Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **9**, No. 4, pp. 795-817

DOI: 10.4208/aamm.2015.m1049
August 2017

A Time Second-Order Mass-Conserved Implicit-Explicit Domain Decomposition Scheme for Solving the Diffusion Equations

Zhongguo Zhou[1,2] and Dong Liang[3,*]

[1] *School of Mathematics, Shandong University, Jinan, Shandong 250100, China*
[2] *School of Information Science and Engineering, Shandong Agricultural University, Taian, Shandong 271018, China*
[3] *Department of Mathematics and Statistics, York University, Toronto, Ontario, M3J 1P3, Canada*

Received 13 April 2015; Accepted (in revised version) 6 June 2016

Abstract. In the paper, a new time second-order mass-conserved implicit/explicit domain decomposition method (DDM) for the diffusion equations is proposed. In the scheme, firstly, we calculate the interface fluxes of sub-domains from the obtained solutions and fluxes at the previous time level, for which we apply high-order Taylor's expansion and transfer the time derivatives to spatial derivatives to improve the accuracy. Secondly, the interior solutions and fluxes in sub-domains are computed by the implicit scheme and using the relations between solutions and fluxes, without any correction step. The mass conservation is proved and the convergence order of the numerical solutions is proved to be second-order in both time and space steps. The super-convergence of numerical fluxes is also proved to be second-order in both time and space steps. The scheme is stable under the stable condition $r \leq 3/5$. The important feature is that the proposed domain decomposition scheme is mass-conserved and is of second order convergence in time. Numerical experiments confirm the theoretical results.

AMS subject classifications: 65M06, 65M12, 65M55, 76S05

Key words: Diffusion equations, time second-order, mass-conserved, implicit-explicit domain decomposition, interface fluxes.

1 Introduction

Time dependent diffusion equations are widely used in science and engineering (see, for example, [1, 2], etc.). Due to the computational complexities and huge computational costs in applications, parallel computing based on domain decomposition methods

*Corresponding author.
Email: zhouzhongguo@mail.sdu.edu.cn (Z. G. Zhou), dliang@mathstat.yorku.ca (D. Liang)

 ©2017 Global Science Press

https://doi.org/10.4208/aamm.2015.m1049 Published online by Cambridge University Press

is playing an important role in scientific computation. The non-overlapping implicit-explicit domain decomposition methods, explicitly calculating the interface values of sub-domains and implicitly solving interior values, were developed (see [3–8, 10–12]). However, these domain decomposition methods did not satisfy mass conservation. Developing mass-conserved domain decomposition schemes is being greatly required in applications, especially in the engineering. [13] gave a scheme of the block-centered finite difference domain decomposition procedure and proved the unconditional stability, mass conservation and second-order accuracy in the space step. But the convergence order of the method is only first order in time step.

In the paper, for improving the accuracy of the mass-conserved schemes, we propose a new time second-order mass-conserved implicit-explicit domain decomposition scheme for the diffusion equations with Neumann boundary conditions, based on the block-centered finite differences. In the scheme, we take the two-step technique to solve the solutions without any correction step. Firstly, the interface fluxes of sub-domains are explicitly calculated from the obtained solutions and fluxes at the previous time level, where we approximate the interface fluxes by applying high-order Taylor's expansion and transferring the time derivatives to spatial derivatives through the original equations to improve its accuracy and its stability condition. Secondly, the interior solutions and fluxes are computed by the implicit scheme in sub-domains using the relation between solutions and fluxes. The new and important feature is that the proposed domain decomposition scheme is mass-conserved and is of second order accuracy in time step. We prove the proposed scheme satisfies mass conservation. We analyze theoretically its stability and convergence and obtain that the scheme is stable under the condition

$$r = a\frac{\tau}{h^2} \leq \frac{3}{5}$$

and the error estimate of the numerical solutions is second order in both time and space steps. We also prove the super-convergence of numerical fluxes to have second order convergence in both time and space steps. Numerical experiments are given to confirm our theoretical results.

The paper is divided into five sections. We give numerical scheme and the algorithm of the time second-order mass-conserved implicit-explicit domain decomposition scheme in Section 2. We analyze the mass conservation, stability and convergence in Section 3 and Section 4. The numerical experiments are given in Section 5. Finally, conclusions are addressed in Section 6.

2 Diffusion equations and the time second-order mass-conserved domain decomposition scheme

We consider the one dimensional diffusion equations,

https://doi.org/10.4208/aamm.2015.m1049 Published online by Cambridge University Press

$$\begin{cases} \dfrac{\partial c}{\partial t}-a\dfrac{\partial^2 c}{\partial x^2}=f(x,t), & x\in[0,1], \quad t\in[0,T], \\ \dfrac{\partial c}{\partial x}\Big|_{x=0}=0, \quad \dfrac{\partial c}{\partial x}\Big|_{x=1}=0, & t\in[0,T], \\ c(x,0)=c_0(x), \end{cases} \tag{2.1}$$

where $c(x,t)$ is the solution, $a>0$ is constant coefficient and $f(x,t)$ is the source function. Define the flux variable

$$q(x,t)=-ac_x(x,t), \tag{2.2}$$

and (2.1) can be simplified to

$$\begin{cases} \dfrac{\partial c}{\partial t}+\dfrac{\partial q}{\partial x}=f(x,t), & x\in[0,1], \quad t\in[0,T], \\ q(0,t)=q(1,t)=0, & 0\le t\le T, \\ c(x,0)=c_0(x). \end{cases} \tag{2.3}$$

From (2.2), we can obtain that if $f(x,t)=0$,

$$\frac{d}{dt}\int_0^1 c\mathrm{d}x=0, \tag{2.4}$$

which expresses that the solution of the diffusion equation satisfies mass conservation.

Let $h>0$ be the space step length and $x_{i+\frac{1}{2}}$ $(i=0,1,\cdots,I)$ be the uniform partition points of $[0,1]$

$$x_{\frac{1}{2}}=0, \quad x_{I+\frac{1}{2}}=1, \quad x_{i+\frac{1}{2}}=ih, \qquad i=1,2,\cdots,I-1, \tag{2.5a}$$

$$x_i=\Big(i-\frac{1}{2}\Big)h, \qquad i=1,2,\cdots,I. \tag{2.5b}$$

Let $\tau>0$ be the time step length and t^n $(n=0,1,\cdots,M)$ be the uniform partition points of $[0,T]$.

$$t^0=0, \quad t^M=T, \quad t^n=n\tau, \quad n=1,2,\cdots,M-1. \tag{2.6}$$

For functions $f(x,t)$, $g(x,t)$ and $l(x,t)$, let

$$f_i^n=f(x_i,t^n), \quad f_{i+\frac{1}{2}}^n=f(x_{i+\frac{1}{2}},t^n), \quad l_i^{n-\frac{1}{2}}=l(x_i,t^{n-\frac{1}{2}}), \tag{2.7}$$

and define the discrete inner products,

$$(f^n,g^n)=\sum_{i=1}^{I}f_i^n g_i^n h, \quad \langle f^n,g^n\rangle=\sum_{i=1}^{I-1}f_{i+\frac{1}{2}}^n g_{i+\frac{1}{2}}^n h, \tag{2.8}$$

and the corresponding norms,

$$\|f^n\|^2=(f^n,f^n), \quad |||f|||^2=\langle f^n,f^n\rangle. \tag{2.9}$$

https://doi.org/10.4208/aamm.2015.m1049 Published online by Cambridge University Press

Introducing the difference operators as follows,

$$\delta_+ f_i^n = \frac{f_{i+1}^n - f_i^n}{h}, \qquad i=1,2,\cdots,I-1, \tag{2.10a}$$

$$\delta_- f_{i+\frac{1}{2}}^n = \frac{f_{i+\frac{1}{2}}^n - f_{i-\frac{1}{2}}^n}{h}, \qquad i=1,2,\cdots,I, \tag{2.10b}$$

$$\partial_t f_i^n = \frac{f_i^n - f_i^{n-1}}{\tau}, \qquad n=1,2,\cdots,M. \tag{2.10c}$$

For the simplicity of description, we divide $[0,1]$ into two sub-domains $[0,x_{k+\frac{1}{2}}]$ and $[x_{k+\frac{1}{2}},1]$, and $x_{k+\frac{1}{2}}$ is the interface point of the sub-domains. $c(x,t)$, $q(x,t)$ are assumed to be smooth enough. Let C_i^n and $Q_{i+\frac{1}{2}}^n$ be the numerical approximations of the exact solutions c_i^n and $q_{i+\frac{1}{2}}^n$.

We propose the time second-order mass-conserved domain decomposition scheme for approximating Eqs. (2.2)-(2.3) with $f(x,t)=0$ as

$$\partial_t C_i^n + \frac{1}{2}\left(\delta_- Q_{i+\frac{1}{2}}^n + \delta_- Q_{i+\frac{1}{2}}^{n-1}\right) = 0, \quad i=1,2,\cdots,I, \quad n=1,2,\cdots,M, \tag{2.11a}$$

$$Q_{i+\frac{1}{2}}^n = -a\delta_+ C_i^n, \quad i=1,\cdots,k-1,k+1,\cdots,I-1, \quad n=1,2,\cdots,M, \tag{2.11b}$$

$$Q_{k+\frac{1}{2}}^n = -a\delta_+ C_k^{n-1} + a\tau\frac{1}{h^2}\left(\check{Q}_{k+\frac{3}{2}}^n - 2Q_{k+\frac{1}{2}}^n + \check{Q}_{k-\frac{1}{2}}^n\right), \quad n=1,2,\cdots,M, \tag{2.11c}$$

where

$$\check{Q}_{k+\frac{3}{2}}^n = \left(1 + a\tau\delta_+\delta_- + \frac{1}{2}a^2\tau^2\delta_+\delta_-\delta_+\delta_-\right)Q_{k+\frac{3}{2}}^{n-1}, \qquad n=1,2,\cdots,M, \tag{2.12a}$$

$$\check{Q}_{k-\frac{1}{2}}^n = \left(1 + a\tau\delta_+\delta_- + \frac{1}{2}a^2\tau^2\delta_+\delta_-\delta_+\delta_-\right)Q_{k-\frac{1}{2}}^{n-1}, \qquad n=1,2,\cdots,M. \tag{2.12b}$$

Let $\check{L} = \delta_+\delta_- + \frac{1}{2}a\tau\delta_+\delta_-\delta_+\delta_-$. (2.12) can be simplified as

$$\check{Q}_{k+\frac{3}{2}}^n = (1 + a\tau\check{L})Q_{k+\frac{3}{2}}^{n-1}, \qquad n=1,2,\cdots,M, \tag{2.13a}$$

$$\check{Q}_{k-\frac{1}{2}}^n = (1 + a\tau\check{L})Q_{k-\frac{1}{2}}^{n-1}, \qquad n=1,2,\cdots,M. \tag{2.13b}$$

The initial and boundary values are given by

$$C_i^0 = c^0(x_i), \qquad i=1,2,\cdots,I, \tag{2.14a}$$

$$Q_{i+\frac{1}{2}}^0 = -a\delta_+ C_i^0, \qquad i=1,2,\cdots,I-1, \tag{2.14b}$$

$$Q_{\frac{1}{2}}^n = Q_{I+\frac{1}{2}}^n = 0, \qquad n=0,1,\cdots,M. \tag{2.14c}$$

Now we describe the algorithm of our time second-order mass-conserved domain decomposition scheme in details as below.

https://doi.org/10.4208/aamm.2015.m1049 Published online by Cambridge University Press

Algorithm 2.1. The algorithm of the time second-order mass-conserved domain decomposition scheme.

Step 1 Initialization: Set

$$C_i^0 = c_0(x_i), \quad i=1,\cdots,I, \quad Q_{1/2}^n = Q_{I+1/2}^n = 0, \quad n=0,1,\cdots,M, \tag{2.15a}$$

$$Q_{i+\frac{1}{2}}^0 = -a\frac{C_{i+1}^0 - C_i^0}{h}, \quad i=1,2,\cdots,I-1. \tag{2.15b}$$

Step 2 Time Stepping Procedure: For $1 \le n \le M$, do Steps 3-4.

Step 3 Along the spatial direction, do Steps (a)-(c).

Step (a) Find the interface values $Q_{k+\frac{1}{2}}^n$ by the local explicit scheme. Set

$$\delta_+ C_k^{n-1} = \frac{C_{k+1}^{n-1} - C_k^{n-1}}{h}, \tag{2.16a}$$

$$\begin{aligned} Q_{k+\frac{1}{2}}^n = \frac{1}{1+2r}\Big(\Big(&\frac{r^3}{2}(Q_{k+\frac{7}{2}}^{n-1} + Q_{k-\frac{5}{2}}^{n-1}) + (r^2-2r^3)(Q_{k+\frac{5}{2}}^{n-1} + Q_{k-\frac{3}{2}}^{n-1}) \\ &+ r\Big(1-2r+\frac{7r^2}{2}\Big)(Q_{k+\frac{3}{2}}^{n-1} + Q_{k-\frac{1}{2}}^{n-1}) + 2r(r-2r^2)Q_{k+\frac{1}{2}}^{n-1} - a\delta_+ C_k^{n-1}\Big)\Big), \end{aligned} \tag{2.16b}$$

where, $r = a\frac{\tau}{h^2}$. Let

$$f_q = -\tau\frac{Q_{k+\frac{1}{2}}^n + Q_{k+\frac{1}{2}}^{n-1}}{2h}. \tag{2.17}$$

Step (b) The interior values C_i^n $(\forall i)$ are obtained in the implicit C-N scheme, where $Q_{i+\frac{1}{2}}^n = -a\delta_+ C_i^n$ $(\forall i,\, i \ne k)$. Set

$$\mathbf{F}_{k\times 1} = [0,0,\cdots,0,f_q]' + \mathbf{M}_2 C_{(1:k)}^{n-1}, \tag{2.18a}$$

$$\mathbf{F}_{(I-k)\times 1} = [-f_q,0,\cdots,0,0]' + \mathbf{M}_4 C_{(k+1:I)}^{n-1}, \tag{2.18b}$$

$$\mathbf{M}_1 C_{(1:k)}^n = \mathbf{F}_{k\times 1}, \quad \mathbf{M}_3 C_{(k+1:I)}^n = \mathbf{F}_{(I-k)\times 1}. \tag{2.18c}$$

Where

$$\mathbf{M}_1 = \begin{pmatrix} 1+\frac{r}{2} & -\frac{r}{2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -\frac{r}{2} & 1+r & -\frac{r}{2} & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{r}{2} & 1+r & -\frac{r}{2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\frac{r}{2} & 1+\frac{r}{2} \end{pmatrix}_{k\times k}, \tag{2.19a}$$

https://doi.org/10.4208/aamm.2015.m1049 Published online by Cambridge University Press

$$\mathbf{M}_2=\begin{pmatrix} 1-\frac{r}{2} & \frac{r}{2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \frac{r}{2} & 1-r & \frac{r}{2} & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{r}{2} & 1-r & \frac{r}{2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{r}{2} & 1-\frac{r}{2} \end{pmatrix}_{k\times k}, \tag{2.19b}$$

$$\mathbf{M}_3=\begin{pmatrix} 1+\frac{r}{2} & -\frac{r}{2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -\frac{r}{2} & 1+r & -\frac{r}{2} & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{r}{2} & 1+r & -\frac{r}{2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\frac{r}{2} & 1+\frac{r}{2} \end{pmatrix}_{(I-k)\times(I-k)}, \tag{2.19c}$$

and

$$\mathbf{M}_4=\begin{pmatrix} 1-\frac{r}{2} & \frac{r}{2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \frac{r}{2} & 1-r & \frac{r}{2} & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{r}{2} & 1-r & \frac{r}{2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{r}{2} & 1-\frac{r}{2} \end{pmatrix}_{(I-k)\times(I-k)}. \tag{2.20}$$

Step (c) The fluxes $Q^n_{i+\frac{1}{2}}$, $i\neq k$ are computed as follows,

$$Q^n_{i+\frac{1}{2}}=-a\frac{C^n_{i+1}-C^n_i}{h},\quad i=1,\cdots,k-1,k+1,\cdots,I. \tag{2.21}$$

Step (d) end

Step 4 Set $n=n+1$.

Step 5 End

Remark 2.1. The time second-order mass-conserved implicit-explicit domain decomposition scheme can be implemented easily. Firstly, we use the obtained solutions and fluxes at the previous time level to calculate the interface flux. Secondly, the solutions of two tridiagonal systems (2.18c) can be solved easily by Thomas' algorithm. Lastly, we solve the fluxes by using the relations between solutions and fluxes.

https://doi.org/10.4208/aamm.2015.m1049 Published online by Cambridge University Press

3 Mass conservation and stability

In this section, we analyze the mass conservation and stability for the proposed scheme (2.11a)-(2.14c).

Theorem 3.1. *The proposed scheme satisfies mass conservation, i.e., if* $f(x,t)=0$, *then*

$$\sum_{i=1}^{I} C_i^n h = \sum_{i=1}^{I} C_i^0 h, \quad n=1,2,\cdots,M. \tag{3.1}$$

Proof. When $f(x,t)=0$, multiplying (2.11a) by h and summing up from $i=1$ to I, we can obtain that

$$\sum_{i=1}^{I} \partial_t C_i^n h + \frac{1}{2}\sum_{i=1}^{I}\left(\delta_- Q^n_{i+\frac{1}{2}} + \delta_- Q^{n-1}_{i+\frac{1}{2}}\right)h = 0, \quad n=1,2,\cdots,M. \tag{3.2}$$

Using the definition of δ_- and the boundary condition (2.14c), we can obtain that

$$\sum_{i=1}^{I} \delta_- Q^n_{i+\frac{1}{2}} h = 0, \quad \sum_{i=1}^{I} \delta_- Q^{n-1}_{i+\frac{1}{2}} h = 0, \quad n=1,2,\cdots,M. \tag{3.3}$$

Substituting (3.3) into (3.2), we have that

$$\sum_{i=1}^{I} \partial_t C_i^n h = 0, \quad n=1,2,\cdots,M. \tag{3.4}$$

Thus, we get that

$$\sum_{i=1}^{I} C_i^n h = \sum_{i=1}^{I} C_i^{n-1} h = \cdots = \sum_{i=1}^{I} C_i^0 h. \tag{3.5}$$

This ends the proof of the theorem. □

Theorem 3.2. *Assume* $r = a\frac{\tau}{h^2} \le \frac{3}{5}$, *then the proposed scheme* (2.11a)-(2.11c) *is* L^2 *stable, i.e., if* $f(x,t)=0$, *there exists a positive constant* K *such that*

$$\max_{1\le n\le M} \|C^n\|^2 \le K\|C^0\|^2. \tag{3.6}$$

Proof. Multiplying (2.11a) by $(C_i^n + C_i^{n-1})h$ and summing up from $i=1$ to I, we then get that

$$\frac{\|C^n\|^2 - \|C^{n-1}\|^2}{\tau} + \sum_{i=1}^{I}\frac{1}{2}\left(\delta_- Q^n_{i+\frac{1}{2}} + \delta_- Q^{n-1}_{i+\frac{1}{2}}\right)\left(C_i^n + C_i^{n-1}\right)h = 0. \tag{3.7}$$

https://doi.org/10.4208/aamm.2015.m1049 Published online by Cambridge University Press

Using (2.11b) and (2.14c), we can obtain that

$$\begin{aligned}
&\sum_{i=1}^{I}\frac{1}{2}\left(\delta_- Q^n_{i+\frac12}+\delta_- Q^{n-1}_{i+\frac12}\right)\left(C^n_i+C^{n-1}_i\right)h\\
=&-\sum_{i=1}^{I-1}\frac{1}{2a}\left(Q^n_{i+\frac12}+Q^{n-1}_{i+\frac12}\right)\left(a\delta_+ C^n_i+a\delta_+ C^{n-1}_i\right)h\\
=&\sum_{i=1,i\neq k}^{I-1}\frac{1}{2a}\left(Q^n_{i+\frac12}+Q^{n-1}_{i+\frac12}\right)^2 h-\frac{1}{2a}\left(Q^n_{k+\frac12}+Q^{n-1}_{k+\frac12}\right)\left(a\delta_+ C^n_k+a\delta_+ C^{n-1}_k\right)h\\
\equiv&A_1+A_2. \qquad (3.8)
\end{aligned}$$

Let $r=a\frac{\tau}{h^2}$, for $n=1$, substituting (2.11a)-(2.13) and (2.14b) into A_2, we have that

$$\begin{aligned}
A_2\equiv&-\frac{1}{2a}\left(Q^1_{k+\frac12}+Q^0_{k+\frac12}\right)\left(a\delta_+ C^1_k+a\delta_+ C^0_k\right)h\\
=&-\frac{1}{2a}\left(Q^1_{k+\frac12}+Q^0_{k+\frac12}\right)\left(a\tau\delta_+\partial_t C^1_k+a\delta_+ C^0_k+a\delta_+ C^0_k\right)h\\
=&\frac{1}{2a}\left(Q^1_{k+\frac12}+Q^0_{k+\frac12}\right)\left(\frac12 a\tau\delta_+\left(\delta_- Q^1_{k+\frac12}+\delta_- Q^0_{k+\frac12}\right)-a\delta_+ C^0_k+Q^0_{k+\frac12}\right)h\\
=&\frac{1}{2a}\left(Q^1_{k+\frac12}+Q^0_{k+\frac12}\right)\Big(\frac{r}{2}\left(Q^1_{k+\frac32}+Q^1_{k-\frac12}+Q^0_{k+\frac32}+Q^0_{k-\frac12}\right)\\
&+(1+r)Q^1_{k+\frac12}-r\left(\check{Q}^1_{k+\frac32}+\check{Q}^1_{k-\frac12}\right)+(1-r)Q^0_{k+\frac12}\Big)h\\
=&\frac{1}{2a}\left(Q^1_{k+\frac12}+Q^0_{k+\frac12}\right)^2 h+\frac{r}{2a}\left(Q^1_{k+\frac12}\right)^2 h+\frac{r}{4a}\left(Q^1_{k+\frac12}+Q^0_{k+\frac12}\right)\\
&\left(Q^1_{k+\frac32}+Q^0_{k+\frac32}\right)h+\frac{r}{4a}\left(Q^1_{k+\frac12}+Q^0_{k+\frac12}\right)\left(Q^1_{k-\frac12}+Q^0_{k-\frac12}\right)h\\
&-\frac{r}{2a}\left(Q^0_{k+\frac12}\right)^2 h-\frac{r}{2a}\left(Q^1_{k+\frac12}+Q^0_{k+\frac12}\right)\left(Q^0_{k+\frac32}\right)h-\frac{r}{2a}\left(Q^1_{k+\frac12}\right.\\
&\left.+Q^0_{k+\frac12}\right)\left(Q^0_{k-\frac12}\right)h-\frac{r}{2a}\left(a\tau\check{L}\left(Q^0_{k+\frac32}+Q^0_{k-\frac12}\right)\right)\left(Q^1_{k+\frac12}+Q^0_{k+\frac12}\right)h. \qquad (3.9)
\end{aligned}$$

Substituting (3.8)-(3.9) into Eq. (3.7), we have that for $n=1$

$$\begin{aligned}
0=&\frac{\|C^1\|^2-\|C^0\|^2}{\tau}+\sum_{i=1}^{I}\frac{1}{2}\left(\delta_- Q^1_{i+\frac12}+\delta_- Q^0_{i+\frac12}\right)\left(C^1_i+C^0_i\right)h\\
=&\frac{\|C^1\|^2-\|C^0\|^2}{\tau}+\sum_{i=1}^{I-1}\frac{1}{2a}\left(Q^1_{i+\frac12}+Q^0_{i+\frac12}\right)^2 h+\frac{r}{2a}\left(Q^1_{k+\frac12}\right)^2 h\\
&+\frac{r}{4a}\left(Q^1_{k+\frac12}+Q^0_{k+\frac12}\right)\left(Q^1_{k+\frac32}+Q^0_{k+\frac32}\right)h+\frac{r}{4a}\left(Q^1_{k+\frac12}+Q^0_{k+\frac12}\right)\left(Q^1_{k-\frac12}+Q^0_{k-\frac12}\right)h\\
&-\frac{r}{2a}\left(Q^0_{k+\frac12}\right)^2 h-\frac{r}{2a}\left(Q^1_{k+\frac12}+Q^0_{k+\frac12}\right)\left(Q^0_{k+\frac32}\right)h-\frac{r}{2a}\left(Q^1_{k+\frac12}+Q^0_{k+\frac12}\right)\left(Q^0_{k-\frac12}\right)h\\
&-\frac{r}{2a}\left(a\tau\check{L}\left(Q^0_{k+\frac32}+Q^0_{k-\frac12}\right)\right)\left(Q^1_{k+\frac12}+Q^0_{k+\frac12}\right)h. \qquad (3.10)
\end{aligned}$$

https://doi.org/10.4208/aamm.2015.m1049 Published online by Cambridge University Press

Applying Hölder inequality,

$$fg \le \frac{f^2}{2\nu} + \frac{\nu g^2}{2}, \quad \nu > 0. \tag{3.11}$$

(3.10) can further be simplified into the following equation

$$\begin{aligned}
&\frac{\|C^1\|^2 - \|C^0\|^2}{\tau} + \sum_{i=1}^{I-1} \frac{1}{2a} \left(Q^1_{i+\frac{1}{2}} + Q^0_{i+\frac{1}{2}}\right)^2 h + \frac{r}{4a} \left(Q^1_{k+\frac{1}{2}} + Q^0_{k+\frac{1}{2}}\right)^2 h \\
&\quad + \frac{r}{4a} \left(Q^1_{k+\frac{1}{2}} + Q^0_{k+\frac{1}{2}}\right) \left(Q^1_{k+\frac{3}{2}} + Q^0_{k+\frac{3}{2}}\right) h + \frac{r}{4a} \left(Q^1_{k+\frac{1}{2}} + Q^0_{k+\frac{1}{2}}\right) \\
&\quad \left(Q^1_{k-\frac{1}{2}} + Q^0_{k-\frac{1}{2}}\right) h - \frac{r}{2a} \left(a\tau \check{L} \left(Q^0_{k+\frac{3}{2}} + Q^0_{k-\frac{1}{2}}\right)\right) \left(Q^1_{k+\frac{1}{2}} + Q^0_{k+\frac{1}{2}}\right) h \\
=& \frac{r}{a} \left(Q^0_{k+\frac{1}{2}}\right)^2 h + \frac{r}{2a} \left(Q^1_{k+\frac{1}{2}} + Q^0_{k+\frac{1}{2}}\right) \left(Q^0_{k+\frac{3}{2}}\right) h + \frac{r}{2a} \left(Q^1_{k+\frac{1}{2}} + Q^0_{k+\frac{1}{2}}\right) \left(Q^0_{k-\frac{1}{2}}\right) h \\
\le& \frac{r}{a} \left(Q^0_{k+\frac{1}{2}}\right)^2 h + \frac{r}{8a} \left(Q^1_{k+\frac{1}{2}} + Q^0_{k+\frac{1}{2}}\right)^2 h + \frac{2r}{a} \left(Q^0_{k+\frac{3}{2}}\right)^2 h + \frac{2r}{a} \left(Q^0_{k-\frac{1}{2}}\right)^2 h.
\end{aligned} \tag{3.12}$$

Note that

$$\begin{aligned}
&- \frac{r}{2a} \left(a\tau \check{L} \left(Q^0_{k+\frac{3}{2}} + Q^0_{k-\frac{1}{2}}\right)\right) \left(Q^1_{k+\frac{1}{2}} + Q^0_{k+\frac{1}{2}}\right) h \\
=& - \frac{r^2}{2a} \left(\frac{r}{2} Q^0_{k+\frac{7}{2}} + (1-2r) Q^0_{k+\frac{5}{2}} + \left(\frac{7r}{2} - 2\right) Q^0_{k+\frac{3}{2}} + 2(1-2r) Q^0_{k+\frac{1}{2}} \right. \\
&\left. + \left(\frac{7r}{2} - 2\right) Q^0_{k-\frac{1}{2}} + (1-2r) Q^0_{k-\frac{3}{2}} + \frac{r}{2} Q^0_{k-\frac{5}{2}} \right) \left(Q^1_{k+\frac{1}{2}} + Q^0_{k+\frac{1}{2}}\right) h \\
\ge& - \left(\frac{7r^5}{16a} \left(Q^0_{k+\frac{7}{2}}\right)^2 + \frac{7r^3(1-2r)^2}{4a} \left(Q^0_{k+\frac{5}{2}}\right)^2 + \frac{7r^3\left(\frac{7r}{2} - 2\right)^2}{4a} \left(Q^0_{k+\frac{3}{2}}\right)^2 \right. \\
&+ \frac{7r^3(1-2r)^2}{a} \left(Q^0_{k+\frac{1}{2}}\right)^2 + \frac{7r^3\left(\frac{7r}{2} - 2\right)^2}{4a} \left(Q^0_{k-\frac{1}{2}}\right)^2 + \frac{7r^3(1-2r)^2}{4a} \left(Q^0_{k-\frac{3}{2}}\right)^2 \\
&\left. + \frac{7r^5}{16a} \left(Q^0_{k-\frac{5}{2}}\right)^2 \right) h - \frac{r}{4a} \left(Q^1_{k+\frac{1}{2}} + Q^0_{k+\frac{1}{2}}\right)^2 h.
\end{aligned} \tag{3.13}$$

We can further get that

$$\begin{aligned}
&\frac{\|C^1\|^2 - \|C^0\|^2}{\tau} + \sum_{i=1, i\neq k, i\neq k\pm 1}^{I-1} \frac{1}{2a} \left(Q^1_{i+\frac{1}{2}} + Q^0_{i+\frac{1}{2}}\right)^2 h + \frac{2-r}{4a} \left(Q^1_{k-\frac{1}{2}} + Q^0_{k-\frac{1}{2}}\right)^2 h \\
&\quad + \frac{2-r}{4a} \left(Q^1_{k+\frac{1}{2}} + Q^0_{k+\frac{1}{2}}\right)^2 h + \frac{2-r}{4a} \left(Q^1_{k+\frac{3}{2}} + Q^0_{k+\frac{3}{2}}\right)^2 h \\
\le& \frac{r + 7r^3(1-2r)^2}{a} \left(Q^0_{k+\frac{1}{2}}\right)^2 h + \frac{8r + 7r^3\left(\frac{7r}{2} - 2\right)^2}{4a} \left(Q^0_{k+\frac{3}{2}}\right)^2 h + \frac{8r + 7r^3\left(\frac{7r}{2} - 2\right)^2}{4a} \left(Q^0_{k-\frac{1}{2}}\right)^2 h \\
&\quad + \frac{7r^5}{16a} \left(Q^0_{k+\frac{7}{2}}\right)^2 h + \frac{7r^3(1-2r)^2}{4a} \left(Q^0_{k+\frac{5}{2}}\right)^2 h + \frac{7r^3(1-2r)^2}{4a} \left(Q^0_{k-\frac{3}{2}}\right)^2 h + \frac{7r^5}{16a} \left(Q^0_{k-\frac{5}{2}}\right)^2 h.
\end{aligned} \tag{3.14}$$

https://doi.org/10.4208/aamm.2015.m1049 Published online by Cambridge University Press

For $n\geq 2$, substituting (2.11a)-(2.13) into A_2, we have that

$$\begin{aligned}
A2\equiv&-\frac{1}{2a}\left(Q^n_{k+\frac12}+Q^{n-1}_{k+\frac12}\right)\left(a\delta_+C^n_k+a\delta_+C^{n-1}_k\right)h\\
=&-\frac{1}{2a}\left(Q^n_{k+\frac12}+Q^{n-1}_{k+\frac12}\right)\left(a\tau\delta_+\partial_tC^n_k+a\delta_+C^{n-1}_k+a\tau\delta_+\partial_tC^{n-1}_k+a\delta_+C^{n-2}_k\right)h\\
=&\frac{1}{2a}\left(Q^n_{k+\frac12}+Q^{n-1}_{k+\frac12}\right)\Big(\frac12a\tau\delta_+\left(\delta_-Q^n_{k+\frac12}+\delta_-Q^{n-1}_{k+\frac12}\right)+(1+2r)Q^n_{k+\frac12}-r\left(\check{Q}^n_{k+\frac32}+\check{Q}^n_{k-\frac12}\right)\\
&+\frac12a\tau\delta_+\left(\delta_-Q^{n-1}_{k+\frac12}+\delta_-Q^{n-2}_{k+\frac12}\right)+(1+2r)Q^{n-1}_{k+\frac12}-r\left(\check{Q}^{n-1}_{k+\frac32}+\check{Q}^{n-1}_{k-\frac12}\right)\Big)h\\
=&\frac{1}{2a}\left(Q^n_{k+\frac12}+Q^{n-1}_{k+\frac12}\right)\Big(\frac{r}{2}\left(Q^n_{k+\frac32}+Q^{n-1}_{k+\frac32}+Q^n_{k-\frac12}+Q^{n-1}_{k-\frac12}\right)\\
&+(1+r)\left(Q^n_{k+\frac12}+Q^{n-1}_{k+\frac12}\right)-r\left(\check{Q}^n_{k+\frac32}+\check{Q}^n_{k-\frac12}\right)-r\left(Q^{n-1}_{k+\frac12}+Q^{n-2}_{k+\frac12}\right)\\
&+\frac{r}{2}\left(Q^{n-1}_{k+\frac32}+Q^{n-2}_{k+\frac32}+Q^{n-1}_{k-\frac12}+Q^{n-2}_{k-\frac12}\right)-r\left(\check{Q}^{n-1}_{k+\frac32}+\check{Q}^{n-1}_{k-\frac12}\right)\Big)h\\
=&\frac{1}{2a}\left(Q^n_{k+\frac12}+Q^{n-1}_{k+\frac12}\right)^2h+\frac{r}{4a}\left(Q^n_{k+\frac12}+Q^{n-1}_{k+\frac12}\right)^2h-\frac{r}{4a}\left(Q^{n-1}_{k+\frac12}+Q^{n-2}_{k+\frac12}\right)^2h\\
&+\frac{r}{4a}\left(Q^n_{k+\frac12}-Q^{n-2}_{k+\frac12}\right)^2h+\frac{r}{4a}\left(Q^n_{k+\frac12}+Q^{n-1}_{k+\frac12}\right)\left(Q^n_{k+\frac32}+Q^{n-1}_{k+\frac32}\right)h\\
&+\frac{r}{4a}\left(Q^n_{k+\frac12}+Q^{n-1}_{k+\frac12}\right)\left(Q^n_{k-\frac12}+Q^{n-1}_{k-\frac12}\right)h-\frac{r}{4a}\left(Q^n_{k+\frac12}+Q^{n-1}_{k+\frac12}\right)\left(Q^{n-1}_{k+\frac32}+Q^{n-1}_{k-\frac12}\right)h\\
&-\frac{r}{4a}\left(Q^n_{k+\frac12}+Q^{n-1}_{k+\frac12}\right)\left(Q^{n-2}_{k+\frac32}+Q^{n-2}_{k-\frac12}\right)h\\
&-\frac{r}{2a}\left(a\tau\check{L}\left(Q^{n-1}_{k+\frac32}+Q^{n-1}_{k-\frac12}+Q^{n-2}_{k+\frac32}+Q^{n-2}_{k-\frac12}\right)\right)\left(Q^n_{k+\frac12}+Q^{n-1}_{k+\frac12}\right)h. \qquad (3.15)
\end{aligned}$$

By (3.8) and (3.15), (3.7) can be simplified as that for $n\geq 2$

$$\begin{aligned}
0=&\frac{\|C^n\|^2-\|C^{n-1}\|^2}{\tau}+\sum_{i=1}^{I}\frac12(\delta_-Q^n_{i+\frac12}+\delta_-Q^{n-1}_{i+\frac12})(C^n_i+C^{n-1}_i)h\\
=&\frac{\|C^n\|^2-\|C^{n-1}\|^2}{\tau}+\sum_{i=1,i\neq k}^{I-1}\frac{1}{2a}(Q^n_{i+\frac12}+Q^{n-1}_{i+\frac12})^2h-\frac{1}{2a}(Q^n_{k+\frac12}+Q^{n-1}_{k+\frac12})(a\delta_+C^n_k+a\delta_+C^{n-1}_k)h\\
=&\frac{\|C^n\|^2-\|C^{n-1}\|^2}{\tau}+\sum_{i=1}^{I-1}\frac{1}{2a}(Q^n_{i+\frac12}+Q^{n-1}_{i+\frac12})^2h+\frac{r}{4a}(Q^n_{k+\frac12}-Q^{n-2}_{k+\frac12})^2h\\
&+\frac{r}{4a}(Q^n_{k+\frac12}+Q^{n-1}_{k+\frac12})^2h+\frac{r}{4a}(Q^n_{k+\frac12}+Q^{n-1}_{k+\frac12})(Q^n_{k+\frac32}+Q^{n-1}_{k+\frac32})h\\
&+\frac{r}{4a}(Q^n_{k+\frac12}+Q^{n-1}_{k+\frac12})(Q^n_{k-\frac12}+Q^{n-1}_{k-\frac12})h-\frac{r}{4a}(Q^{n-1}_{k+\frac12}+Q^{n-2}_{k+\frac12})^2h\\
&-\frac{r}{4a}(Q^n_{k+\frac12}+Q^{n-1}_{k+\frac12})(Q^{n-1}_{k+\frac32}+Q^{n-1}_{k+\frac32})h-\frac{r}{4a}(Q^n_{k+\frac12}+Q^{n-1}_{k+\frac12})(Q^{n-2}_{k-\frac12}\\
&+Q^{n-2}_{k-\frac12})h-\frac{r}{2a}(a\tau\check{L}(Q^{n-1}_{k+\frac32}+Q^{n-1}_{k-\frac12}+Q^{n-2}_{k+\frac32}+Q^{n-2}_{k-\frac12}))(Q^n_{k+\frac12}+Q^{n-1}_{k+\frac12})h. \qquad (3.16)
\end{aligned}$$

https://doi.org/10.4208/aamm.2015.m1049 Published online by Cambridge University Press

Applying Hölder inequality, we can obtain that for $n \geq 2$

$$\begin{aligned}
&\frac{\|C^n\|^2-\|C^{n-1}\|^2}{\tau}+\sum_{i=1}^{I-1}\frac{1}{2a}\big(Q^n_{i+\frac{1}{2}}+Q^{n-1}_{i+\frac{1}{2}}\big)^2h+\frac{r}{4a}\big(Q^n_{k+\frac{1}{2}}-Q^{n-2}_{k+\frac{1}{2}}\big)^2h+\frac{r}{4a}\big(Q^n_{k+\frac{1}{2}}+Q^{n-1}_{k+\frac{1}{2}}\big)^2h\\
&\quad+\frac{r}{4a}\big(Q^n_{k+\frac{1}{2}}+Q^{n-1}_{k+\frac{1}{2}}\big)\big(Q^n_{k+\frac{3}{2}}+Q^{n-1}_{k+\frac{3}{2}}\big)h+\frac{r}{4a}\big(Q^n_{k+\frac{1}{2}}+Q^{n-1}_{k+\frac{1}{2}}\big)\big(Q^n_{k-\frac{1}{2}}+Q^{n-1}_{k-\frac{1}{2}}\big)h\\
&\quad-\frac{r}{2a}\big(a\tau\check{L}\big(Q^{n-1}_{k+\frac{3}{2}}+Q^{n-1}_{k-\frac{1}{2}}+Q^{n-2}_{k+\frac{3}{2}}+Q^{n-2}_{k-\frac{1}{2}}\big)\big)\big(Q^n_{k+\frac{1}{2}}+Q^{n-1}_{k+\frac{1}{2}}\big)h\\
=&\frac{r}{4a}\big(Q^{n-1}_{k+\frac{1}{2}}+Q^{n-2}_{k+\frac{1}{2}}\big)^2h+\frac{r}{4a}\big(Q^n_{k+\frac{1}{2}}+Q^{n-1}_{k+\frac{1}{2}}\big)\big(Q^{n-1}_{k+\frac{3}{2}}+Q^{n-2}_{k+\frac{3}{2}}\big)h\\
&\quad+\frac{r}{4a}\big(Q^n_{k+\frac{1}{2}}+Q^{n-1}_{k+\frac{1}{2}}\big)\big(Q^{n-1}_{k-\frac{1}{2}}+Q^{n-2}_{k-\frac{1}{2}}\big)h\\
\leq&\frac{r}{4a}\big(Q^{n-1}_{k+\frac{1}{2}}+Q^{n-2}_{k+\frac{1}{2}}\big)^2h+\frac{r}{8a}\big(Q^n_{k+\frac{1}{2}}+Q^{n-1}_{k+\frac{1}{2}}\big)^2h\\
&\quad+\frac{r}{4a}\big(Q^{n-1}_{k+\frac{3}{2}}+Q^{n-2}_{k+\frac{3}{2}}\big)^2h+\frac{r}{4a}\big(Q^{n-1}_{k-\frac{1}{2}}+Q^{n-2}_{k-\frac{1}{2}}\big)^2h.
\end{aligned}\tag{3.17}$$

Let

$$\begin{aligned}
F\equiv&\sum_{i=1}^{I-1}\frac{1}{2a}\big(Q^n_{i+\frac{1}{2}}+Q^{n-1}_{i+\frac{1}{2}}\big)^2h+\frac{r}{4a}\big(Q^n_{k+\frac{1}{2}}-Q^{n-2}_{k+\frac{1}{2}}\big)^2h+\frac{r}{4a}\big(Q^n_{k+\frac{1}{2}}+Q^{n-1}_{k+\frac{1}{2}}\big)^2h\\
&+\frac{r}{4a}\big(Q^n_{k+\frac{1}{2}}+Q^{n-1}_{k+\frac{1}{2}}\big)\big(Q^n_{k+\frac{3}{2}}+Q^{n-1}_{k+\frac{3}{2}}\big)h+\frac{r}{4a}\big(Q^n_{k+\frac{1}{2}}+Q^{n-1}_{k+\frac{1}{2}}\big)\big(Q^n_{k-\frac{1}{2}}+Q^{n-1}_{k-\frac{1}{2}}\big)h\\
&-\frac{r}{2a}\big(a\tau\check{L}\big(Q^{n-1}_{k+\frac{3}{2}}+Q^{n-1}_{k-\frac{1}{2}}+Q^{n-2}_{k+\frac{3}{2}}+Q^{n-2}_{k-\frac{1}{2}}\big)\big)\big(Q^n_{k+\frac{1}{2}}+Q^{n-1}_{k+\frac{1}{2}}\big)h-\frac{r}{8a}\big(Q^n_{k+\frac{1}{2}}+Q^{n-1}_{k+\frac{1}{2}}\big)^2h\\
&-\frac{r}{4a}\big(Q^{n-1}_{k+\frac{1}{2}}+Q^{n-2}_{k+\frac{1}{2}}\big)^2h-\frac{r}{4a}\big(Q^{n-1}_{k+\frac{3}{2}}+Q^{n-2}_{k+\frac{3}{2}}\big)^2h-\frac{r}{4a}\big(Q^{n-1}_{k-\frac{1}{2}}+Q^{n-2}_{k-\frac{1}{2}}\big)^2h\\
\geq&\sum_{i=1,i\neq k,k\pm1}^{I-1}\frac{1}{2a}\big(Q^n_{i+\frac{1}{2}}+Q^{n-1}_{i+\frac{1}{2}}\big)^2h+\frac{2-r}{4a}\big(Q^n_{k+\frac{3}{2}}+Q^{n-1}_{k+\frac{3}{2}}\big)^2h+\frac{2-r}{4a}\big(Q^n_{k+\frac{1}{2}}+Q^{n-1}_{k+\frac{1}{2}}\big)^2h\\
&+\frac{2-r}{4a}\big(Q^n_{k-\frac{1}{2}}+Q^{n-1}_{k-\frac{1}{2}}\big)^2h-\frac{7r^3\big(\frac{7r}{2}-2\big)^2+r}{4a}\big(Q^{n-1}_{k+\frac{3}{2}}+Q^{n-2}_{k+\frac{3}{2}}\big)^2h\\
&-\frac{28r^3(1-2r)^2+r}{4a}\big(Q^{n-1}_{k+\frac{1}{2}}+Q^{n-2}_{k+\frac{1}{2}}\big)^2h-\frac{7r^3\left(\frac{7r}{2}-2\right)^2+r}{4a}\big(Q^{n-1}_{k-\frac{1}{2}}+Q^{n-2}_{k-\frac{1}{2}}\big)^2h\\
&-\frac{7r^5}{16a}\big(Q^{n-1}_{k+\frac{7}{2}}+Q^{n-2}_{k+\frac{7}{2}}\big)^2h-\frac{7r^3(1-2r)^2}{4a}\big(Q^{n-1}_{k+\frac{5}{2}}+Q^{n-2}_{k+\frac{5}{2}}\big)^2h\\
&-\frac{7r^3(1-2r)^2}{4a}\big(Q^{n-1}_{k-\frac{3}{2}}+Q^{n-2}_{k-\frac{3}{2}}\big)^2h-\frac{7r^5}{16a}\big(Q^{n-1}_{k-\frac{5}{2}}+Q^{n-2}_{k-\frac{5}{2}}\big)^2h,
\end{aligned}\tag{3.18}$$

where, in (3.18), we have applied the following inequality,

$$\begin{aligned}
&-\frac{r}{2a}\big(a\tau\check{L}\big(Q^{n-1}_{k+\frac{3}{2}}+Q^{n-1}_{k-\frac{1}{2}}+Q^{n-2}_{k+\frac{3}{2}}+Q^{n-2}_{k-\frac{1}{2}}\big)\big)\big(Q^n_{k+\frac{1}{2}}+Q^{n-1}_{k+\frac{1}{2}}\big)h\\
=&-\frac{r^2}{2a}\Big(\frac{r}{2}\big(Q^{n-1}_{k+\frac{7}{2}}+Q^{n-2}_{k+\frac{7}{2}}\big)+(1-2r)\big(Q^{n-1}_{k+\frac{5}{2}}+Q^{n-2}_{k+\frac{5}{2}}\big)
\end{aligned}$$

https://doi.org/10.4208/aamm.2015.m1049 Published online by Cambridge University Press

$$
\begin{aligned}
&+\Big(\frac{7r}{2}-2\Big)\big(Q^{n-1}_{k+\frac{3}{2}}+Q^{n-2}_{k+\frac{3}{2}}\big)+2(1-2r)\big(Q^{n-1}_{k+\frac{1}{2}}+Q^{n-2}_{k+\frac{1}{2}}\big)\\
&+\Big(\frac{7r}{2}-2\Big)\big(Q^{n-1}_{k-\frac{1}{2}}+Q^{n-2}_{k-\frac{1}{2}}\big)+(1-2r)\big(Q^{n-1}_{k-\frac{3}{2}}+Q^{n-2}_{k-\frac{3}{2}}\big)\\
&+\frac{r}{2}\big(Q^{n-1}_{k-\frac{5}{2}}+Q^{n-2}_{k-\frac{5}{2}}\big)\Big)\big(Q^{n}_{k+\frac{1}{2}}+Q^{n-1}_{k+\frac{1}{2}}\big)h\\
\geq&-\frac{7r^3}{4a}\Big(\frac{r^2}{4}\big(Q^{n-1}_{k+\frac{7}{2}}+Q^{n-2}_{k+\frac{7}{2}}\big)^2+(1-2r)^2\big(Q^{n-1}_{k+\frac{5}{2}}+Q^{n-2}_{k+\frac{5}{2}}\big)^2\\
&+\Big(\frac{7r}{2}-2\Big)^2\big(Q^{n-1}_{k+\frac{3}{2}}+Q^{n-2}_{k+\frac{3}{2}}\big)^2+4(1-2r)^2\big(Q^{n-1}_{k+\frac{1}{2}}+Q^{n-2}_{k+\frac{1}{2}}\big)^2\\
&+\Big(\frac{7r}{2}-2\Big)^2\big(Q^{n-1}_{k-\frac{1}{2}}+Q^{n-2}_{k-\frac{1}{2}}\big)^2+(1-2r)^2\big(Q^{n-1}_{k-\frac{3}{2}}+Q^{n-2}_{k-\frac{3}{2}}\big)^2\\
&+\frac{r^2}{4}\big(Q^{n-1}_{k-\frac{5}{2}}+Q^{n-2}_{k-\frac{5}{2}}\big)^2\Big)h-\frac{r}{4a}\big(Q^{n}_{k+\frac{1}{2}}+Q^{n-1}_{k+\frac{1}{2}}\big)^2h. \qquad (3.19)
\end{aligned}
$$

Assume that $r\leq\frac{3}{5}$ and it holds that

$$
\frac{2-r}{4a}>D_1\equiv\max\Big\{\frac{28r^3(1-2r)^2+r}{4a},\frac{7r^3(\frac{7r}{2}-2)^2+r}{4a}\Big\}, \qquad (3.20a)
$$

$$
\frac{1}{2a}>D_2\equiv\max\Big\{\frac{7r^3(1-2r)^2}{4a},\frac{7r^5}{16a}\Big\}. \qquad (3.20b)
$$

Applying (3.20), we can obtain that from (3.18)

$$
\begin{aligned}
&\|C^n\|^2+\tau D_1\big(\big(Q^{n}_{k+\frac{1}{2}}+Q^{n-1}_{k+\frac{1}{2}}\big)^2h+\big(Q^{n}_{k+\frac{3}{2}}+Q^{n-1}_{k+\frac{3}{2}}\big)^2h+\big(Q^{n}_{k-\frac{1}{2}}+Q^{n-1}_{k-\frac{1}{2}}\big)^2h\big)\\
&+\tau D_2\big(\big(Q^{n}_{k+\frac{7}{2}}+Q^{n-1}_{k+\frac{7}{2}}\big)^2h+\big(Q^{n}_{k+\frac{5}{2}}+Q^{n-1}_{k+\frac{5}{2}}\big)^2h+\big(Q^{n}_{k-\frac{3}{2}}+Q^{n-1}_{k-\frac{3}{2}}\big)^2h+\big(Q^{n}_{k-\frac{5}{2}}+Q^{n-1}_{k-\frac{5}{2}}\big)^2h\big)\\
\leq&\|C^{n-1}\|^2+\tau D_1\big(\big(Q^{n-1}_{k+\frac{1}{2}}+Q^{n-2}_{k+\frac{1}{2}}\big)^2h+\big(Q^{n-1}_{k+\frac{3}{2}}+Q^{n-2}_{k+\frac{3}{2}}\big)^2h+\big(Q^{n-1}_{k-\frac{1}{2}}+Q^{n-2}_{k-\frac{1}{2}}\big)^2h\big)\\
&+\tau D_2\big(\big(Q^{n-1}_{k+\frac{7}{2}}+Q^{n-2}_{k+\frac{7}{2}}\big)^2h+\big(Q^{n-1}_{k+\frac{5}{2}}+Q^{n-2}_{k+\frac{5}{2}}\big)^2h\\
&+\big(Q^{n-1}_{k-\frac{3}{2}}+Q^{n-2}_{k-\frac{3}{2}}\big)^2h+\big(Q^{n-1}_{k-\frac{5}{2}}+Q^{n-2}_{k-\frac{5}{2}}\big)^2h\big). \qquad (3.21)
\end{aligned}
$$

Thus, multiplying (3.7) by τ and summing up from $n=1$ to M, from (3.14) and (3.21), we can get that

$$
\begin{aligned}
\max_{1\leq n\leq M}\|C^n\|^2\leq&\|C^0\|^2+\frac{r+7r^3(1-2r)^2}{a}\tau\big(Q^0_{k+\frac{1}{2}}\big)^2h+\frac{8r+7r^3\big(\frac{7r}{2}-2\big)^2}{4a}\tau\big(Q^0_{k+\frac{3}{2}}\big)^2h\\
&+\frac{7r^3(1-2r)^2}{4a}\tau\big(Q^0_{k+\frac{5}{2}}\big)^2h+\frac{7r^5}{16a}\tau\big(Q^0_{k+\frac{7}{2}}\big)^2h+\frac{8r+7r^3\big(\frac{7r}{2}-2\big)^2}{4a}\tau\big(Q^0_{k-\frac{1}{2}}\big)^2h\\
&+\frac{7r^3(1-2r)^2}{4a}\tau\big(Q^0_{k-\frac{3}{2}}\big)^2h+\frac{7r^5}{16a}\tau\big(Q^0_{k-\frac{5}{2}}\big)^2h. \qquad (3.22)
\end{aligned}
$$

https://doi.org/10.4208/aamm.2015.m1049 Published online by Cambridge University Press

Finally, noting that

$$\tau(Q^0_{i+\frac{1}{2}})^2 \le a^2\tau\Big(\frac{|C^0_{i+1}|+|C^0_i|}{h}\Big)^2 \le 2r^2\big(|C^0_{i+1}|^2+|C^0_i|^2\big), \quad i=k-3,\cdots,k+3. \tag{3.23}$$

Applying the stability condition $r\le 3/5$, we can have that

$$\max_{1\le n\le M}\|C^n\|^2 \le K\|C^0\|^2. \tag{3.24}$$

This ends the proof. □

Remark 3.1. In the section, we prove that the scheme satisfies mass conservation and obtain the stability condition $a\frac{\tau}{h^2}\le\frac{3}{5}$.

4 Convergence

In this section, we analyze the error estimate of the proposed scheme (2.11a)-(2.14c).

Theorem 4.1. *Under the condition of Theorem 3.2, let c^n be the exact solutions and C^n be the solution of the scheme* (2.11a)-(2.14c)*, then we have the following error estimate*

$$\max_{1\le n\le M}\|c^n-C^n\|+\Big(\sum_{n=1}^{M}|||q^n-Q^n|||^2\tau\Big)^{1/2}\le\mathcal{O}(\tau^2+h^2). \tag{4.1}$$

Proof. Let $e_i^{n-\frac{1}{2}}$ and $\varepsilon^n_{i+\frac{1}{2}}$ be the truncation errors of (4.2a)-(4.2c). Then we have

$$\partial_t c^n_i+\frac{1}{2}(\delta_- q^n_{i+\frac{1}{2}}+\delta_- q^{n-1}_{i+\frac{1}{2}})=e_i^{n-\frac{1}{2}}, \quad i=1,2,\cdots,I, \quad n=1,2,\cdots,M, \tag{4.2a}$$

$$q^n_{i+\frac{1}{2}}=-a\delta_+ c^n_i+\varepsilon^n_{i+\frac{1}{2}}, \quad i=1,2,\cdots,k-1,k+1,\cdots,I, \quad n=0,1,\cdots,M, \tag{4.2b}$$

$$q^n_{k+\frac{1}{2}}=-a\delta_+ c^{n-1}_k+a\tau\frac{1}{h^2}(\check{q}^n_{k+\frac{3}{2}}-2q^n_{k+\frac{1}{2}}+\check{q}^n_{k-\frac{1}{2}})+\varepsilon^n_{k+\frac{1}{2}}, \quad n=1,2,\cdots,M, \tag{4.2c}$$

where

$$\check{q}^n_{k+\frac{3}{2}}=(1+a\tau\check{L})q^{n-1}_{k+\frac{3}{2}}, \qquad \check{q}^n_{k-\frac{1}{2}}=(1+a\tau\check{L})q^{n-1}_{k-\frac{1}{2}}, \qquad n=1,2,\cdots,M. \tag{4.3}$$

As is known that

$$q^n_{i+\frac{1}{2}}=-ac^n_x|_{i+\frac{1}{2}}=-a\delta_+ c^n_i+\mathcal{O}(h^2), \quad i\ne k, \tag{4.4a}$$

$$\frac{c^n_i-c^{n-1}_i}{\tau}+\frac{1}{2}(\delta_- q^n_{i+\frac{1}{2}}+\delta_- q^{n-1}_{i+\frac{1}{2}})=\frac{\partial c^{n-1/2}}{\partial t}\Big|_i+\frac{\partial q^{n-1/2}}{\partial x}\Big|_i+\mathcal{O}(\tau^2+h^2). \tag{4.4b}$$

https://doi.org/10.4208/aamm.2015.m1049 Published online by Cambridge University Press

Thus, it is hold that $\varepsilon^n_{i+\frac{1}{2}}=\mathcal{O}(h^2)$, $i\neq k$, $e_i^{n-\frac{1}{2}}=\mathcal{O}(\tau^2+h^2)$. Now we derive the truncation errors of $\varepsilon^n_{k+\frac{1}{2}}$, $n=1,2,\cdots,M$.

$$\begin{aligned}
q^n_{k+\frac{1}{2}}=&q^{n-1}_{k+\frac{1}{2}}+\tau(q_t)^n_{k+\frac{1}{2}}+\mathcal{O}(\tau^2)=-a(c_x)^{n-1}_{k+\frac{1}{2}}+a\tau(q_{xx})^n_{k+\frac{1}{2}}+\mathcal{O}(\tau^2)\\
=&-a\delta_+c_k^{n-1}+a\tau\frac{1}{h^2}\left(q^n_{k+\frac{3}{2}}-2q^n_{k+\frac{1}{2}}+q^n_{k-\frac{1}{2}}\right)+\mathcal{O}(\tau^2+h^2+\tau h^2)\\
=&a\tau\frac{1}{h^2}\Big(q^{n-1}_{k+\frac{3}{2}}+\tau(q_t)^{n-1}_{k+\frac{3}{2}}+\frac{\tau^2}{2}(q_{tt})^{n-1}_{k+\frac{3}{2}}-2q^n_{k+\frac{1}{2}}+q^{n-1}_{k-\frac{1}{2}}\\
&+\tau(q_t)^{n-1}_{k-\frac{1}{2}}+\frac{\tau^2}{2}(q_{tt})^{n-1}_{k-\frac{1}{2}}\Big)-a\delta_+c_k^{n-1}+\mathcal{O}\Big(\tau^2+h^2+\tau h^2+\frac{\tau^4}{h^2}\Big)\\
=&a\tau\frac{1}{h^2}\Big(q^{n-1}_{k+\frac{3}{2}}+a\tau(q_{xx})^{n-1}_{k+\frac{3}{2}}+a^2\frac{\tau^2}{2}(q_{xxxx})^{n-1}_{k+\frac{3}{2}}-2q^n_{k+\frac{1}{2}}+q^{n-1}_{k-\frac{1}{2}}\\
&+a\tau(q_{xx})^{n-1}_{k-\frac{1}{2}}+a^2\frac{\tau^2}{2}(q_{xxxx})^{n-1}_{k-\frac{1}{2}}\Big)-a\delta_+c_k^{n-1}+\mathcal{O}\Big(\tau^2+h^2+\tau h^2+\frac{\tau^4}{h^2}\Big)\\
=&a\tau\frac{1}{h^2}\Big(q^{n-1}_{k+\frac{3}{2}}+a\tau\delta_+\delta_-q^{n-1}_{k+\frac{3}{2}}+a^2\frac{\tau^2}{2}\delta_+\delta_-\delta_+\delta_-q^{n-1}_{k+\frac{3}{2}}-2q^n_{k+\frac{1}{2}}+q^{n-1}_{k-\frac{1}{2}}\\
&+a\tau\delta_+\delta_-q^{n-1}_{k-\frac{1}{2}}+a^2\frac{\tau^2}{2}\delta_+\delta_-\delta_+\delta_-q^{n-1}_{k-\frac{1}{2}}\Big)-a\delta_+c_k^{n-1}+\mathcal{O}\Big(\tau^2+h^2+\tau h^2+\frac{\tau^4}{h^2}+\tau^3\Big)\\
=&-a\delta_+c_k^{n-1}+a\tau\frac{1}{h^2}\Big((1+a\tau\check{L})q^{n-1}_{k+\frac{3}{2}}-2q^n_{k+\frac{1}{2}}+(1+a\tau\check{L})q^{n-1}_{k-\frac{1}{2}}\Big)+\mathcal{O}\Big(\tau^2+h^2+\tau h^2+\frac{\tau^4}{h^2}+\tau^3\Big).
\end{aligned}\tag{4.5}$$

From (4.2c) and (4.5), we obtain that

$$\varepsilon^n_{k+\frac{1}{2}}=\mathcal{O}\Big(\tau^2+h^2+\tau h^2+\frac{\tau^4}{h^2}+\tau^3\Big).$$

Define

$$\theta_i^n=c_i^n-C_i^n,\qquad i=1,2,\cdots,I,\qquad n=0,1,\cdots,M,\tag{4.6a}$$

$$\eta^n_{i+\frac{1}{2}}=q^n_{i+\frac{1}{2}}-Q^n_{i+\frac{1}{2}},\qquad i=0,1,\cdots,I,\qquad n=0,1,\cdots,M.\tag{4.6b}$$

Subtracting (4.2a)-(4.2c) from (2.11a)-(2.11c), we can obtain that

$$\partial_t\theta_i^n+\frac{1}{2}\left(\delta_-\eta^n_{i+\frac{1}{2}}+\delta_-\eta^{n-1}_{i+\frac{1}{2}}\right)=e_i^{n-\frac{1}{2}},\quad i=1,\cdots,I,\quad n=1,2,\cdots,M,\tag{4.7a}$$

$$\eta^n_{i+\frac{1}{2}}=-a\delta_+\theta_i^n+\varepsilon^n_{i+\frac{1}{2}},\quad i=1,\cdots,k-1,k+1,\cdots,I,\quad n=1,2,\cdots,M,\tag{4.7b}$$

$$\eta^n_{k+\frac{1}{2}}=-a\delta_+\theta_k^{n-1}+r\left(\check{\eta}^n_{k+\frac{3}{2}}-2\eta^n_{k+\frac{1}{2}}+\check{\eta}^n_{k-\frac{1}{2}}\right)+\varepsilon^n_{k+\frac{1}{2}},\quad n=1,2,\cdots,M,\tag{4.7c}$$

$$\eta^n_{\frac{1}{2}}=\eta^n_{I+\frac{1}{2}}=0,\quad n=0,\cdots,M,\tag{4.7d}$$

$$\theta_i^0=0,\quad \eta^0_{i+\frac{1}{2}}=\varepsilon^0_{i+\frac{1}{2}},\quad i=1,2,\cdots,I.\tag{4.7e}$$

Multiplying (4.7a) by $(\theta_i^n+\theta_i^{n-1})h$ and summing up from $i=1$ to I, we get that

$$\frac{\|\theta^n\|^2-\|\theta^{n-1}\|^2}{\tau}-\sum_{i=1}^I\frac{1}{2}\left(\eta^n_{i+\frac{1}{2}}+\eta^{n-1}_{i+\frac{1}{2}}\right)\left(\delta_+\theta_i^n+\delta_+\theta_i^{n-1}\right)h=\left(\theta^n+\theta^{n-1},e^{n-\frac{1}{2}}\right).\tag{4.8}$$

https://doi.org/10.4208/aamm.2015.m1049 Published online by Cambridge University Press

For the second term of (4.8), it holds that

$$\begin{aligned}
&-\sum_{i=1}^{I-1}\frac{1}{2a}\left(\eta_{i+\frac{1}{2}}^{n}+\eta_{i+\frac{1}{2}}^{n-1}\right)\left(a\delta_{+}\theta_{i}^{n}+a\delta_{+}\theta_{i}^{n-1}\right)h\\
=&\sum_{i=1,i\neq k}^{I-1}\frac{1}{2a}\left(\eta_{i+\frac{1}{2}}^{n}+\eta_{i+\frac{1}{2}}^{n-1}\right)^{2}h-\frac{1}{2a}\left(\eta_{k+\frac{1}{2}}^{n}+\eta_{k+\frac{1}{2}}^{n-1}\right)\left(a\delta_{+}\theta_{k}^{n}+a\delta_{+}\theta_{k}^{n-1}\right)h\\
&-\sum_{i=1,i\neq k}^{I-1}\frac{1}{2a}\left(\eta_{i+\frac{1}{2}}^{n}+\eta_{i+\frac{1}{2}}^{n-1}\right)\left(\varepsilon_{i+\frac{1}{2}}^{n}+\varepsilon_{i+\frac{1}{2}}^{n-1}\right)h\\
\equiv&B_{1}+B_{2}.
\end{aligned}\tag{4.9}$$

For $n=1$, we have that

$$\begin{aligned}
B_{2}\equiv&-\frac{1}{2a}\left(\eta_{k+\frac{1}{2}}^{1}+\eta_{k+\frac{1}{2}}^{0}\right)\left(a\delta_{+}\theta_{k}^{1}+a\delta_{+}\theta_{k}^{0}\right)h-\sum_{i=1,i\neq k}^{I-1}\frac{1}{2a}\left(\eta_{i+\frac{1}{2}}^{n}+\eta_{i+\frac{1}{2}}^{0}\right)\left(\varepsilon_{i+\frac{1}{2}}^{1}+\varepsilon_{i+\frac{1}{2}}^{0}\right)h\\
=&\frac{1}{2a}\left(\eta_{k+\frac{1}{2}}^{1}+\eta_{k+\frac{1}{2}}^{0}\right)\Big(\frac{r}{2}\left(\eta_{k+\frac{3}{2}}^{1}+\eta_{k-\frac{1}{2}}^{1}+\eta_{k+\frac{3}{2}}^{0}+\eta_{k-\frac{1}{2}}^{0}\right)+(1+r)\eta_{k+\frac{1}{2}}^{1}+(1-r)\eta_{k+\frac{1}{2}}^{0}\\
&-r\left(\check{\eta}_{k+\frac{3}{2}}^{1}+\check{\eta}_{k-\frac{1}{2}}^{1}\right)\Big)h-\frac{1}{2a}\left(\eta_{k+\frac{1}{2}}^{1}+\eta_{k+\frac{1}{2}}^{0}\right)\left(a\tau\delta_{+}e_{k}^{\frac{1}{2}}\right)h-\frac{1}{2a}\langle\eta^{1}+\eta^{0},\varepsilon^{1}+\varepsilon^{0}\rangle\\
=&\frac{1}{2a}\left(\eta_{k+\frac{1}{2}}^{1}+\eta_{k+\frac{1}{2}}^{0}\right)^{2}h-\frac{r}{2a}\left(\eta_{k+\frac{1}{2}}^{1}\right)^{2}h-\frac{r}{2a}\left(\eta_{k+\frac{1}{2}}^{0}\right)^{2}h\\
&+\frac{r}{4a}\left(\eta_{k+\frac{1}{2}}^{1}+\eta_{k+\frac{1}{2}}^{0}\right)\left(\eta_{k+\frac{3}{2}}^{1}+\eta_{k+\frac{3}{2}}^{0}\right)h+\frac{r}{4a}\left(\eta_{k+\frac{1}{2}}^{1}+\eta_{k+\frac{1}{2}}^{0}\right)\left(\eta_{k-\frac{1}{2}}^{1}+\eta_{k-\frac{1}{2}}^{0}\right)h\\
&-\frac{r}{2a}\left(\eta_{k+\frac{1}{2}}^{1}+\eta_{k+\frac{1}{2}}^{0}\right)\left(\eta_{k+\frac{3}{2}}^{0}\right)h-\frac{r}{2a}\left(\eta_{k+\frac{1}{2}}^{1}+\eta_{k+\frac{1}{2}}^{0}\right)\left(\eta_{k-\frac{1}{2}}^{0}\right)h\\
&-\frac{r}{2a}\left(a\tau\check{L}\left(\eta_{k+\frac{3}{2}}^{0}+\eta_{k-\frac{1}{2}}^{0}\right)\right)\left(\eta_{k+\frac{1}{2}}^{1}+\eta_{k+\frac{1}{2}}^{0}\right)h\\
&-\frac{1}{2a}\left(\eta_{k+\frac{1}{2}}^{1}+\eta_{k+\frac{1}{2}}^{0}\right)\left(a\tau\delta_{+}e_{k}^{\frac{1}{2}}\right)h-\frac{1}{2a}\langle\eta^{1}+\eta^{0},\varepsilon^{1}+\varepsilon^{0}\rangle.
\end{aligned}\tag{4.10}$$

Substituting (4.9)-(4.10) into Eq. (4.8), we have that for $n=1$

$$\begin{aligned}
&\frac{\|\theta^{1}\|^{2}-\|\theta^{0}\|^{2}}{\tau}+\sum_{i=1}^{I}\frac{1}{2}\left(\delta_{-}\eta_{i+\frac{1}{2}}^{1}+\delta_{-}\eta_{i+\frac{1}{2}}^{0}\right)\left(\theta_{i}^{1}+\theta_{i}^{0}\right)h\\
=&\frac{\|\theta^{1}\|^{2}-\|\theta^{0}\|^{2}}{\tau}-\sum_{i=1,i\neq k}^{I-1}\frac{1}{2a}\left(\eta_{i+\frac{1}{2}}^{1}+\eta_{i+\frac{1}{2}}^{0}\right)\left(\varepsilon_{i+\frac{1}{2}}^{1}+\varepsilon_{i+\frac{1}{2}}^{0}\right)\\
&+\sum_{i=1,i\neq k}^{I-1}\frac{1}{2a}\left(\eta_{i+\frac{1}{2}}^{1}+\eta_{i+\frac{1}{2}}^{0}\right)^{2}h-\frac{1}{2a}\left(\eta_{k+\frac{1}{2}}^{1}+\eta_{k+\frac{1}{2}}^{0}\right)\left(a\delta_{+}\theta_{k}^{1}+a\delta_{+}\theta_{k}^{0}\right)h\\
=&\frac{\|\theta^{1}\|^{2}-\|\theta^{0}\|^{2}}{\tau}+\frac{1}{2a}|||\eta^{1}+\eta^{0}|||^{2}+\frac{r}{2a}(\eta_{k+\frac{1}{2}}^{1})^{2}h\\
&+\frac{r}{4a}\left(\eta_{k+\frac{1}{2}}^{1}+\eta_{k+\frac{1}{2}}^{0}\right)\left(\eta_{k+\frac{3}{2}}^{1}+\eta_{k+\frac{3}{2}}^{0}\right)h+\frac{r}{4a}\left(\eta_{k+\frac{1}{2}}^{1}+\eta_{k+\frac{1}{2}}^{0}\right)\left(\eta_{k-\frac{1}{2}}^{1}+\eta_{k-\frac{1}{2}}^{0}\right)h
\end{aligned}$$

https://doi.org/10.4208/aamm.2015.m1049 Published online by Cambridge University Press

$$-\frac{r}{2a}(\eta^0_{k+\frac12})^2h-\frac{r}{2a}(\eta^1_{k+\frac12}+\eta^0_{k+\frac12})(\eta^0_{k+\frac32})h-\frac{r}{2a}(\eta^1_{k+\frac12}+\eta^0_{k+\frac12})(\eta^0_{k-\frac12})h$$
$$-\frac{1}{2a}(\eta^1_{k+\frac12}+\eta^0_{k+\frac12})(a\tau\delta_+e^{\frac12}_k)h-\frac{1}{2a}\langle\eta^1+\eta^0,\varepsilon^1+\varepsilon^0\rangle$$
$$-\frac{r}{2a}(a\tau\check{L}(\eta^0_{k+\frac32}+\eta^0_{k-\frac12}))(\eta^1_{k+\frac12}+\eta^0_{k+\frac12})h. \tag{4.11}$$

Further, we can get that

$$\begin{aligned}
&\frac{\|\theta^1\|^2-\|\theta^0\|^2}{\tau}+\frac{1}{2a}\|\|\eta^1+\eta^0\|\|^2+\frac{r}{4a}(\eta^1_{k+\frac12}+\eta^0_{k+\frac12})^2h+\frac{r}{4a}(\eta^1_{k+\frac12}+\eta^0_{k+\frac12})(\eta^1_{k+\frac32}+\eta^0_{k+\frac32})h\\
&+\frac{r}{4a}(\eta^1_{k+\frac12}+\eta^0_{k+\frac12})(\eta^1_{k-\frac12}+\eta^0_{k-\frac12})h-\frac{r}{2a}(a\tau\check{L}(\eta^0_{k+\frac32}+\eta^0_{k-\frac12}))(\eta^1_{k+\frac12}+\eta^0_{k+\frac12})h\\
&-\frac{1}{2a}(\eta^1_{k+\frac12}+\eta^0_{k+\frac12})(a\tau\delta_+e^{\frac12}_k)h-\frac{1}{2a}\langle\eta^1+\eta^0,\varepsilon^1+\varepsilon^0\rangle\\
=&\frac{r}{a}(\eta^0_{k+\frac12})^2h+\frac{r}{2a}(\eta^1_{k+\frac12}+\eta^0_{k+\frac12})(\eta^0_{k+\frac32})h+\frac{r}{2a}(\eta^1_{k+\frac12}+\eta^0_{k+\frac12})(\eta^0_{k-\frac12})h+(\theta^1+\theta^0,e^{\frac12})\\
\le&\frac{r}{a}(\eta^0_{k+\frac12})^2h+\frac{r}{8a}(\eta^1_{k+\frac12}+\eta^0_{k+\frac12})^2h+\frac{2r}{a}(\eta^0_{k+\frac32})^2h+\frac{2r}{a}(\eta^0_{k-\frac12})^2h+(\theta^1+\theta^0,e^{\frac12}).
\end{aligned}\tag{4.12}$$

Similar as the proof of stability, we further have that

$$\begin{aligned}
&\frac{\|\theta^1\|^2-\|\theta^0\|^2}{\tau}+\sum_{i=1,i\neq k,i\neq k\pm1}^{I-1}\frac{1}{2a}(\eta^1_{i+\frac12}+\eta^0_{i+\frac12})^2h+\frac{2-r}{4a}(\eta^1_{k+\frac12}+\eta^0_{k+\frac12})^2h\\
&+\frac{2-r}{4a}(\eta^1_{k+\frac32}+\eta^0_{k+\frac32})^2h+\frac{2-r}{4a}(\eta^1_{k-\frac12}+\eta^0_{k-\frac12})^2h\\
\le&\frac{r+7r^3(1-2r)^2}{a}(\eta^0_{k+\frac12})^2h+\frac{8r+7r^3(\frac{7r}{2}-2)^2}{4a}(\eta^0_{k+\frac32})^2h\\
&+\frac{8r+7r^3\left(\frac{7r}{2}-2\right)^2}{4a}(\eta^0_{k-\frac12})^2h+\frac{7r^5}{16a}(\eta^0_{k+\frac72})^2h+\frac{7r^3(1-2r)^2}{4a}(\eta^0_{k+\frac52})^2h\\
&+\frac{7r^3(1-2r)^2}{4a}(\eta^0_{k-\frac32})^2h+\frac{7r^5}{16a}(\eta^0_{k-\frac52})^2h+\frac{1}{2a}(\eta^1_{k+\frac12}+\eta^0_{k+\frac12})(a\tau\delta_+e^{\frac12}_k)h\\
&+\frac{1}{2a}\langle\eta^1+\eta^0,\varepsilon^1+\varepsilon^0\rangle+(\theta^1+\theta^0,e^{\frac12}).
\end{aligned}\tag{4.13}$$

For $n\ge2$, we get that

$$\begin{aligned}
B_2\equiv&-\frac{1}{2a}(\eta^n_{k+\frac12}+\eta^{n-1}_{k+\frac12})(a\delta_+\theta^n_k+a\delta_+\theta^{n-1}_k)h-\sum_{i=1,i\neq k}^{I-1}\frac{1}{2a}(\eta^n_{i+\frac12}+\eta^{n-1}_{i+\frac12})(\varepsilon^n_{i+\frac12}+\varepsilon^{n-1}_{i+\frac12})h\\
=&\frac{1}{2a}(\eta^n_{k+\frac12}+\eta^{n-1}_{k+\frac12})^2h+\frac{r}{4a}(\eta^n_{k+\frac12}+\eta^{n-1}_{k+\frac12})^2h-\frac{r}{4a}(\eta^{n-1}_{k+\frac12}+\eta^{n-2}_{k+\frac12})^2h+\frac{r}{4a}(\eta^n_{k+\frac12}-\eta^{n-2}_{k+\frac12})^2h\\
&+\frac{r}{4a}(\eta^n_{k+\frac12}+\eta^{n-1}_{k+\frac12})(\eta^n_{k+\frac32}+\eta^{n-1}_{k+\frac32})h+\frac{r}{4a}(\eta^n_{k+\frac12}+\eta^{n-1}_{k+\frac12})(\eta^n_{k-\frac12}+\eta^{n-1}_{k-\frac12})h\\
&-\frac{r}{4a}(\eta^n_{k+\frac12}+\eta^{n-1}_{k+\frac12})(\eta^{n-1}_{k+\frac32}+\eta^{n-1}_{k-\frac12})h-\frac{r}{4a}(\eta^n_{k+\frac12}+\eta^{n-1}_{k+\frac12})(\eta^{n-2}_{k+\frac32}+\eta^{n-2}_{k-\frac12})h
\end{aligned}$$

https://doi.org/10.4208/aamm.2015.m1049 Published online by Cambridge University Press

$$-\frac{r}{2a}\left(a\tau\check{L}\left(\eta_{k+\frac{3}{2}}^{n-1}+\eta_{k-\frac{1}{2}}^{n-1}\right)\right)\left(\eta_{k+\frac{1}{2}}^{n}+\eta_{k+\frac{1}{2}}^{n-1}\right)h-\frac{r}{2a}\left(a\tau\check{L}\left(\eta_{k+\frac{3}{2}}^{n-2}+\eta_{k-\frac{1}{2}}^{n-2}\right)\right)\left(\eta_{k+\frac{1}{2}}^{n}+\eta_{k+\frac{1}{2}}^{n-1}\right)h$$
$$-\frac{1}{2a}\left(\eta_{k+\frac{1}{2}}^{n}+\eta_{k+\frac{1}{2}}^{n-1}\right)\left(a\tau\delta_{+}\left(e_{k}^{n-\frac{1}{2}}+e_{k}^{n-\frac{3}{2}}\right)\right)h-\frac{1}{2a}\langle\eta^{n}+\eta^{n-1},\varepsilon^{n}+\varepsilon^{n-1}\rangle. \tag{4.14}$$

From (4.9) and (4.14), Eq. (4.8) can be simplified as that for $n\geq 2$

$$\begin{aligned}
&\frac{\|\theta^{n}\|^{2}-\|\theta^{n-1}\|^{2}}{\tau}+\sum_{i=1}^{I}\frac{1}{2}(\delta_{-}\eta_{i+\frac{1}{2}}^{n}+\delta_{-}\eta_{i+\frac{1}{2}}^{n-1})(\theta_{i}^{n}+\theta_{i}^{n-1})h\\
=&\frac{\|\theta^{n}\|^{2}-\|\theta^{n-1}\|^{2}}{\tau}+\sum_{i=1,i\neq k}^{I-1}\frac{1}{2a}\left(\eta_{i+\frac{1}{2}}^{n}+\eta_{i+\frac{1}{2}}^{n-1}\right)^{2}h\\
&-\frac{1}{2a}\left(\eta_{k+\frac{1}{2}}^{n}+\eta_{k+\frac{1}{2}}^{n-1}\right)\left(a\delta_{+}\theta_{k}^{n}+a\delta_{+}\theta_{k}^{n-1}\right)h-\sum_{i=1,i\neq k}^{I-1}\frac{1}{2a}\left(\eta_{i+\frac{1}{2}}^{n}+\eta_{i+\frac{1}{2}}^{n-1}\right)\left(\varepsilon_{i+\frac{1}{2}}^{n}+\varepsilon_{i+\frac{1}{2}}^{n-1}\right)\\
=&\frac{\|\theta^{n}\|^{2}-\|\theta^{n-1}\|^{2}}{\tau}+\frac{1}{2a}\||\eta^{n}+\eta^{n-1}|\|^{2}+\frac{r}{4a}\left(\eta_{k+\frac{1}{2}}^{n}-\eta_{k+\frac{1}{2}}^{n-2}\right)^{2}h\\
&+\frac{r}{4a}\left(\eta_{k+\frac{1}{2}}^{n}+\eta_{k+\frac{1}{2}}^{n-1}\right)^{2}h+\frac{r}{4a}\left(\eta_{k+\frac{1}{2}}^{n}+\eta_{k+\frac{1}{2}}^{n-1}\right)\left(\eta_{k+\frac{3}{2}}^{n}+\eta_{k+\frac{3}{2}}^{n-1}\right)h\\
&+\frac{r}{4a}\left(\eta_{k+\frac{1}{2}}^{n}+\eta_{k+\frac{1}{2}}^{n-1}\right)\left(\eta_{k-\frac{1}{2}}^{n}+\eta_{k-\frac{1}{2}}^{n-1}\right)h-\frac{r}{4a}\left(\eta_{k+\frac{1}{2}}^{n-1}+\eta_{k+\frac{1}{2}}^{n-2}\right)^{2}h\\
&-\frac{r}{4a}\left(\eta_{k+\frac{1}{2}}^{n}+\eta_{k+\frac{1}{2}}^{n-1}\right)\left(\eta_{k+\frac{3}{2}}^{n-1}+\eta_{k+\frac{3}{2}}^{n-2}\right)h-\frac{r}{4a}\left(\eta_{k+\frac{1}{2}}^{n}+\eta_{k+\frac{1}{2}}^{n-1}\right)\left(\eta_{k-\frac{1}{2}}^{n-1}+\eta_{k-\frac{1}{2}}^{n-2}\right)h\\
&-\frac{r}{2a}\left(a\tau\check{L}\left(\eta_{k+\frac{3}{2}}^{n-2}+\eta_{k-\frac{1}{2}}^{n-2}\right)\right)\left(\eta_{k+\frac{1}{2}}^{n}+\eta_{k+\frac{1}{2}}^{n-1}\right)h-\frac{r}{2a}\left(a\tau\check{L}\left(\eta_{k+\frac{3}{2}}^{n-1}+\eta_{k-\frac{1}{2}}^{n-1}\right)\right)\left(\eta_{k+\frac{1}{2}}^{n}+\eta_{k+\frac{1}{2}}^{n-1}\right)h\\
&-\frac{1}{2a}\left(\eta_{k+\frac{1}{2}}^{n}+\eta_{k+\frac{1}{2}}^{n-1}\right)\left(a\tau\delta_{+}\left(e_{k}^{n-\frac{1}{2}}+e_{k}^{n-\frac{3}{2}}\right)\right)h-\frac{1}{2a}\langle\eta^{n}+\eta^{n-1},\varepsilon^{n}+\varepsilon^{n-1}\rangle.
\end{aligned}\tag{4.15}$$

Further, we can get that

$$\begin{aligned}
&\frac{\|\theta^{n}\|^{2}-\|\theta^{n-1}\|^{2}}{\tau}+\frac{1}{2a}\||\eta^{n}+\eta^{n-1}|\|^{2}+\frac{r}{4a}\left(\eta_{k+\frac{1}{2}}^{n}-\eta_{k+\frac{1}{2}}^{n-2}\right)^{2}h\\
&+\frac{r}{4a}\left(\eta_{k+\frac{1}{2}}^{n}+\eta_{k+\frac{1}{2}}^{n-1}\right)^{2}h+\frac{r}{4a}\left(\eta_{k+\frac{1}{2}}^{n}+\eta_{k+\frac{1}{2}}^{n-1}\right)\left(\eta_{k+\frac{3}{2}}^{n}+\eta_{k+\frac{3}{2}}^{n-1}\right)h\\
&+\frac{r}{4a}\left(\eta_{k+\frac{1}{2}}^{n}+\eta_{k+\frac{1}{2}}^{n-1}\right)\left(\eta_{k-\frac{1}{2}}^{n}+\eta_{k-\frac{1}{2}}^{n-1}\right)h-\frac{r}{2a}\left(a\tau\check{L}\left(\eta_{k+\frac{3}{2}}^{n-2}+\eta_{k-\frac{1}{2}}^{n-2}\right)\right)\\
&\left(\eta_{k+\frac{1}{2}}^{n}+\eta_{k+\frac{1}{2}}^{n-1}\right)h-\frac{r}{2a}\left(a\tau\check{L}\left(\eta_{k+\frac{3}{2}}^{n-1}+\eta_{k-\frac{1}{2}}^{n-1}\right)\right)\left(\eta_{k+\frac{1}{2}}^{n}+\eta_{k+\frac{1}{2}}^{n-1}\right)h\\
&-\frac{1}{2a}\left(\eta_{k+\frac{1}{2}}^{n}+\eta_{k+\frac{1}{2}}^{n-1}\right)\left(a\tau\delta_{+}\left(e_{k}^{n-\frac{1}{2}}+e_{k}^{n-\frac{3}{2}}\right)\right)h-\frac{1}{2a}\langle\eta^{n}+\eta^{n-1},\varepsilon^{n}+\varepsilon^{n-1}\rangle\\
=&\frac{r}{4a}\left(\eta_{k+\frac{1}{2}}^{n-1}+\eta_{k+\frac{1}{2}}^{n-2}\right)^{2}h+\frac{r}{4a}\left(\eta_{k+\frac{1}{2}}^{n}+\eta_{k+\frac{1}{2}}^{n-1}\right)\left(\eta_{k+\frac{3}{2}}^{n-1}+\eta_{k+\frac{3}{2}}^{n-2}\right)h\\
&+\frac{r}{4a}\left(\eta_{k+\frac{1}{2}}^{n}+\eta_{k+\frac{1}{2}}^{n-1}\right)\left(\eta_{k-\frac{1}{2}}^{n-1}+\eta_{k-\frac{1}{2}}^{n-2}\right)h+\left(\theta^{n}+\theta^{n-1},e^{n-\frac{1}{2}}\right)
\end{aligned}$$

https://doi.org/10.4208/aamm.2015.m1049 Published online by Cambridge University Press

$$\leq \frac{r}{4a}\left(\eta_{k+\frac{1}{2}}^{n-1}+\eta_{k+\frac{1}{2}}^{n-2}\right)^2 h+\frac{r}{8a}\left(\eta_{k+\frac{1}{2}}^{n}+\eta_{k+\frac{1}{2}}^{n-1}\right)^2 h+\frac{r}{4a}\left(\eta_{k+\frac{3}{2}}^{n-1}+\eta_{k+\frac{3}{2}}^{n-2}\right)^2 h$$
$$+\frac{r}{4a}\left(\eta_{k-\frac{1}{2}}^{n-1}+\eta_{k-\frac{1}{2}}^{n-2}\right)^2 h+\left(\theta^n+\theta^{n-1},e^{n-\frac{1}{2}}\right). \tag{4.16}$$

Let

$$\begin{aligned}
F\equiv&\frac{1}{2a}\|\|\eta_{i+\frac{1}{2}}^{n}+\eta_{i+\frac{1}{2}}^{n-1}\|\|^2+\frac{r}{4a}\left(\eta_{k+\frac{1}{2}}^{n}-\eta_{k+\frac{1}{2}}^{n-2}\right)^2 h-\frac{r}{4a}\left(\eta_{k+\frac{1}{2}}^{n}+\eta_{k+\frac{1}{2}}^{n-1}\right)^2 h\\
&+\frac{r}{4a}\left(\eta_{k+\frac{1}{2}}^{n}+\eta_{k+\frac{1}{2}}^{n-1}\right)\left(\eta_{k+\frac{3}{2}}^{n}+\eta_{k+\frac{3}{2}}^{n-1}\right)h+\frac{r}{4a}\left(\eta_{k+\frac{1}{2}}^{n}+\eta_{k+\frac{1}{2}}^{n-1}\right)\left(\eta_{k-\frac{1}{2}}^{n}+\eta_{k-\frac{1}{2}}^{n-1}\right)h\\
&-\frac{r}{2a}\left(a\tau\check{L}\left(\eta_{k+\frac{3}{2}}^{n-1}+\eta_{k-\frac{1}{2}}^{n-1}+\eta_{k+\frac{3}{2}}^{n-2}+\eta_{k-\frac{1}{2}}^{n-2}\right)\right)\left(\eta_{k+\frac{1}{2}}^{n}+\eta_{k+\frac{1}{2}}^{n-1}\right)h\\
&-\frac{1}{2a}\left(\eta_{k+\frac{1}{2}}^{n}+\eta_{k+\frac{1}{2}}^{n-1}\right)\left(a\tau\delta_+\left(e_k^{n-\frac{1}{2}}+e_k^{n-\frac{3}{2}}\right)\right)h-\frac{1}{2a}\left\langle\eta^n+\eta^{n-1},\varepsilon^n+\varepsilon^{n-1}\right\rangle\\
\geq&\sum_{i=1,i\neq k,k\pm1}^{I-1}\frac{1}{2a}\left(\eta_{i+\frac{1}{2}}^{n}+\eta_{i+\frac{1}{2}}^{n-1}\right)^2 h+\frac{2-r}{4a}\left(\eta_{k+\frac{1}{2}}^{n}+\eta_{k+\frac{1}{2}}^{n-1}\right)^2 h\\
&+\frac{2-r}{4a}\left(\eta_{k+\frac{3}{2}}^{n}+\eta_{k+\frac{3}{2}}^{n-1}\right)^2 h+\frac{2-r}{4a}\left(\eta_{k-\frac{1}{2}}^{n}+\eta_{k-\frac{1}{2}}^{n-1}\right)^2 h+\frac{r}{4a}\left(\eta_{k+\frac{1}{2}}^{n}-\eta_{k+\frac{1}{2}}^{n-2}\right)^2 h\\
&-\frac{5r^3\left(\frac{7r}{2}-2\right)^2+r}{4a}\left(\eta_{k+\frac{3}{2}}^{n-1}+\eta_{k+\frac{3}{2}}^{n-2}\right)^2 h-\frac{20r^3(1-2r)^2+r}{4a}\left(\eta_{k+\frac{1}{2}}^{n-1}+\eta_{k+\frac{1}{2}}^{n-2}\right)^2 h\\
&-\frac{5r^3\left(\frac{7r}{2}-2\right)^2+r}{4a}\left(\eta_{k-\frac{1}{2}}^{n-1}+\eta_{k-\frac{1}{2}}^{n-2}\right)^2 h-\frac{5r^5}{16a}\left(\eta_{k+\frac{7}{2}}^{n-1}+\eta_{k+\frac{7}{2}}^{n-2}\right)^2 h\\
&-\frac{5r^3(1-2r)^2}{4a}\left(\eta_{k+\frac{5}{2}}^{n-1}+\eta_{k+\frac{5}{2}}^{n-2}\right)^2 h-\frac{5r^3(1-2r)^2}{4a}\left(\eta_{k-\frac{3}{2}}^{n-1}+\eta_{k-\frac{3}{2}}^{n-2}\right)^2 h\\
&-\frac{5r^5}{16a}\left(\eta_{k-\frac{5}{2}}^{n-1}+\eta_{k-\frac{5}{2}}^{n-2}\right)^2 h-\frac{1}{2a}\left(\eta_{k+\frac{1}{2}}^{n}+\eta_{k+\frac{1}{2}}^{n-1}\right)\left(a\tau\delta_+\left(e_k^{n-\frac{1}{2}}+e_k^{n-\frac{3}{2}}\right)\right)h\\
&-\frac{1}{2a}\left\langle\eta^n+\eta^{n-1},\varepsilon^n+\varepsilon^{n-1}\right\rangle\\
\geq&\sum_{i=1,i\neq k,k\pm1}^{I-1}\frac{1}{2a}\left(\eta_{i+\frac{1}{2}}^{n}+\eta_{i+\frac{1}{2}}^{n-1}\right)^2 h+\frac{2-r}{4a}\left(\eta_{k+\frac{1}{2}}^{n}+\eta_{k+\frac{1}{2}}^{n-1}\right)^2 h\\
&+\frac{2-r}{4a}\left(\eta_{k+\frac{3}{2}}^{n}+\eta_{k+\frac{3}{2}}^{n-1}\right)^{n-1}\right)^2 h+\frac{2-r}{4a}\left(\eta_{k-\frac{1}{2}}^{n}+\eta_{k-\frac{1}{2}}^{n-1}\right)^2 h+\frac{r}{4a}\left(\eta_{k+\frac{1}{2}}^{n}-\eta_{k+\frac{1}{2}}^{n-2}\right)^2 h\\
&-D_1\left(\eta_{k+\frac{3}{2}}^{n-1}+\eta_{k+\frac{3}{2}}^{n-2}\right)^2 h-D_1\left(\eta_{k+\frac{1}{2}}^{n-1}+\eta_{k+\frac{1}{2}}^{n-2}\right)^2 h-D_1\left(\eta_{k-\frac{1}{2}}^{n-1}+\eta_{k-\frac{1}{2}}^{n-2}\right)^2 h\\
&-D_2\left(\eta_{k+\frac{7}{2}}^{n-1}+\eta_{k+\frac{7}{2}}^{n-2}\right)^2 h-D_2\left(\eta_{k+\frac{5}{2}}^{n-1}+\eta_{k+\frac{5}{2}}^{n-2}\right)^2 h-D_2\left(\eta_{k-\frac{3}{2}}^{n-1}+\eta_{k-\frac{3}{2}}^{n-2}\right)^2 h\\
&-D_2\left(\eta_{k-\frac{5}{2}}^{n-1}+\eta_{k-\frac{5}{2}}^{n-2}\right)^2 h-\frac{1}{2a}\left(\eta_{k+\frac{1}{2}}^{n}+\eta_{k+\frac{1}{2}}^{n-1}\right)\left(a\tau\delta_+\left(e_k^{n-\frac{1}{2}}+e_k^{n-\frac{3}{2}}\right)\right)h\\
&-\frac{1}{2a}\left\langle\eta^n+\eta^{n-1},\varepsilon^n+\varepsilon^{n-1}\right\rangle
\end{aligned}$$

https://doi.org/10.4208/aamm.2015.m1049 Published online by Cambridge University Press

$$
\begin{aligned}
&\geq D\||\eta^n+\eta^{n-1}|\|^2-\frac{1}{8a\epsilon}\big(a\tau\delta_+\big(e_k^{n-\frac{1}{2}}+e_k^{n-\frac{3}{2}}\big)\big)^2h-\frac{1}{8a\epsilon}\||\varepsilon^n+\varepsilon^{n-1}|\|^2\\
&\quad+D_1\big(\eta^n_{k+\frac{3}{2}}+\eta^{n-1}_{k+\frac{3}{2}}\big)^2h+D_1\big(\eta^n_{k+\frac{1}{2}}+\eta^{n-1}_{k+\frac{1}{2}}\big)^2h\\
&\quad+D_1\big(\eta^n_{k-\frac{1}{2}}+\eta^{n-1}_{k-\frac{1}{2}}\big)^2h+D_2\big(\eta^n_{k+\frac{7}{2}}+\eta^{n-1}_{k+\frac{7}{2}}\big)^2h+D_2\big(\eta^n_{k+\frac{5}{2}}+\eta^{n-1}_{k+\frac{5}{2}}\big)^2h\\
&\quad+D_2\big(\eta^n_{k-\frac{3}{2}}+\eta^{n-1}_{k-\frac{3}{2}}\big)^2h+D_2\big(\eta^n_{k-\frac{5}{2}}+\eta^{n-1}_{k-\frac{5}{2}}\big)^2h-D_1\big(\eta^{n-1}_{k+\frac{3}{2}}+\eta^{n-2}_{k+\frac{3}{2}}\big)^2h\\
&\quad-D_1\big(\eta^{n-1}_{k+\frac{1}{2}}+\eta^{n-2}_{k+\frac{1}{2}}\big)^2h-D_1\big(\eta^{n-1}_{k-\frac{1}{2}}+\eta^{n-2}_{k-\frac{1}{2}}\big)^2h-D_2\big(\eta^{n-1}_{k+\frac{7}{2}}+\eta^{n-2}_{k+\frac{7}{2}}\big)^2h\\
&\quad-D_2\big(\eta^{n-1}_{k+\frac{5}{2}}+\eta^{n-2}_{k+\frac{5}{2}}\big)^2h-D_2\big(\eta^{n-1}_{k-\frac{3}{2}}+\eta^{n-2}_{k-\frac{3}{2}}\big)^2h-D_2\big(\eta^{n-1}_{k-\frac{5}{2}}+\eta^{n-2}_{k-\frac{5}{2}}\big)^2h,
\end{aligned}
\tag{4.17}
$$

where $D=1/8a$ is a constant, independently of h and τ and $\epsilon=(\frac{7}{20a}-D_1)/4$ is small positive constant. Substituting (4.17) into (4.16), we can get that

$$
\begin{aligned}
&\frac{\|\theta^n\|^2-\|\theta^{n-1}\|^2}{\tau}+D\||\eta^n+\eta^{n-1}|\|^2+D_1\big(\eta^n_{k+\frac{3}{2}}+\eta^{n-1}_{k+\frac{3}{2}}\big)^2h+D_1\big(\eta^n_{k+\frac{1}{2}}+\eta^{n-1}_{k+\frac{1}{2}}\big)^2h\\
&+D_1\big(\eta^n_{k-\frac{1}{2}}+\eta^{n-1}_{k-\frac{1}{2}}\big)^2h-D_1\big(\eta^{n-1}_{k+\frac{3}{2}}+\eta^{n-2}_{k+\frac{3}{2}}\big)^2h-D_1\big(\eta^{n-1}_{k+\frac{1}{2}}+\eta^{n-2}_{k+\frac{1}{2}}\big)^2h\\
&-D_1\big(\eta^{n-1}_{k-\frac{1}{2}}+\eta^{n-2}_{k-\frac{1}{2}}\big)^2h+D_2\big(\eta^n_{k+\frac{7}{2}}+\eta^{n-1}_{k+\frac{7}{2}}\big)^2h+D_2\big(\eta^n_{k+\frac{5}{2}}+\eta^{n-1}_{k+\frac{5}{2}}\big)^2h\\
&+D_2\big(\eta^n_{k-\frac{3}{2}}+\eta^{n-1}_{k-\frac{3}{2}}\big)^2h+D_2\big(\eta^n_{k-\frac{5}{2}}+\eta^{n-1}_{k-\frac{5}{2}}\big)^2h-D_2\big(\eta^{n-1}_{k+\frac{7}{2}}+\eta^{n-2}_{k+\frac{7}{2}}\big)^2h\\
&-D_2\big(\eta^{n-1}_{k+\frac{5}{2}}+\eta^{n-2}_{k+\frac{5}{2}}\big)^2h-D_2\big(\eta^{n-1}_{k-\frac{3}{2}}+\eta^{n-2}_{k-\frac{3}{2}}\big)^2h-D_2\big(\eta^{n-1}_{k-\frac{5}{2}}+\eta^{n-2}_{k-\frac{5}{2}}\big)^2h\\
&\leq\big(\theta^n+\theta^{n-1},e^{n-\frac{1}{2}}\big)+\frac{1}{8a\epsilon}\big(a\tau\delta_+\big(e_k^{n-\frac{1}{2}}+e_k^{n-\frac{3}{2}}\big)\big)^2h+\frac{1}{8a\epsilon}\||\varepsilon^n+\varepsilon^{n-1}|\|^2\\
&\leq\|\theta^n\|^2+\|\theta^{n-1}\|^2+\frac{1}{2}\|e^{n-\frac{1}{2}}\|^2+\frac{r\tau}{4\epsilon}\big(|e_k^{n-\frac{1}{2}}|^2+|e_k^{n-\frac{3}{2}}|^2+|e_{k+1}^{n-\frac{1}{2}}|^2+|e_{k+1}^{n-\frac{3}{2}}|^2\big)h\\
&\quad+\frac{1}{4a\epsilon}\big(\||\varepsilon^n|\|^2+\||\varepsilon^{n-1}|\|^2\big).
\end{aligned}
\tag{4.18}
$$

Dealing with (4.13) as same as (4.18), we can get that

$$
\begin{aligned}
&\frac{\|\theta^1\|^2-\|\theta^0\|^2}{\tau}+\sum_{i=1,i\neq k,i\neq k\pm1}^{I-1}\frac{1-\varepsilon}{2a}\big(\eta^1_{i+\frac{1}{2}}+\eta^0_{i+\frac{1}{2}}\big)^2h+\frac{2-r-2\varepsilon}{4a}\big(\eta^1_{k+\frac{1}{2}}+\eta^0_{k+\frac{1}{2}}\big)^2h\\
&+\frac{2-r-\varepsilon}{4a}\big(\eta^1_{k+\frac{3}{2}}+\eta^0_{k+\frac{3}{2}}\big)^2h+\frac{2-r-\varepsilon}{4a}\big(\eta^1_{k-\frac{1}{2}}+\eta^0_{k-\frac{1}{2}}\big)^2h\\
&\leq\frac{8}{a}\|\eta^0\|^2+\big(\theta^1+\theta^0,e^{\frac{1}{2}}\big)+\frac{1}{8a\epsilon}\big(a\tau\delta_+e_k^{\frac{1}{2}}\big)^2h+\frac{1}{8a\epsilon}\||\varepsilon^1+\varepsilon^0|\|^2\\
&\leq\|\theta^1\|^2+\|\theta^0\|^2+\frac{1}{2}\|e^{\frac{1}{2}}\|^2+\frac{r\tau}{4\epsilon}\big(|e_k^{\frac{1}{2}}|^2+|e_{k+1}^{\frac{1}{2}}|^2\big)h+\frac{1}{4a\epsilon}\big(\||\varepsilon^1|\|^2+\||\varepsilon^0|\|^2\big)+\frac{8}{a}\|\eta^0\|^2.
\end{aligned}
\tag{4.19}
$$

Thus, we have that

$$
\frac{\|\theta^1\|^2-\|\theta^0\|^2}{\tau}+D\||\eta^1+\eta^0|\|^2+D_1\big(\eta^1_{k+\frac{3}{2}}+\eta^0_{k+\frac{3}{2}}\big)^2h
$$

https://doi.org/10.4208/aamm.2015.m1049 Published online by Cambridge University Press

$$+D_1\big(\eta^1_{k+\frac{1}{2}}+\eta^0_{k+\frac{1}{2}}\big)^2h+D_1\big(\eta^1_{k-\frac{1}{2}}+\eta^0_{k-\frac{1}{2}}\big)^2h+D_2\big(\eta^1_{k+\frac{7}{2}}+\eta^0_{k+\frac{7}{2}}\big)^2h$$
$$+D_2\big(\eta^1_{k+\frac{5}{2}}+\eta^0_{k+\frac{5}{2}}\big)^2h+D_2\big(\eta^1_{k-\frac{3}{2}}+\eta^0_{k-\frac{3}{2}}\big)^2h+D_2\big(\eta^1_{k-\frac{5}{2}}+\eta^0_{k-\frac{5}{2}}\big)^2h$$
$$\le\|\theta^1\|^2+\|\theta^0\|^2+\frac{1}{2}\|e^{\frac{1}{2}}\|^2+\frac{\tau}{4\epsilon}\|e^{\frac{1}{2}}\|^2+\Big(\frac{1}{4a\epsilon}+\frac{8}{a}\Big)\big(|||\varepsilon^1|||^2+|||\varepsilon^0|||^2\big). \tag{4.20}$$

Multiplying τ and summing up from $n=1$ to M and substituting (4.18) and (4.20) into (4.9), we have that

$$\|\theta^n\|^2+D\sum_{n=1}^{M}|||\eta^n+\eta^{n-1}|||^2\tau$$
$$\le 2\sum_{n=0}^{M}\|\theta^n\|^2\tau+\Big(\frac{1}{2}+\frac{\tau}{4\varepsilon}\Big)\sum_{n=0}^{M}\|e^n\|^2\tau+\Big(\frac{1}{4a\varepsilon}+\frac{8}{a}\Big)\sum_{n=0}^{M}\big(|||\varepsilon^n|||^2\big)\tau. \tag{4.21}$$

By discrete Gronwall Lemma and satisfy the condition of Theorem 3.2, we can obtain that

$$\max_{1\le n\le M}\|\theta^n\|+D\Big(\sum_{n=1}^{M}|||\eta^n+\eta^{n-1}|||^2\tau\Big)^{\frac{1}{2}}\le\mathcal{O}(\tau^2+h^2). \tag{4.22}$$

This ends the proof of the theorem. □

Remark 4.1. In the section, we have proven that the error estimates of the numerical solutions and numerical fluxes are second order in time step under the stable condition $r\le 3/5$.

5 Numerical experiment

In this section, we give the numerical experiments of the second-order mass-conserved domain decompostion scheme. We consider the parabolic equations as follows,

$$\begin{cases}\dfrac{\partial c}{\partial t}-a\dfrac{\partial^2 c}{\partial x^2}=0, & x\in[0,1],\quad t\in[0,T],\\ \dfrac{\partial c}{\partial x}\Big|_{x=0}=0,\quad \dfrac{\partial c}{\partial x}\Big|_{x=1}=0, & t\in[0,T],\\ c_0=\cos(\pi x), & x\in[0,1].\end{cases} \tag{5.1}$$

The domain $[0,1]$ is divided into the two sub-sections $[0,0.5]$ and $[0.5,1]$. We focus on the study of mass conservation and convergence of proposed scheme.

Define

$$e_h=\max_{1\le n\le M}\|C^n-c^n\|,\quad \gamma_h=\Big(\sum_{n=1}^{M}|||Q^n-q^n|||^2\tau\Big)^{\frac{1}{2}}, \tag{5.2a}$$
$$\check{e}_h=\max_{1\le n\le M}\|C^n-c^n\|_\infty=\max_{1\le n\le M}\max_{1\le i\le I}|C_i^n-c_i^n|, \tag{5.2b}$$
$$\check{\gamma}_h=\max_{1\le n\le M}\||Q-q|\|_\infty=\max_{1\le n\le M}\max_{1\le i\le I-1}|Q_i^n-q_i^n|. \tag{5.2c}$$

https://doi.org/10.4208/aamm.2015.m1049 Published online by Cambridge University Press

Table 1: Mass errors of the mass-conserved domain decomposition scheme.

τ	a	e_h	$\check{e}_h$	γ_h	$\check{\gamma}_h$	Mass Errors
$\frac{1}{100}$	0.03	3.1981E-4	4.5097E-4	4.4525E-5	9.5676E-5	6.4393E-16
	0.01	1.2991E-4	1.8316E-4	1.9683E-5	3.2179E-5	8.3267E-16
	0.003	4.1760E-5	5.8875E-5	6.5497E-6	9.6775E-6	3.6082E-16
$\frac{1}{50}$	0.03	3.1928E-4	4.5041E-4	4.4237E-6	9.4047E-6	8.7153E-16
	0.01	1.2988E-4	1.8314E-4	1.9650E-5	3.2045E-6	6.6613E-16
	0.003	4.1759E-5	5.8875E-5	6.5466E-6	9.6677E-6	7.7716E-16
$\frac{1}{20}$	0.03	3.1553E-4	4.4655E-4	4.3181E-5	9.0594E-5	2.5535E-16
	0.01	1.2972E-4	1.8298E-4	1.9537E-5	3.1542E-5	3.9968E-16
	0.003	4.1754E-5	5.8870E-5	6.5371E-6	9.6344E-6	1.8874E-16

Table 2: The space orders of the mass-conserved domain decomposition scheme.

h	1/20	1/40	1/60	1/80
e_h	5.3493E-4	1.3371E-4	5.9426E-5	3.3427E-5
Ratio	-	2.0002	2.0000	2.0000
$\check{e}_h$	7.5417E-4	1.8895E-4	8.4013E-5	4.7264E-5
Ratio	-	1.9969	1.9990	1.9995
γ_h	9.1906E-5	2.2980E-5	1.0214E-5	5.7453E-6
Ratio	-	1.9998	1.9998	2.0000
$\check{\gamma}_h$	3.2283-004	8.0727E-5	3.5880E-5	2.0183E-5
Ratio	-	1.9997	1.9998	1.9999

In Table 1, we examine the mass conservation with different steps at $T=1$. Take $h=1/20$, $N=1/h$. Set

$$\text{Mass}_0=\sum_{i=1}^{N}C_i^0 h, \tag{5.3a}$$

$$\text{Mass}_k=\sum_{i=1}^{N}C_i^k h, \quad \text{MassError}=|\text{Mass}_\text{k}-\text{Mass}_0|, \tag{5.3b}$$

where $k=1,2,\cdots,M$. Mass Error denotes the mass difference between the initial and some state. We will observe the variations of Mass-Error by selecting different diffusion coefficients a.

In Table 1, it is clear that the time second-order mass-conserved domain decomposition scheme satisfies mass conversation with different time steps at $T=1$. Meanwhile, we can see that the accuracy of the scheme is better when the diffusion coefficient decreases.

In Table 2, we verify the spatial-orders of the time second-order mass-conserved domain decomposition scheme. In order to avoid the the effect of time step length τ, we take τ is sufficiently small. Take $T=1$, $\tau=1/20000$, $a=0.1$.

We can see clearly that the spatial-orders of the solutions and fluxes both are second order in the L_2-norm in Table 2. Moreover, the spatial orders of the solutions and fluxes in the L_∞-norm are also second order.

https://doi.org/10.4208/aamm.2015.m1049 Published online by Cambridge University Press

Table 3: The time orders of the mass-conserved domain decomposition scheme.

τ	1/20	1/40	1/60	1/80	1/100
e_h	6.8215E-5	1.7065E-5	7.5853E-6	4.2670E-6	2.7309E-6
Ratio	-	1.9990	1.9997	1.9998	2.0000
$\check{e}_h$	9.6186E-5	2.4116E-5	1.0724E-5	6.0334E-6	3.8617E-6
Ratio	-	1.9958	1.9987	1.9993	1.9996
γ_h	1.0561E-5	2.6462E-6	1.1769E-6	6.6225E-7	4.2393E-7
Ratio	-	1.9968	1.9983	1.9987	1.9990
$\check{\gamma}_h$	1.5985E-5	4.0135E-6	1.7870E-6	1.0064E-6	6.4452E-7
Ratio	-	1.9938	1.9955	1.9958	1.9971

In Table 3, we will test the time-order of the new proposed scheme. Because the scheme exists the stability constraint, we cann't assume the space step h be enough small. We take that $\tau = h$ and let diffusion coefficient be small to meet the stability condition. Take $T = 1$, $\tau = h$ and $a = 0.005$.

We can see clearly that the errors of the solutions and fluxes become smaller and smaller as the time step decreases. The time-orders of the solutions and fluxes are second-order in both the L_2-norm and the L_∞-norm in Table 3.

6 Conclusions

In the paper, a new time second-order mass-conserved implicit-explicit domain decomposition scheme for the parabolic equations with Neumann boundary conditions was developed. The important feature is that the developed scheme satisfies mass conservation and is of second-order accuracy in both time step and space step. Meanwhile, because the scheme based on block-centered finite differences, the convergence order of the numerical fluxes was proved to have the super convergence that is same as the accuracy of the numerical solutions. The scheme can be extended to use in applications in engineering, which improves the accuracy in time and keeps mass conservation.

Acknowledgements

The authors would like to thank referees for their suggestions which have helped to improve the paper. D. Liang was supported partially by Natural Sciences and Engineering Research Council of Canada and by the National Natural Science Foundation of China under grant 11271232.

References

[1] K. AZIZ AND A. SETTARI, Petroleum Reservoir Simulation, Applied Science Publisher, Ltd., London, 1979.

https://doi.org/10.4208/aamm.2015.m1049 Published online by Cambridge University Press

[2] J. BEAR, Hydraulics of Groundwater, McGraw-Hill, Nwe York, 1978.
[3] D. AMITAI, A. AVERBUCH, M. ISRAELI AND S. ITZIKOWITZ, *Implicit-explicit parallel asynchronous solver of parabolic PDEs,* SIAM J. Sci. Comput., 19 (1998), pp. 1366–1404.
[4] C. N. DAWSON, Q. DU AND T. F. DUPONT, *A finite difference domain decomposition algorithm for numerical solution of the heat equation,* Math. Comput., 57 (1991), pp. 63–71.
[5] M. DRYJA AND X. TU, *A domain decomposition discretization of parabolic problems*, Numerische Mathematik, 107 (2007), pp. 625–640.
[6] Q. DU, M. MU AND Z. WU, *Efficient parallel algorithms for parbolic problems,* SIAM J. Numer. Anal., 39 (2001), pp. 1469–1487.
[7] C. DU AND D. LIANG, *An efficient S-DDM iterative approach for compressible contamination fluid flows in porous media,* J. Comput. Phys., 229 (2010), pp. 4501–4521.
[8] D. EVANS AND A. ABDULLAH, *Group explicit methods for parabolic equations,* Int. J. Comput. Math., 14 (1983), pp. 73–105.
[9] S. GAIFFE, R. GLOWINSKI AND R. LEMONNIER, *Domain decomposition and splitting methods for parabolic equations via a mixed formula,* in the 12th International Conference on Domain Decomposition, Chiba, Japan, 1999.
[10] Y. KUZNETSOV, *New algorithms for approximate realization of implicit difference scheme,* Soviet J. Numer. Anal. Math. Model., 3 (1988), pp. 99–114.
[11] D. LIANG AND C. DU, *The efficient S-DDM scheme and its analysis for solving parabolic equations,* I. Comput. Phys., 272 (2014), pp. 46–69.
[12] H. SHI AND H. LIAO, *Unconditional stability of corrected explicit-implicit domain decomposition algorithms for parallel approximation of heat equations,* SIAM J. Numer. Anal., 44 (2006), pp. 1584–1611.
[13] S. ZHU, *Conservative domain decomposition procedure with unconditional stability and second-order accuracy*, Appl. Math. Comput., 216 (2010), pp. 3275–3282.

https://doi.org/10.4208/aamm.2015.m1049 Published online by Cambridge University Press