrm{H}^{0}(A)=\inf R$. □

4. Some applications of the Intersection Theorem

In this section, we explore some implications of our previous results. More precisely, the amplitude inequality of DG-modules is improved; Minamoto's conjecture is proved; the DG-version of the Vasconcelos conjecture about Gorenstein rings is obtained.

The next proposition improves Jørgensen's results, that is to say, eliminates the hypothesis on local DG-rings in [12, Theorem 3.1].

Proposition 4.1. *Let* $(A,\bar{\mathfrak{m}},\bar{\kappa})$ *be a local DG-ring with* $\mathrm{amp}A<\infty$ *and* $0\ncong X,Y\in D^{b}_{f}(A)$ *with* $\mathrm{projdim}_{A}X<\infty$. *One has an inequality*

$$\mathrm{amp}(X\otimes^{L}_{A}Y)\geq \mathrm{amp}Y.$$

In particular, we have $\mathrm{amp}X\geq\mathrm{amp}A$.

Proof. Let $B=\mathrm{L}\Lambda(A,\bar{\mathfrak{m}})$. Then $X\otimes^{\mathrm{L}}_{A}B, Y\otimes^{\mathrm{L}}_{A}B\in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(B)$ with $\mathrm{projdim}_{B}(X\otimes^{\mathrm{L}}_{A}B)<\infty$ by [3, Lemma 4.9] and $\mathrm{amp}(X\otimes^{\mathrm{L}}_{A}Y)=\mathrm{amp}((X\otimes^{\mathrm{L}}_{A}B)\otimes^{\mathrm{L}}_{B}(Y\otimes^{\mathrm{L}}_{A}B))$, $\mathrm{amp}Y=\mathrm{amp}(Y\otimes^{\mathrm{L}}_{A}B)$. Thus, we may assume that A is derived $\bar{\mathfrak{m}}$-adic complete, and A has a normalized dualizing DG-module R by [20, Proposition 7.21]. We have the following isomorphisms

$$\begin{aligned}\mathrm{RHom}_{A}(X\otimes^{\mathrm{L}}_{A}Y,R) &\simeq \mathrm{RHom}_{A}(X,\mathrm{RHom}_{A}(Y,R))\\ &\simeq \mathrm{RHom}_{A}(\mathrm{RHom}_{A}(\mathrm{RHom}_{A}(X,A),A),\mathrm{RHom}_{A}(Y,R))\\ &\simeq \mathrm{RHom}_{A}(X,A)\otimes^{\mathrm{L}}_{A}\mathrm{RHom}_{A}(Y,R).\end{aligned}$$

where the second one is by $X\simeq\mathrm{RHom}_{A}(\mathrm{RHom}_{A}(X,A),A)$ and the third one is by Lemma 2.2(1) as $\mathrm{projdim}_{A}\mathrm{RHom}_{A}(X,A)<\infty$. So we have the next (in)equalities

$$\begin{aligned}\mathrm{amp}(X\otimes^{\mathrm{L}}_{A}Y) &= \mathrm{lc.dim}_{A}\mathrm{RHom}_{A}(X\otimes^{\mathrm{L}}_{A}Y,R)-\mathrm{depth}_{A}\mathrm{RHom}_{A}(X\otimes^{\mathrm{L}}_{A}Y,R)\\ &\geq \mathrm{lc.dim}_{A}\mathrm{RHom}_{A}(Y,R)-\mathrm{depth}_{A}\mathrm{RHom}_{A}(Y,R)\\ &= \mathrm{amp}Y,\end{aligned}$$

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where the equalities are by [20, Theorem 7.26] and the inequality is by the above isomorphism and Corollary 3.3, as desired. □

REMARK 4.2. Let $(A, \bar{\mathfrak{m}}, \bar{\kappa})$ be a local DG-ring with $\operatorname{amp} A < \infty$.

(1) If $\operatorname{amp} A > 0$, then $\bar{\kappa}$ never has finite projective dimension. Thus, there are DG-modules in $\mathrm{D_f}(A)$ which are not compact.

(2) Let $\bar{x}$ be a sequence in $\bar{\mathfrak{m}}$ and $X \in \mathrm{D^b_f}(A)$ with $\operatorname{amp} X = \operatorname{amp} A$. If $\operatorname{projdim}_A X < \infty$, then $\operatorname{amp} X//\bar{\mathbf{x}} = \operatorname{amp}(A//\bar{\mathbf{x}} \otimes^{\mathrm{L}}_A X) \geq \operatorname{amp} A//\bar{\mathbf{x}} \geq \operatorname{amp} A$ by Proposition 4.1, it follows by [13, Lemma 2.13] that $\operatorname{amp} A//\bar{\mathbf{x}} = \operatorname{amp} A$ and $\bar{\mathbf{x}}$ is A-regular. This is the DG-version of Auslander's zerodivisor conjecture.

(3) The example after [12, Theorem 0.2] shows that the condition $\operatorname{amp} A < \infty$ is crucial.

The next result improves the amplitude inequality in Proposition 4.1.

PROPOSITION 4.3. *Let $(A, \bar{\mathfrak{m}}, \bar{\kappa})$ be a sequence-regular DG-ring and $X, Y \in D^b_f(A)$ with $\operatorname{projdim}_A X < \infty$. One has an inequality*

$$\operatorname{amp}(X \otimes^L_A Y) \geq \operatorname{amp} X + \operatorname{amp} Y + \inf A.$$

Proof. Let $B = \mathrm{L}\Lambda(A, \bar{\mathfrak{m}})$. It follows by [24, Corollary 4.5] that A is sequence-regular if and only if B is sequence-regular. Also $\operatorname{projdim}_B(X \otimes^{\mathrm{L}}_A B) < \infty$, we may assume that A is derived $\bar{\mathfrak{m}}$-adic complete and has a normalized dualizing DG-module R. Note that $X \simeq \mathrm{RHom}_A(R, R \otimes^{\mathrm{L}}_A X)$, one has $\inf X = \inf \mathrm{RHom}_A(R, R \otimes^{\mathrm{L}}_A X) \geq \inf(R \otimes^{\mathrm{L}}_A X) - \sup R$, so $\inf(R \otimes^{\mathrm{L}}_A X) \leq \inf X + \sup R$, and $\sup(R \otimes^{\mathrm{L}}_A X) = \sup X + \sup R$ by Lemma 2.3(3). We have the next inequalities

$$\begin{aligned} \operatorname{amp}(X \otimes^{\mathrm{L}}_A Y) &\geq \operatorname{amp}(X \otimes^{\mathrm{L}}_A R) + \operatorname{amp} Y + \inf A \\ &\geq \operatorname{amp} X + \operatorname{amp} Y + \inf A, \end{aligned}$$

where the first inequality is by Corollary 3.8, the second one is by the preceding proof. □

We denote by $X^\natural$ the graded $A^\natural$-module obtained by forgetting the differential of X in $\mathrm{D}(A)$. Let A be a local Cohen–Macaulay DG-ring and $\bar{x}$ a maximal A-regular sequence. Then the DG-module $X = A//\bar{x}$ has finite projective dimension, $\mathrm{H}(X)$ has finite length and $\operatorname{amp} A = \operatorname{amp} X$. The next corollary proves that its opposite is also true.

COROLLARY 4.4. *Let $(A, \bar{\mathfrak{m}}, \bar{\kappa})$ be a local DG-ring with $\operatorname{amp} A < \infty$ and $0 \not\simeq X \in D^b(A)$. If $\operatorname{projdim}_A X < \infty$ and $H(X)$ is finite length, then $\operatorname{lc.dim} A \leq \operatorname{projdim}_A X + \sup X$. In addition, if $\operatorname{amp} X = \operatorname{amp} A$ then A is local Cohen–Macaulay.*

Proof. Note that $\operatorname{projdim}_A X[\sup X] = \operatorname{projdim}_A X + \sup X$ and $\sup X[\sup X] = 0$, it follows that $X[\sup X]$ has a minimal semi-free resolution F with $F^\natural = \coprod_{-n \leq j \leq 0} A^\natural[-j]^{(\beta_j)}$ by [7, Section 0.5], where $n = \operatorname{projdim}_A X[\sup X]$ and each β_j is finite. As $\mathrm{H}(F) \cong \mathrm{H}(X)$ is finite length, it follows by Theorem 3.1(2) that $\operatorname{lc.dim} A \leq n = \operatorname{projdim}_A X + \sup X$.

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If $\mathrm{amp}X = \mathrm{amp}A$, then $\mathrm{lc.dim}A \leq \mathrm{projdim}_A X + \sup X = \mathrm{depth}A - \mathrm{depth}_A X - \inf A + \inf X = \mathrm{depth}A - \inf A$, and so A is local Cohen–Macaulay. □

Shaul [22, Theorem 5.22(2)] showed the DG-version of Bass Conjecture under the noetherian model assumption, and proved the conjecture of Minamoto (see [13, Conjecture 2.36]). He also conjectured that this assumption is redundant (see [22, Remark 5.25]). The affirmative answer to the DG-version of Bass conjecture is the special case of the following proposition, which answers Minamoto's Conjecture or Shaul's Conjecture.

PROPOSITION 4.5. *Let $(A, \bar{\mathfrak{m}}, \bar{\kappa})$ be a local DG-ring with $\mathrm{amp}A < \infty$, and let $0 \not\simeq X \in D_f^+(A)$ with $\mathrm{injdim}_A X < \infty$ and $0 \not\simeq Y \in D_f^b(A)$. One has an inequality,*

$$\mathrm{ampRHom}_A(Y, X) \geq \mathrm{lc.dim}_A Y - \mathrm{depth}_A Y.$$

In particular, A is local Cohen–Macaulay if and only if there is such X with $\mathrm{amp}X = \mathrm{amp}A$.

Proof. Let $B = \mathrm{L}\Lambda(A, \bar{\mathfrak{m}})$. As $\mathrm{H}^0(B) = \mathrm{H}^0(A) \otimes^{\mathrm{L}}_A B$ by [19, Lemma 6.3], we have

$$\mathrm{RHom}_B(\mathrm{H}^0(B)/\bar{\mathfrak{m}}\mathrm{H}^0(B), X \otimes^{\mathrm{L}}_A B) \simeq \mathrm{RHom}_A(\mathrm{H}^0(A)/\bar{\mathfrak{m}}, X) \otimes^{\mathrm{L}}_A B,$$

it follows by [13, Corollary 2.31] that $\mathrm{injdim}_B(X \otimes^{\mathrm{L}}_A B) < \infty$. By [22, Propositions 2.10 and 3.3 and Theorem 2.15], one has $\mathrm{depth}_A Y = \mathrm{depth}_B(Y \otimes^{\mathrm{L}}_A B)$ and $\mathrm{lc.dim}_A Y = \mathrm{lc.dim}_B(Y \otimes^{\mathrm{L}}_A B)$. Also $\mathrm{ampRHom}_A(Y, X) = \mathrm{amp}(\mathrm{RHom}_A(Y, X) \otimes^{\mathrm{L}}_A B)$ by [3, Lemma 4.8]. Thus we may assume that A is derived $\bar{\mathfrak{m}}$-adic complete, and A has a normalized dualizing DG-module R. Then $\mathrm{RHom}_A(X, R) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ by [29, Proposition 7.2] and Lemma 2.1(2), and $\mathrm{projdim}_A \mathrm{RHom}_A(X, R) < \infty$. As $\mathrm{injdim}_A X < \infty$ and $X, Y \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$, $\mathrm{RHom}_A(Y, X) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ and $\mathrm{RHom}_A(X, R) \otimes^{\mathrm{L}}_A Y \simeq \mathrm{RHom}_A(\mathrm{RHom}_A(Y, X), R)$, it follows that

$$\begin{aligned} \mathrm{lc.dim}_A Y - \mathrm{depth}_A Y &\leq \mathrm{lc.dim}_A(\mathrm{RHom}_A(X, R) \otimes^{\mathrm{L}}_A Y) \\ &\quad - \mathrm{depth}_A(\mathrm{RHom}_A(X, R) \otimes^{\mathrm{L}}_A Y) \\ &= \mathrm{ampRHom}_A(Y, X). \end{aligned}$$

where the inequality is by Corollary 3.3 and the equality is by [20, Theorem 7.26].

Let A be local Cohen–Macaulay and $\bar{x}$ a maximal A-regular sequence. Then the DG-module $X = \mathrm{Hom}_A(A//\bar{x}, E(A, \bar{\mathfrak{m}}))$ has finite injective dimension and $\mathrm{amp}A = \mathrm{amp}X$. Conversely, one has $\mathrm{amp}A \leq \mathrm{lc.dim}A - \mathrm{depth}A \leq \mathrm{amp}X = \mathrm{amp}A$ by [22, Theorem 4.1(1)]. Thus, A is local Cohen–Macaulay. □

The next corollary is a DG-setting of Bass characterization of Cohen–Macaulay rings.

COROLLARY 4.6. *Let $(A, \bar{\mathfrak{m}}, \bar{\kappa})$ be a local DG-ring with $ampA < \infty$. The next are equivalent:*

(1) *A is local Cohen–Macaulay;*

(2) *There exists $0 \not\simeq Y \in D_f^b(A)$ with $ampY = ampA$ so that $lc.dim_A RHom_A(X, Y) < \infty$ for all $X \in D_f^b(A)$;*

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(3) *There is* $0 \not\simeq Y \in D^b_f(A)$ *with* $\mathrm{amp}Y = \mathrm{amp}A$ *so that* $\mathrm{lc.dim}_A\mathrm{RHom}_A(\bar{\kappa}, Y) < \infty$.

Proof. (1) $\Rightarrow$ (2) By [22, Theorem 5.22(1)], there exists $0 \not\simeq Y \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ such that $\mathrm{amp}Y = \mathrm{amp}A$ and $\mathrm{injdim}_A Y < \infty$. Hence Theorem 3.5(i) yields the statement (2).

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Since $\mathrm{supRHom}_A(\bar{\kappa}, Y) = \mathrm{lc.dim}_A\mathrm{RHom}_A(\bar{\kappa}, Y) < \infty$ by [22, (2.14)], $\mathrm{injdim}_A Y < \infty$ by [13, Corollary 2.31]. Hence, Proposition 4.5 implies that A is local Cohen–Macaulay. □

Let $(A, \bar{\mathfrak{m}}, \bar{\kappa})$ be a local DG-ring. For $X \in \mathrm{D}(A)$, set

$$\mu^n_A(X) = \mathrm{rank}_{\bar{\kappa}}\mathrm{H}^n(\mathrm{RHom}_A(\bar{\kappa}, X)).$$

Vasconcelos [25] conjectured the following result for a local ring.

PROPOSITION 4.7. *Let* $(A, \bar{\mathfrak{m}}, \bar{\kappa})$ *be a local DG-ring with* $\mathrm{amp}A < \infty$ *and set* $d = \mathrm{dimH}^0(A)$. *If* $\mu^{d+infA}_A(A) = 1$, *then* A *is local Gorenstein.*

Proof. We use induction on d. If $d = 0$, then $\mathrm{inf}A = \mathrm{depth}A$ by assumption and [22, Propositions 3.3] and so A is local Cohen–Macaulay. Consequently, $\mathrm{injdim}_A A = \mathrm{supRHom}_A(\bar{\kappa}, A) = \mathrm{inf}A$ by [26, Propositions 4.7] and A is local Gorenstein. Now assume that $d > 0$. By [22, Theorem 5.18], there is an A-regular element $\bar{x} \in \bar{\mathfrak{m}}$ such that $\mathrm{lc.dim}A//\bar{x} = \mathrm{dimH}^0(A)/(\bar{x}) = d - 1$. Also by [13, Lemma 2.9], one has an isomorphism

$$\mathrm{RHom}_{A//\bar{x}}(\bar{\kappa}, A//\bar{x})[-1] \simeq \mathrm{RHom}_A(\bar{\kappa}, A),$$

which implies that $\mu^{d-1+\mathrm{inf}A//\bar{x}}_{A//\bar{x}}(A//\bar{x}) = 1$ as $\mathrm{inf}A//\bar{x} = \mathrm{inf}A$. By induction, $A//\bar{x}$ is local Gorenstein, and so $\mathrm{lc.dim}_A\mathrm{RHom}_{A//\bar{x}}(\bar{\kappa}, A//\bar{x}) = \mathrm{lc.dim}_{A//\bar{x}}\mathrm{RHom}_{A//\bar{x}}(\bar{\kappa}, A//\bar{x}) < \infty$ by Corollary 3.6. Thus $\mathrm{lc.dim}_A\mathrm{RHom}_A(\bar{\kappa}, A) < \infty$ and A is local Gorenstein. □

COROLLARY 4.8. *Let* $(A, \bar{\mathfrak{m}}, \bar{\kappa})$ *be a local DG-ring with* $ampA < \infty$ *and set* $d = dimH^0(A)$ *and* $E = E(A, \bar{\mathfrak{m}})$. *Then the following are equivalent:*

(1) A *is local Gorenstein;*
(2) $\mu^{d+infA}_A(A) = 1$;
(3) $\mathrm{RHom}_A(E, \bar{\kappa}) \simeq \bar{\kappa}[-d - \mathrm{inf}A]$;
(4) $\bar{\kappa} \otimes^L_A E \simeq \bar{\kappa}[d + \mathrm{inf}A]$.

Proof. (1) $\Leftrightarrow$ (2) follows from Proposition 4.7 and [13, Theorem 3.9].

(1) $\Leftrightarrow$ (4) By [13, Theorem 3.9] and the isomorphism $\bar{\kappa} \otimes^{\mathrm{L}}_A E \simeq \mathrm{RHom}_A(\mathrm{RHom}_A(\bar{\kappa}, A), E)$.

(3) $\Leftrightarrow$ (4) follows from the isomorphism $\mathrm{RHom}_A(E, \bar{\kappa}) \simeq \mathrm{RHom}_A(\bar{\kappa} \otimes^{\mathrm{L}}_A E, E)$. □

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Data availability statement

No data, models, or code were generated or used during the study.

Conflict of Interest

The authors declared no potential conflict of interest with respect to the research, authorship, and/or publication of this article.

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