Probability in the Engineering and Informational Sciences, **20**, 2006, 271–286. Printed in the U.S.A.

MOVE-FORWARD RULES AND *f*-SWAP RULES APPLIED TO A COMMUNICATION PROBLEM

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In a communication network, one might attempt to route calls from an origin to a destination through n paths that will be tried one by one, each having a success probability $p_i \in (0,1)$, $i = 1,2,\ldots,n$. The order of trying is controlled by a routing table. The number of attempts made is defined as the cost of the routing table. Move-forward self-organizing rules are applied to the routing table and comparisons of expected equilibrium costs are performed when $p_2 = p_3 = \cdots = p_n$. Stationary distributions for a subset of f-swap rules are obtained for general p_i's.

1. INTRODUCTION

In a communication network, one might attempt to route calls from an origin to a destination. Suppose that there are n paths that can be tried one by one, each having a success probability $p_i \in (0,1)$, $i = 1,2,\ldots,n$. Note that the p_i's do not necessarily sum to one. The order of trying is controlled by a routing table represented by π, a permutation of $\{1,2,\ldots,n\}$. Each time there is a need of connection from the origin to the destination, the path $\pi(1)$ is attempted first. If path $\pi(i)$ $(i = 1,2,\ldots,n-1)$ is attempted and fails, path $\pi(i+1)$ is attempted next. This process continues until one of the paths succeeds or all of the paths fail. We say that the call is completed or blocked, respectively. Whether a path succeeds or fails is assumed to be independent of the other paths and other calls. The number of attempts made is defined as the cost of this communication problem. Topkis [12] suggested that self-organizing rules can be applied to reorganize the routing table each time a call is completed.

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https://doi.org/10.1017/S0269964806060177 Published online by Cambridge University Press

Specifically, he considered the following move-to-front rule that moves the successful path to the front of the routing table so that it will be tried first for the next call. Topkis [12] showed that under the move-to-front self-organization rule, the ordering in the routing table will reach an equilibrium. He showed that the stationary distribution for a routing table π under the move-to-front rule is given by

$$P(\pi) \propto \sum_{i=2}^{n} \frac{q_{\pi(i)} q_{\pi(i+1)} \cdots q_{\pi(n)}}{1 - q_{\pi(i)} q_{\pi(i+1)} \cdots q_{\pi(n)}},$$

where $q_i \equiv 1 - p_i$ for $i = 1,2,\ldots,n$. The expected cost is then given by

$$\sum_{\pi} P(\pi) \text{cost}(\pi)$$

when $\text{cost}(\pi)$ is the expected number of attempts for a call to be connected under the routing table π.

Self-organizing rules are most commonly used under an information retrieval setting when the items under search are linearly ordered. Here, we consider n records $R_1, R_2, \ldots, R_n$ in a linear list. The configuration of these records can be represented by a permutation $\pi \equiv (\pi(1), \pi(2), \ldots, \pi(n))$ of the set $\{1,2,\ldots,n\}$. From time to time, there are successive independent demands for them. Each time there is a request for a certain record, we start at the front (first position) of the list and compare the records in the list one by one until we find a record $R_{\pi(i)}$, which is in the ith position, being the exact record requested. The time spent or the number of comparisons required to find the requested record is the search cost of this problem. We assume that the record R_i has probability p_i of being requested. Unlike the communication problem, the p_i's sum up to unity. The move-to-front rule moves the requested record to the first position in the linear list. The stationary distribution and the expected equilibrium cost under the move-to-front rule is well known in the self-organizing literature.

There are many self-organizing rules for a linear list other than the move-to-front rule. There is the famous transposition rule, considered by Hendricks [3,4] and Rivest [9], which moves the requested record to one position before its original position without changing the relative positions of the other records. A *move-forward* rule (Kan and Ross [5], Phelps and Thomas [8]) moves the requested record, which is in position i, to position l_i, with $l_1 = 1$, $l_i < i$ for $i = 2,3,\ldots,n$, and $l_i \leq l_{i+1}$ for $i = 1,2,\ldots,n-1$, and does not change the relative ordering of all other records. Examples for move-forward rules include the famous move-to-front rule and transposition rule, the family of move-up-k rules, SWITCH(k) rules, and POS(k) rules (Rivest [9], Tenenbaum and Nemes [11], Chong and Lam [1]). Rules that do not necessarily move things up, like the f-swap rules, are also considered in the literature (Lam and Cowan [7]). Note that each self-organizing rule under the information retrieval setting can also be applied to the communication setting.

One interesting problem is to compute the stationary distributions for various self-organizing rules. Under the information retrieval setting, stationary distribu-

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tions are obtained for a large range of self-organizing rules, like the SWITCH(k) rules, POS(k) rules, and the f-swap rules; see Tenenbaum and Nemes [11], Chong and Lam [1], and Lam and Cowan [7] for references.

Another problem is to compare the cost of two self-organizing rules. It is well known that for an information retrieval problem, the transposition rule outperforms the move-to-front rule, which, in turn, outperforms the random rule that simply arranges the record randomly. Among the family of move-forward rules, Lam [6] obtained some cost comparison results under the condition that $p_2 = p_3 = \cdots = p_n$.

As we have pointed out earlier, the same self-organizing rule can be applied in the information retrieval and the communication settings. However, the stationary distribution under these settings are different. This is because the transition probabilities for the underlying Markov chain are not exactly the same. For example, when we consider three records under the move-to-front rule, the transition probability from the state (231) to (123) equals p_1 and the transition probability from (123) to itself is also p_1. However, under the communication setting, the former transition probability equals $(1 - p_2)(1 - p_3)p_1$ and the latter equals $p_1 + (1 - p_1)(1 - p_2)(1 - p_3)$. Because the communication problem has more complicated transition probabilities, its stationary distributions might not have an analytic solution even though its counterpart has one.

The expected cost of a self-organizing rule at equilibrium is given by $\sum_\pi P(\pi)\text{cost}(\pi)$, where $P(\pi)$ is the stationary distribution and $\text{cost}(\pi)$ is the cost of a given configuration π. Like $P(\pi)$, which might be different under the two settings, $\text{cost}(\pi)$ is also different. Under the information retrieval setting, $\text{cost}(\pi) = \sum_{i=1}^{n} ip_{\pi(i)}$. However, under the communication setting,

$$\text{cost}(\pi) = 1 + \sum_{j=1}^{n-1} \prod_{k=1}^{j} (1 - p_{\pi(k)}).$$

Because of the above-mentioned differences, it comes as no surprise that the communication problem has less theoretical results than the record problem. The first step in narrowing this gap can be found in Ross [10], where the stationary distribution for the transposition rule under the communication setting can be given analytically and it is proved that the connection cost for the transposition rule is not more than that for the move-to-front rule when all but one of the p_i's are the same. Chong and Lam [2] found the stationary distributions for the family of POS(k) rules and showed that for the case of three paths, the transposition rule is never worse than the move-to-front rule.

This article attempts to narrow the gap between the theoretical results for self-organizing rules under the information retrieval or communication setting. In Section 2 we consider a family of self-organizing rules called the f-swap rules under the communication settings. When the f-swap rules satisfy some mild conditions, we are able to derive the stationary distribution for general $p = (p_1, p_2, \ldots, p_n)$. In Section 3 we consider the family of move-up rules under the communication setting. When two move-up rules are ordered according to a particular manner, we are able to compare their expected cost when the probabilities

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satisfy $p_2 = p_3 = \cdots = p_n$. In Section 4 the expected cost of a given move-forward rule is shown to be monotone in p_1.

2. *f*-SWAP RULES

Consider the following family of *f*-swap rules: If the successful attempt of a path is in position j, $j = 1,2,\ldots,n$, then this path exchanges its position with the path in position $f(j)$, where f is a given mapping from the set $\{1,2,\ldots,n\}$ into itself. The mapping f can be represented by a directed graph with every vertex having out-degree unity. Throughout this section we will assume that the graph has exactly one directed cycle. Under this assumption, we will generalize the result about the stationary distributions in Lam and Cowan [7].

Let R be all the vertices in the unique directed cycle of the graph. Define

$$A = \{j | f(j) > j, j \in \{1,2,\ldots,n\} \backslash R\}$$

and

$$B = \{j | f(j) < j, j \in \{1,2,\ldots,n\} \backslash R\}.$$

For a position $j \in A \cup B$, let $g(j)$ be the generation number of j (i.e., the smallest number of mappings applied such that $f \circ f \circ \cdots \circ f(j) \in R$). For $j \in R$, let $g(j) = 0$. Furthermore, let

$$w(j) = |\{i | f^{(i)}(j) > f^{(i+1)}(j), i = 0,1,\ldots,g(j) - 1\}|$$

for $j \in A \cup B$ and $w(j) = 0$ for $j \in R$.

For convenience we will let $q_j = 1 - p_j$, $a_j = p_j/q_j$ and

$$\pi^k(j) := \begin{cases} \pi(f(k)) & \text{if } j = k \\ \pi(k) & \text{if } j = f(k) \\ \pi(j) & \text{otherwise,} \end{cases}$$

which represents the routing table immediately preceding the routing table π when a call is completed through the path in position k. We will also make use of the conventions that for any function F,

$$\sum_{j \in \emptyset} F(j) = 0, \qquad \prod_{j \in \emptyset} F(j) = 1, \qquad \prod_{j=1}^{0} F(j) = 1.$$

The following lemma is needed for finding the stationary distributions.

LEMMA 1:

$$\sum_{j=k}^{n} a_{\pi(j)} \prod_{m=1}^{j} q_{\pi(m)} + \prod_{j=1}^{n} q_j = \prod_{j=1}^{k-1} q_{\pi(j)}, \qquad k = 1,2,\ldots,n. \tag{1}$$

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PROOF: When $k = n$, the left-hand side of Eq. (1) is

$$a_{\pi(n)} \prod_{m=1}^{n} q_{\pi(m)} + \prod_{j=1}^{n} q_j = (a_{\pi(n)} + 1) \prod_{j=1}^{n} q_j = \frac{1}{q_{\pi(n)}} \prod_{j=1}^{n} q_j,$$

which equals the right-hand side of Eq. (1). Now suppose that Eq. (1) is true when $k = K + 1$. Then when $k = K$, by assumption of induction

$$\begin{aligned}\sum_{j=K}^{n} a_{\pi(j)} \prod_{m=1}^{j} q_{\pi(m)} + \prod_{j=1}^{n} q_j &= a_{\pi(K)} \prod_{m=1}^{K} q_{\pi(m)} + \sum_{j=K+1}^{n} a_{\pi(j)} \prod_{m=1}^{j} q_{\pi(m)} + \prod_{j=1}^{n} q_j \\ &= a_{\pi(K)} \prod_{m=1}^{K} q_{\pi(m)} + \prod_{m=1}^{K} q_{\pi(m)} \\ &= \prod_{m=1}^{K-1} q_{\pi(m)}.\end{aligned}$$

Hence, Eq. (1) is true when $k = 1,2,\ldots,n$. ■

THEOREM 1: *Under the swap-forward rule with*

(a) $R = \{z\}$, *where* $z = 1,2,\ldots,n$ *or*
(b) $R = \{z, z + 1\}$, *where* $z = 1,2,\ldots,n - 1$,

the stationary probability of state π *is proportional to*

$$\frac{1}{\prod_{j\in A} p_{\pi(j)}^{g(j)-w(j)} \prod_{j\in B} a_{\pi(j)}^{w(j)}} = \prod_{j=1}^{n} \frac{q_{\pi(j)}^{w(j)}}{p_{\pi(j)}^{g(j)}}.$$

Especially when $z = 1$ *and* $f(k) = k - 1$ *for* $k \geq 2$, *the swap-forward rule becomes the transposition rule and the stationary probability of state* π *is proportional to* $1/\prod_{j=2}^{n} a_{\pi(j)}^{j-1} \propto \prod_{j=1}^{n} a_{\pi(j)}^{n-j}$ (*see Chong and Lam* [2]).

PROOF: First note that $g(f(k)) = g(k) - 1$ for $k \in A \cup B$ and

$$j \in A \Leftrightarrow f(j) > j \Rightarrow w(f(j)) = w(j),$$

$$j \in B \Leftrightarrow f(j) < j \Rightarrow w(f(j)) = w(j) - 1.$$

Let

$$c(\pi) = \frac{1}{\prod_{j\in A} p_{\pi(j)}^{g(j)-w(j)} \prod_{j\in B} a_{\pi(j)}^{w(j)}}.$$

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(a) Suppose $R = \{z\}$. Let $t(\sigma, \pi)$ denote the transition probability from state σ to state π. Then

$$t(\sigma,\pi) = \begin{cases} p_{\sigma(z)} \prod_{j=1}^{z-1} q_{\sigma(j)} + \prod_{j=1}^{n} q_j & \text{if } \pi = \sigma \\ p_{\sigma(k)} \prod_{j=1}^{k-1} q_{\sigma(j)} & \text{if } \pi = \sigma^k, k \neq z \\ 0 & \text{otherwise,} \end{cases}$$

or, equivalently,

$$t(\sigma,\pi) = \begin{cases} p_{\pi(z)} \prod_{j=1}^{z-1} q_{\pi(j)} + \prod_{j=1}^{n} q_j & \text{if } \sigma = \pi \\ p_{\pi(f(k))} \prod_{j=1}^{k-1} q_{\pi(j)} & \text{if } \sigma = \pi^k, k \in A \\ a_{\pi(f(k))} \prod_{j=1}^{k} q_{\pi(j)} & \text{if } \sigma = \pi^k, k \in B \\ 0 & \text{otherwise.} \end{cases}$$

Since $\pi^z = \pi$, we have

$$\begin{aligned} &\sum_{j=1}^{n} t(\pi^j,\pi) \frac{c(\pi^j)}{c(\pi)} \\ &\quad = \sum_{j\in A} \left[p_{\pi(f(j))} \prod_{m=1}^{j-1} q_{\pi(m)} \right] \frac{p_{\pi(j)}}{p_{\pi(f(j))}} \\ &\qquad + \sum_{j\in B} \left[a_{\pi(f(j))} \prod_{m=1}^{j} q_{\pi(m)} \right] \frac{a_{\pi(j)}}{a_{\pi(f(j))}} + p_{\pi(z)} \prod_{m=1}^{z-1} q_{\pi(m)} + \prod_{m=1}^{n} q_m \\ &\quad = \sum_{j\in A} a_{\pi(j)} \prod_{m=1}^{j} q_{\pi(m)} + \sum_{j\in B} a_{\pi(j)} \prod_{m=1}^{j} q_{\pi(m)} + a_{\pi(z)} \prod_{m=1}^{z} q_{\pi(m)} \\ &\qquad + \prod_{m=1}^{n} q_m \\ &\quad = \sum_{j=1}^{n} a_{\pi(j)} \prod_{m=1}^{j} q_{\pi(m)} + \prod_{m=1}^{n} q_m \\ &\quad = 1, \end{aligned}$$

where the last equality is due to Lemma 1. Hence, $\sum_{j=1}^{n} c(\pi^j) t(\pi^j, \pi) = c(\pi)$.

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(b) Suppose $R = \{z, z+1\}$ (i.e., $f(z) = z+1$ and $f(z+1) = z$). Let $u(\sigma, \pi)$ denote the transition probability from state σ to state π. Then

$$u(\sigma,\pi) = \begin{cases} \prod_{j=1}^{n} q_j & \text{if } \pi = \sigma \\ p_{\sigma(z)} \prod_{j=1}^{z-1} q_{\sigma(j)} + p_{\sigma(z+1)} \prod_{j=1}^{z} q_{\sigma(j)} & \text{if } \pi = \sigma^z \\ p_{\sigma(k)} \prod_{j=1}^{k-1} q_{\sigma(j)} & \text{if } \pi = \sigma^k,\ k \neq z, \text{ and } k \neq z+1 \\ 0 & \text{otherwise} \end{cases}$$

or, equivalently,

$$u(\sigma,\pi) = \begin{cases} \prod_{j=1}^{n} q_j & \text{if } \sigma = \pi \\ p_{\pi(z+1)} \prod_{j=1}^{z-1} q_{\pi(j)} + a_{\pi(z)} \prod_{j=1}^{z+1} q_{\pi(j)} & \text{if } \sigma = \pi^z \\ p_{\pi(f(k))} \prod_{j=1}^{k-1} q_{\pi(j)} & \text{if } \sigma = \pi^k,\ k \in A \\ a_{\pi(f(k))} \prod_{j=1}^{k} q_{\pi(j)} & \text{if } \sigma = \pi^k,\ k \in B \\ 0 & \text{otherwise.} \end{cases}$$

Since now $c(\pi^z) = c(\pi)$, we have

$$\begin{aligned} &\sum_{j \in A \cup B} u(\pi^j, \pi) \frac{c(\pi^j)}{c(\pi)} + u(\pi^z, \pi) \frac{c(\pi^z)}{c(\pi)} + u(\pi, \pi) \\ &\quad = \sum_{j \in A} \left[p_{\pi(f(j))} \prod_{m=1}^{j-1} q_{\pi(m)} \right] \frac{p_{\pi(j)}}{p_{\pi(f(j))}} + \sum_{j \in B} \left[a_{\pi(f(j))} \prod_{m=1}^{j} q_{\pi(m)} \right] \frac{a_{\pi(j)}}{a_{\pi(f(j))}} \\ &\qquad + \left[p_{\pi(z+1)} \prod_{m=1}^{z-1} q_{\pi(m)} + a_{\pi(z)} \prod_{m=1}^{z+1} q_{\pi(m)} \right] + \prod_{m=1}^{n} q_m \\ &\quad = \sum_{j \in A} a_{\pi(j)} \prod_{m=1}^{j} q_{\pi(m)} + \sum_{j \in B} a_{\pi(j)} \prod_{m=1}^{j} q_{\pi(m)} \\ &\qquad + \left[a_{\pi(z)} \prod_{m=1}^{z} q_{\pi(m)} + a_{\pi(z+1)} \prod_{m=1}^{z+1} q_{\pi(m)} \right] + \prod_{m=1}^{n} q_m \\ &\quad = \sum_{j=1}^{n} a_{\pi(j)} \prod_{m=1}^{j} q_{\pi(m)} + \prod_{m=1}^{n} q_m \\ &\quad = 1, \end{aligned}$$

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where the last equality is due to Lemma 1 and the second equality is from

$$p_{\pi(z+1)} + p_{\pi(z)} q_{\pi(z+1)} = p_{\pi(z)} + p_{\pi(z+1)} - p_{\pi(z)} p_{\pi(z+1)}$$
$$= p_{\pi(z)} + p_{\pi(z+1)} q_{\pi(z)}.$$

Hence, $\sum_{j \in A \cup B} u(\pi^j, \pi) c(\pi^j) + u(\pi^z, \pi) c(\pi^z) + u(\pi, \pi) c(\pi) = c(\pi)$. ■

Example 1: Suppose $n = 3$, $f(1) = 3$, $f(2) = 1$, and $f(3) = 1$. Then the transition probability from σ to (1,2,3) is

$$\begin{cases} q_1 q_2 q_3 & \text{if } \sigma = (1,2,3) \\ p_1 q_2 & \text{if } \sigma = (2,1,3) \\ p_3 + p_1 q_2 q_3 & \text{if } \sigma = (3,2,1) \\ 0 & \text{otherwise.} \end{cases}$$

Note that $g(1) = g(3) = w(1) = w(3) = 0$ and $g(2) = w(2) = 1$. Hence, $c(\pi) = 1/a_{\pi(2)}$, where $c(\pi)$ is defined in the proof of Theorem 1, and

$$q_1 q_2 q_3 \frac{c((1,2,3))}{c((1,2,3))} + p_1 q_2 \frac{c((2,1,3))}{c((1,2,3))} + (p_3 + p_1 q_2 q_3) \frac{c((3,2,1))}{c((1,2,3))}$$

$$= q_1 q_2 q_3 + p_1 q_2 \frac{a_2}{a_1} + p_3 + p_1 q_2 q_3$$

$$= q_2 q_3 + q_1 p_2 + p_3 \not\equiv 1.$$

Therefore, Theorem 1 cannot be generalized to the case when $R = \{y, z\}$ with $|y - z| > 1$.

Example 2: Suppose $n = 3$, $f(1) = 2$, $f(2) = 3$, and $f(3) = 1$. Then the transition probability from σ to (1,2,3) is

$$\begin{cases} q_1 q_2 q_3 & \text{if } \sigma = (1,2,3) \\ q_1 p_3 & \text{if } \sigma = (1,3,2) \\ p_2 & \text{if } \sigma = (2,1,3) \\ p_1 q_2 q_3 & \text{if } \sigma = (3,2,1) \\ 0 & \text{otherwise.} \end{cases}$$

Now $g(i) = w(i) = 0$ for $i = 1,2,3$ and $c(\pi) = 1$ for all permutations π. Obviously,

$$q_1 q_2 q_3 + q_1 p_3 + p_2 + p_1 q_2 q_3 = q_2 q_3 + q_1 p_3 + p_2 \not\equiv 1.$$

Therefore, Theorem 1 cannot be generalized to the case when $|R| > 2$.

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3. COMPARING COSTS UNDER TWO MOVE-FORWARD RULES WHEN $p_2 = p_3 = \cdots = p_n$

In this section we compare the move-forward rules applied to a communication problem with the assumption that $p_2 = p_3 = \cdots = p_n$. For a move-forward rule characterized by $l_1, l_2, \ldots, l_n$, let

$$k(j) = \max\{i : l_{i+j} \le j,\ i = 1,2,\ldots,n-j\}, \qquad j = 1,2,\ldots,n-1, \tag{2}$$

which uniquely characterizes a move-forward rule (Lam [6]).

Given that $p_1 = p$, $p_2 = p_3 = \cdots = p_n = r$, $0 < p < 1$, $0 < r < 1$. Let $q = 1 - p$ and $s = 1 - r$. Similar to Lam [6], the arrangement of the paths can be denoted by state i, $i = 1,2,\ldots,n$, where i is the position of the unique path with success probability p. Let p_{ij} be the one-step transition probability from state i to state j. Then

$$p_{1j} = \begin{cases} p + q(1 - r - sr - \cdots - s^{k(2)-1}r) & \text{if } j = 1 \\ q(r + sr + \cdots + s^{k(2)-1}r) & \text{if } j = 2 \\ 0 & \text{otherwise;} \end{cases}$$

$$p_{ij} = \begin{cases} s^{i-1}p & \text{if } j = l_i \\ r + sr + \cdots + s^{i-2}r + s^{i-1}q(1 - r - sr - \cdots - s^{k(i)-1}r) & \text{if } j = i \\ s^{i-1}q(r + sr + \cdots + s^{k(i)-1}r) & \text{if } j = i+1 \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 2,3,\ldots,n$.

Let the equilibrium probability vector be $(h(1), h(2), \ldots, h(n))$, $H(0)$ be 0, and $H(j)$ be $\sum_{i=1}^{j} h(i)$ for $j = 1,2,\ldots,n$. Then the $H(j)$'s satisfy

$$\begin{aligned} H(j) = H(j-1) + h(j)[1 - s^{j-1}q(r + sr + \cdots + s^{k(j)-1}r)] \\ + h(j+1)s^j p + h(j+2)s^{j+1}p + \cdots + h(j+k(j))s^{j+k(j)-1}p, \\ j = 1,2,\ldots,n-1 \end{aligned} \tag{3}$$

or, equivalently,

$$\begin{aligned} h(j)qr[s^{j-1} + s^j + \cdots + s^{j+k(j)-2}] \\ = h(j+1)ps^j + h(j+2)ps^{j+1} + \cdots + h(j+k(j))ps^{j+k(j)-1}, \\ j = 1,2,\ldots,n-1. \end{aligned} \tag{4}$$

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LEMMA 2:

(a) If $p \geq r$, then $h(1) \geq h(2) \geq \cdots \geq h(n)$.
(b) If $p \leq r$, then $h(1) \leq h(2) \leq \cdots \leq h(n)$.
(c) If $p \geq r$, then $h(j + k(j))/h(j) \leq qr/ps$ for $j = 1,2,\cdots,n-1$.
(d) If $p \leq r$, then $h(j + k(j))/h(j) \geq qr/ps$ for $j = 1,2,\cdots,n-1$.

PROOF: Suppose $p \geq r$.

From the $(n-1)$st equation of Eq. (4), we have $h(n-1)qrs^{n-2} = h(n)ps^{n-1}$. Hence, $h(n) \leq h(n-1)$.

Now suppose

$$h(j+1) \geq h(j+2) \geq \cdots \geq h(n) \quad \text{for some } j \leq n-2$$

and we need to prove $h(j) \geq h(j+1)$. If $k(j) = 1$, the jth equation of Eq. (4) is $h(j)qrs^{j-1} = h(j+1)ps^{j}$ and $h(j) \geq h(j+1)$ follows. If $k(j) \geq 2$, from the $(j+1)$st equation of Eq. (4) we get

$$\begin{aligned} &h(j+1)qr[1 + s + \cdots + s^{k(j+1)-1}] \\ &\quad = h(j+2)ps + h(j+3)ps^2 + \cdots + h(j + k(j+1) + 1)ps^{k(j+1)}. \end{aligned} \tag{5}$$

By the induction assumption, we have

$$\begin{aligned} &h(j+1)[s + s^2 + \cdots + s^{k(j+1)}] \\ &\quad \geq h(j+2)s + h(j+3)s^2 \\ &\qquad + \cdots + h(j + k(j+1) + 1)s^{k(j+1)} \end{aligned}$$

and

$$\begin{aligned} &h(j+2)[s^{k(j)+1} + s^{k(j)+2} + \cdots + s^{k(j+1)+1}] \\ &\quad \geq h(j + k(j) + 1)s^{k(j)+1} + h(j + k(j) + 2)s^{k(j)+2} \\ &\qquad + \cdots + h(j + k(j+1) + 1)s^{k(j+1)+1}, \\ &h(j+3)[s^{k(j)+2} + s^{k(j)+3} + \cdots + s^{k(j+1)+2}] \\ &\quad \geq h(j + k(j) + 1)s^{k(j)+2} + h(j + k(j) + 2)s^{k(j)+3} \\ &\qquad + \cdots + h(j + k(j+1) + 1)s^{k(j+1)+2}, \\ &\qquad \cdots\cdots, \\ &h(j + k(j))[s^{2k(j)-1} + s^{2k(j)} + \cdots + s^{k(j)+k(j+1)-1}] \\ &\quad \geq h(j + k(j) + 1)s^{2k(j)-1} + h(j + k(j) + 2)s^{2k(j)} \\ &\qquad + \cdots + h(j + k(j+1) + 1)s^{k(j)+k(j+1)-1}. \end{aligned}$$

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Adding

$$h(j+2)[s^2+s^3+\cdots+s^{k(j)}]+h(j+3)[s^3+s^4+\cdots+s^{k(j)+1}]$$
$$+\cdots+h(j+k(j))[s^{k(j)}+s^{k(j)+1}+\cdots+s^{2k(j)-2}]$$

to both sides of the sum of the above $k(j)$ inequalities, we have

$$[h(j+1)s+h(j+2)s^2+\cdots+h(j+k(j))s^{k(j)}][1+s+\cdots+s^{k(j+1)-1}]$$
$$\geq [h(j+2)s+h(j+3)s^2+\cdots+h(j+k(j))s^{k(j)-1}$$
$$+h(j+k(j)+1)s^{k(j)}+h(j+k(j)+2)s^{k(j)+1}$$
$$+\cdots+h(j+k(j+1)+1)s^{k(j+1)}]$$
$$\times[1+s+\cdots+s^{k(j)-1}].$$

Combining with Eq. (5), we obtain

$$h(j+1)s+h(j+2)s^2+\cdots+h(j+k(j))s^{k(j)}$$
$$\geq [1+s+\cdots+s^{k(j)-1}]h(j+1)\frac{qr}{p}. \tag{6}$$

Additionally, the jth equation of Eq. (4) is

$$h(j)[1+s+\cdots+s^{k(j)-1}]\frac{qr}{p}$$
$$=h(j+1)s+h(j+2)s^2+\cdots+h(j+k(j))s^{k(j)}. \tag{7}$$

Combining Eqs. (6) and (7), we get $h(j)\geq h(j+1)$, which completes the proof of part (a).

The proof of part (b) is similar to the proof of part (a).

For the proof of part (c), we make use of Eq. (7) and part (a) to obtain

$$h(j)[1+s+\cdots+s^{k(j)-1}]\frac{qr}{p}\geq h(j+1)[s+s^2+\cdots+s^{k(j)}].$$

Thus, $h(j)qr\geq h(j+1)ps$.

The proof of part (d) is similar to the proof of part (c). ■

For state i, the expected cost, which is the expected number of paths to attempt before a call is completed or blocked, is

$$1+s+s^2+\cdots+s^{i-1}+s^{i-1}q+s^iq+\cdots+s^{n-2}q=\frac{1-s^i+q(s^{i-1}-s^{n-1})}{1-s}.$$

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Thus, the expected cost corresponding to the equilibrium probability vector $(h(1), h(2), \ldots, h(n))$ is

$$C := \sum_{i=1}^{n} h(i) \frac{1 - qs^{n-1} + s^{i-1}(q - s)}{1 - s} = \frac{1 - qs^{n-1}}{r} + \frac{r - p}{r} \sum_{i=1}^{n} h(i) s^{i-1}.$$

For the probability vectors $b = (b(1), b(2), \ldots, b(n))$ and $b' = (b'(1), b'(2), \ldots, b'(n))$, define $b \prec b'$ if and only if

$$\sum_{j=1}^{i} b(j) \geq \sum_{j=1}^{i} b'(j) \quad \text{for all } i = 1, 2, \ldots, n-1.$$

LEMMA 3:

(a) If $p \geq r$, then $b \prec b'$ implies that the expected cost of b is not more than that for b'.

(b) If $p \leq r$, then $b \prec b'$ implies that the expected cost of b is not less than that for b'.

PROOF: By Abel's partial summation formula,

$$\sum_{i=1}^{n} b(i) s^{i-1} = \left[\sum_{j=1}^{n} b(j) \right] s^{n-1} + \sum_{i=1}^{n-1} \left(\sum_{j=1}^{i} b(j) \right) (s^{i-1} - s^{i}).$$

Therefore, $b \prec b' \Rightarrow \sum_{i=1}^{n} b(i) s^{i-1} \geq \sum_{i=1}^{n} b'(i) s^{i-1}$. The conclusions can then be derived from the above expression for C. ■

For a transition probability matrix $(p_{mj}, m, j = 1, 2, \ldots, n)$, let

$$P_{m,i} = \sum_{j=1}^{i} p_{mj}, \qquad m, i = 1, 2, \ldots, n.$$

LEMMA 4: *For every move-forward rule, $P_{m,i} \geq P_{m+1,i}$, $m = 1, 2, \ldots, n-1$, $i = 1, 2, \ldots, n$.*

PROOF: In fact,

$$P_{m,i} = \begin{cases} 1 & \text{if } m < i \\ 1 - s^{i-1} q (r + sr + \cdots + s^{k(i)-1} r) & \text{if } m = i \\ s^{m-1} p & \text{if } i < m \leq i + k(i) \\ 0 & \text{if } i + k(i) < m, \end{cases}$$

and

$$\begin{aligned} P_{i,i} &= 1 - s^{i-1} q (1 - s^{k(i)}) \\ &\geq 1 - s^{i-1} q \\ &\geq s^{i-1} p \geq s^{i} p = P_{i+1,i}. \end{aligned}$$

■

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COROLLARY: *For every move-forward rule, $b \prec b'$ implies $bP \prec b'P$.*

PROOF:

$$\begin{aligned}
\sum_{j=1}^{i}\left[\sum_{m=1}^{n} b(m)p_{mj}\right] &= \sum_{m=1}^{n} b(m)P_{m,i} \\
&= \left[\sum_{m=1}^{n} b(m)\right]P_{n,i} + \sum_{m=1}^{n-1}\left(\sum_{j=1}^{m} b(j)\right)(P_{m,i} - P_{m+1,i}) \\
&\geq \left[\sum_{m=1}^{n} b'(m)\right]P_{n,i} + \sum_{m=1}^{n-1}\left(\sum_{j=1}^{m} b'(j)\right)(P_{m,i} - P_{m+1,i}) \\
&= \sum_{j=1}^{i}\left[\sum_{m=1}^{n} b'(m)p_{mj}\right],
\end{aligned}$$

where the second equality comes from the Abel's partial summation formula and the inequality is due to $b \prec b'$ and Lemma 4. ■

THEOREM 2: *Suppose that we have two move-forward rules characterized by $\{l_1, l_2, \ldots, l_n\}$ and $\{l'_1, l'_2, \ldots, l'_n\}$, respectively, and $l_i \leq l'_i$ for all $i = 1,2,\ldots,n$. Denote their equilibrium probability vectors by h and h', respectively, and the corresponding expected costs by C and C'. Then the following hold:*

(a) $p \geq r$ implies $h' \prec h$.
(b) $p \leq r$ implies $h \prec h'$.
(c) $C' \leq C$.

PROOF: Note that $l_i \leq l'_i$ for all $i = 1,2,\ldots,n$ is equivalent to $k'(j) \leq k(j)$ for all $j = 1,2,\ldots,n-1$ (Lam [6]), where the $k(j)$'s and $k'(j)$'s are defined by Eq. (2).

For the proof of part (a), suppose $p \geq r$. First we will prove that $h' \prec h'P$, where P is the transition probability matrix according to the move-forward rule with $k(1), k(2), \ldots, k(n-1)$. The sum of the first i entries of $h'P$ is given by

$$\begin{aligned}
D(i) &:= \sum_{j=1}^{i}\sum_{m=1}^{n} h'(m)p_{mj} \\
&= \sum_{m=1}^{n} h'(m)P_{m,i} \\
&= H'(i-1) + h'(i)[1 - s^{i-1}qr - s^{i}qr - \cdots - s^{i+k(i)-2}qr] \\
&\quad + h'(i+1)s^{i}p + h'(i+2)s^{i+1}p + \cdots + h'(i+k(i))s^{i+k(i)-1}p,
\end{aligned}$$

where $H'(i)$ is the sum of the first i entries of h'. Note that $H'(i)$ satisfies the ith equation of Eq. (3): that is,

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$$H'(i) = H'(i-1) + h'(i)[1 - s^{i-1}qr - s^{i}qr - \cdots - s^{i+k'(i)-2}qr]$$
$$+ h'(i+1)s^{i}p + h'(i+2)s^{i+1}p + \cdots + h'(i+k'(i))s^{i+k'(i)-1}p.$$

Thus, $D(i) \leq H'(i)$, is equivalent to

$$h'(i)[s^{i+k'(i)-1} + s^{i+k'(i)} + \cdots + s^{i+k(i)-2}]qr$$
$$\geq [h'(i+k'(i)+1)s^{i+k'(i)-1} + h'(i+k'(i)+2)s^{i+k'(i)}$$
$$+ \cdots + h'(i+k(i))s^{i+k(i)-2}]ps.$$

Since $p \geq r$, this is true by parts (c) and (a) of Lemma 2. Thus, $h' \prec h'P$ is proved. Then by the corollary to Lemma 4, $h' \prec h'P \prec h'P^2 \prec h'P^3 \prec \cdots$ and so $h' \prec h$.

The proof of part (b) is similar to that of part (a).

For the proof of part (c), $C' \leq C$ comes immediately from parts (a) and (b) and Lemma 3. ■

4. COMPARING COSTS UNDER A MOVE-FORWARD RULE FOR DIFFERENT SETS OF SUCCESS PROBABILITIES

In this section, we consider only one move-forward rule and assume that there are two sets of success probabilities, $p_1 = p$, $p_2 = p_3 = \cdots = p_n = r$ and $p_1 = p'$, $p_2 = p_3 = \cdots = p_n = r'$, where $0 < p < 1$, $0 < r < 1$, $0 < p' < 1$, and $0 < r' < 1$. Let P and P' be the corresponding transition matrices and let h and h' be the corresponding equilibrium probability vectors. Let also $q = 1 - p$, $s = 1 - r$, $q' = 1 - p'$, and $s' = 1 - r'$.

LEMMA 5: *If $p \geq p'$ and $r = r'$, then $h \prec h'$.*

PROOF: Let $P_{m,i} = \sum_{j=1}^{i} p_{mj}$ and $P'_{m,i} = \sum_{j=1}^{i} p'_{mj}$ for $m, i = 1,2,\ldots,n$, where the p_{mj}'s and p'_{mj}'s are the one-step transition probabilities for the two sets of success probabilities, respectively. For any probability vectors b and b' such that $b \prec b'$, we have $bP' \prec b'P'$ by the corollary to Lemma 4. On the other hand, for every $i = 1,2,\ldots,n$,

$$\sum_{j=1}^{i}\left[\sum_{m=1}^{n} b(m)p_{mj}\right] = \sum_{m=1}^{n} b(m)P_{m,i} \geq \sum_{m=1}^{n} b(m)P'_{m,i} = \sum_{j=1}^{i}\left[\sum_{m=1}^{n} b(m)p'_{mj}\right]$$

because $P_{m,i} \geq P'_{m,i}$ for $m, i = 1,2,\ldots,n$ (refer to the expression for $P_{m,i}$ in the proof of Lemma 4). Therefore, $bP \prec bP'$. Hence, $h = hP \prec hP' \prec h(P')^2 \prec h(P')^3 \prec \cdots$ and so $h \prec h'$. ■

Denote the expected costs corresponding to h and h' by C and C', respectively.

THEOREM 3: *If $p \geq p' \geq r = r'$ or $p \geq r = r' \geq p'$, then $C \leq C'$.*

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PROOF: Suppose that $p \geq p' \geq r = r'$. Let

$$c_i = 1 + s + \cdots + s^{i-1} + s^{i-1}q + s^i q + \cdots + s^{n-2}q,$$

$$c_i' = 1 + s + \cdots + s^{i-1} + s^{i-1}q' + s^i q' + \cdots + s^{n-2}q'.$$

Then $C = \sum_{i=1}^{n} h(i)c_i$ and $C' = \sum_{i=1}^{n} h'(i)c_i'$. Now by Abel's partial summation formula and Lemma 5,

$$\begin{aligned} C &= c_n \sum_{i=1}^{n} h(i) + \sum_{i=1}^{n-1} \left(\sum_{j=1}^{i} h(j) \right)(c_i - c_{i+1}) \\ &= 1 + s + \cdots + s^{n-1} - (p - r) \sum_{i=1}^{n-1} s^{i-1} \left(\sum_{j=1}^{i} h(j) \right) \\ &\leq 1 + s + \cdots + s^{n-1} - (p' - r) \sum_{i=1}^{n-1} s^{i-1} \left(\sum_{j=1}^{i} h'(j) \right) = C'. \end{aligned}$$

For the case $p \geq r = r' \geq p'$,

$$\begin{aligned} C &= 1 + s + \cdots + s^{n-1} - (p - r) \sum_{i=1}^{n-1} s^{i-1} \left(\sum_{j=1}^{i} h(j) \right) \\ &\leq 1 + s + \cdots + s^{n-1} \\ &\leq 1 + s + \cdots + s^{n-1} + (r - p') \sum_{i=1}^{n-1} s^{i-1} \left(\sum_{j=1}^{i} h'(j) \right) = C'. \end{aligned}$$

■

When $r = r' \geq p \geq p'$, C might be greater or smaller than C'. As an example, consider cases when $n = 4$ and $r = 0.9$. For the move-to-front rule, the expected equilibrium costs are 1.11957414, 1.11957444, and 1.11957383 for $p = 0.814$, 0.815, and 0.816, respectively. For the POS(2) rule, the expected equilibrium costs are 1.11899514, 1.11899612, and 1.11899606 for $p = 0.821$, 0.822, and 0.823, respectively. For the transposition rule, the expected equilibrium costs are 1.11853856, 1.11853924, and 1.11853865 for $p = 0.828$, 0.829, and 0.83, respectively.

Acknowledgment

The authors would like to acknowledge the support provided by the Hong Kong University Grants Council (grant No. RGC 2011/98H).

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