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ON IMMUNISATION AND *S*-CONVEX EXTREMAL DISTRIBUTIONS

By C. Courtois and M. Denuit

ABSTRACT

The paper concerns the interest risk management of insurance companies or banks. Classes of stochastic order relations for arbitrary discrete random variables are used to find extremal strategies of immunisation in the context of deterministic immunisation theory. In a special case, the results obtained by Hürlimann (2002) are extended to conditions for immunisation under arbitrary *s*-convex or *s*-concave shift factors of the term structure of interest rates. The notion of the Shiu measure is generalised to an immunisation risk measure, accounting for more moments of the asset and liability risks.

KEYWORDS

Immunisation; Extremal Strategies; *S*-Convex Order; Interest Rate Risk

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1. Introduction

A financial institution, such as an insurance company, faces different types of risks, the major one being interest rate fluctuation. If interest rates change, in either the level of interest rates or in the shape of the yield curve, the insurance company may be confronted by a risk of making losses. In fact, as the net cashflow is defined as the difference between the asset and the liability cashflows, a positive net cashflow implies that the asset cashflow surpasses the liability cashflow. This leads to an excess of cash available for reinvestment. If interest rates rise when the net cashflows are positive, there is no problem. However, if interest rates fall, the insurance company will have to reinvest this surplus of cash at rates which are lower than the initial rates. This is called the *reinvestment risk*. On the other hand, if negative net cashflows occur, then the insurance company does not have enough cash to meet its liability obligations (i.e. it is not able to pay insurance losses as promised), and the liquidation of assets or borrowing is needed. Consequently, if interest rates rise when such a situation appears, *market risk* or *price risk* arises, because capital losses can occur due to liquidation assets, whose values have fallen because of the increase in interest rates.

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Immunisation is a technique used by actuaries in asset/liability management, to protect the portfolio value V against the interest rate risk. An excellent review of immunisation theory can be found in Panjer (1998). See also the seminal papers by Moll (1909), Haynes & Kirton (1952), Redington (1952), Wallas (1960) and, for a good historical review, Chapter 4 of Poitras (2007). The first author to find strategies for matching assets and liabilities seems to be Redington (1952). He found that the strategy to immunise the portfolio value against interest rate fluctuations was to equate the duration of assets to that of the liabilities while requiring the cashflows from the assets to be more spread out than those from liabilities. Since then, the importance of the maturity structure of the liabilities in selecting appropriate assets has been widely recognised. However, problems with Redington's theory of immunisation are that the yield curves are assumed to be flat. Consequently, the interest rate fluctuations have to be small, and the model is not arbitrage free. Other models include that of Shiu (1988), which provides necessary and sufficient conditions so that the change in the portfolio value ΔV is non-negative for any convex shift function; under the assumption of convex interest shifts, a portfolio is immunised if, and only if, a decomposition of asset inflows exists such that each component separately immunises each liability outflow.

As mentioned in Hürlimann (2002a, 2003), there exists a quite elementary connection between convex ordering and immunisation which leads to an improvement of the technical understanding of this theory and, for instance, to the derivation of a necessary and sufficient condition for immunisation under arbitrary convex shift factors of the term structure of interest rates. However, to the knowledge of the authors, there is only one evocation of the relationship between financial immunisation theory and other integral stochastic orderings than the convex one, see Hürlimann (2002b). Here, we extend the necessary and sufficient condition for immunisation under arbitrary convex shift factors to a necessary and sufficient condition under arbitrary s-convex or s-concave shift factors. To that end, use is made of the s-convex or s-concave integral stochastic orderings on an arbitrary grid, defined and studied in Denuit *et al.* (1999) and Courtois & Denuit (2006). We also demonstrate that, in the Nelson-Siegel framework, many shift functions are indeed s-convex or s-concave for realistic values of their parameters. This makes the theoretical results attractive for practical implementation. Moreover, for the class of s-convex or s-concave shifts, we give interesting results which extend Theorems 5.1, 5.2 and 5.3 of Hürlimann (2002). Precisely, we define an immunisation risk measure $R^s(X, Y)$, which depends on the sth moments of the asset and liability risks, and which extends the *Shiu measure* (using the M-square index). Then we see that the s-convex (or s-concave) extremal distributions constitute the appropriate tool to provide immunisation strategies which are maximal with respect to $R^s(X, Y)$.

The paper is organised as follows. For the sake of completeness, we first

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recall, in Section 2, the general context of deterministic immunisation theory. Section 3 recaps, quickly, the notion of *s*-convex or *s*-concave stochastic orderings to compare random variables valued in an arbitrary grid. Extremal distributions in moment spaces are briefly presented. Section 4 justifies the *s*-convexity or *s*-concavity assumption made on the class of shift factors. Section 5 is devoted to the derivation of strategies, such that the portfolio of assets and liabilities is immunised, while Section 6 gives examples of applications of such strategies, and compares the results to those using duration vectors. The final Section 7 concludes.

2. Immunisation Problem

The context of this paper is deterministic immunisation theory, i.e. deterministic interest rates, assets and liabilities. Let us consider a portfolio of n non-negative liability outflows $\{L_{s_1}, \ldots, L_{s_n}\}$, due at dates $\{s_1, \ldots, s_n\}$, and a stream of m non-negative asset inflows $\{A_{t_1}, \ldots, A_{t_m}\}$, occurring at dates $\{t_1, \ldots, t_m\}$. Throughout the paper, we allow some assets or liabilities to be equal to zero, and we assume that the same discount function is used to value the assets and the liabilities. We assume, furthermore, that there are no interest rate options embedded in the assets and the liabilities, so that they are independent of interest rate movements.

Now, for a given term structure of interest rates (TSIR), denote by $P(t)$ the price at the current time $t = 0$ of a non-callable and default-free zero-coupon bond with value one at maturity t, i.e. the present value of one monetary unit due at time t. Note that, if r_t (resp. f_t) is the current spot rate (resp. forward rate) of term t ($t \geq 0$), then the discount function $P(t)$ can be written as $1/(1 + r_t)^t$ (resp. $1/(1 + f_1) \cdots (1 + f_t)$). If continuous compounding is used and δ_t denotes the instantaneous forward rate, then:

$$P(t) = \mathrm{e}^{-\int_0^t \delta_t dt} = \mathrm{e}^{-ts_t}$$

where $\{s_t \mid t \geq 0\}$ is the term structure of interest rates. These expressions will be used later in the paper.

With this notation, $\alpha_{t_k} = A_{t_k} P(t_k)$ and $\lambda_{s_l} = L_{s_l} P(s_l)$ are the current prices of the asset and liability flows. Let the present value of the whole portfolio of assets (resp. liabilities) be denoted by A (resp. L), i.e. $A = \sum_{k=1}^{m} \alpha_{t_k}$ (resp. $L = \sum_{l=1}^{n} \lambda_{s_l}$). Then the present value V of the portfolio at time $t = 0$ is given by $V = A - L = \sum_{k=1}^{m} \alpha_{t_k} - \sum_{l=1}^{n} \lambda_{s_l}$. We furthermore make the assumption that V vanishes at time $t = 0$, i.e. that the present value of the assets is equal to the present value of the liabilities. A practical justification of the relevance of this hypothesis can, for instance, be found in Panjer (1998).

We recall that the durations of assets D_A and liabilities D_L are given by:

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$$D_A = \frac{\sum_{k=1}^{m} t_k \alpha_{t_k}}{\sum_{k=1}^{m} \alpha_{t_k}} \quad \text{and} \quad D_L = \frac{\sum_{l=1}^{n} s_l \lambda_{s_l}}{\sum_{l=1}^{n} \lambda_{s_l}}.$$

The convexities are defined as:

$$C_A = \frac{\sum_{k=1}^{m} t_k^2 \alpha_{t_k}}{\sum_{k=1}^{m} \alpha_{t_k}} \quad \text{and} \quad C_L = \frac{\sum_{l=1}^{n} s_l^2 \lambda_{s_l}}{\sum_{l=1}^{n} \lambda_{s_l}}$$

while the corresponding M-squared indexes are $C_A - D_A^2$ and $C_L - D_L^2$.

As this paper is concerned with the immunisation of the portfolio value V against the interest rate risk, one is thus interested in how the value of the portfolio at a time immediately following the current time $t = 0$ can change when the basic TSIR changes. If a shock to the TSIR occurs, which changes the zero-coupon bond price from $P(t)$ to $P^*(t)$, then the 'new' present value of the portfolio is:

$$V^* = A^* - L^* = \sum_{k=1}^{m} \alpha_{t_k}^* - \sum_{l=1}^{n} \lambda_{s_l}^*$$

where $\alpha_{t_k}^* = A_{t_k} P^*(t_k)$ and $\lambda_{s_l}^* = L_{s_l} P^*(s_l)$ are, respectively, the present values of the assets' and the liabilities' portfolios under the new TSIR. Note that here the yield curve is not assumed to be flat and the interest rate shocks are not necessarily small, extending the models of Redington (1952) and Shiu (1988, 1990). Finally, if $f(t) = P^*(t)/P(t)$ is the *shift factor*, the post-shift change in value immediately following time $t = 0$ is given by:

$$\Delta V = V^* - V = \sum_{k=1}^{m} \alpha_{t_k}(f(t_k) - 1) - \sum_{l=1}^{n} \lambda_{s_l}(f(s_l) - 1)$$

or, equivalently, by:

$$\Delta V = \sum_{k=1}^{m} \alpha_{t_k} f(t_k) - \sum_{l=1}^{n} \lambda_{s_l} f(s_l)$$

because we assumed that $V = 0$. The classical *immunisation problem* was first formulated by Redington (1952), and it consists of finding conditions under which $\Delta V \geq 0$ for any change of TSIR. Unfortunately, if one also postulates that there does not exist any arbitrage opportunity in the market, the portfolio value V cannot be immunised for all interest rates movements unless $V^* = V$ for all shocks. Therefore, ΔV can become negative for some types of change of TSIR, and, in such a case, it is good to get precise bounds on the change of value. Here we consider arbitrary generalised convex or

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concave shift factors, and we give some possible immunisation strategies with the help of *s*-convex extremal distributions (see Section 3 for a definition of *s*-convex or *s*-concave stochastic orders and their associated extrema).

As mentioned in Shiu (1988) and Hürlimann (2002), immunisation theory can be studied using an ordering of risk approach. That way of viewing the problem is quite elementary, and provides straightforward proofs of some of the main results of the immunisation theory. The ordering of risk is based on the following connection.

Definition 2.1. The asset risk X is the random variable with support $\mathcal{T} = \{t_1, \ldots, t_m\}$ and probabilities $\{q_1, \ldots, q_m\}$, where $q_k = P[X = t_k] = \alpha_{t_k}/A$ is the normalised asset inflow at time t_k. Similarly, the liability risk Y is the random variable with support $\mathcal{S} = \{s_1, \ldots, s_n\}$ and probabilities $\{p_1, \ldots, p_n\}$, where $p_l = P[Y = s_l] = \lambda_{s_l}/L$ is the normalised liability outflow at time s_l.

Without loss of generality, we set $A = L = 1$. With these definitions, the durations, convexities and M-square indices of the asset and liability cashflows can be expressed as the moments of the random variables X and Y. Specifically, $D_A = \mathrm{E}[X]$, $D_L = \mathrm{E}[Y]$, $C_A = \mathrm{E}[X^2]$, $C_L = \mathrm{E}[Y^2]$, $M_A^2 = \mathrm{Var}[X]$ and $M_L^2 = \mathrm{Var}[Y]$. Similarly, we consider:

$$\Delta V = \mathrm{E}[f(X)] - \mathrm{E}[f(Y)] \tag{1}$$

which means that the change in portfolio value can be identified with the mean difference between transformed asset and liability risks. As is mentioned in the introduction, the difference (1) is representative for integral stochastic orderings generated by a class of shift functions $\mathcal{C}$. Considering immunisation under the class of convex shift factors, Hürlimann (2002) studied this difference using the notion of convex order. This author pointed out that many of the immunisation results by Shiu (1988, 1990), Montrucchio & Peccati (1991) and Uberti (1997) are direct consequences of well-known properties of the convex order. For instance, Theorem 2.1 in Hürlimann (2002) states that, under the hypothesis of convex shift factor, if the durations of assets and liabilities are equal (i.e. the asset and liability risks have equal means) and $V = 0$, then the portfolio of assets and liabilities is immunised if, and only if, Y is smaller than X in the convex order. In this paper, instead of convex shift factors, we consider *s*-convex or *s*-concave shift factors, and we use the extremal distributions of the related stochastic orders to find immunisation strategies. Furthermore, we will see that this approach dealing with extremal distribution provides straightforward proofs of Theorems 5.1, 5.2 and 5.3 in Hürlimann (2002).

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3. *S*-Convex and *S*-Concave Functions, and Related Orderings

For clarity, we briefly recall the definition of the *s*-convex and *s*-concave stochastic orders which will be used in Section 5 to study the problem of portfolio immunisation under arbitrary shift factors of (convex/concave) type.

A stochastic ordering is an order relation which allows probability measures to be compared. In this paper, we will be interested in some classes of integral stochastic orderings. Let us recall that an integral stochastic ordering $\preceq_{\mathcal{C}}^{\mathcal{S}}$ is a stochastic order relation generated by a class $\mathcal{C}$ of measurable functions $f : \mathcal{S} \to \mathbb{R}$. To be specific, given two random variables X and Y valued in $\mathcal{S}$, X is said to be smaller than Y in the $\preceq_{\mathcal{C}}^{\mathcal{S}}$-sense, written as $X \preceq_{\mathcal{C}}^{\mathcal{S}} Y$, if, and only if:

$$\mathrm{E}[f(X)] \leq \mathrm{E}[f(Y)] \text{ for all the functions } f \in \mathcal{C}$$

provided that the expectations exist. Usually $\mathcal{S}$ is taken to be the union of the supports of X and Y. Note that such orderings obviously rely on the classical expected utility theory (by considering $\mathcal{C}$ as a class of utility functions of 'rational decision-makers').

Taking for $\mathcal{C}$ the class of the functions $f : \mathbb{R}^+ \to \mathbb{R}$, with non-negative first derivative ($f^{1/} \geq 0$), yields the well-known stochastic dominance $\preceq_{\mathrm{st}}$. Taking for $\mathcal{C}$ the class of the functions $f : \mathbb{R}^+ \to \mathbb{R}$ with non-negative (resp. non-positive) second derivative ($f^{2/} \geq 0$) (resp. $-f^{2/} \geq 0$), yields the convex (resp. concave) order $\preceq_{\mathrm{cx}}$ (resp. $\preceq_{\mathrm{cv}}$). Taking for $\mathcal{C}$ the class of the functions $f : \mathbb{R}^+ \to \mathbb{R}$ with non-negative sth derivative ($f^{s/} \geq 0$) (resp. such that $(-1)^{s+1} f^{s/} \geq 0$), Denuit *et al.* (1998) defined the *s*-convex (resp. *s*-concave) order $\preceq_{s-\mathrm{cx}}^{\mathbb{R}^+}$ (resp. $\preceq_{s-\mathrm{cv}}^{\mathbb{R}^+}$). The orderings with $s = 1$ and $s = 2$ are $\preceq_{\mathrm{st}}$, $\preceq_{\mathrm{cx}}$ and $\preceq_{\mathrm{cv}}$, respectively.

In this paper, we consider random variables taking on values in an arbitrary ordered finite grid of non-negative points, denoted by $\mathcal{E}_n = \{e_0, \ldots, e_n\}$ say. Stochastic orderings specific for comparing such random variables have been proposed by Denuit *et al.* (1999), and also studied in Courtois & Denuit (2006). As is mentioned in those two papers, the *s*-convex orders on an arbitrary grid are of direct interest in various fields of applications, especially for problems of risky decision making, portfolio selection, insurance premium evaluation and of option pricing. In the context of immunisation theory, the asset and liability risks defined in Definition 2.1 are good examples of situations where random variables valued in an arbitrary grid arise naturally. To be able to define the order $\preceq_{s-\mathrm{cx}}^{\mathcal{E}_n}$, we need a more general tool than just derivatives, called divided differences.

A general approach to the *s*-convex orders uses the concept of divided differences which extend the well-known derivatives. More precisely, let $f : \mathcal{S} \to \mathbb{R}$ and $x_0 < x_1 < \ldots < x_s \in \mathcal{S}$. Starting from:

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$$[x_i]f = f(x_i), \quad i = 0, \ldots, s$$

the sth divided differences are defined recursively by:

$$[x_0, \ldots, x_s]f = \frac{[x_1, \ldots, x_s]f - [x_0, \ldots, x_{s-1}]f}{x_s - x_0} = \sum_{i=0}^{s} \frac{f(x_i)}{\prod_{j=0; j\neq i}^{s}(x_i - x_j)}.$$

A function $f : \mathcal{S} \to \mathbb{R}$ is said to be s-convex if $[x_0, \ldots, x_s]f \geq 0$ for any $x_0 < x_1 < \ldots < x_s \in \mathcal{S}$. It is called s-concave if $(-1)^{s+1}[x_0, \ldots, x_s]f \geq 0$ for any $x_0 < x_1 < \ldots < x_s \in \mathcal{S}$. The order $\preceq_{s-\text{cx}}^{\mathcal{S}}$ (resp. $\preceq_{s-\text{cv}}^{\mathcal{S}}$) can then be defined by taking for $\mathcal{C}$ the class of all the s-convex (resp. s-concave) functions $f : \mathcal{S} \to \mathbb{R}$. This general definition of $\preceq_{s-\text{cx}}^{\mathcal{S}}$ is valid whatever the form of the support $\mathcal{S}$ of the random variables to be compared. Note that, if f possesses an sth derivative, then for any $x_0 \in \mathbb{R}$, $h_0, h_1, \ldots, h_k \geq 0$:

$$[x_0, x_0 + h_1, \ldots, x_0 + h_k]f$$
$$= \int_{\xi_1=0}^{1} \int_{\xi_2=0}^{\xi_1} \ldots \int_{\xi_k=0}^{\xi_{k-1}} f^{k)}(\xi_k(h_k - h_{k-1}) + \ldots + \xi_2(h_2 - h_1) + \xi_1 h_1 + x_0) d\xi_k \ldots d\xi_2 d\xi_1.$$

Since the functions x^k and $-x^k$ are both s-convex for $k = 1, \ldots, s-1$, whatever $\mathcal{S}$, we see that:

$$X \preceq_{s-\text{cx}}^{\mathcal{S}} Y \Rightarrow \text{E}[X^k] = \text{E}[Y^k] \text{ for } k = 1, \ldots, s-1.$$

The relation $\preceq_{s-\text{cx}}^{\mathcal{S}}$ can therefore only be used to compare random variables with the same first $(s-1)$ moments. The relation $\preceq_{s-\text{cx}}^{\mathcal{S}}$ is therefore restricted to moment spaces.

At this stage, we could wonder whether there is anything to gain by considering the specific form of the support of the random variables to be compared (instead of viewing all of them valued in $\mathbb{R}^+$). For $s = 1, 2$, the form of the support of the random variables to be compared is not relevant, in the sense that they can all be seen as valued in $\mathbb{R}^+$: $\preceq_{1-\text{cx}}^{\mathcal{S}} \Leftrightarrow \preceq_{1-\text{cx}}^{\mathbb{R}^+} \Leftrightarrow \preceq_{\text{st}}$ and $\preceq_{2-\text{cx}}^{\mathcal{S}} \Leftrightarrow \preceq_{2-\text{cx}}^{\mathbb{R}^+} \Leftrightarrow \preceq_{\text{cx}}$ for any $\mathcal{S} \subseteq \mathbb{R}^+$. For $s \geq 3$, however, the discrete and real cases are no longer equivalent. Having two random variables valued in $\mathcal{E}_n$, the implication $X \preceq_{s-\text{cx}}^{\mathcal{E}_n} Y \Rightarrow X \preceq_{s-\text{cx}}^{\mathbb{R}^+} Y$ always holds true, but the reciprocal is false in general. Various counterexamples can be found in Denuit *et al.* (1999). We thus get finer stochastic inequalities, taking into account the particular form of the support. More generally, whatever $\mathcal{S}_1 \subset \mathcal{S}_2$, the implication $X \preceq_{s-\text{cx}}^{\mathcal{S}_1} Y \Rightarrow X \preceq_{s-\text{cx}}^{\mathcal{S}_2} Y$ is true in general, but not its reciprocal. For example, in the context of decision analysis, if the decision-maker's preferences agree with some s-convex ordering, when comparing two alternatives, it is safer to consider them valued in a smaller set of outcomes

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rather than in a larger one (because any such comparison can be extended to a larger set, but not reciprocally).

To end with, let us mention that, whatever the form of the support of the random variables in presence, the s-convex and s-concave are closely related through the relation:

$$X \preceq^{\mathcal{S}}_{s-\mathrm{cv}} Y \Leftrightarrow \begin{cases} X \preceq^{\mathcal{S}}_{s-\mathrm{cx}} Y, & \text{if } s \text{ is odd} \\ Y \preceq^{\mathcal{S}}_{s-\mathrm{cx}} X, & \text{if } s \text{ is even.} \end{cases}$$

We are thus allowed to restrict our study to the convex case.

4. *S*-Convex and *S*-Concave Shift Factors

In this section, we show that considering the class of s-convex or s-concave shift factors is a reasonable assumption.

Let us first look at the behaviour of the TSIR in Redington's model, i.e. where there is no distinction between short-term and long-term interest rates and the yield curves are always assumed to be flat. Within this model and with the notations of Section 2, r (resp. δ) denotes the spot rate (resp. the force of interest) and the discount function is $P(t) = 1/(1+r)^t$ (resp. $P(t) = \mathrm{e}^{-\delta t}$), $t \geq 0$. If a shock to the TSIR appears, then the zero-coupon bond price changes from $P(t)$ to $P^*(t) = 1/(1+r^*)^t$ (resp. $P^*(t) = \mathrm{e}^{-\delta^* t}$), $t \geq 0$, where r^* (resp. δ^*) is the new spot rate (resp. new force of interest). For $t \geq 0$, the shift factor becomes:

$$f(t) = \frac{P^*(t)}{P(t)} = \left(\frac{1+r}{1+r^*}\right)^t$$

(resp. $f(t) = \mathrm{e}^{(\delta-\delta^*)t}$) and it is easily seen that:

$$f(\cdot) \text{ is } \begin{cases} s\text{-convex if } \begin{cases} s \text{ is even} \\ s \text{ is odd and } r \geq r^* (\text{resp. } \delta \geq \delta^*) \end{cases} \\ s\text{-concave if } s \text{ is odd and } r \geq r^* (\text{resp. } \delta \geq \delta^*). \end{cases} \tag{2}$$

Consequently, we see that, in the context of parallel shifts of the yield curve, considering s-convex or s-concave shift factors is reasonable.

Let us now look at the behaviour of the shift factors when the whole yield curve does not move in a parallel fashion. In the sequel, the Nelson-Siegel model (for a description of the model, see, for instance, Nawalkha *et al.* (2005)). Recall that, in this case, the zero-coupon rates are given as:

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$$s_t = b_0 + (b_1 + b_2)\frac{\tau}{t}(1 - e^{-t/\tau}) - b_2 e^{-t/\tau} \quad t > 0$$

and $s_0 = b_0 + b_1$, where $\{s_t \mid t \geq 0\}$ is the term structure of interest rates consistent with its definition in Section 2. The Nelson-Siegel model is based on four parameters. $\tau > 0$ is the speed of convergence of the term structure toward $b_0 = s_\infty$. Given a constant τ, a variation in the parameter b_0 can be interpreted as a height change, while a change in b_1 can be interpreted as a slope change (though this parameter also affects the curvature change slightly). A change in b_2 can be interpreted as a curvature change in the TSIR. The following values for those parameters are taken from Chapter 5 of Nawalkha *et al.* (2005), and will be considered for all the illustrations throughout the paper:

$$b_0 = 0.07 \quad b_1 = -0.02 \quad b_2 = 0.001 \quad \tau = 2.$$

The corresponding term structure is shown in Figure 1.

Now, in order to justify the assumption which we made on the shift factors $f(\cdot)$, let us study their behaviour where a change in height, slope and curvature of the yield curve occurs. Height change is modelled by parameter b_0 in the Nelson-Siegel model. It is easily verified that $f^{k/}(t) = f(t)(-\Delta b_0)^k$ $(k \geq 1)$, and that yields to statement (2) in the case of parallel shifts of the yield curve. Now, let us study the influence of a variation in the b_1 parameter (i.e. of a variation in the slope of the yield curve). The following behaviour can be observed:

(1) If s is even and $\Delta b_1 \geq 0$, then the shift function is s-convex.

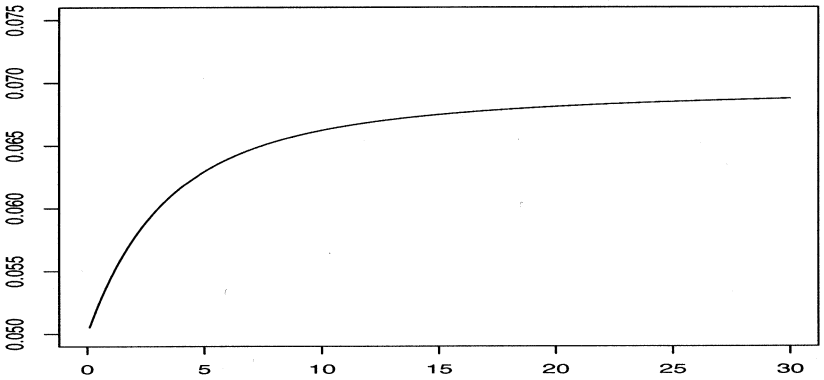


Figure 1. Term structure of instantaneous zero-coupon yields

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Table 1. *s*-convex and *s*-concave properties of the shift factor with $b_0 = 0.7$, $b_1 = -0.02$, $b_2 = 0.001$ and $\tau = 2$

Δb_1	0.50	0.10	0.05	0.01	0.001	−0.001	−0.01	−0.05	−0.10	−0.50
$b_1^* = b_1 + \Delta b_1$	0.48	0.08	0.03	0.01	−0.019	−0.021	−0.03	−0.07	−0.12	−0.52
1-convex	No	No	No	No	No	Yes	Yes	Yes	Yes	Yes
2-convex	Yes	Yes	Yes	Yes	Yes	No	No	No	No	No
3-convex	No	No	No	No	No	Yes	Yes	Yes	Yes	No
4-convex	Yes	Yes	Yes	Yes	Yes	No	No	No	No	No
5-convex	No	No	No	No	No	Yes	Yes	No	No	No
6-convex	Yes	Yes	Yes	Yes	Yes	No	No	No	No	No
7-convex	No	No	No	No	No	Yes	No	No	No	No
8-convex	Yes	Yes	Yes	Yes	Yes	No	No	No	No	No
9-convex	No	No	No	No	No	Yes	No	No	No	No
10-convex	Yes	Yes	Yes	Yes	Yes	No	No	No	No	No
2-convex	No	No	No	No	No	Yes	Yes	Yes	Yes	Yes
4-convex	No	No	No	No	No	Yes	Yes	Yes	No	No
6-convex	No	No	No	No	No	Yes	Yes	No	No	No
8-convex	No	No	No	No	No	Yes	No	No	No	No
10-convex	No	No	No	No	No	Yes	No	No	No	No

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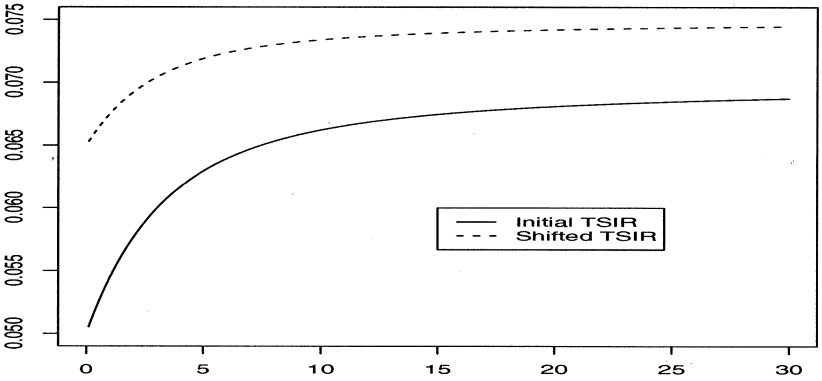


Figure 2. Shock in the zero-coupon yields

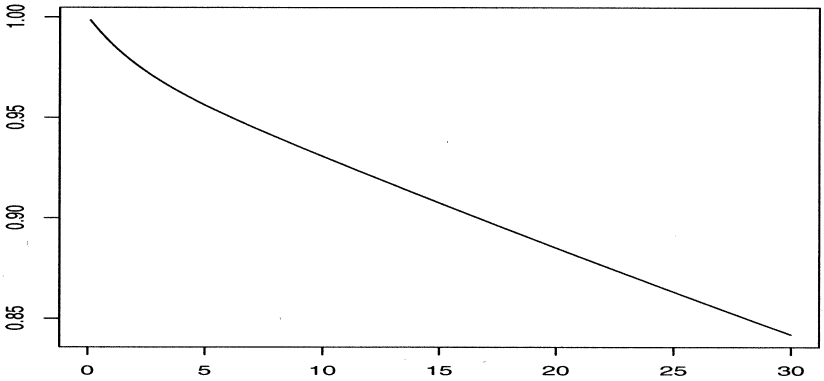


Figure 3. Shift factor corresponding to the shock in the yields of Figure 2

(2) If $\Delta b_1 \leq 0$, then there exists an odd $s \geq 0$ such that the shift function is k-convex for all odd numbers $k \leq s$. The smallest is $|\Delta b_1|$, the biggest is s.
(3) If $\Delta b_1 \leq 0$, then there exists an even $s \geq 0$ such that the shift function is k-concave for all even numbers $k \leq s$. The smallest is $|\Delta b_1|$, the biggest is s.

To illustrate the behaviour of the property of s-(convexity/concavity) of the shift factor up to order $s = 10$, and in response to variations of b_1, see Table 1. Remark that an inverted curve is obtained for $\Delta b_1 = -2b_1$.

Unfortunately, no comparable observations could be made with regard to

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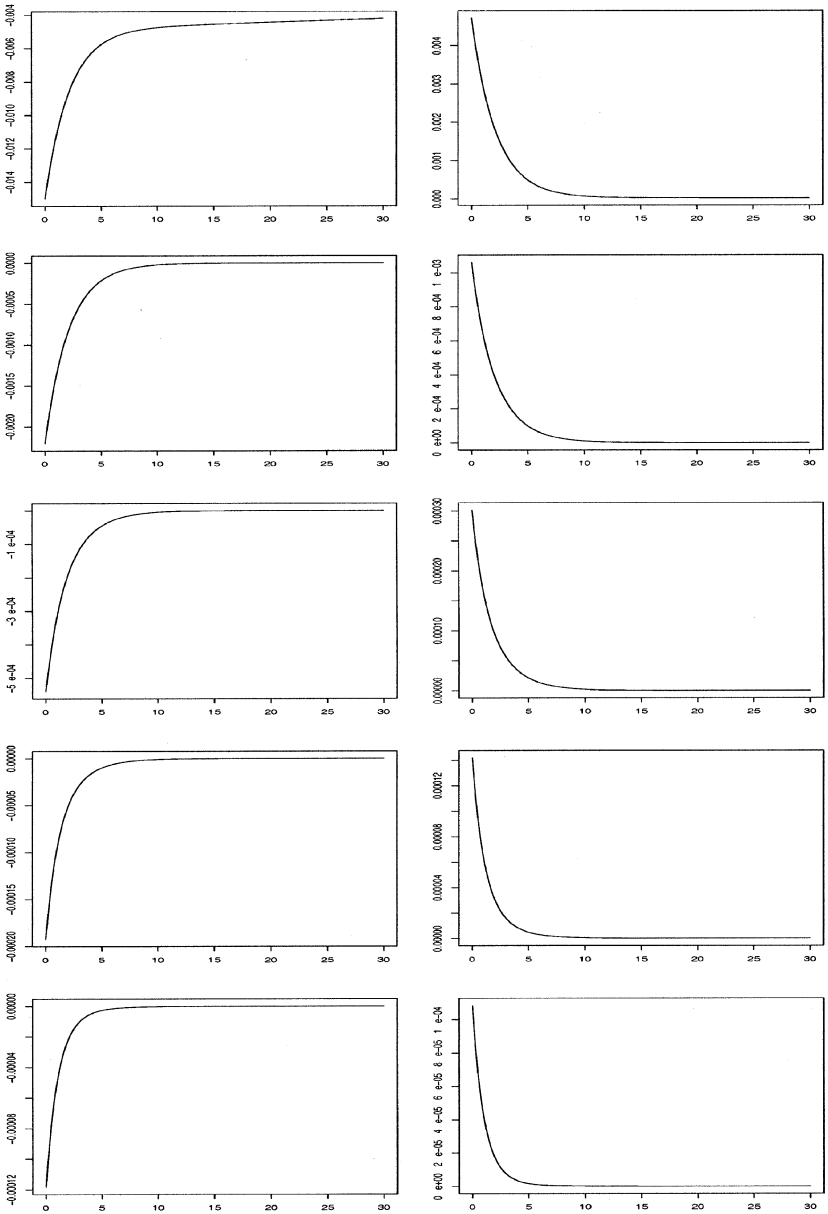


Figure 4. Derivatives of the shift factor

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curvature change in the yield curve. However, note that this does not imply that a shift in the yield curve which is a composition of a curvature change and of any other change is not s-convex or s-concave. To illustrate this claim, let us consider the shift of the yield curve given in Example 5.2 of Nawalkha *et al.* (2005), i.e.:

$$b_0^* = b_0 + \Delta b_0 = 0.075 \quad b_1^* = b_1 + \Delta b_1 = -0.01$$
$$b_2^* = b_2 + \Delta b_2 = 0.002 \quad \tau^* = \tau + \Delta\tau = 2.$$

Such a scenario is the most likely to happen in reality, because, if short rates increase more than the long rates, then the slope of the yield curve will experience a negative shift, while the curvature will most likely experience a positive shift (from a high negative curvature to a low negative curvature). The shock in the yield curve is illustrated in Figure 2 and the corresponding shift factor in Figure 3. Derivatives of the shift factor are easily computed, which leads to the conclusion that $f(\cdot)$ is s-convex for all even $s \geq 1$. See Figure 4 for graphs of the derivatives up to order ten.

5. Immunisation and Extremal Distributions

Let us denote as $\mathcal{M}_s(\mathcal{S}; \mu_1, \mu_2, \ldots, \mu_{s-1})$ the moment space of all the random variables valued in $\mathcal{E}_n$ and with prescribed first $(s-1)$ moments $\mu_k = \mathrm{E}[X^k]$, $k = 1, \ldots, s-1$. Henceforth, the moment sequence $(\mu_1, \mu_2, \ldots, \mu_{s-1})$ is supposed to be such that $\mathcal{M}_s(\mathcal{S}; \mu_1, \mu_2, \ldots, \mu_{s-1})$ is non-void. Conditions for such a space to be non-void can be found, e.g., in Karlin & Studden (1966).

In this section, we will make use of the random variables extremal with respect to the $\preceq^{\mathcal{S}}_{s-\mathrm{cx}}$ and $\preceq^{\mathcal{S}}_{s-\mathrm{cv}}$ orders introduced in Section 3; that is random variables $Z^{(s)}_{\min}$, $Z^{(s)}_{\max}$, $W^{(s)}_{\min}$ and $W^{(s)}_{\max}$ belonging to $\mathcal{M}_s(\mathcal{S}; \mu_1, \mu_2, \ldots, \mu_{s-1})$ and such that either:

$$Z^{(s)}_{\min} \preceq^{\mathcal{S}}_{s-\mathrm{cx}} Z \preceq^{\mathcal{S}}_{s-\mathrm{cx}} Z^{(s)}_{\max}$$

or

$$W^{(s)}_{\min} \preceq^{\mathcal{S}}_{s-\mathrm{cv}} Z \preceq^{\mathcal{S}}_{s-\mathrm{cv}} W^{(s)}_{\max}$$

for all $Z \in \mathcal{M}_s(\mathcal{S}; \mu_1, \mu_2, \ldots, \mu_{s-1})$. For an explicit form of these extrema when $\mathcal{S}$ is an arbitrary grid of non-negative points we refer to Courtois & Denuit (2006). Precisely, it is shown how the s-convex and s-concave orderings can be useful in connection with their respective extrema to find strategies such that the portfolio of assets and liabilities is immunised.

Consider asset and liability risks X and Y, as in Section 2, and the

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immunisation problem under s-convex or s-concave shift factors. A straightforward application of relation (1) gives the following result.

Theorem 5.1. Let X and Y be asset and liability risks, such that $\mathrm{E}[X^k] = \mathrm{E}[Y^k]$ $(k = 1, \ldots, s-1)$, and an s-convex shift factor $f(\cdot)$ of the TSIR. If $V = 0$, the portfolio of assets and liabilities is immunised, that is $\Delta V = \mathrm{E}[f(X)] - \mathrm{E}[f(Y)] \geq 0$ if, and only if, the relation $Y \preceq_{s-\mathrm{cx}}^{\mathcal{T}\cup\mathcal{S}} X$ holds.

Remarks 5.1.

(1) As a particular case of Theorem 5.1, we find Theorem 2.1 of Hürlimann (2002), which prescribes a necessary and sufficient condition for immunisation under arbitrary convex shift factors of the term structure of interest rates.
(2) If, instead of being s-convex, the considered shift factors are s-concave, then the portfolio is immunised if, and only if, the relation $Y \preceq_{s-\mathrm{cv}}^{\mathcal{T}\cup\mathcal{S}} X$ holds; that is $Y \preceq_{s-\mathrm{cx}}^{\mathcal{T}\cup\mathcal{S}} X$ if s is odd and $X \preceq_{s-\mathrm{cx}}^{\mathcal{T}\cup\mathcal{S}} Y$ if s is even.
(3) If $V \geq 0$, the necessary and sufficient condition in Theorem 5.1 becomes sufficient. In fact, we obviously have that $\Delta V = A \cdot \mathrm{E}[f(X)] - L \cdot \mathrm{E}[f(Y)] + V$. Consequently, $V \geq 0$ and $Y \preceq_{s-\mathrm{cx}}^{\mathcal{T}\cup\mathcal{S}} X$ imply that $\Delta V \geq 0$.

Let us now assume that the portfolio corresponding to asset and liability risks X and Y is such that $Y \preceq_{s-\mathrm{cx}}^{\mathcal{T}\cup\mathcal{S}} X$. The remainder of this section will emphasise the utility of the extremal distributions with respect to $\preceq_{s-\mathrm{cx}}^{\mathcal{T}\cup\mathcal{S}}$ and $\preceq_{s-\mathrm{cv}}^{\mathcal{T}\cup\mathcal{S}}$ in immunisation theory. As the $\preceq_{s-\mathrm{cv}}^{\mathcal{T}\cup\mathcal{S}}$-order and extrema are directly deduced from their s-convex anologs (see Section 3), the end of the paper will be restricted to the convex case.

First, let us give an appropriate measure of immunisation risk which is a generalisation of the *Shiu measure* $R(X, Y) = \frac{1}{2}(M_A^2 - M_L^2)$ (see Shiu, 1986). Proceeding as in Panjer (1998, Section 3.8), a positive number ξ exists such that the change in value of the assets and liabilities portfolio due to an instantaneous interest rate shock is given by:

$$\begin{aligned}
\Delta V &= f^{1/}(0)\left(\sum_{k=1}^{m} t_k \alpha_{t_k} - \sum_{l=1}^{n} s_l \lambda_{s_l}\right) + \frac{f^{2/}(0)}{2!}\left(\sum_{k=1}^{m} t_k^2 \alpha_{t_k} - \sum_{l=1}^{n} s_l^2 \lambda_{s_l}\right) + \cdots \\
&\quad + \frac{f^{s-1/}(0)}{(s-1)!}\left(\sum_{k=1}^{m} t_k^{s-1} \alpha_{t_k} - \sum_{l=1}^{n} s_l^{s-1} \lambda_{s_l}\right) + \frac{f^{s/}(\xi)}{s!}\left(\sum_{k=1}^{m} t_k^{s} \alpha_{t_k} - \sum_{l=1}^{n} s_l^{s} \lambda_{s_l}\right) \\
&= f^{1/}(0)(D_A - D_L) + \frac{f^{2/}(0)}{2}(M_A^2 - M_L^2) + \cdots + \frac{f^{s-1/}(0)}{(s-1)}(\mathrm{E}[X^{s-1}] - \mathrm{E}[Y^{s-1}]) \\
&\quad + \frac{f^{s/}(\xi)}{s}(\mathrm{E}[X^s] - \mathrm{E}[Y^s])
\end{aligned}$$

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which is equivalent to:

$$\Delta V = \frac{f^{s/}(\xi)}{s!}(\mathrm{E}[X^s] - \mathrm{E}[Y^s]) \tag{3}$$

if the first $(s-1)$ moments of the asset and liability risks are equal. Consequently, as $f^{s/}(\xi)$ is positive for all s-convex functions, the quantity $R^s(X, Y) = 1/s!(\mathrm{E}[X^s] - \mathrm{E}[Y^s])$ is an appropriate measure of immunisation risk. Note that, for $s = 2$, this measure reduces to the Shiu measure. Note also that a vanishing measure $R^s(X, Y) = 0$ is attainable only if X and Y are identically distributed. In fact, two random variables ordered with respect to the s-convex ordering and with identical sth non-central moment are identically distributed.

Now, three types of extremal variables are considered, the first two being with maximum sth moment increase (i.e. maximising the risk measure $R^s(X, Y)$). Many financial institutions, including insurance companies, have definite and certain future commitments so that their liability outflows are fixed and certain. In such a case, strategies of immunisation will make use of random variables extremal for the set $\{X : Y \preceq_{s-\mathrm{cx}}^{\mathcal{T}\cup\mathcal{S}} X\}$ for a fixed Y and with fixed support $\mathcal{T} = \{t_1, \ldots, t_m\}$ of X. Other financial companies, like banks, may have fixed and certain asset inflows (repayments from loans for instance), so that, in this case, immunisation of the portfolio will have recourse to variables extremal for the set $\{Y : Y \preceq_{s-\mathrm{cx}}^{\mathcal{T}\cup\mathcal{S}} X\}$ for a fixed X and with fixed support $\mathcal{S} = \{s_1, \ldots, s_n\}$ of Y. The third type of extremal variables used will be those which yield the absolute sth moment increase when Y and X vary over the supports $\mathcal{S}$ and $\mathcal{T}$.

We first examine in detail the cases $s = 2$, 3 and 4. Let $s = 2$.

Proposition 5.2.

(1) For fixed $\mu_1 = D_A = D_L$, the maximum variance increase for finite discrete random variables with support $\mathcal{T} = \{t_1, \ldots, t_m\}$, under the restriction $Y \preceq_{2-\mathrm{cx}}^{\mathcal{T}\cup\mathcal{S}} X$ is:

$$\frac{1}{2}(D_L(t_1 + t_m) - t_1 t_m - C_L)$$

and is attained at:

$$X_{\max}^{(2)} = \begin{cases} t_1 & \text{with probability } \frac{t_m - \mu_1}{t_m - t_1} \\ t_m & \text{with probability } \frac{\mu_1 - t_1}{t_m - t_1}. \end{cases}$$

(2) For fixed $\mu_1 = D_A = D_L$, the maximum variance increase for finite discrete random variables with support $\mathcal{S} = \{s_1, \ldots, s_n\}$, under the

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restriction $Y \preceq_{2-\mathrm{cx}}^{\mathcal{T}\cup\mathcal{S}} X$ is:

$$\frac{1}{2}(C_A - D_A(s_l + s_{l+1}) + s_l s_{l+1})$$

and is attained at:

$$Y_{\min}^{(2)} = \begin{cases} s_l & \text{with probability } \frac{s_{l+1}-\mu_1}{s_{l+1}-s_l} \\ s_{l+1} & \text{with probability } \frac{\mu_1 - s_l}{s_{l+1}-s_l} \end{cases}$$

where $l \in \{1, \ldots, n-1\}$ is the integer such that $s_l < \mu_1 \leq s_{l+1}$.

(3) For fixed $\mu_1 = D_A = D_L$, the absolute maximum variance increase for finite discrete random variables with supports $\mathcal{T}$ and $\mathcal{S}$, under the restriction $Y \preceq_{2-\mathrm{cx}}^{\mathcal{T}\cup\mathcal{S}} X$, is attained at $(X_{\max}^{(2)}, Y_{\min}^{(2)})$.

Proof. It is well known that the random variable maximising the second non-central moment over the moment space $\mathcal{M}_2(\mathcal{T}; \mu_1)$ is the 2-convex maximum on $\mathcal{T}$. As $\mu_1 = D_A = D_L$, the random variable extremal for the set $\{X : Y \preceq_{2-\mathrm{cx}}^{\mathcal{T}\cup\mathcal{S}} X\}$ for a fixed Y and with fixed support $\mathcal{T}$ of X is $X_{\max}^{(2)}$. The corresponding maximal variance increase is:

$$R_{\max}^{(2)}(\mathcal{T}, Y) = \max_X R^{(2)}(X, Y) = \frac{1}{2}(\mathrm{E}[(X_{\max}^{(2)})^2] - C_L) = \frac{1}{2}(D_L(t_1 + t_m) - t_1 t_m - C_L).$$

Equivalently, the random variable minimising the second non-central moment over the moment space $\mathcal{M}_2(\mathcal{S}; \mu_1)$ is the 2-convex minimum on $\mathcal{S}$. As $\mu_1 = D_A = D_L$, the random variable extremal for the set $\{Y : Y \preceq_{2-\mathrm{cx}}^{\mathcal{T}\cup\mathcal{S}} X\}$, for a fixed X and with fixed support $\mathcal{S}$ of Y, is $Y_{\min}^{(2)}$. The corresponding maximal variance increase is:

$$R_{\max}^{(2)}(\mathcal{S}, X) = \max_Y R^{(2)}(X, Y) = \frac{1}{2}(C_A - \mathrm{E}[(Y_{\min}^{(2)})^2]) = \frac{1}{2}(C_A - D_A(s_l + s_{l+1}) + s_l s_{l+1})$$

where $l \in \{1, \ldots, n-1\}$ is the integer such that $s_l < D_A \leq s_{l+1}$.

Now, the maximum variance increase for finite discrete random variables with supports $\mathcal{T}$ and $\mathcal{S}$, under the restriction $Y \preceq_{2-\mathrm{cx}}^{\mathcal{T}\cup\mathcal{S}} X$, is attained at $X_{\max}^{(2)}$ and $Y_{\min}^{(2)}$. If we let μ_1 vary, then the absolute maximum variance increase:

$$R_{\max}^{(2)} = \max_X \max_Y R^{(2)}(X, Y) = \max_{\mu_1} \frac{1}{2}(\mu_1(t_1 + t_m - s_l - s_{l+1}) - (t_1 t_m - s_l s_{l+1}))$$

is attained when $\mu_1 = s_{(n+1)/2}$.

Theorems 5.1, 5.2 and 5.3 in Hürlimann (2002) are particular cases of the

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previous theorem when $\mathcal{T} = \mathcal{S} = \{1, \ldots, n\}$. Consequently, the random variable attaining the *maximum Shiu measure* is obviously $X^{(2)}_{\max}$ and the *absolute maximum Shiu measure* is attained when the liability risk is $Y^{(2)}_{\min}$ with $D_L = s_{(n+1)/2}$; where $s_{(n+1)/2}$ is equal to $(n/2)+1$ if n is even and to $(n+1)/2$ if n is odd; the corresponding extremal asset risk being $X^{(2)}_{\max}$, with $D_A = D_L$.

Note also that, as $X \preceq^{\mathcal{T}}_{2-\mathrm{cx}} X^{(2)}_{\max}$ and $Y^{(2)}_{\min} \preceq^{\mathcal{S}}_{2-\mathrm{cx}} Y$, the following upper bound on the change of the portfolio value ΔV is obvious:

$$\Delta V \leq \mathrm{E}[f(X^{(2)}_{\max})] - \mathrm{E}[f(Y^{(2)}_{\min})]$$

for all shift factors f that are 2-convex on $\mathcal{T} \cup \mathcal{S}$. A lower bound can also easily be obtained using the minimum asset risk with respect to $\preceq^{\mathcal{T}}_{2-\mathrm{cx}}$ and the maximum liability risk with respect to $\preceq^{\mathcal{S}}_{2-\mathrm{cx}}$. Now, let $s = 3$.

Proposition 5.3. Let $\mu_1 = D_A = D_L$ and $\mu_2 = M_A = M_L$ be fixed.

(1) Let $k \in \{1, \ldots, m-2\}$ be the integer such that $t_k < (\mu_1 t_m - \mu_2)/(t_m - \mu_1) \leq t_{k+1}$. The maximum third moment increase for finite discrete random variables with support $\mathcal{T} = \{t_1, \ldots, t_m\}$, under the restriction $Y \preceq^{\mathcal{T}\cup\mathcal{S}}_{3-\mathrm{cx}} X$ is attained at:

$$X^{(3)}_{\max} = \begin{cases} t_k & \text{with probability } \frac{\mu_2 - \mu_1(t_{k+1}+t_m)+t_{k+1}t_m}{(t_{k+1}-t_k)(t_m-t_k)} \\ t_{k+1} & \text{with probability } \frac{-\mu_2+\mu_1(t_k+t_m)-t_k t_m}{(t_{k+1}-t_k)(t_m-t_{k+1})} \\ t_m & \text{with probability } \frac{\mu_2-\mu_1(t_k+t_{k+1})+t_k t_{k+1}}{(t_m-t_k)(t_m-t_{k+1})} \end{cases}$$

and is equal to $(1/3!)(\mathrm{E}[(X^{(3)}_{\max})^3] - \mathrm{E}[Y^3])$.

(2) Let $l \in \{2, \ldots, n-1\}$ be the integer such that $s_l < (\mu_2 - \mu_1 s_1)/(\mu_1 - s_1) \leq s_{l+1}$. The maximum third moment increase for finite discrete random variables with support $\mathcal{S} = \{s_1, \ldots, s_n\}$, under the restriction $Y \preceq^{\mathcal{T}\cup\mathcal{S}}_{3-\mathrm{cx}} X$ is attained at:

$$Y^{(3)}_{\min} = \begin{cases} s_1 & \text{with probability } \frac{\mu_2-\mu_1(s_l+s_{l+1})+s_l s_{l+1}}{(s_l-s_1)(s_{l+1}-s_1)} \\ s_l & \text{with probability } \frac{-\mu_2+\mu_1(s_1+s_{l+1})-s_1 s_{l+1}}{(s_l-s_1)(s_{l+1}-s_l)} \\ s_{l+1} & \text{with probability } \frac{\mu_2-\mu_1(s_1+s_l)+s_1 s_l}{(s_{l+1}-s_l)(s_{l+1}-s_l)} \end{cases}$$

and is equal to $(1/3!)(\mathrm{E}[X^3] - \mathrm{E}[(Y^{(3)}_{\min})^3])$.

(3) The absolute maximum third moment increase for finite discrete random variables with supports $\mathcal{T}$ and $\mathcal{S}$, under the restriction $Y \preceq^{\mathcal{T}\cup\mathcal{S}}_{3-\mathrm{cx}} X$ is attained at $(X^{(3)}_{\max}, Y^{(3)}_{\min})$.

Proof. It is well-known that the random variable maximising the third non-central moment over the moment space $\mathcal{M}_3(\mathcal{T}; \mu_1, \mu_2)$ is the 3-convex

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maximum on $\mathcal{T}$. As $\mu_1 = D_A = D_L$ and $\mu_2 = M_A = M_L$, the random variable extremal for the set $\{X : Y \preceq_{3-\mathrm{cx}}^{\mathcal{T}\cup\mathcal{S}} X\}$ for a fixed Y and with fixed support $\mathcal{T}$ of X is $X_{\max}^{(3)}$. The corresponding maximal third moment increase is:

$$R_{\max}^{(3)}(\mathcal{T}, Y) = \max_X R^{(3)}(X, Y) = \frac{1}{3!}(\mathrm{E}[(X_{\max}^{(3)})^3] - \mathrm{E}[Y^3]).$$

Equivalently, the random variable minimising the second non-central moment over the moment space $\mathcal{M}_3(\mathcal{S}; \mu_1, \mu_2)$ is the 3-convex minimum on $\mathcal{S}$. In this case, the random variable extremal for the set $\{Y : Y \preceq_{3-\mathrm{cx}}^{\mathcal{T}\cup\mathcal{S}} X\}$ for a fixed X and with fixed support $\mathcal{S}$ of Y is $Y_{\min}^{(3)}$. The corresponding maximal third moment increase is:

$$R_{\max}^{(3)}(\mathcal{S}, X) = \max_Y R^{(3)}(X, Y) = \frac{1}{3!}(\mathrm{E}[X^3] - \mathrm{E}[(Y_{\min}^{(3)})^3]).$$

Finally, let $s = 4$.

Proposition 5.4. Let $\mu_1 = D_A = D_L$, $\mu_2 = M_A = M_L$ and $\mu_3 = \mathrm{E}[X^3] = \mathrm{E}[Y^3]$ be fixed.

(1) Let $k \in \{2, \ldots, m-2\}$ be the integer such that $t_k < [\mu_3 - \mu_2(t_1 + t_m) + \mu_1 t_1 t_m]/[\mu_2 - \mu_1(t_1 + t_m) + t_1 t_m] \le t_{k+1}$. The maximum fourth moment increase for finite discrete random variables with support $\mathcal{T} = \{t_1, \ldots, t_m\}$, under the restriction $Y \preceq_{4-\mathrm{cx}}^{\mathcal{T}\cup\mathcal{S}} X$, is attained at:

$$X_{\max}^{(4)} = \begin{cases} t_1 & \text{with probability } \frac{-\mu_3+\mu_2(t_k+t_{k+1}+t_m)-\mu_1(t_kt_{k+1}+t_kt_m+t_{k+1}t_m)+t_kt_{k+1}t_m}{(t_k-t_1)(t_{k+1}-t_1)(t_m-t_1)} \\ t_k & \text{with probability } \frac{\mu_3-\mu_2(t_1+t_{k+1}+t_m)+\mu_1(t_1t_{k+1}+t_1t_m+t_{k+1}t_m)-t_1t_{k+1}t_m}{(t_k-t_1)(t_{k+1}-t_k)(t_m-t_k)} \\ t_{k+1} & \text{with probability } \frac{-\mu_3+\mu_2(t_1+t_k+t_m)-\mu_1(t_1t_k+t_1t_m+t_kt_m)+t_1t_kt_m}{(t_{k+1}-t_1)(t_{k+1}-t_k)(t_m-t_{k+1})} \\ t_m & \text{with probability } \frac{\mu_3-\mu_2(t_1+t_k+t_{k+1})+\mu_1(t_1t_k+t_1t_{k+1}+t_kt_{k+1})-t_1t_kt_{k+1}}{(t_m-t_1)(t_m-t_k)(t_m-t_{k+1})} \end{cases}$$

and equal to $(1/4!)(\mathrm{E}[(X_{\max}^{(4)})^4] - \mathrm{E}[Y^2])$.

(2) Let l_1 and l_2 be integers such that $1 \le l_1 < l_1 + 1 < l_2 < l_2 + 1 \le n$, and define:

$$\begin{aligned}
\alpha_1 &:= -\mu_3 + \mu_2(s_{l_1+1} + s_{l_2} + s_{l_2+1}) - \mu_1(s_{l_1+1}s_{l_2} + s_{l_1+1}s_{l_2+1} + s_{l_2}s_{l_2+1}) + s_{l_1+1}s_{l_2}s_{l_2+1} \\
\alpha_2 &:= \mu_3 - \mu_2(s_{l_1} + s_{l_2} + s_{l_2+1}) + \mu_1(s_{l_1}s_{l_2} + s_{l_1}s_{l_2+1} + s_{l_2}s_{l_2+1}) - s_{l_1}s_{l_2}s_{l_2+1} \\
\alpha_3 &:= -\mu_3 + \mu_2(s_{l_1} + s_{l_1+1} + s_{l_2+1}) - \mu_1(s_{l_1}s_{l_1+1} + s_{l_1}s_{l_2+1} + s_{l_1+1}s_{l_2+1}) + s_{l_1}s_{l_1+1}s_{l_2+1} \\
\alpha_4 &:= \mu_3 - \mu_2(s_{l_1} + s_{l_1+1} + s_{l_2}) + \mu_1(s_{l_1}s_{l_1+1} + s_{l_1}s_{l_2} + s_{l_1+1}s_{l_2}) - s_{l_1}s_{l_1+1}s_{l_2}
\end{aligned}$$

which are positive. Then the maximum fourth moment increase for

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finite discrete random variables with support $\mathcal{S} = \{s_1, \ldots, s_n\}$, under the restriction $Y \preceq_{4-\mathrm{cx}}^{\mathcal{T}\cup\mathcal{S}} X$, is attained at:

$$Y_{\min}^{(4)} = \begin{cases} s_{l_1} & \text{with probability } \frac{\alpha_1}{(s_{l_1+1}-s_{l_1})(s_{l_2}-s_{l_1})(s_{l_2+1}-s_{l_1})} \\ s_{l_1+1} & \text{with probability } \frac{\alpha_2}{(s_{l_1+1}-s_{l_1})(s_{l_2}-s_{l_1+1})(s_{l_2+1}-s_{l_1+1})} \\ s_{l_2} & \text{with probability } \frac{\alpha_3}{(s_{l_2}-s_{l_1})(s_{l_2}-s_{l_1+1})(s_{l_2+1}-s_{l_2})} \\ s_{l_2+1} & \text{with probability } \frac{\alpha_4}{(s_{l_2+1}-s_{l_1})(s_{l_2+1}-s_{l_1+1})(s_{l_2+1}-s_{l_2})} \end{cases}$$

and is equal to $(1/4!)(\mathrm{E}[X^4] - \mathrm{E}[(Y_{\min}^{(4)})^4])$.

(3) The absolute maximum third moment increase for finite discrete random variables with supports $\mathcal{T}$ and $\mathcal{S}$, under the restriction $Y \preceq_{3-\mathrm{cx}}^{\mathcal{T}\cup\mathcal{S}} X$, is attained at $(X_{\max}^{(4)}, Y_{\min}^{(4)})$.

Proof. It is well-known that the random variable maximising the third non-central moment over the moment space $\mathcal{M}_4(\mathcal{T}; \mu_1, \mu_2, \mu_3)$ is the 4-convex maximum on $\mathcal{T}$. The random variable extremal for the set $\{X : Y \preceq_{4-\mathrm{cx}}^{\mathcal{T}\cup\mathcal{S}} X\}$ for a fixed Y and with fixed support $\mathcal{T}$ of X is $X_{\max}^{(4)}$. The corresponding maximal fourth moment increase is:

$$R_{\max}^{(4)}(\mathcal{T}, Y) = \max_X R^{(4)}(X, Y) = \frac{1}{4!}(\mathrm{E}[(X_{\max}^{(4)})^4] - \mathrm{E}[Y^2]).$$

Equivalently, the random variable minimising the second non-central moment over the moment space $\mathcal{M}_4(\mathcal{S}; \mu_1, \mu_2, \mu_3)$ is the 4-convex minimum on $\mathcal{S}$. Now, the random variable extremal for the set $\{Y : Y \preceq_{4-\mathrm{cx}}^{\mathcal{T}\cup\mathcal{S}} X\}$ for a fixed X and with fixed support $\mathcal{S}$ of Y is $Y_{\min}^{(4)}$. The corresponding maximal fourth moment increase is:

$$R_{\max}^{(4)}(\mathcal{S}, X) = \max_Y R^{(4)}(X, Y) = \frac{1}{4!}(\mathrm{E}[X^4] - \mathrm{E}[(Y_{\min}^{(4)})^4]).$$

The general case $s \geq 5$ can also be treated easily. However, we leave this generalisation to the interested reader, and just refer to Courtois & Denuit (2006), where the general form of the s-convex extrema on an arbitrary ordered grid of non-negative points can be found. Summarising, as the sign of $f^{s/}(\cdot)$ in (3) is known to be non-negative (resp. non-positive) for s-convex (resp. s-concave) shift factors, the proposed strategies of immunisation are those which maximise the difference between the sth moments of the asset and liability risks. However, if it happens that $f(\cdot)$ is neither s-convex nor s-concave for some s, then we are not sure about the sign of $f^{s/}(\cdot)$, and a cautious strategy could be to minimise the absolute value of the difference between the sth moments of the asset and liability risks.

Note finally that, generalising the notion of s-convex function, bounds on

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the change in portfolio value can be derived in case the shift factor of the TSIR is not s-convex. Such bounds are comparable to those of Uberti (1997). Moreover, immunisation results can be obtained.

Given real numbers α and β, and an interval $I \subseteq \mathbb{R}$, a real function $f(\cdot)$ is called α-(s-convex) on I if $f(x) - (1/s!)\alpha x^s$ is s-convex on I. It is called (s-convex)-β on I if $(1/s!)\beta x^s - f(x)$ is s-convex on I. Note that any shift $f(\cdot)$ which is s times differentiable is α-(s-convex) on the asset risk support $\mathcal{T}$ with $\alpha = \inf_{x\in\mathcal{T}} f^{s/}(x)$ and (s-convex)-β on the liability risk support $\mathcal{S}$ with $\beta = \sup_{x\in\mathcal{S}} f^{s/}(x)$.

With these definitions in mind, let X and Y be asset and liability risks such that $Y \preceq_{s-\mathrm{cx}}^{\mathcal{T}\cup\mathcal{S}} X$ and $V = 0$. If the shift factor $f(t)$ of the TSIR is α-(s-convex) on $\mathcal{T}$ and (s-convex)-β on $\mathcal{S}$, then the following bounds on the change in portfolio value are obvious:

$$\alpha R^{(s)}(X, Y) \leq \Delta V = \mathrm{E}[f(X)] - \mathrm{E}[f(Y)] \leq \beta R^{(s)}(X, Y).$$

Now, consider asset and liability risks X and Y, defined as in Section 2. We assume the corresponding portfolio to be such that $Y \preceq_{s-\mathrm{cx}}^{\mathcal{T}\cup\mathcal{S}} X$. Then the previous bounds lead to an immunisation strategy to control the change in porfolio value for arbitrary shifts in the TSIR. Fong & Vasicek (1984), Shiu (1986, 1988, 1990) and Uberti (1997) suggested such a strategy for $s = 2$, and it can be extended as follows to any value of $s \geq 2$. First, it consists in fixing bounds on the tolerable change in value of ΔV by choosing a real non-negative number R. Then, it remains to analyse under which conditions the equation $R^{(s)}(X, Y) = R$ has solutions in the set of all feasible asset and liability risks X and Y, such that $Y \preceq_{s-\mathrm{cx}}^{\mathcal{T}\cup\mathcal{S}} X$. The term 'feasible' requires, of course, a precise definition, depending on the particular problem which is considered and the constraints which the unknown quantities have to verify.

As already stated, $R = 0$ is possible only if X and Y are identically distributed. Moreover, it is easily seen that the maximum value $R_{\max} > 0$ of the measure $R^{(s)}(X, Y)$, for which $R^{(s)}(X, Y) = R$ has feasible solutions for all $R \in (0, R_{\max}]$, is attained at $(X_{\max}^{(s)}, Y_{\min}^{(s)})$.

6. Application

In applications, the non-negative deterministic liabilities due at dates $s_1, \ldots, s_n$ are known, that is that the liability risk Y is determined. On the other side, the asset risk X is unknown, but subject to condition $Y \preceq_{s-\mathrm{cx}}^{\mathcal{T}\cup\mathcal{S}} X$, as well as to financial market feasibility. To describe the last economic restriction, suppose that there are at disposal J non-callable and default-free fixed-income financial assets, which pay the non-negative amount A_{j,t_k} at the end of the kth period ($k = 1, \ldots, m$) for an initial investment of one unit of money in the jth fixed-income security ($j = 1, \ldots, J$). Let $x_j \geq 0$ denote the

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amount of money to be invested in the jth security, i.e. $A_{t_k} = \sum_{j=1}^{J} x_j A_{j,t_k}$ $(k = 1, \ldots, m)$. Set $\alpha_{j,t_k} = A_{j,t_k} P(t_k)$, $A^{(j)} = \sum_{k=1}^{m} \alpha_{j,t_k}$ the present value of the jth security inflows, and let us define the jth asset risk X_j with support $\mathcal{T}$ and probabilities $P[X_j = t_k] = q_{j,k} = \alpha_{j,t_k}/A^{(j)}$ $(k = 1, \ldots, m)$. The normalisation assumption $A = L = 1$ says that $\sum_{j=1}^{J} x_j = 1$ and $A^{(j)} = 1$, and the present value of the kth asset inflow is given by the linear constraint:

$$\sum_{j=1}^{J} x_j q_{j,k} = q_k \quad k = 1, \ldots, m$$

where $q_k = P[X = t_k]$ $(k = 1, \ldots, m)$. Note that, with these definitions, we have:

$$\mathrm{E}[X^k] = \sum_{j=1}^{J} x_j \mathrm{E}[X_j^k], \quad k = 0, 1, 2 \ldots$$

As stated in Section 5, the asset risk $X_{\max}^{(s)}$ $(s \geq 2)$ is such that $\mathrm{E}[X^k] = \mathrm{E}[Y^k]$ $(k = 1, \ldots, s-1)$ and $Y \preceq_{s-\mathrm{cx}}^{\mathcal{T}\cup\mathcal{S}} X_{\max}^{(s)}$, leading to immunising strategies against shifts in the TSIR which imply s-convex shift factors. As an illustration, let us take $s = 3$. In this case, the possible immunising strategies are constructed as follows. As the support of $X_{\max}^{(3)}$ is $\{t_K, t_{K+1}, t_m\}$, with K such that $t_K < (D_L t_m - C_L)/(t_m - D_L) \leq t_{K+1}$, the J financial assets in which the financial company may invest have to pay non-negative amounts at times t_K and/or t_{K+1} and/or t_m. Moreover, the unknown amounts $(x_1, \ldots, x_J)$ invested in those securities are such that:

$$\begin{cases} (x_1 A_{1,t_K} + \cdots + x_J A_{J,t_K}) P(t_K) = \frac{C_L - D_L(t_{K+1} + t_m) + t_{K+1} t_m}{(t_{K+1} - t_K)(t_m - t_K)} \\ (x_1 A_{1,t_{K+1}} + \cdots + x_J A_{J,t_{K+1}}) P(t_{K+1}) = \frac{-C_L + D_L(t_K + t_m) - t_K t_m}{(t_{K+1} - t_K)(t_m - t_{K+1})} \\ (x_1 A_{1,t_m} + \cdots + x_J A_{J,t_m}) P(t_m) = \frac{C_L - D_L(t_K + t_{K+1}) + t_K t_{K+1}}{(t_m - t_K)(t_m - t_{K+1})} \\ \sum_{j=1}^{J} x_j = 1. \end{cases} \tag{4}$$

Note that $(x_1, \ldots, x_J)$ verifying the previous conditions is such that:

$$\begin{cases} x_1 D_1 + \cdots + x_J D_J = D_L \\ x_1 C_1 + \cdots + x_J C_J = C_L \end{cases}$$

where D_j and C_j are the durations and convexities of the jth security $(j = 1, \ldots, J)$. If the number of securities J equals the number of constraints $(s + 1)$, then a unique solution exists for the security proportions $(x_1, \ldots, x_J)$.

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If $J < s+1$, then the system does not have any solutions. If $J > s+1$ (which is the most common case), the system of equations has an infinite number of solutions.

Gains in immunisation performance have also been made possible by the concept of a duration vector model. This model uses a vector of higher order duration measures to immunise against changes in the shape parameters (i.e. height, slope, curvature) of the yield curve. It is well known that the duration vector model provides a high level of immunisation performance, using only three to five risk measures (instead of two for the duration/convexity model). We refer to Chapter 5 of Nawalkha *et al.* (2005) for further details.

Let us denote $(D_A(1), \ldots, D_A(s-1))$ and $(D_L(1), \ldots, D_L(s-1))$, respectively, as the duration vectors of the assets and the liabilities. These vectors are exactly the vectors of the $(s-1)$ first moments of the asset and liability risks X and Y. Then, to immunise the portfolio value, proportions $(x_1, \ldots, x_J)$ of investments in different assets are chosen such that the duration vector of the porfolio of assets is set equal to the duration vector of the liabilities:

$$\begin{cases} D_A(1) = x_1 D_1(1) + \cdots + x_J D_J(1) = D_L(1) \\ D_A(2) = x_1 D_1(2) + \cdots + x_J D_J(2) = D_L(2) \\ \vdots \\ D_A(s-1) = x_1 D_1(s-1) + \cdots + x_J D_J(s-1) = D_L(s-1) \\ x_1 + \cdots + x_J = 1. \end{cases} \tag{5}$$

Note that the system of s equations (5) is, in fact, equivalent to the one which we would obtain with the immunising strategy proposed in this paper which uses the maximum $X^{(s)}_{\max}$, but adding one constraint which fixes the s occurrence dates of the immunising asset inflows. As the system of equations (5) can have an infinite number of solutions (if $J > s$), the proposed strategy to select a unique immunising solution is to optimise the quadratic function:

$$\min \sum_{j=1}^{J} x_j^2$$

subject to the set of constraints given in (5). As it is mentioned in Nawalkha *et al.* (2005), this objective function "is used for achieving diversification across all bonds, and reduces bond-specific idiosyncratic risks that are not captured (e.g., liquidity risk) by the systematic term structure movements." This could also be a solution to solve systems like (4) which have an infinite number of solutions (i.e. with $J > s+1$).

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7. Conclusion

In this paper, we note that there exists a quite elementary connection between s-convex orderings and immunisation, which leads to the derivation of a necessary and sufficient condition of immunisation under arbitrary s-convex shift factors of the TSIR. We note that, for $s = 2$, this amounts to the following well-known immunisation strategy under a convex shift factor of the TSIR; equate the duration of the assets to that of the liabilities while requiring the cashflows from the assets to be more spread out than those from the liabilities. Moreover, we define an immunisation risk measure which extends the Shiu measure and is a function of the sth moments of the asset and liability risks. We also see that the s-convex extremal distributions are good tools to provide immunisation strategies which are maximal with respect to this new measure.

An improvement which could be considered is to let the assets A_{t_k} $(k = 1, \ldots, m)$ and the liabilities L_{s_l} $(l = 1, \ldots, n)$ be random variables instead of real numbers. In such a case, the post-shift change of the portfolio becomes a random variable, and the immunisation problem consists in finding conditions under which the expectation of ΔV remains non-negative. To that end, we suppose that the expected present value of the assets is equal to the expected present value of the liabilities, and we define new asset and liability risks X_A and Y_L with mixture distributions in such a way that $P(X_A = t_k \mid A_{t_k}) = \alpha_{t_k}/\mathrm{E}[A]$ $(k = 1, \ldots, m)$ and $P(Y_L = s_l \mid L_{s_l}) = \lambda_{s_l}/\mathrm{E}[L]$ $(l = 1, \ldots, n)$.

Without loss of generality, we can make the assumption that $\mathrm{E}[A] \geq \mathrm{E}[L]$ under the historical measure, and we obtain that $\mathrm{E}[\Delta V] = \mathrm{E}[A]\mathrm{E}[f(X_A)] - \mathrm{E}[L]\mathrm{E}[f(Y_L)] + \mathrm{E}[V]$; which means that the expected change in portfolio value can be linked with the mean difference between transformed asset and liability mixture risks. As for fixed asset and liability flows, this difference is representative for integral stochastic orderings generated by a class of shift functions f. Consequently, considering asset and liability cashflows such that $\mathrm{E}[X_A^k] = \mathrm{E}[Y_L^k]$ $(k = 1, \ldots, s-1)$ and an s-convex shift factor of the TSIR, the portfolio of assets and liabilities is immunised (i.e. $\mathrm{E}[\Delta V] \geq 0$) if the relation $Y_L \preceq_{s-cx}^{\mathcal{T} \cup \mathcal{S}} X_A$ holds. It now remains to see how the previous result can be used to find reliable immunisation strategies. This will be the topic of a forthcoming paper.

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