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Generating partitions for random transformations

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Abstract. We prove a relative version of Krieger's Theorem constructing relative generating partitions with cardinality estimated via the relative entropy. We introduce relative asymptotically entropy expansive random transformations and construct for them relative topological generators with cardinality estimated via the relative topological entropy, which enables us to represent such systems by means of random symbolic subshifts.

0. *Introduction*

The Kolmogorov–Sinai Theorem says that the entropy of a measure-preserving transformation is equal to its entropy with respect to any generating partition. Rohlin showed in [**Ro**] that any ergodic transformation T of finite entropy has a countable generating partition whose entropy is arbitrarily close to the entropy of T. Krieger proved (see [**Kr**] and [**DGS**]) that there exist finite generators with such properties and if the entropy of T is less than $\log \ell$, then there exists a generator of cardinality ℓ.

In this paper we study generating partitions in the relative ergodic theory setup. Here T is an invertible skew product transformation acting by $T(\omega, x) = (\vartheta\omega, T_\omega x)$ where ϑ is an invertible ergodic map of a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and T_ω, often called a random transformation, maps a Polish space X into itself. The relative measure entropy of T was first studied in [**AR**] under the name 'mixed entropy of the fiber of a skew product' where it was obtained as a difference of entropies of T and ϑ, which makes sense when these entropies are finite; but in many interesting cases both these entropies are infinite though the relative entropy is still finite. Later, a general setup was considered and other notions of the relativized ergodic theory such as the relative topological entropy and pressure were introduced and studied (see [**LW, Ki1, Bog, Ki2**]). A recent account of many other results on random transformations can be found in [**Ar**].

A relative version of the Kolmogorov–Sinai Theorem is not difficult to prove (see, for instance, [**Ki1**] and [**Bog**]), but the construction of relative generating partitions has not appeared in the literature so far. Of course, if the usual entropy of T is finite then

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Rohlin's countable and Krieger's finite generators will serve as relativized generators as well. However, the more precise analogs of Krieger's result which we prove in this paper, namely, that if T is ergodic then there is a relative generating partition with ℓ elements provided that the relative entropy is less than $\log \ell$, cannot be obtained in this way whenever the usual entropy of ϑ is larger than $\log 2$. Moreover, if the usual entropy of T is infinite although the relative entropy is finite the known results cannot be employed at all. Actually, we prove a truly relative result which complements Krieger's Theorem so that both together yield the general theorem on skew products. We stress that our relative results do not require extra assumptions on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which would undermine the probabilistic philosophy behind random transformations. On the other hand, Rohlin's Countable and Krieger's Finite Generators Theorems rely on separability and they are valid only for Lebesgue spaces.

The finite generator of the Relative Krieger Theorem provides a coding of the system for one ergodic invariant measure. In this paper we will also be interested in relative topological generators $\mathcal{Q}$, which are not only generating partitions for all T-invariant measures with the marginal $\mathbb{P}$ on Ω, but also that $\mathcal{Q}$ separate orbits of T_ω in the sense that $\mathbb{P}$-almost surely (a.s.) for any $x, y \in X$, $x \neq y$ there exists $n \in \mathbb{Z}$ such that $T^n(\omega, x)$ and $T^n(\omega, y)$ belong to different elements of $\mathcal{Q}$. We introduce the notion of asymptotically entropy expansive random transformations, which include as very particular cases expansive random transformations and those which have zero relative topological entropy. Assuming that ϑ is aperiodic, X is compact, T_ω are homeomorphisms and that the relative topological entropy of T is less than $\log \ell$ for an integer $\ell > 1$, we construct for asymptotically entropy expansive random transformations relative topological generators $\mathcal{Q}$ with ℓ elements. Actually, we construct such partitions in a more general situation when T is defined on a T-invariant set $\mathcal{E} \subset \Omega \times X$ with compact fibers $\mathcal{E}_\omega$, which are mapped by T_ω homeomorphically. This result does not have a counterpart in the usual deterministic theory without strong additional assumptions since, for instance, the only generator for the identity transformation is the partition into points. The relative generating partitions described above enable us to represent the system as a random symbolic shift.

The structure of this paper is the following. In the next section we formulate precisely our setup and state our main results. In §2 we derive a relative version of Krieger's Theorem. In §3 we study relative versions of entropy expansiveness, asymptotic entropy expansiveness and conditional topological entropy, which were introduced in the deterministic case in [**Bow**] and [**Mi**]. In §4 we construct relative topological generators as described above.

1. *Preliminaries and main results*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space together with a $\mathbb{P}$-preserving ergodic invertible transformation ϑ and $(X, \mathcal{B})$ be a Polish space with the Borel σ-algebra $\mathcal{B}$. Let $\mathcal{E}$ be a measurable, with respect to the product σ-algebra $\mathcal{F} \times \mathcal{B}$, subset of $\Omega \times X$ and T be a measurable invertible map of $\mathcal{E}$ to itself acting by $T(\omega, x) = (\vartheta\omega, T_\omega x)$. The family $\{T_\omega, \omega \in \Omega\}$ is called a random transformation and each T_ω maps the fiber $\mathcal{E}_\omega = \{x \in X : (\omega, x) \in \mathcal{E}\}$ to $\mathcal{E}_{\vartheta\omega}$.

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Denote by $\mathcal{P}_{\mathbb{P}}(\Omega \times X)$ the space of probability measures on $\Omega \times X$ having the marginal $\mathbb{P}$ on Ω and set $\mathcal{P}_{\mathbb{P}}(\mathcal{E}) = \{\mu \in \mathcal{P}_{\mathbb{P}}(\Omega \times X) : \mu(\mathcal{E}) = 1\}$. Any $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E})$ on $\mathcal{E}$ factorizes as $d\mu(\omega, x) = d\mu_\omega(x)\, d\mathbb{P}(\omega)$ (see, for instance, [**Du**, §10.2]) where μ_ω are regular conditional probabilities with respect to the σ-algebra $\mathcal{F}_{\mathcal{E}}$ formed by all sets $(A \times X) \cap \mathcal{E}$ with $A \in \mathcal{F}$. This means that μ_ω is a probability measure on $\mathcal{E}_\omega$ for $\mathbb{P}$-a.a. ω and for any measurable set $R \subset \mathcal{E}$, $\mathbb{P}$-a.s. $\mu_\omega(R_\omega) = \mu(R|\mathcal{F}_{\mathcal{E}})$, where $R_\omega = \{x : (\omega, x) \in R\}$, and so $\mu(R) = \int \mu_\omega(R_\omega)\, d\mathbb{P}(\omega)$. It is easy to see that $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E})$ is T-invariant if and only if $T_\omega \mu_\omega = \mu_{\vartheta\omega}$ $\mathbb{P}$-a.s. and the space of such measures will be denoted by $\mathcal{I}_{\mathbb{P}}(\mathcal{E})$. Observe that $T^n(\omega, x) = (\vartheta^n\omega, T^n_\omega x)$, where $T^n_\omega = T_{\vartheta^{n-1}\omega} \circ \cdots \circ T_{\vartheta\omega} \circ T_\omega$ for $n \geq 0$, $T^0_\omega = \mathrm{id}$, and $T^n_\omega = T^{-1}_{\vartheta^{-n}\omega} \circ \cdots \circ T^{-1}_{\vartheta^{-2}\omega} \circ T^{-1}_{\vartheta^{-1}\omega}$ for $n \leq -1$.

Let $\mathcal{R} = \{R_i\}$ be a finite or countable partition of $\mathcal{E}$ into measurable sets and $\mathcal{R}(\omega) = \{R_i(\omega)\}$, where $R_i(\omega) = \{x \in \mathcal{E}_\omega : (\omega, x) \in R_i\}$ is a partition of $\mathcal{E}_\omega$. The conditional entropy of $\mathcal{R}$ given σ-algebra $\mathcal{F}_{\mathcal{E}}$ is defined by

$$H_\mu(\mathcal{R}|\mathcal{F}_{\mathcal{E}}) = -\int \sum_i \mu(R_i|\mathcal{F}_{\mathcal{E}}) \log \mu(R_i|\mathcal{F}_{\mathcal{E}})\, d\mathbb{P} = \int H_{\mu_\omega}(\mathcal{R}(\omega))\, d\mathbb{P}(\omega) \tag{1.1}$$

where $H_{\mu_\omega}(\mathcal{A})$ denotes the usual entropy of a partition $\mathcal{A}$. The relative (also called *fiber*) entropy $h^{(r)}_\mu(T)$ of T with respect to $\mu \in \mathcal{I}_{\mathbb{P}}(\mathcal{E})$ is defined by the formula

$$h^{(r)}_\mu(T) = \sup_{\mathcal{Q}} h^{(r)}_\mu(T, \mathcal{Q}) \tag{1.2}$$

where

$$h^{(r)}_\mu(T, \mathcal{Q}) = \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{Q} \,\middle|\, \mathcal{F}_{\mathcal{E}}\right) \tag{1.3}$$

is called the relative entropy of a partition $\mathcal{Q}$ and $\bigvee$ denotes the join of partitions. The supremum in (1.2) is taken over all finite or countable measurable partitions $\mathcal{Q} = \{Q_i\}$ of $\mathcal{E}$ with finite conditional entropy

$$H_\mu(\mathcal{Q}|\mathcal{F}_{\mathcal{E}}) < \infty, \tag{1.4}$$

and the $\mathbb{P}$-a.s. limit in (1.3) exists in view of subadditivity of the conditional entropy (see [**Ro**, §5] or [**Ki1**, §2.1]).

Observe that if $\mathcal{Q} = \{Q_i\}$ is a partition of $\mathcal{E}$, then $\mathcal{Q}^{(n)} = \bigvee_{i=0}^{n-1} (T^i)^{-1}\mathcal{Q}$ is a partition of $\mathcal{E}$ consisting of sets $\{Q^{(n)}_j\}$ such that the corresponding partition $\mathcal{Q}^{(n)}(\omega) = \{Q^{(n)}_j(\omega)\}$, $Q^{(n)}_j(\omega) = \{x : (\omega, x) \in Q^{(n)}_j\}$ of $\mathcal{E}_\omega$ has the form $\mathcal{Q}^{(n)}(\omega) = \bigvee_{i=0}^{n-1} (T^i_\omega)^{-1}\mathcal{Q}(\vartheta^i\omega)$, where $\mathcal{Q}(\omega) = \{Q_j(\omega)\}$, $Q_j(\omega) = \{x \in \mathcal{E}_\omega : (\omega, x) \in Q_j\}$ partitions $\mathcal{E}_\omega$. This together with (1.1) yields

$$h^{(r)}_\mu(T, \mathcal{Q}) = \lim_{n\to\infty} \frac{1}{n} \int H_{\mu_\omega}\left(\bigvee_{i=0}^{n-1} (T^i_\omega)^{-1}\mathcal{Q}(\vartheta^i\omega)\right) d\mathbb{P}(\omega). \tag{1.5}$$

This way of obtaining the fiber entropy is more in line with the spirit of random dynamical systems. It is not difficult to see (see [**Ki1**, §2.1] and [**Bog**]) that the resulting entropy remains the same taking the supremum in (1.2) only over partitions $\mathcal{Q}$ of $\mathcal{E}$ into sets Q_i of the form $Q_i = (\Omega \times P_i) \cap \mathcal{E}$, where $\mathcal{P} = \{P_i\}$ is a partition of X into measurable

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sets, so that $Q_i(\omega) = P_i \cap \mathcal{E}_\omega$. The Abramov–Rohlin formula $h_\mu(T) = h_\mu^{(r)}(T) + h_{\mathbb{P}}(\vartheta)$ (see [**AR**]), where $h_\mu(T)$ and $h_{\mathbb{P}}(\vartheta)$ are usual entropies of the corresponding measure preserving transformations, relates different entropies emerging in this setup, although in many interesting cases only $h_\mu^{(r)}(T)$ is finite among these three (see [**Ki1**, Theorem II.1.2]). It is easy to see that $a_n(\omega) = H_{\mu_\omega}(\bigvee_{i=0}^{n-1}(T_\omega^i)^{-1}\mathcal{Q}(\vartheta^i\omega))$ is a subadditive process and (1.5) together with the subadditive ergodic theorem (see, for instance, [**Ki1**, §A.2]) yields $h_\mu^{(r)}(T, \mathcal{Q}) = \int \lim_{n\to\infty} n^{-1}a_n \, d\mathbb{P}$. If, in addition, $\mathbb{P}$ is ergodic then this formula remains true $\mathbb{P}$-a.s. without integrating against $\mathbb{P}$.

Definition 1.1. A countable or finite measurable partition $\mathcal{Q}_\mathcal{E}$ of $\mathcal{E}$ is called a relative (or fiber) μ-generator for $\mu \in \mathcal{I}_\mathbb{P}(\mathcal{E})$ if $\mathbb{P}$-a.s. $\bigvee_{i=-\infty}^{\infty}(T_\omega^i)^{-1}\mathcal{Q}_\mathcal{E}(\vartheta^i\omega)$ generates the restriction of the σ-algebra $\mathcal{B}$ to $\mathcal{E}_\omega$ up to sets of μ_ω-measure zero. If $\mathcal{Q}_\mathcal{E} = \{Q_i\}$ with $\mathcal{Q}_\mathcal{E}(\omega) = \{Q_i\}$ and $-\infty \le m \le n \le \infty$, then a sequence $(\xi_m, \xi_{m+1}, \dots, \xi_n)$ will be called the $(\omega, m, n, \mathcal{Q})$-name of $x \in \mathcal{E}_\omega$ if $T_\omega^i x \in Q_{\xi_i}(\vartheta^i\omega)$ for all $i = m, m+1, \dots, n$. A $(\omega, -\infty, \infty, \mathcal{Q})$-name will just be called a $(\omega, \mathcal{Q})$-name.

In light of this definition, it is clear that $\mathcal{Q}$ is a relative μ-generator if there exists a measurable set $\tilde{\mathcal{E}} \subset \mathcal{E}$ with $\mu(\tilde{\mathcal{E}}) = 1$, such that any $x, y \in \tilde{\mathcal{E}}_\omega = \{z \in \mathcal{E}_\omega : (\omega, z) \in \tilde{\mathcal{E}}\}$, $x \ne y$ have different $(\omega, \mathcal{Q})$-names. If $\mathcal{Q}$ is a relative μ-generator satisfying (1.4), then using general properties of the conditional entropy it is not difficult to derive a corresponding version of the Kolmogorov–Sinai Theorem (cf. [**Ki1**, Lemma II.1.5], [**Bog**, Theorem 2.4]) saying that $h_\mu^{(r)}(T) = h_\mu^{(r)}(T, \mathcal{Q})$. As usual, we say that ϑ is aperiodic (with respect to $\mathbb{P}$) if $\mathbb{P}\{\omega \in \Omega : \exists n \ \vartheta^n\omega = \omega\} = 0$. We will prove in the next section the following relative version of Krieger's Theorem [**Kr**] (see also [**DGS**]).

THEOREM 1.2. *Suppose that $\mu \in \mathcal{I}_\mathbb{P}(\mathcal{E})$ is ergodic, ϑ is aperiodic and $h_\mu^{(r)}(T) < \log \ell$ for some integer $\ell > 1$. Then there exists a relative μ-generator $\mathcal{Q}$ consisting of ℓ sets.*

The aperiodicity assumption on ϑ means that our random transformations are, indeed, random, at least mildly. If $(\Omega, \mathcal{F}, \mathbb{P})$ is, in addition, a Lebesgue space (which is not a very natural assumption from the probabilistic point of view), then we do not have to assume aperiodicity of ϑ and in this case Theorem 1.2 is a generalization of Krieger's Theorem. Indeed, since ϑ is ergodic it is either aperiodic or purely periodic. In the former case we use Theorem 1.2 and in the latter case $h_\mathbb{P}(\vartheta) = 0$, and so by [**AR**], $h_\mu(T) = h_\mu^{(r)}(T) < \log \ell$. Then Theorem 1.2 follows from Krieger's original result, since, of course, any absolute generating partition for T is also its relative generating partition. Under the Lebesgue space assumption on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ there is another proof of Theorem 1.2 based on orbit equivalence considerations suggested by A. Danilenko following the talk by the first author about this work at a conference in Toruń in September 2000. We discuss these arguments at the end of the next section.

For each $x \in \mathcal{E}_\omega$ set $\phi_\omega(x) = \xi = \{\xi_i\}_{i\in\mathbb{Z}} \in \{1, 2, \dots, \ell\}^\mathbb{Z}$ provided $T^j(\omega, x) \in Q_{\xi_j}(\vartheta^j\omega)$ for all $j \in \mathbb{Z}$. This yields the map ϕ_ω from $\mathcal{E}_\omega$ to the sequence space $\Sigma(\omega) = \phi_\omega\mathcal{E}_\omega$ which is injective μ_ω-almost everywhere $\mathbb{P}$-a.s. Let σ be the left shift acting by $(\sigma\xi)_l = \xi_{l+1}$ then

$$\sigma\,\Sigma(\omega) = \Sigma(\vartheta\omega) \quad \text{and} \quad \phi_{\vartheta\omega}T_\omega = \sigma\,\vartheta_\omega, \tag{1.6}$$

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which provides a μ-a.s. random (or relative) symbolic representation of T. Namely, this yields an isomorphism between $(\mathcal{E}, T, \mu)$ and $(\Sigma, \vartheta \times \sigma, \vartheta\mu)$ where $\Sigma = \{(\omega, \xi) : \xi \in \Sigma(\omega)\}$.

Next, assume that X is a compact metric space with a distance function d and all fibers $\mathcal{E}_\omega \subset X$ are compact as well. For any positive random variable $\varepsilon = \varepsilon(\omega)$, $x, y \in \mathcal{E}_\omega$ and $m, n \in \mathbb{Z}$, $m < n$ introduce a family of metrics on each $\mathcal{E}_\omega$ by

$$d^\omega_{\varepsilon;m,n}(x, y) = \max_{m \le k < n} (d(T^k_\omega x, T^k_\omega y)(\varepsilon(\vartheta^k\omega))^{-1})$$

and denote also $d^\omega_{\varepsilon,n} = d^\omega_{\varepsilon;0,n}$, $d^\omega_{\varepsilon,\pm n} = d^\omega_{\varepsilon;-n,n}$. For $F, E \subset \mathcal{E}_\omega$, we say that E (ω, ε, n)-spans F if for any $y \in F$ there is $x \in E$ so that $d^\omega_{\varepsilon,n}(x, y) \le 1$. Denote by $r_n(F, \omega, \varepsilon)$ the minimum cardinality of a set which (ω, ε, n)-spans F. If $K \subset \mathcal{E}_\omega$ is compact then $r_n(K, \omega, \varepsilon) < \infty$.

For any compact subset $Y \subset X$ denote by $N_Y(r)$ the minimal number of closed balls of diameter r which cover Y. Observe that $N_Y(r)$ is non-increasing and right continuous in r and is lower semi-continuous in Y on the space $\mathcal{K}(X)$ of compact subsets of X considered with the Hausdorff topology. It follows that $\{(Y, r) \in \mathcal{K}(X) \times (0, \infty) : N_Y(r) \le n\}$ is a closed set in the product topology of $\mathcal{K}(X) \times (0, \infty)$ for any $n \in \mathbb{N}_+$ and, in particular, $N_Y(r)$ is measurable in the pair (Y, r) with respect to the product Borel σ-algebra. Since for any positive random variable $\varepsilon = \varepsilon(\omega)$ the map $\psi : \Omega \mapsto \mathcal{K}(X) \times (0, \infty)$ defined by $\psi(\omega) = (\mathcal{E}_\omega, \varepsilon(\omega))$ is measurable then $N \circ \psi(\omega) = N_{\mathcal{E}_\omega}(\varepsilon(\omega))$ is measurable. Following [**Ki2**] we say that a positive random variable ε belongs to the class $\mathcal{N}$ and write $\varepsilon \in \mathcal{N}$ if

$$\int \log N_{\mathcal{E}_\omega}(\varepsilon(\omega))\, d\mathbb{P}(\omega) < \infty. \tag{1.7}$$

Since $N_X(r) \ge N_{\mathcal{E}_\omega}(r)$ for all ω this holds true if, in particular, $\int \log N_X(\varepsilon(\omega))\, d\mathbb{P}(\omega) < \infty$. The latter is satisfied if, for instance, the upper Minkowski (box) dimension

$$D_X = \limsup_{r \to 0} \frac{\log N_X(r)}{-\log r}$$

of X is finite and $-\int \log \varepsilon(\omega)\, d\mathbb{P}(\omega) < \infty$. Note also that $\log N_Y(\min(r_1, r_2)) = \max(\log N_Y(r_1), \log N_Y(r_2)) \le \log N_Y(r_1) + \log N_Y(r_2)$, and so if $\varepsilon, \delta \in \mathcal{N}$ then $\min(\varepsilon, \delta) \in \mathcal{N}$. Thus $\mathcal{N}$ is a directed set and, clearly, it also contains all positive constants.

For each compact $K \subset \mathcal{E}_\omega$ and any positive random variable $\varepsilon = \varepsilon(\omega) \in \mathcal{N}$ define

$$r(K, \omega, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log r_n(K, \omega, \varepsilon) \tag{1.8}$$

and

$$h_T(K, \omega) = \lim_{\varepsilon \xrightarrow{\mathcal{N}} 0} r(K, \omega, \varepsilon) \tag{1.9}$$

where the limit is taken over the directed set $\mathcal{N}$ and it exists in view of monotonicity of $r(K, \omega, \varepsilon)$ in ε. Similarly to [**Ki2**] it follows that the limit remains the same if we take it over only constant $\varepsilon \to 0$. Let $B_x(\omega, \varepsilon, n)$, $B_x(\omega, \varepsilon, \pm n)$, and $B_x(\omega, \varepsilon; m, n)$ be the closed ball in $\mathcal{E}_\omega$ centered at x and having the radius 1 with respect to the metrics $d^\omega_{\varepsilon,n}$,

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$d^{\omega}_{\varepsilon,\pm n}$ and $d^{\omega}_{\varepsilon;m,n}$, respectively. For $d^{\omega}_{\varepsilon,1}$ and $B_x(\omega,\varepsilon,1)$ we write simply d^{ω}_{ε} and $B_x(\omega,\varepsilon)$. Set

$$\Gamma_\varepsilon(x,\omega)=\bigcap_{n\ge 1} B_x(\omega,\varepsilon,\pm n)$$

and

$$h^*_T(\omega,\varepsilon)=\sup_{x\in\mathcal{E}_\omega} h_T(\Gamma_\varepsilon(x,\omega),\omega). \tag{1.10}$$

Observe, that

$$T^{-1}_\omega\Gamma_\varepsilon(T_\omega x,\vartheta\omega)=\Gamma_\varepsilon(x,\omega),$$

and so

$$r_n(\Gamma_\varepsilon(x,\omega),\omega,\delta)\ge r_{n-1}(\Gamma_\varepsilon(T_\omega x,\vartheta\omega),\omega,\delta)$$

for any positive random variable δ. Hence

$$h_T(\Gamma_\varepsilon(x,\omega),\omega)\ge h_T(\Gamma_\varepsilon(T_\omega x,\vartheta\omega),\vartheta\omega)$$

and since $T_\omega\mathcal{E}_\omega=\mathcal{E}_{\vartheta\omega}$ we conclude that $\mathbb{P}$-a.s.

$$h^*_T(\omega,\varepsilon)\ge h^*_T(\vartheta\omega,\varepsilon)$$

and by ergodicity of ϑ it follows that $h^*_T(\omega,\varepsilon)=$ constant $\mathbb{P}$-a.s. and from now on it will be denoted by $h^*_T(\varepsilon)$. Observe, that $h^*_T(\varepsilon)$ is the same for all ε such that $\varepsilon(\omega)\ge\operatorname{diam}\mathcal{E}_\omega$ $\mathbb{P}$-a.s. and it coincides then with the relative topological entropy $h^{(r)}_{\mathrm{top}}(T)$ of T. In view of monotonicity of $h^*_T(\varepsilon)$ in ε the limit

$$h^*_T=\lim_{\varepsilon\xrightarrow{\mathcal{N}}0} h^*_T(\varepsilon) \tag{1.11}$$

exists and $h^*_T\le h^{(r)}_{\mathrm{top}}(T)$. Similarly to [**Ki2**] it follows that this limit remains the same if it is taken only over constant $\varepsilon\to 0$.

Definition 1.3. T will be called relatively expansive with an expansivity characteristic $\varepsilon\in\mathcal{N}$ if $\mathbb{P}$-a.s. $\Gamma_\varepsilon(x,\omega)=x$ for all $x\in\mathcal{E}_\omega$. T will be called relatively h-expansive if $h^*_T(\varepsilon)=0$ for some $\varepsilon\in\mathcal{N}$ and T will be called relatively asymptotically h-expansive if $h^*_T=0$.

In the deterministic case any C^∞ diffeomorphism of a compact Riemannian manifold is asymptotically h-expansive (see [**Bu**]). It seems that extending methods of [**Bu**] to the relative situation in the spirit of [**KY**] and §3 below, it should be possible to show that under some integrability assumption on C^∞ norms any random C^∞ diffeomorphism of a compact Riemannian manifold is relatively asymptotically h-expansive.

Definition 1.4. A finite or countable partition $\mathcal{Q}=\{Q_i\}$ of $\mathcal{E}$ into measurable sets Q_i will be called a relative topological generator provided $\mathbb{P}$-a.s. whenever $T^n(\omega,x)$ and $T^n(\omega,y)$ belong to the same element of $\mathcal{Q}$ for all $n\in\mathbb{Z}$, it follows that $x=y$. In other words, $\mathbb{P}$-a.s. any points $x,y\in\mathcal{E}_\omega$, $x\ne y$ have different $(\omega,\mathcal{Q})$-names.

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The emphasis here is that all points $x \neq y$ on the same fiber $\mathcal{E}_\omega$ must be separated. This implies that $\mathcal{Q}$ is a generator for any invariant measure whose marginal on Ω is $\mathbb{P}$ but, in fact, it is a much more stringent requirement which cannot be satisfied in many non-relative situations.

THEOREM 1.5. *Suppose that ϑ is aperiodic, $h_T^* = 0$ and $h_{\text{top}}^{(r)}(T) < \log \ell$ for some integer $\ell > 1$ then there exists a relative topological generator consisting of ℓ sets.*

The ideology of the proofs of Theorems 1.2 and 1.5 are somewhat similar. They consist of the first step which yields ℓ-element partitions of $\mathcal{E}_\omega$ for ω belonging to a set of $\mathbb{P}$-measure close to one. This is done via coding using most but not all levels of Rohlin's towers constructed in Ω. These partitions mostly do not change on further steps and we do modifications mainly on the complement ω-set. The partition constructed on the first step carries already most of the entropy. After that we repeatedly employ the second step which enables us to encode points more and more precisely, i.e. to increase the resolution of partitions, and, on the other hand, (in the case of Theorem 1.5) to make smaller the set of ω for which $\mathcal{E}_\omega$ has been partitioned into more than ℓ subsets. This is done using the few levels of Rohlin's towers whose number can be estimated in the measure case via conditional entropies of refinements of partitions and in the topological case via ε-entropies $h^*(\varepsilon)$.

2. *Relativized Krieger's Theorem*

First, we state the following relative version of the Shannon–McMillan–Breiman Theorem which in a bit less general setup can be found, for instance, in [**Ki1**] and [**Bog**]. For any finite or countable measurable partition $\mathcal{Q}$ of $\mathcal{E}$ with fiber partitions $\mathcal{Q}(\omega)$ of $\mathcal{E}_\omega$, recall that the fibers of the partition $\mathcal{Q}^{(n)} = \bigvee_{i=0}^{n-1}(T^i)^{-1}\mathcal{Q}$ have the form $\mathcal{Q}^{(n)}(\omega) = \bigvee_{i=0}^{n-1}(T_\omega^i)^{-1}\mathcal{Q}(\vartheta^i\omega)$. For any $x \in \mathcal{E}_\omega$ denote by $Q^{(n)}(\omega, x)$ the element of the partition $\mathcal{Q}^{(n)}(\omega)$ which contains x.

THEOREM 2.1. *Suppose that $\mu \in \mathcal{I}_{\mathbb{P}}(\mathcal{E})$ with factorisations $\{\mu_\omega\}$ is ergodic then for any finite or countable measurable partition $\mathcal{Q}$ of $\mathcal{E}$ with $H_\mu(\mathcal{Q}|\mathcal{F}_\mathcal{E}) < \infty$, μ-a.s.*

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_\omega(Q^{(n)}(\omega, x)) = -h_\mu^{(r)}(T, \mathcal{Q}). \tag{2.1}$$

Actually, we will need the following.

COROLLARY 2.2. *Suppose that the conditions of Theorem 2.1 hold true. Let a finite or countable measurable partition $\mathcal{R}$ of $\mathcal{E}$ with $H_\mu(\mathcal{R}|\mathcal{F}_\mathcal{E}) < \infty$ be a refinement of $\mathcal{Q}$ (i.e. each element of $\mathcal{Q}$ is a union of some elements of $\mathcal{R}$) and set $\eta = h_\mu^{(r)}(T, \mathcal{Q})$, $\zeta = h_\mu^{(r)}(T, \mathcal{R})$. Given $\varepsilon > 0$ there is an N_ε and a measurable set $E = E(\varepsilon) \subset \mathcal{E}$ with fibers $E_\omega = \{x : (\omega, x) \in E\}$ such that:*

(i) *$\mathbb{P}(\pi E) > 1 - \varepsilon$ and $\mu_\omega(E_\omega) > 1 - \varepsilon$ $\forall \omega \in \pi E$ where $\pi : \mathcal{E} \to \Omega$ denotes the natural projection, i.e. $\pi E = \{\omega \in \Omega : (\omega, x) \in E \text{ for some } x \in \mathcal{E}_\omega\}$;*

(ii) *if $n \geq N_\varepsilon$ and $(\omega, x) \in E$ then $2^{-n(\eta+\varepsilon)} < \mu_\omega(Q^{(n)}(\omega, x)) < 2^{-n(\eta-\varepsilon)}$ and $2^{-n(\zeta+\varepsilon)} < \mu_\omega(R^{(n)}(\omega, x)) < 2^{-n(\zeta-\varepsilon)}$;*

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(iii) *for any $n \ge N_\varepsilon$ and each $\omega \in \pi E$ the set E_ω is covered by less than $2^{n(\eta+\varepsilon)}$ elements ($2^{n(\zeta+\varepsilon)}$ elements) of $\mathcal{Q}^{(n)}(\omega)$ (of $\mathcal{R}^{(n)}(\omega)$);*

(iv) *for any $n \ge N_\varepsilon$ and each $(\omega, x) \in E$ the set $Q^{(n)}(\omega, x) \cap E_\omega$ is covered by less than $2^{n(\zeta-\eta+\varepsilon)}$ elements of $\mathcal{R}^{(n)}(\omega)$.*

Proof. Set

$$G^{(n)} = \{(\omega, x) \in \mathcal{E} : 2^{-k(\eta+\varepsilon)} < \mu_\omega(Q^{(k)}(\omega, x)) < 2^{-k(\eta-\varepsilon)}\, \forall k \ge n\}$$

and $G^{(n)}_\omega = \{x \in \mathcal{E}_\omega : (\omega, x) \in G^{(n)}\}$. Then $G^{(n)} \uparrow G$ and $G^{(n)}_\omega \uparrow G_\omega$ as $n \uparrow \infty$ and by Theorem 2.1 $\mu(G) = 1$. Hence, by Fubini's Theorem $\mu_\omega(G_\omega) = 1$ $\mathbb{P}$-a.s. It follows that there exists an N_ε such that $\Gamma^{(n)} = \{\omega :\ \mu_\omega(G^{(n)}_\omega) > 1 - \varepsilon/2\}$ satisfies $\mathbb{P}(\Gamma^{(n)}) \ge 1 - \varepsilon/2$ for all $n \ge N_\varepsilon$. Define $E = G^{(N_\varepsilon)} \cap (\Gamma^{(N_\varepsilon)} \times X)$ then

$$\mu(E) = \int_{\Gamma^{(N_\varepsilon)}} \mu_\omega(G^{(n)}_\omega)\, d\mathbb{P}(\omega) > (1 - \varepsilon/2)^2 > 1 - \varepsilon.$$

This proves both (i) and (ii) (repeating the argument for the partition $\mathcal{R}$). For $n \ge N_\varepsilon$ each element Q of $\mathcal{Q}^{(n)}(\omega)$, which intersects $E_\omega = G^{(N_\varepsilon)}_\omega$, satisfies $\mu_\omega(Q) > 2^{-n(\eta+\varepsilon)}$. Since these elements are disjoint, and so their total measure cannot exceed 1 we derive (iii) (repeating the argument for $\mathcal{R}$). The same argument for $\mathcal{Q}$ and $\mathcal{R}$ with $\varepsilon/2$ in place of ε enables us to choose N_ε and to construct the set E so that it satisfies (i) and for any elements $Q \in \mathcal{Q}^{(n)}(\omega)$ and $R \in \mathcal{R}^{(n)}(\omega)$ having non-empty intersections with E_ω,

$$\mu_\omega(Q) < 2^{-n(\eta-\varepsilon/2)} \quad \text{and} \quad \mu_\omega(R) > 2^{-(\zeta+\varepsilon/2)}. \tag{2.2}$$

Now for the first time we use that $\mathcal{R}$ is a refinement of $\mathcal{Q}$, and so each $Q^{(n)}(\omega, x)$ is a disjoint union of some elements $R^{(n)}_i$ of $\mathcal{R}^{(n)}(\omega)$. If $(\omega, x) \in E$ then $Q = Q^{(n)}(\omega, x)$ satisfies the first part of (2.2) and if $Q = \bigcup_i R^{(n)}_i$ then $Q \cap E_\omega \subset \bigcup_i \{R^{(n)}_i : R^{(n)}_i \cap E_\omega \ne \emptyset\}$. All $R^{(n)}_i$ in the last union are disjoint subsets of Q and they satisfy the second part of (2.2) which yields (iv). □

Next, we prove Theorem 1.2. Let $\tilde{\mathcal{Q}}_1 \prec \tilde{\mathcal{Q}}_2 \prec \cdots$ be a sequence of refining finite measurable partitions of X such that $\bigvee_{i=1}^\infty \tilde{\mathcal{Q}}_i$ generates the whole σ-algebra $\mathcal{B}$. Set $\mathcal{Q}_i = (\Omega \times \tilde{\mathcal{Q}}_i) \cap \mathcal{E}$. We can and do choose $\tilde{\mathcal{Q}}_i$ so that

$$h^{(r)}_\mu(T) - h^{(r)}_\mu(T, \mathcal{Q}_i) < \frac{\alpha}{10^i} \tag{2.3}$$

where $\alpha = \log \ell - h^{(r)}_\mu(T) > 0$.

Pick a small $\varepsilon_1 < \alpha/10$ (which will be further specified below) and apply Corollary 2.2 to find for $\mathcal{Q}_1$ and this ε_1 the set $E = E(\varepsilon_1)$ and $N_1 = N_{\varepsilon_1}$ as promised. By the strong Rohlin Lemma (see [**KW**, Theorem 2]) there exists a measurable set $B_1 \subset \pi E$ such that the sets $\vartheta^j B_1$, $j = 0, 1, \dots, N_1$ are pairwise disjoint and $\mathbb{P}\big(\bigcup_{j=0}^{N_1} \vartheta^j B_1\big) > 1 - 2\varepsilon_1$. The collection $\{\vartheta^j B_1\}_0^{N_1}$ is called a Rohlin tower of height N_1 with the base B_1 and the jth level $\vartheta^j B_1$, $j = 0, 1, \dots, N_1$.

Let K_1 be the integral part of $(1 - (9\alpha/10 \log \ell))(N_1 + 1) + 1$ and set $\mathcal{A}_{K_1} = \{1, 2, \dots, \ell\}^{K_1}$. Since we chose $\varepsilon_1 < \alpha/10$ then by Corollary 2.2(iii) each $E_\omega, \omega \in B_1$ is covered by less than $2^{N_1(h^{(r)}_\mu(T)+\varepsilon_1)} < 2^{K_1 \log \ell}$ elements of $\mathcal{Q}_1^{(N_1)}(\omega)$. Thus these elements

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can be encoded by words from $\mathcal{A}_{K_1}$ so that, say, the word $(\ell, \ell, \dots, \ell)$ is left unused. We denote by $Q_{\boldsymbol{a}}$ such an element encoded by $\boldsymbol{a} \in \mathcal{A}_{K_1}$ and we write also $Q_{(\ell,\dots,\ell)}$ to denote the union of elements of $\mathcal{Q}_1^{(N_1)}(\omega)$ which do not intersect E_ω. This encoding can be done measurably in ω. Indeed, each element of $\mathcal{Q}_1^{(n)}$ is determined by a word of length n with letters $1, 2, \dots, \operatorname{card} \mathcal{Q}_1$. When assigning anew words from $\mathcal{A}_{K_1}$ to elements of $\mathcal{Q}_1^{(N_1)}(\omega)$ which intersect E_ω, we keep the same lexicographic order as before which yields a well-defined measurable encoding. For each $\omega \in B_1$, $j = N_1 - K_1 + 1, \dots, N_1 - 1, N_1$ and $k = 1, 2, \dots, \ell$ define the sets $P_k(\vartheta^j\omega) = \{x \in \mathcal{E}_{\vartheta^j\omega} : (T_\omega^j)^{-1}x \in Q_{\boldsymbol{a}}, \boldsymbol{a} = (a_1, \dots, a_{K_1}), a_{j-N_1+K_1} = k\}$. Now we obtain a partition $\mathcal{P}_1(\omega') = \{P_1(\omega'), \dots, P_\ell(\omega')\}$ of the whole $\mathcal{E}_{\omega'}$ for each $\omega' \in D_1 \stackrel{\text{def}}{=} \bigcup_{N_1-K_1<j\le N_1} \vartheta^j B_1$. For $\omega \in \Omega \setminus D_1$ we define $\mathcal{P}_1(\omega)$ as the one-element partition of $\mathcal{E}_\omega$. In view of our choices we also have that $\mathbb{P}(\Omega \setminus D_1) > 4\alpha/5 \log \ell$.

Observe, that for any $\omega' \in \vartheta^j B_1$ with $\vartheta^{-j}\omega' = \omega$ and each $y \in T_\omega^j E_\omega$ the $(\omega', -j, N_1 - j, \mathcal{P}_1)$-name of y determines its $(\omega', -j, N_1 - j, \mathcal{Q}_1)$-name. Hence, on the set $\tilde{E}_1 = \bigcup_{j=0}^{N_1} T^j(E \cap (B_1 \times X))$ with $\mu(\tilde{E}_1) > (1 - \varepsilon_1)(1 - 2\varepsilon_1) > 1 - 3\varepsilon_1$ the partition $\mathcal{P}_1^{(-N_1,N_1)} = \bigvee_{i=-N_1}^{N_1} T^i\mathcal{P}_1$ refines $\mathcal{Q}_1$. Taking into account that

$$h_\mu^{(r)}(T, \mathcal{P}_1^{(-N_1,N_1)}) = h_\mu^{(r)}(T, \mathcal{P}_1)$$

and

$$h_\mu^{(r)}(T) - \frac{\alpha}{10} < h_\mu^{(r)}(T, \mathcal{Q}_1) \le h_\mu^{(r)}(T, \mathcal{P}_1^{(-N_1,N_1)}) + \int H_{\mu_\omega}(\mathcal{Q}_1|\mathcal{P}_1^{(-N_1,N_1)})\, d\mathcal{P}(\omega) \tag{2.4}$$

(cf. [**Ki1**] and [**Bog**]) we easily derive that if ε_1 has been chosen small enough (which we assume!) then

$$|h_\mu^{(r)}(T, \mathcal{P}_1) - h_\mu^{(r)}(T, \mathcal{Q}_1)| < \frac{\alpha}{10}. \tag{2.5}$$

Next, we describe the second step of our construction. We choose small $\varepsilon_2 < \alpha/10^2$ and by Corollary 2.2(iv), applied to $\mathcal{R} = \mathcal{Q}_2 \vee \mathcal{P}_1$ and $\mathcal{Q} = \mathcal{P}_1$, we obtain $N_2 = N_{\varepsilon_2}$ and $E = E(\varepsilon_2)$ as promised. Actually, we take N_2 so large that the ergodic theorem enables us to find a measurable set $C \subset \Omega$ with $\mathbb{P}(C) > 1 - \varepsilon_2$ so that for any $\omega \in C$,

$$\frac{1}{N_2} \sum_{i=0}^{N_2} \mathbb{I}_{\Omega\setminus D_1}(\vartheta^i\omega) > \frac{4\alpha}{5 \log \ell}. \tag{2.6}$$

Now we construct a Rohlin tower of height N_2 with a base $B_2 \subset \pi E \cap C$ and such that $\mathbb{P}\big(\bigcup_{j=0}^{N_2} \vartheta^j B_2\big) > 1 - 2\varepsilon_2$. By Corollary 2.2(iv) each $Q^{(N_2)}(\omega, x) \cap E_\omega$ is covered by less than $2^{(\alpha/4)N_2}$ elements of $\mathcal{R}^{(N_2)}(\omega)$. We encode these elements by words from $\mathcal{A}_r = \{1, \dots, \ell\}^r$ with $r = [\alpha N_2/4 \log \ell] + 1$ and we do so by partitioning $E_{\vartheta^j\omega}$ into ℓ sets on levels j belonging to a set $J_\omega \subset \{j : \vartheta^j\omega \in \Omega \setminus D_1\}$ of cardinality r. Namely, to each such element $R^{(N_2)}(\omega, x)$, $x \in E_\omega$ we assign (measurably in ω) a word $\boldsymbol{a} \in \mathcal{A}_r$ and denote this element by $R_{\boldsymbol{a}}$ reserving the notation $R_{\ell,\dots,\ell}$ for the union of elements of $\mathcal{R}^{(N_2)}(\omega)$ which do not intersect E_ω. Now, as on the first step, for $j \in J_\omega$ and $k = 1, 2, \dots, \ell$ we set $P_k(\vartheta^j\omega) = \{x \in \mathcal{E}_{\vartheta^j\omega} : (T_\omega^j)^{-1}x \in R_{\boldsymbol{a}}, \boldsymbol{a} = (a_1, \dots, a_\ell), a_j = k\}$.

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Next, we define a new partition $\mathcal{P}_2(\omega') = \{P_1(\omega'), \dots, P_\ell(\omega')\}$ of $\mathcal{E}_{\omega'}$ into ℓ subsets for all $\omega' \in D_2 \stackrel{\text{def}}{=} \cup\{\vartheta^j B_2 : 0 \le j \le N_2$ and either $\vartheta^j\omega \in D_1$ or $j \in J_\omega\}$ so that it coincides with $\mathcal{P}_1(\omega')$ for $\omega' \in D_1$. For $\omega \in D_1 \setminus D_2$ we put $\mathcal{P}_2(\omega) = \mathcal{P}_1(\omega)$ and for $\omega \notin D_1 \cup D_2$ we define $\mathcal{P}_2(\omega)$ as the one-element partition of $\mathcal{E}_\omega$. Similarly to the first step the partition $\mathcal{P}_2^{(-N_2,N_2)} = \bigvee_{i=-N_2}^{N_2} T^i\mathcal{P}_2$ refines $\mathcal{Q}_2$ on a set $\tilde{E}_2$ of μ-measure greater than $1 - 3\varepsilon_2$, and so we can and do choose ε_2 so small that

$$|h_\mu^{(r)}(T, \mathcal{P}_2) - h_\mu^{(r)}(T, \mathcal{Q}_2)| < \frac{\alpha}{10^2}. \tag{2.7}$$

Repeatedly applying this second step with $\varepsilon_i < \varepsilon_{i-1}/10$, we obtain a sequence of integers $N_k = N_{\varepsilon_k}$, a sequence of sets $\tilde{E}_k$ with $\mu(\tilde{E}_k) > 1 - 3\varepsilon_k$ and a sequence of partitions $\mathcal{P}_1 \prec \mathcal{P}_2 \prec \cdots$ into at most ℓ measurable subsets such that

$$|h_\mu^{(r)}(T, \mathcal{P}_k) - h_\mu^{(r)}(T, \mathcal{Q}_k)| < \frac{\alpha}{10^k}. \tag{2.8}$$

and $\mathcal{P}_k^{(-N_k,N_k)} = \bigvee_{i=-N_k}^{N_k} T^i\mathcal{P}_k$ refines $\mathcal{Q}_k$ on the set $\tilde{E}_k$. A simple computation shows that our choices guarantee that on each step k we have still enough levels of the corresponding N_k-tower with the one-element partition to proceed with our refining encoding. Finally, it is clear that the partition $\mathcal{P} = \bigvee_{k=1}^{\infty} \mathcal{P}_k$ is the required relative μ-generator, completing the proof of Theorem 1.2. □

As mentioned in the previous section, after hearing the ideas of the proof of Theorem 1.2 in a talk by the first author, A. Danilenko suggested an alternate proof which is based on an orbit equivalence of ϑ to a $\mathbb{P}$-preserving invertible ergodic transformation ϑ' with $h_{\mathbb{P}}(\vartheta') = 0$, which holds true assuming that $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space (see [**Dy**]). If $\vartheta'(\omega) = \vartheta^{k(\omega)}\omega$ then defining

$$T'(\omega, x) = (\vartheta^{k(\omega)}\omega, T_\omega^{k(\omega)}x)$$

we obtain from [**AR**] together with [**RW**] that

$$h_\mu(T') = h_\mu^{(r)}(T') = h_\mu^{(r)}(T).$$

Thus $h_\mu(T') < \log \ell$ and applying the original Krieger Theorem we obtain an absolute generating partition $\mathcal{P}$ for T' with ℓ elements. The proof concludes by the crucial observation that while this partition may not be an absolute generator for T it is always a relative generator for T. This follows, since knowing ω and the $\mathcal{P}$-name of (ω, x) with respect to T also gives us the $\mathcal{P}$-name of (ω, x) with respect to T' (the orbit is $\mathcal{F}$-measurable) and this, in turn, determines (ω, x) uniquely.

Although brief, this proof relies on several highly non-trivial theorems including Krieger's Theorem itself. On the other hand, our proof of the relative theorem is simpler than the proof of the absolute result since no care needs to be taken to encode the bases of Rohlin towers. Another advantage of our approach is that essentially the same technique works for the construction of topological generators in Theorem 1.5, where the orbit equivalence considerations do not help. Finally, the above orbit equivalence arguments require that $(\Omega, \mathcal{F}, \mathbb{P})$ be a Lebesgue space, whereas in our proof the random mechanism is quite arbitrary and, in particular, this probability space can be non-separable, which is natural from the probabilistic point of view.

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3. *Relative topological entropy-like characteristics*

We start this section with the following result, which will be crucial in the proof of Theorem 1.5 and which is a relative version of Proposition 2.2 from [**Bow**].

PROPOSITION 3.1. *Let $\varepsilon \in \mathcal{N}$ and $a = h_T^*(\varepsilon)$. Then for any $\beta > 0$ and a random variable $\delta > 0$ with $\frac{1}{2}\delta \in \mathcal{N}$ there exists a random variable $C = C(\omega)$ such that*

$$r_n(B_x(\omega, \varepsilon, n), \omega, \delta) \leq C(\omega)C(\vartheta^n\omega)e^{(a+\beta)n} \tag{3.1}$$

for all $x \in \mathcal{E}_\omega$.

Proof. Since $\mathbb{P}$-a.s.

$$h_T^*(\varepsilon) \geq h_T(\Gamma_\varepsilon(y, \omega), \omega) \geq r(\Gamma_\varepsilon(y, \omega), \omega, \gamma)$$

for any $y \in \mathcal{E}_\omega$ and $\gamma \in \mathcal{N}$ then $\mathbb{P}$-a.s. for each $y \in \mathcal{E}_\omega$ we have that $\mathbb{P}$-a.s.

$$m(y, \omega) = \min\left\{m : a + \beta \geq \frac{1}{m}\log r_m\left(\Gamma_\varepsilon(y, \omega), \omega, \frac{1}{4}\delta\right)\right\} + 1 < \infty.$$

Using the technique of measurable multifunctions (see [**CV**]) we can choose finite sets $E(y, \omega)$ similarly to [**Ki2**] depending measurably in ω and having cardinality $r_m(\Gamma_\varepsilon(y, \omega), \omega, \frac{1}{4}\delta)$ which $(\omega, \frac{1}{4}\delta, m(y, \omega))$-span $\Gamma_\varepsilon(y, \omega)$. Set

$$U(y, \omega) = \bigcup_{z \in E(y,\omega)} \operatorname{int}(B_z(\omega, \tfrac{1}{2}\delta, m(\omega, y)))$$

where int denotes the interior. Now, $U(y, \omega)$ is an open neighborhood of the compact set $\Gamma_\varepsilon(y, \omega)$. Since $B_y(\omega, \varepsilon, \pm n)$ is a decreasing as $n \to \infty$ chain of compact sets with intersection $\Gamma_\varepsilon(y, \omega)$, we can define random variables

$$N(y, \omega) = \min\{n \geq 1 : B_y(\omega, \varepsilon, \pm n) \subset U(y, \omega)\} + 1.$$

Observe, that

$$\bigcap_{\kappa > 0} B_y(\omega, \varepsilon + \kappa, \pm N(y, \omega)) = B_y(\omega, \varepsilon, \pm N(\omega, y)) \subset U(y, \omega),$$

and so we can define a random variable $\kappa < \infty$ $\mathbb{P}$-a.s. by

$$\kappa(\omega) = \min\{n \geq 1 : B_y(\omega, \varepsilon + 1/n, \pm N(y, \omega)) \subset U(y, \omega)\}.$$

Set $V(y, \omega) = \operatorname{int} B_y(\omega, 1/\kappa, \pm N(y, \omega))$ then $B_{T_\omega^j u}(\omega, \varepsilon) \subset B_{T_\omega^j y}(\omega, \varepsilon + 1/\kappa)$ for any $u \in V(y, \omega)$, and $|j| < N(y, \omega)$, and so for each $u \in V(y, \omega)$,

$$B_u(\omega, \varepsilon, \pm N(y, \omega)) \subset U(y, \omega).$$

Next, we can choose points $y_1(\omega), \ldots, y_{s(\omega)}(\omega) \in \mathcal{E}_\omega$ in a measurable in ω way so that

$$\bigcup_{1 \leq i \leq s(\omega)} V(y_i(\omega), \omega) = \mathcal{E}_\omega.$$

Set

$$N(\omega) = \max\{N(y_1(\omega), \omega), \ldots, N(y_{s(\omega)}(\omega), \omega), m(y_1(\omega), \omega), \ldots, m(y_{s(\omega)}(\omega), \omega)\} + 1.$$

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Take an arbitrary $x \in \mathcal{E}_\omega$. For any $t, n \in \mathbb{Z}$,

$$T^t_\omega B_x(\omega, \varepsilon, n) = B_{T^t_\omega x}(\vartheta^t\omega, \varepsilon; -t, n-t)$$

and if $t \in [N(\vartheta^t\omega), n - N(\vartheta^t\omega)]$ then

$$B_{T^t_\omega x}(\vartheta^t\omega, \varepsilon; -t, n-t) \subset B_{T^t_\omega x}(\vartheta^t\omega, \varepsilon; \pm N(\vartheta^t\omega)) \subset U(y_i(\vartheta^t\omega), \vartheta^t\omega)$$

if $T^t_\omega x \in V(y_i(\vartheta^t\omega), \vartheta^t\omega)$. Now $E(y_i(\vartheta^t\omega), \vartheta^t\omega)$ $(\vartheta^t\omega, \frac{1}{2}\delta, m(y_i(\vartheta^t\omega), \vartheta^t\omega))$-spans $U(y_i(\vartheta^t\omega), \vartheta^t\omega)$ and also $T^t_\omega B_x(\omega, \varepsilon, n)$.

Let $\gamma_M = \{\omega : N(\omega) \le M\}$. If $\vartheta^t\omega \in \Gamma_M$ and $t \in [M, n-M]$, then we obtain that $E(y_i(\vartheta^t\omega), \vartheta^t\omega)$ $(\vartheta^t\omega, \frac{1}{2}\delta, m(y_i(\vartheta^t\omega), \vartheta^t\omega))$-spans $T^t_\omega B_x(\omega, \varepsilon, n)$. Now define $0 = t_0(\omega) < t_1(\omega) < \cdots < t_{r(\omega)}(\omega) = n$ in the following way. Set $s_1(\omega) = \min\{l \ge M : \vartheta^l\omega \in \Gamma_M\}$. If $s_1(\omega) \ge n$ then we put $r(\omega) = 1$. If $s_1(\omega) < n$, then we put $t_1(\omega) = s_1(\omega)$ and we pick $V(y_i(\vartheta^{t_1}\omega), \vartheta^{t_1}\omega)$ containing $T^{t_1}_\omega x$. Suppose we have chosen $s_1(\omega), \dots, s_k(\omega), t_1(\omega), \dots, t_k(\omega)$ and $y_{i_1}(\vartheta^{t_1}\omega), \dots, y_{i_k}(\vartheta^{t_k}\omega)$ with $t_k(\omega) < n$. Set

$$s_{k+1}(\omega) = \min\{l \ge t_k(\omega) + m(y_i(\vartheta^{t_k}\omega), (\vartheta^{t_k}\omega)) : \vartheta^l \in \Gamma_M\}.$$

If $s_{k+1}(\omega) > n - M$ we set $r(\omega) = k+1$ and $t_r(\omega) = n$. If $s_{k+1}(\omega) \le n - M$ then set $t_{k+1}(\omega) = s_{k+1}(\omega)$ and choose $V(y_i(\vartheta^{t_{k+1}}\omega), \vartheta^{t_{k+1}}\omega)$ containing $T^{t_{k+1}}_\omega x$. Eventually this process stops. Set $q(\omega) = \min\{l \ge M : \vartheta^{-l}\omega \in \Gamma_M\}$,

$$\tilde{R}(\omega) = \prod_{i=-q(\omega)+1}^{t_1(\omega)-1} N_{\mathcal{E}_{\vartheta^i\omega}}(\tfrac{1}{2}\delta(\vartheta^i\omega)),$$

and

$$Q_n(\omega) = \prod_{i:\vartheta^i\omega\notin\Gamma_M} N_{\mathcal{E}_{\vartheta^i\omega}}(\tfrac{1}{2}\delta(\vartheta^i\omega)).$$

It is easy to see from the construction that

$$\begin{aligned} r_n(B_x(\omega, \varepsilon, n), \omega, \delta) &\le \tilde{R}(\omega)\tilde{R}(\vartheta^n\omega)Q_n(\omega) \\ &\quad \times \prod_{0<j<r(\omega)-1} \exp((a + \tfrac{1}{2}\beta)m(y_{i_j}(\vartheta^{t_j(\omega)}\omega), \vartheta^{t_j(\omega)}\omega)) \\ &\le \tilde{R}(\omega)\tilde{R}(\vartheta^n\omega)Q_n(\omega)e^{(a+1/2\beta)n}. \end{aligned} \tag{3.2}$$

Since $\frac{1}{2}\delta \in \mathcal{N}$ then $\mathbb{P}$-a.s.

$$\lim_{n\to\infty} \frac{1}{n} \log Q_n(\omega) = \int_{\Omega\setminus\Gamma_M} \log N_{\mathcal{E}_\omega}\left(\frac{1}{2}\delta(\omega)\right) d\mathbb{P}(\omega) \tag{3.3}$$

and the latter integral tends to zero as $M \to \infty$. Thus, (3.1) follows from (3.2) and (3.3) choosing M large enough. □

Next we will deal with relative versions of notions and results from [**Mi**]. Let $\mathcal{P}(X)$ denote the set of all covers of the space X containing a finite subcover and let $\mathcal{U}(X)$ be the set of all open finite covers of X. For $A \in \mathcal{P}(X)$ we write $A^n_T(\omega) = \bigvee_{k=0}^{n-1}(T^k_\omega)^{-1}A(\vartheta^k\omega)$ where $A(\omega) = A \cap \mathcal{E}_\omega$ is the induced cover of $\mathcal{E}_\omega$ by elements $a \cap \mathcal{E}_\omega$, $a \in A$. For any non-empty set $Y \subset X$ and a cover $A \in \mathcal{P}$ write $N(Y, A) = \min\left\{\text{card}(C) : C \subset A, Y \subset \bigcup C\right\}$

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and $N(\emptyset, A) = 1$. For $A, B \in \mathcal{P}(X)$ we also set $N(A(\omega)|B(\omega)) = \max_{b\in B} N(b \cap \mathcal{E}_\omega, A)$. As usual, for $A, B \in \mathcal{P}(X)$ we also write $A \succ B$ if A is a refinement of B, i.e. elements of B are unions of some elements of A. Similarly to (1.2)–(1.10) in [**Mi**] we have that for any $A, B, C, D \in \mathcal{P}(X)$,

$$N(A(\omega)|B(\omega)) \le N(C(\omega)|D(\omega)) \tag{3.4}$$

provided $C \succ A$ and $B \succ D$,

$$N(T_\omega^{-1}A(\vartheta\omega)|T_\omega^{-1}B(\vartheta\omega)) \le N(A(\vartheta\omega)|B(\vartheta\omega)), \tag{3.5}$$

$$N(A(\omega) \vee B(\omega)|C(\omega)) \le N(A(\omega)|C(\omega))N(B(\omega)|A(\omega) \vee C(\omega)), \tag{3.6}$$

and

$$N(A(\omega) \vee B(\omega)|C(\omega) \vee D(\omega)) \le N(A(\omega)|C(\omega))N(B(\omega)|D(\omega)). \tag{3.7}$$

By (3.5) and (3.7) it is easy to see that the sequence $a_n(\omega) = \log N(A_T^n(\omega)|B_T^n(\omega))$ is subadditive, and so the subadditive ergodic theorem (see, for instance, [**Ki1**]) implies that $\mathbb{P}$-a.s. the limit

$$h_T(A|B) = \lim_{n\to\infty} \frac{1}{n} \log N(A_T^n(\omega)|B_T^n(\omega)) \tag{3.8}$$

exists and is not random. The number $h_T(A|B)$ will be called the relative conditional entropy of T on a cover A with respect to a cover B. If B is the trivial cover and A is an open cover of X then $h_T(A|B)$ is the relative topological entropy $h_{\text{top}}^{(r)}(T, A)$ of T with respect to the cover A, and so by (3.4),

$$h_{\text{top}}^{(r)}(T, A) \ge h_T(A|B) \tag{3.9}$$

for any $B \in \mathcal{P}(X)$.

Observe, that by (3.4),

$$h_T(A|B) \le h_T(C|D) \quad \text{if } C \succ A \text{ and } B \succ D. \tag{3.10}$$

Thus there exists a limit (which may be finite or infinite) over the directed set $\mathcal{U}(X)$,

$$h_T(B) = \lim_{A\in\mathcal{U}(X)} h_T(A|B) = \sup_{A\in\mathcal{U}(X)} h_T(A|B) \tag{3.11}$$

which will be called the relative conditional entropy of T with respect to a cover B. By (3.10),

$$h_T(A) \le h_T(B) \quad \text{for } A \succ B, \tag{3.12}$$

and so we can take the limit once more

$$h_T = \lim_{B\in\mathcal{U}(X)} h_T(B) = \inf_{B\in\mathcal{U}(X)} h_T(B) \tag{3.13}$$

which is called the relative conditional entropy of T. In view of (3.9),

$$h_{\text{top}}^{(r)} \ge h_T. \tag{3.14}$$

The following result connects $h_T^*(\varepsilon)$ studied in the first part of this section with relative conditional entropies $h_T(A)$.

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PROPOSITION 3.2. *Let $\varepsilon \in \mathcal{N}$ and $A, E \in \mathcal{U}(X)$ satisfy* $\operatorname{diam} A(\omega) \le \varepsilon(\omega) < \frac{1}{2}L(\omega)$ *where $L = L(\omega)$ is a random variable such that $L(\omega)$ is a Lebesgue number (see, for instance, [**Ki1**]) of the cover $E(\omega)$. Then*

$$h_T(A) \le h_T^*(\varepsilon) \le h_T(E). \tag{3.15}$$

It follows that T is asymptotically relative h-expansive if and only if $h_T = 0$.

Proof. Let $\delta > 0$ be a number and $D \in \mathcal{U}(X)$ be a cover of X with a Lebesgue number greater than 2δ. Since $\operatorname{diam} A(\omega) \le \varepsilon(\omega)$ then every element of $A_T^n(\omega)$ is contained in some $B_x(\omega, \varepsilon, n)$, and so

$$N(D_T^n(\omega)|A_T^n(\omega)) \le \sup_{x \in \mathcal{E}_\omega} N(B_x(\omega, \varepsilon, n), D^n). \tag{3.16}$$

We claim that

$$N(B_x(\omega, \varepsilon, n), D^n) \le r_n(B_x(\omega, \varepsilon, n), \omega, \delta). \tag{3.17}$$

Indeed, let E be a (ω, δ, n)-spanning set of $B_x(\omega, \varepsilon, n)$ of minimal cardinality. Then $\bigcup_{y \in E} B_y(\omega, \delta, n) \supset B_x(\omega, \varepsilon, n)$. Since 2δ is the Lebesgue number of D then each $B_z(\omega, \delta)$ is contained in some element of $D(\omega)$, and so each $B_y(\omega, \delta, n)$ is contained in some element of $D^n(\omega)$; in particular, this is true for y's from E. Thus, for each $y \in E$ we can choose $d(y) \in D^n(\omega)$ so that $d(y) \supset B_y(\omega, \delta, n)$, and so $\bigcup_{y \in E} d(y) \supset B_x(\omega, \varepsilon, n)$ (although not all $d(y)$'s may be different). Hence, (3.17) follows. Now, (3.16) and (3.17) together with Proposition 3.1 yield

$$N(D_T^n(\omega)|A_T^n(\omega)) \le C(\omega)C(\vartheta^n\omega)e^{(h_T^*(\varepsilon)+\beta)n}. \tag{3.18}$$

Next, taking log, dividing by n and passing to the limit along $n_i = n_i(\omega)$ of successive k such that $\vartheta^k\omega \in \tilde{\Gamma}_L = \{\omega : C(\omega) \le L\}$, we conclude by (3.18) that $\mathbb{P}$-a.s. $h_T(D|A) \le h_T^*(\varepsilon) + \beta$ and as β is arbitrary $h_T(D|A) \le h_T^*(\varepsilon)$. Since δ is arbitrary and D is an arbitrary cover with the Lebesgue number greater than 2δ we conclude that the left-hand side of (3.15) holds true.

In order to derive the second part of (3.15) observe that any $B_x(\omega, \varepsilon, n)$ is contained in some member of $E_T^n(\omega)$ since $\varepsilon < \frac{1}{2}L$. Let A be a minimal (ω, δ, n)-spanning set of $B_x(\omega, \varepsilon, n)$. Then $\bigcup_{z \in A} B_z(\omega, \delta, n) \supset B_x(\omega, \varepsilon, n)$. Let C be a finite open cover of X with $\operatorname{diam} C(\omega) \le \delta(\omega)$. Then every element of $C_T^n(\omega)$ is contained in some $B_z(\omega, \delta, n)$, and so

$$\begin{aligned} N(C_T^n(\omega)|E_T^n(\omega)) &\ge N(B_x(\omega, \varepsilon, n), C^n) \\ &\ge r_n(B_x(\omega, \varepsilon, n), \omega, \delta) \ge r_n(\Gamma_\varepsilon(x, \omega), \omega, \delta). \end{aligned}$$

Taking log, dividing by n and letting $n \to \infty$ we obtain

$$h_T(E) \ge h_T(C|E) \ge r(\Gamma_\varepsilon(x, \omega), \omega, \delta).$$

Letting $\delta \to 0$ along the directed set $\mathcal{N}$ we obtain $h_T(E) \ge h_T(\Gamma_\varepsilon(x, \omega), \omega)$. Taking sup over $x \in \mathcal{E}_\omega$ we arrive at the right-hand side of (3.15). □

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4. *Relative topological generators*

In this section we will prove Theorem 1.5. Set $\alpha = \log \ell - h^{(r)}_{\text{top}}(T)$ and pick small constants $\varepsilon_1 > 0$ and $\eta_1 > 0$ so that

$$h^*_T(2\varepsilon_1) < \frac{\alpha}{10} \quad \text{and} \quad \eta_1 N_X\left(\frac{1}{4}\varepsilon_1\right) < \frac{\alpha}{10} \tag{4.1}$$

where, recall, $N_X(\varepsilon)$ is the minimal number of closed balls of diameter ε needed to cover X. By the definition of the relative topological entropy (see, for instance, [**Ki1, Ki2**]) we can find $L_1 \in \mathbb{N}$ and a measurable set $A_1 \subset \Omega$ with $\mathbb{P}(A_1) > 1 - \eta_1/10$ such that

$$r_{L_1}(\mathcal{E}_\omega, \omega, \tfrac{1}{4}\varepsilon_1) < 2^{(h^{(r)}_{\text{top}}(T)+\alpha/10)L_1} \quad \forall \omega \in A_1. \tag{4.2}$$

By the ergodic theorem there exists an $M_1 > \max(10\eta^{-1}L_1, 6\alpha^{-1}\log \ell)$ and a set $B_1 \subset \Omega$ with $\mathbb{P}(B_1) > 1 - \eta_1/10$ such that

$$\frac{1}{M_1}\sum_{i=0}^{M_1-1} \mathbb{I}_{A_1}(\vartheta^i\omega) > 1 - \frac{\eta_1}{10} \quad \forall \omega \in B_1 \tag{4.3}$$

where $\mathbb{I}_A(\omega) = 1$ if $\omega \in A$ and 0 otherwise.

Using the Strong Rohlin Lemma from [**KW**], we construct a tower of height $M_1 - 1$ with a base $C_1 \subset B_1$ and the disjoint level sets $\vartheta^j C_1$, $j = 0, 1, \ldots, M_1 - 1$ such that $\mathbb{P}\big(\bigcup_{j=0}^{M_1-1} \vartheta^j C_1\big) > 1 - \eta_1/10$. For each $\omega \in C_1$ the column $\{\vartheta^i\omega, i = 0, \ldots, M_1 - 1\}$ based on ω can be divided into disjoint blocks $\{\vartheta^i\omega, i = i_j, i_j + 1, \ldots, i_j + L_1 - 1\}$, $j = 1, \ldots, k$ of size L_1 with starting points $\vartheta^{i_j}\omega$ in A_1, so that these blocks cover all but $(\eta_1/5)$th of the column, i.e. $kL_1 > M_1(1 - \eta_1/5)$. By (4.2) for each starting point $\vartheta^{i_j}\omega$ of the jth L_1-block we can find a covering of $\mathcal{E}_{\vartheta^{i_j}\omega}$ by less than $2^{(h^{(r)}_{\text{top}}(T)+\alpha/10)L_1}$ balls of type $B_x(\vartheta^{i_j}\omega, \frac{1}{4}\varepsilon_1, L_1)$ with $x \in \mathcal{E}_{\vartheta^{i_j}\omega}$. For those $\vartheta^i\omega$ which are not in one of these L_1-blocks, we cover the corresponding $\mathcal{E}_{\vartheta^i\omega}$ by $N_{\mathcal{E}_{\vartheta^i\omega}}(\frac{1}{4}\varepsilon_1)$ balls of type $B_y(\vartheta^i\omega, \frac{1}{4}\varepsilon_1)$. Observe, that if for two points $v, w \in \mathcal{E}_\omega$, $\omega \in C_1$, their iterates $T^{i_j}_\omega v$, $T^{i_j}_\omega w$ belong to the same balls $B_{x_j}(\vartheta^{i_j}\omega, \frac{1}{4}\varepsilon_1, L_1)$, $j = 1, \ldots, k$ on the starting levels of the L_1-blocks above and $T^i_\omega v$, $T^i_\omega w$ belong to the same balls $B_{y_i}(\vartheta^i\omega, \frac{1}{4}\varepsilon_1)$ on the levels outside of these L_1-blocks, then $w \in B_v(\omega, \frac{1}{2}\varepsilon_1, M_1)$. This, together with (4.2) and the second inequality in (4.1), yields that an $(\omega, \frac{1}{2}\varepsilon_1, M_1)$-spanning set $E(\omega)$ of minimal cardinality $r(\omega) = r_{M_1}(\mathcal{E}_\omega, \omega, \frac{1}{2}\varepsilon_1)$ contains less than $2^{(h^{(r)}_{\text{top}}(T)+\alpha/5)M_1}$ points. We can define $E(\omega)$ for each $\omega \in C_1$ and using the technique of [**CV**], similarly to [**Ki2**], the set $E(\omega) = \{v_1(\omega), \ldots, v_{r(\omega)}\}$ can be chosen so that the points $v_i(\omega)$ depend measurably on $\omega \in C_1$. Set $U_i(\omega) = B_{v_i}(\omega, \frac{1}{2}\varepsilon_1, M_1)$, $v_i \in E(\omega)$, $i = 1, \ldots, r(\omega)$ then $\mathcal{U}(\omega) = \{U_1(\omega), \ldots, U_{r(\omega)}(\omega)\}$ form a cover of $\mathcal{E}_\omega$ and the sets $\mathcal{U}_i = \{(\omega, x) : \omega \in C_1, r(\omega) \geq i, x \in U_i(\omega)\}$ are measurable. Next, we transform $\mathcal{U}(\omega)$ into a partition $\tilde{\mathcal{P}}(\omega) = \{\tilde{P}_1(\omega), \ldots, \tilde{P}_{r(\omega)}(\omega)\}$ of $\mathcal{E}_\omega$ in the standard way setting $\tilde{P}_1(\omega) = U_1(\omega)$ and recursively $\tilde{P}_k(\omega) = U_k(\omega) \setminus \bigcup_{i=1}^{k-1} \tilde{P}_i(\omega)$.

Let K_1 be the integral part of $(1 - \alpha/2\log \ell)M_1 + 1$ and set $\mathcal{A}_{K_1} = \{1, 2, \ldots, \ell\}^{K_1}$. Let $\boldsymbol{a}(1), \boldsymbol{a}(2), \ldots, \boldsymbol{a}(\ell^{K_1})$ be the lexicographic ordering of ℓ-words from $\mathcal{A}_{K_1}$. Since

$$\ell^{K_1} = 2^{(h^{(r)}_{\text{top}}(T)+\alpha/2)M_1} > r(\omega)$$

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we can assign to each $\tilde P_i(\omega)$, $i \le r(\omega)$ the word $\boldsymbol{a}(i)$. Now, for each $\omega \in C_1$, $M_1-K_1 \le j < M_1$ and $k = 1, 2, \dots, \ell$, define $P_k(\vartheta^j\omega) = \{x \in \mathcal{E}_{\vartheta^j\omega} : (T^j_\omega)^{-1}x \in \tilde P_i(\omega)$ and $\boldsymbol{a}(i) = (a^{(i)}_1, \dots, a^{(i)}_\ell)$ with $a^{(i)}_j = k\}$. Clearly, $\mathcal{P}_1(\omega') = \{P_1(\omega'), \dots, P_\ell(\omega')\}$ forms a partition of $\mathcal{E}_{\omega'}$ which is defined for all $\omega' \in D^{(1)}_1 \stackrel{\text{def}}{=} \bigcup_{M_1-K_1\le j<M_1} \vartheta^j C_1$.

For each

$$\omega' \in D^{(2)}_1 \stackrel{\text{def}}{=} \bigcup_{M_1-K_1>j\ge 0} \vartheta^j C_1$$

we define $\mathcal{P}_1(\omega')$ as the one-element partition $\mathcal{P}_1(\omega') = \{\mathcal{E}_{\omega'}\}$. Let

$$\omega' \in D^{(3)}_1 \stackrel{\text{def}}{=} \Omega \setminus \bigcup_{M_1>j\ge 0} \vartheta^j C_1.$$

Cover, first, X by $\kappa = N_X(\varepsilon_1)$ closed balls $B_1, B_2, \dots, B_\kappa$ of diameter ε_1 and transform this cover into a partition $\mathcal{Q} = (Q_1, \dots, Q_\kappa)$ of X by taking $Q_1 = B_1$ and recursively $Q_j = B_j \setminus \bigcup_{1\le i<j} Q_i$. For each $\omega' \in D^{(3)}_1$ we define the partition $\mathcal{P}_1(\omega') = \{P_1(\omega'), \dots, P_\kappa(\omega')\}$ setting $P_i(\omega') = Q_i \cap \mathcal{E}_{\omega'}$. The whole construction has been done measurably in ω so that $\mathcal{P}_1 = \{P_i\}$, $P_i = \{(\omega, x) : x \in P_i(\omega)\}$ is a measurable partition of $\mathcal{E}$ whose restriction $\mathcal{P}^{(C)}_1 = \{P_1, \dots, P_\ell\}$ with $P_1 = \{(\omega, x) : \omega \in \vartheta^j C_1$ and $x \in P_1(\vartheta^j\omega)$ if $M_1 - K_1 \le j < M_1 - 1$ or $x \in \mathcal{E}_{\vartheta^j\omega}$ if $0 \le j < M_1 - K_1\}$, $P_i = \{(\omega, x) : \omega \in \vartheta^i C_1, x \in P_i(\omega)$ for some $i = M_1 - K_1, M_1 - K_1 + 1, \dots, M_1 - 1\}$, $i > 1$ consists of, at most, ℓ elements.

Thus, outside of the set $D^{(3)}_1$, having probability of at most $\eta_1/10$, the partitions $\mathcal{P}_1(\omega)$ contain at most ℓ elements. Observe, that if $x, y \in \mathcal{E}_{\vartheta^i\omega}$, $0 \le i < M_1$ and x, y have the same $(\vartheta^i\omega, \mathcal{P}_1)$-names, then $d(T^j_\omega x, T^j_\omega y) \le \varepsilon_1$ for all $j \in \mathbb{Z}$ which we will express by saying that $\mathcal{P}_1$ has a resolution of at least ε_1. In the further steps of our construction, we will not change the partition on $\mathcal{E}_\omega$ if $\omega \in D^{(1)}_1$ and will transform single-element partitions of some of $\mathcal{E}_\omega$'s with $\omega \in D^{(2)}_1$ into ℓ-element partitions in order to both encode points from some of $\mathcal{E}_\omega$'s with $\omega \in D^{(3)}_1$ (which will decrease the set of fibers partitioned into more than ℓ sets) and improve the resolution of the partition $\mathcal{P}_1$.

Next, we describe the second step of our construction. Choose small constants $\varepsilon_2 \in (0, \frac{1}{2}\varepsilon_1)$ and $\eta_2 \in (0, \frac{1}{2}\eta_1)$ such that

$$h^*_T(2\varepsilon_2) < \frac{\alpha}{10^2} \quad \text{and} \quad \eta_2 \log N_X\left(\frac{1}{4}\varepsilon_2\right) < \frac{\alpha}{10^2}. \tag{4.4}$$

In view of the first part of (4.1), Proposition 3.1 enables us to choose an arbitrarily large $L_2 \ge M_1$ and a measurable set $A_2 \subset \Omega$ with $\mathbb{P}(A_2) > 1 - \eta_2/10$ so that

$$r_{L_2}(B_x(\omega, 2\varepsilon_1, L_2), \omega, \varepsilon_2) \le 2^{\alpha N_2/10}. \tag{4.5}$$

By the ergodic theorem there exists an $M_2 > 10\eta_2^{-1}L_2$ and a measurable set $B_2 \subset \Omega$ with $\mathbb{P}(B_2) > 1 - (\eta_2/10)$ such that for any $\omega \in B_2$,

$$\frac{1}{M_2}\sum_{i=0}^{M_2-1} \mathbb{I}_{A_2}(\vartheta^i\omega) > 1 - \frac{\eta_2}{10}, \tag{4.6}$$

$$\frac{1}{M_2}\sum_{i=0}^{M_2-1} \mathbb{I}_{D^{(2)}_1}(\vartheta^i\omega) > \frac{\alpha}{3\log\ell}, \tag{4.7}$$

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and

$$\frac{1}{M_2}\sum_{i=0}^{M_2-1}\mathbb{I}_{D_1^{(3)}}(\vartheta^i\omega) < \frac{\eta_1}{5}. \tag{4.8}$$

Again using the Strong Rohlin Lemma we construct a tower of height M_2-1 with a measurable base $C_2 \subset B_2$ such that T^iC_2, $0 \le i < M_2$ are disjoint and $\mathbb{P}\big(\bigcup_{j=0}^{M_2-1}\vartheta^jC_2\big) > 1-\eta_2/10$.

By (4.8) for any $\omega \in C_2$ there are, at most, $[\frac{1}{5}M_2\eta_1]$ levels i between 0 and M_2-1 for which $\vartheta^i\omega \in D_1^{(3)}$ and by the second part of (4.1) we need less than $2^{M_2\alpha/10}$ names in order to encode for all such i points $T^i_\omega x$, according to elements of partitions $\mathcal{P}_1(\vartheta^i\omega)$ constructed on the first step. For these names we will use words from $\mathcal{A}_q = \{1,2,\dots,\ell\}^q$, where $q = [\alpha M_2/10\log\ell]$. We encode such points partitioning $\mathcal{E}_{\vartheta^j\omega}$ into ℓ sets on q levels j out of $[\alpha M_2/3\log\ell]$ levels for which $\vartheta^j\omega \in D_1^{(2)}$. After that we also change partitions $\mathcal{P}_1(\vartheta^i\omega)$ on levels i with $\vartheta^i\omega \in D_1^{(3)}$, by just taking there the one-element partitions. We denote by $\tilde{\mathcal{P}}_1(\omega')$ the resulting new partition constructed for all $\omega' \in \bigcup_{i=0}^{M_2-1}\vartheta^iC_2$.

Observe, that each element of the partition $\tilde{\mathcal{P}}(\omega) = \bigvee_{i=0}^{M_2-1}(T^i_\omega)^{-1}\tilde{\mathcal{P}}_1(\vartheta^i\omega)$, $\omega \in C_2$ is contained in some ball $B_x(\omega,\varepsilon_1,M_2)$. Similarly to the first step of our construction, divide M_2-columns based on $\omega \in C_2$ into disjoint L_2-blocks with starting points $\vartheta^j\omega \in A_2$ and cover $\mathcal{E}_{\vartheta^i\omega}$ when $\vartheta^i\omega$ is outside of these blocks by $N_{\mathcal{E}_{\vartheta^i\omega}}(\frac{1}{2}\varepsilon_2)$ balls of type $B_y(\vartheta^i\omega,\frac{1}{4}\varepsilon_2)$. Then using (4.5), (4.6) and the second part of (4.4), it is not difficult to see that for all $\omega \in C_2$ and $x \in \mathcal{E}_\omega$,

$$r_{M_2}(B_x(\omega,2\varepsilon_1,M_2),\omega,\varepsilon_2) \le 2^{\alpha M_2/5}. \tag{4.9}$$

It follows that we can partition (measurably in ω) each element of the partition $\tilde{\mathcal{P}}(\omega)$ into at most $2^{\alpha M_2/5}$ subsets, so that each element of the new partition will be contained in some $B_y(\omega,\varepsilon_2,M_2)$. Namely, similarly to [**Ki2**] choose measurably in ω a minimal $(\omega,\varepsilon_2,M_2)$-spanning set $E(\omega)$. By (4.9), $E(\omega)$ contains at most $2^{\alpha M_2/5}$ points $v_1,\dots,v_q$ in each element P of $\tilde{\mathcal{P}}(\omega)$. Subpartition P in the canonical way taking $P_1 = P \cap B_{v_1}(\omega,\varepsilon_2,M_2)$ and recursively $P_k = P \cap B_{v_k}(\omega,\varepsilon_2,M_2)\setminus\bigcup_{k>i\ge1}P_i$. We can encode these new elements by words from $\mathcal{A}_r = \{1,\dots,\ell\}^r$ with $r = [\alpha M_2/5\log\ell]+1$ and we do so by partitioning $\mathcal{E}_{\vartheta^j\omega}$ into ℓ sets on r levels j for which $\vartheta^j\omega \in D_1^{(2)}$ which were not used above. This yields a new partition $\mathcal{P}_2(\omega')$ for all $\omega' \in D_2^{(1)} \cup D_2^{(2)} = \bigcup_{M_2>i\ge0}\vartheta^iC_2$, so that $D_2^{(1)}$ consists of ω' for which $\mathcal{P}_2(\omega')$ has cardinality ℓ and $D_2^{(2)}$ consists of ω' for which $\mathcal{P}_2(\omega')$ is the one-element partition. It is easy to see that $\mathbb{P}(D_2^{(2)}) > \alpha/4\log\ell$, which suffices to use $D_2^{(2)}$ on the next step of our construction.

For

$$\omega' \in D_2^{(3)} \stackrel{\text{def}}{=} \Omega \setminus \bigcup_{M_2>j\ge0}\vartheta^jC_2.$$

we define partitions $\mathcal{P}_2(\omega')$ in the same way as for $\omega' \in D_1^{(3)}$ on the first step, but with ε_2 in place of ε_1. Observe that $\mathbb{P}(D_2^{(3)}) < \eta_2/10$. By the construction, if x, y have the same $(\omega,\mathcal{P}_2)$ names, then $d(T^i_\omega x, T^i_\omega y) \le \varepsilon_2$, i.e. $\mathcal{P}_2$ has a resolution of at least $\varepsilon_2 < \frac{1}{2}\varepsilon_1$.

Repeatedly applying the second step of our construction with $\varepsilon_{i+1} < \frac{1}{2}\varepsilon_i$, $\eta_{i+1} < \frac{1}{2}\eta_i$, $h^*_T(2\varepsilon_i) < \alpha/10^i$, $\eta_i\log N_X(\frac{1}{4}\varepsilon_i) < \alpha/10^i$ for $i = 2,3,\dots$, we will obtain (measurably

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in ω) a sequence of partitions $\mathcal{P}_1(\omega), \mathcal{P}_2(\omega), \dots$ and sequences of sets $D_1^{(1)}, D_2^{(1)}, \dots, D_1^{(2)}, D_2^{(2)}, \dots$ and $D_1^{(3)}, D_2^{(3)}, \dots$ such that the resolution of $\mathcal{P}_i(\omega)$ is at least ε_i, it has cardinality ℓ for $\omega \in D_i^{(1)}$ and cardinality 1 for $\omega \in D_i^{(2)}$ and $\mathbb{P}(D_i^{(3)}) < \eta_i/5$. Since $\eta_i < 2^{-(i-1)}\eta_1$, then employing the Borel–Cantelli Lemma we conclude that $\mathbb{P}$-a.s. either $\omega \in \bigcap_{j\geq 1}\bigcup_{i\geq j} D_i^{(1)}$ and $\mathcal{E}_\omega$ has been partitioned on some step i into ℓ measurable subsets and $\mathcal{P}_j(\omega) = \mathcal{P}_i(\omega)$ for all $j > i$, or there exists i so that $\mathcal{P}_j(\omega)$ is the one-element partition for all $j \geq i$. This means that the sequence $\mathcal{P}_i(\omega)$ does not change starting from some i depending on ω and the limiting partition $\mathcal{P}(\omega)$ (which depends measurably on ω by construction) satisfies our conditions, except that there may exist a measurable set $D \subset \Omega$ such that $\mathcal{P}(\omega)$ is of cardinality 1 for $\omega \in D$. By partitioning such $\mathcal{P}(\omega)$ into ℓ measurable sets in an arbitrary but measurable in ω way, we complete our construction.

References

[Ar] L. Arnold. *Random Dynamical Systems*. Springer, New York, 1998.

[AR] L. M. Abramov and V. A. Rohlin. The entropy of a skew product of measure-preserving transformation. *Amer. Math. Soc. Transl. Ser. 2* **48** (1966), 255–265.

[Bog] T. Bogenschütz. Entropy, pressure, and a variational principle for random dynamical systems. *Random and Computational Dynam.* **1** (1992), 99–116.

[Bow] R. Bowen. Entropy-expansive maps. *Trans. Amer. Math. Soc.* **164** (1972), 323–333.

[Bu] J. Buzzi. Intrinsic ergodicity of smooth interval maps. *Israel J. Math.* **100** (1997), 125–161.

[CV] C. Castaing and M. Valadier. *Convex Analysis and Measurable Multifunctions (Lecture Notes in Mathematics, 580)*. Springer, New York, 1977.

[Du] R. M. Dudley. *Real Analysis and Probability*. Wadsworth & Brooks/Cole, Pacific Grove, 1989.

[Dy] H. A. Dye. On groups of measure preserving transformations I. *Amer. J. Math.* **81** (1959), 119–159.

[DGS] M. Denker, C. Grillenberg and K. Sigmund. *Ergodic Theory on Compact Spaces (Lecture Notes in Mathematics, 527)*. Springer, Berlin, 1976.

[Ki1] Y. Kifer. *Ergodic Theory of Random Transformations*. Birkhäuser, Boston, MA, 1986.

[Ki2] Y. Kifer. On the topological pressure for random bundle transformations. *Rohlin's Memorial*. Eds. V. Turaev and A. Vershik. *Amer. Math. Soc. Transl.* **202**(2) (2001), 197–214.

[Kr] W. Krieger. On entropy and generators of measure-preserving transformations. *Trans. Amer. Math. Soc.* **149** (1970), 453–464. (Erratum **168** (1972), 519.)

[KY] Y. Kifer and Y. Yomdin. Volume growth and topological entropy for random transformations. *Dynamical Systems (Lecture Notes in Mathematics, 1342)*. Ed. J. C. Alexander. Springer, New York, 1988, pp. 361–373.

[KW] Y. Katznelson and B. Weiss. Commuting measure-preserving transformations. *Israel J. Math.* **12** (1972), 161–173.

[LW] F. Ledrappier and P. Walters. A relativized variational principle for continuous transformations. *J. London Math. Soc.* **16**(2) (1977), 568–576.

[Mi] M. Misiurewicz. Topological conditional entropy. *Studia Mathematica* **55** (1976), 175–200.

[Ro] V. A. Rohlin. Lectures on the entropy theory of measure-preserving transformation. *Russian Math. Surveys* **22**(5) (1967), 1–52.

[RW] D. J. Rudolph and B. Weiss. Entropy and mixing for amenable group actions. *Ann. Math.* **151** (2000), 1119–1150.

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