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Entropy continuity of Lyapunov exponents for non-flat one-dimensional maps

HENGYI LI

Laboratoire de Mathématiques d'Orsay, Universite Paris-Saclay, France
(e-mail: hengyi.li@universite-paris-saclay.fr)

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Abstract. We show that the continuity property of Lyapunov exponents proved by Buzzi *et al.* [Continuity properties of Lyapunov exponents for surface diffeomorphisms. *Invent. Math.* **230**(2) (2022), 767–849] for smooth surface diffeomorphisms extends to smooth interval maps in the case when the map only has non-flat critical points and the entropies converging to the topological entropy. The result we obtained is stronger than the continuity of Lyapunov exponents, in particular, we prove the uniform integrability of Lyapunov exponents over entropies.

Key words: one-dimensional systems, Lyapunov exponents, non-uniformly expanding systems, measures of maximal entropy, smooth ergodic theory
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1. *Introduction*

Lyapunov exponents are key invariants of smooth ergodic theory. They are upper semicontinuous, but usually not lower continuous functions of the invariant probability measure [**V2**]. The goal of this work is to understand when the convergence of the entropy implies convergence of the Lyapunov exponents for smooth interval maps, as was established for surface diffeomorphisms in [**BCS**].

In this paper, we consider the simplest case when the map has only non-flat critical points and the entropy converges to the topological entropy. We obtain convergence of the Lyapunov exponents in Theorem 1.1. More precisely, we establish uniform integrability of $\log|f'|$ in Theorem 3.4. For a general survey on this phenomena, see [**V1**, **S**].

The one-dimensional case with non-flat critical points turns out to exhibit quite different difficulties than the surface diffeomorphism case, but the difficulties are quite different. This is also the simplest case of non-invertible and non-uniformly hyperbolic systems [**Q**]. They come mostly from the lack of bounded distortion with respect to the differential. Namely, unlike the diffeomorphism case, we no longer have some uniform ε_0 such that for all $0 < \varepsilon < \varepsilon_0$, for all x, $f(B(x,\varepsilon)) \subseteq B(fx, 2|f'x|\varepsilon)$, where $B(x,\varepsilon)$ is the ball centered at x.

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One possible application of Theorem 1.1 is to prove not only the finiteness [B], but also the exponential mixing of the measures of maximal entropy for C^∞ one-dimensional systems with positive topological entropy and finite critical set [M1].

1.1. *Statement of our main result.* We let $X = [0, 1]$ or $\mathbb{R}/\mathbb{Z}$, and $f : X \to X$ be a C^∞ smooth map. Given $x \in X$, we say f is non-flat at x, or simply that x is non-flat, if there exists $d > 0$ such that $f^{(d)}(x) \neq 0$. We assume that for all $x \in X$ such that $f'(x) = 0$, x is non-flat.

In this paper, a measure will always mean an f-invariant Borel probability measure on X. For a measure μ, we denote by $h(f, \mu)$, or just $h(\mu)$, the entropy of μ. We denote by $h_{\text{top}}(f)$ the topological entropy of f.

We consider sequences of measures of the following type.

Definition 1. Let $(\mu_k)_{k\in\mathbb{N}^*}$ be a sequence of measures on X. We say that it *approximates f in entropy,* if:

(1) each μ_k is ergodic for f;
(2) the sequence converges in the weak topology: $\mu_k \xrightarrow{*} \mu$ for some measure μ;
(3) the entropies converge to the topological entropy: $\lim_{k\to\infty} h(f, \mu_k) = h_{\text{top}}(f)$.

For any probability measure ν on X, define its Lyapunov exponent to be

$$\lambda(\nu) := \int \log |f'| \, d\nu.$$

Our main result is the following theorem.

THEOREM 1.1. *Let $f : X \to X$ be a smooth map with only non-flat critical points and let $(\mu_k)_{k\in\mathbb{N}}$ be a sequence of probability measures on X approximating f in entropy. Denote its weak limit by μ. If $h_{\text{top}}(f) > 0$, then*

$$\lim_{k\to\infty} \lambda(\mu_k) = \lambda(\mu). \tag{$*$}$$

1.2. *Strategy: entropy and uniform integrability defect.* Our overall strategy, like in [BCS], is to deduce from any failure of continuity of the exponent a non-trivial bound on the entropy. As one could expect, the critical points play a role similar to the homoclinic tangencies there. In contrast to the decomposition of [BCS, Theorem D], we are only able to quantify numerically the lack of uniform integrability of $\log |f'|$ with respect to the μ_k, $k \geq 1$ as follows.

Definition 2. The *uniform integrability defect* of $(\mu_k)_{k\in\mathbb{N}^+}$ is

$$\alpha := \lim_{\delta\to 0} \limsup_{k\to\infty} \int_{\{|f'|<\delta\}} -\log |f'| \, d\mu_k.$$

It is well known that $\lambda(\nu)$ is upper semi-continuous with respect to ν. As $-\log |f'|$ is continuous on $\{x : |f'(x)| \geq \delta\}$, $\int_{\{|f'|\geq\delta\}} -\log |f'| \, d\mu_k \to \int_{\{|f'|\geq\delta\}} -\log |f'| \, d\mu$. One then sees the following lemma.

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LEMMA 1.2. *The uniform-defect α of a sequence of measures gives the defect in lower semicontinuity of the Lyapunov exponent:*

$$\alpha = \lambda(\mu) - \liminf_{k\to\infty} \lambda(\mu_k).$$

Setting $\Lambda(f) := \sup |f'|$ and $\lambda(f) := \log \Lambda$, we get the following entropy bound.

THEOREM 1.3. *Let $f : X \to X$ be a smooth map with non-flat critical points. Let $(\mu_k)_{k\in\mathbb{N}^+}$ be a sequence of ergodic measures converging in the weak topology to some measure μ. If α is the uniform integrability defect, then*

$$\limsup_{k\to\infty} h(\mu_k) \le \left(1 - \frac{\alpha}{\lambda(f)}\right) h_{\text{top}}(f).$$

Note that the two previous statements imply our main result, Theorem 1.1.

1.3. *Organization of the paper.* In §2, we define and investigate shadowing intervals. These intervals are orbit segments approximated by critical points. We prove that the portion of these intervals among $[0, n[$ converges to the uniform defect $\alpha/(\lambda(f))$ (Theorem 2.1).

In §3, we deduce an entropy bound (Theorem 3.4) from the portion of shadowing intervals among $[0, n[$.

1.4. *Notation.* Let $C(f) := \{c \in X : f'(c) = 0\}$. Here, $C(f)$ is a finite set, since each c is non-flat. For each $c \in C(f)$, write $d_c := \min\{d \in \mathbb{N}^+ : f^{(d)}(c) \neq 0\}$ and

$$p_c(x) := \frac{f^{(d_c)}(c)}{(d_c - 1)!}(x - c)^{d_c - 1},$$

the $(d_c - 1)$th Taylor polynomial of f' at c. Then, define $g_c(x)$ as the unique smooth function such that

$$f'(x) = p_c(x) \cdot g_c(x). \tag{1.1}$$

Define $K(f) := \max_{c\in C(f)}\{d_c\}$. When f is clear from the context, we abbreviate $K := K(f)$. Note that $g_c(c) = 1$.

We abbreviate $f'x := f'(x)$, $x_a := f^a x := f^a(x)$ for all $x \in X$, for all $a \in \mathbb{N}$.

Since $C(f)$ is a finite set, for $\varepsilon < \frac{1}{2}\min\{d(c_1, c_2) : c_1, c_2 \in C(f)\}$ and for all $x \in X$, the number of points in $(x - \varepsilon, x + \varepsilon) \cap C(f) \le 1$. Given $a \in \mathbb{N}$, when $(x_a - \varepsilon, x_a + \varepsilon) \cap C(f) \neq \emptyset$, we let c_a be the unique point in the intersection.

Fix $\varepsilon_1' > 0$ such that for all $c \in C(f)$, $|f'|_{(c-2\varepsilon_1', c)}|$ is monotone decreasing and $|f'|_{(c,c+2\varepsilon_1')}|$ is monotone increasing. Note that $(c - 2\varepsilon_1', c + \varepsilon_1') \cap C(f) = \{c\}$ for all $c \in C(f)$. Fix $\varepsilon_1'' > 0$ such that for all $c \in C(f)$, for all $x, y \in (c - 2\varepsilon_1'', c + 2\varepsilon_1'')$, $\frac{1}{2} < |g_c(x)|/|g_c(y)| < 2$. Finally, let

$$\varepsilon_1 := \min\{\varepsilon_1', \varepsilon_1''\}. \tag{1.2}$$

Denote by $\mathbb{P}(f)$ the space of f-invariant, Borel, probability measures on X and $\mathbb{P}_{\text{erg}}(f)$ the subspace of ergodic measures in $\mathbb{P}(f)$.

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Throughout this paper, we consider integer intervals, such as $[a, b[:= \{k \in \mathbb{N} : a \le k < b, a \in \mathbb{N}, b \in \mathbb{R}_+\}$ for $-\infty < a < b \le \infty$. For every finite subset $K \subseteq \mathbb{N}$, we denote by $K^- := \min K$ and $K^+ := \max K$.

2. *Shadowing intervals*

Definition 3. Given $\varepsilon > 0$, an integer interval $I = [a, b[$ is an *ε-shadowing interval for* $x \in X$ if there exists $c \in C(f)$, such that for all $0 \le h < b - a$,

$$d(x_{a+h}, f^h c) < \varepsilon.$$

For all $L \in \mathbb{N}^+$, an ε-shadowing interval $[a, b[$ for x is *(L, ε)-shadowing* if $b - a \ge L$.

In this section, we prove the following theorem.

THEOREM 2.1. *For all $\varepsilon > 0$, $L \in \mathbb{N}^+$, there exists $\delta > 0$ such that for all $\nu \in \mathbb{P}_{\mathrm{erg}}(f)$, with $h(\nu) > 0$, for ν-almost every (a.e.) $x \in X$, there exists a collection $\mathcal{S}(x)$ of finite integer intervals, satisfying the following.*

(1) *The map $x \mapsto \mathcal{S}(x)$ is measurable on X: for each $n \in \mathbb{N}$, for each collection J of subsets of $[0, n[$, $\mathcal{S}_n^{-1}(J) := \{x \in X : \{I \cap [0, n[: I \in \mathcal{S}(x)\} = J\}$ is measurable.*

(2) *The intervals in $\mathcal{S}(x)$ are (L, ε)-shadowing and pairwise disjoint.*

(3) *Denote by $\bigcup \mathcal{S}(x) := \bigcup_{I \in \mathcal{S}(x)} I$. For all $n \in \mathbb{N}^+$,*

$$\left| \bigcup \mathcal{S}(x) \cap [0, n[\right| \ge \left(1 - \frac{1}{L}\right) \sum_{i=0}^{n-1} -\frac{\log_\delta |f' x_i|}{\lambda} - g_{\varepsilon,L,n}(x),$$

where

$$\log_\delta t := \begin{cases} 0 & \text{if } t \ge \delta, \\ \log t & \text{otherwise,} \end{cases}$$

and $g_{\varepsilon,L,n} : X \to \mathbb{R}$, depending on ε, L and n, is measurable and satisfies for ν-a.e. x,

$$\lim_{n \to \infty} \frac{1}{n} g_n(x) = 0.$$

2.1. *Basic properties.*

LEMMA 2.2. *For all $L \in \mathbb{N}^+$, $0 < \varepsilon < \varepsilon_1$, there exists $\delta_1 > 0$ such that for all $0 < \delta < \delta_1$, for all $x \in X$, and for all $a \in \mathbb{N}$, if $|f' x_a| < \delta$, the integer interval $[a, a + l(a)[$ is a (L, ε)-shadowing interval, where*

$$l(a) := \frac{-\log |f' x_a|}{\lambda(f)}.$$

Proof. Given $L > 0$, $0 < \varepsilon < \varepsilon_1$, choose $\delta_1' \in (0, 1)$ such that $|f' x| < \delta_1'$ implies there exists a unique $c \in C(f)$, with $d(x, c) < \varepsilon$. Then, define $\delta_1 := \min\{\delta_1' \Lambda(f)^{-L}\}$.

Assume $a \in \mathbb{N}$ satisfies $|f' x_a| < \delta_1$. We check that $[a, a + l(a)[$ is (L, ε)-shadowing.

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First, note that $l(a) \geq L$. Indeed,

$$l(a) = \frac{-\log |f' x_a|}{\lambda(f)} > \frac{-\log \delta_1}{\lambda(f)} > \frac{L \log \Lambda(f)}{\lambda(f)} = L.$$

Second, observe that $[a, a + l(a)[$ is ε-shadowing.

Since $\delta_1 < \delta_1'$, there exists a unique $c_a \in C(f)$ such that $d(x_a, c_a) < \varepsilon$. Now, for all $0 < h < l(a)$,

$$\begin{aligned} d(x_{a+h}, f^h c_a) &\leq \Lambda(f)^{h-1} d(x_{a+1}, f c_a) \\ &\leq \Lambda(f)^{h-1} \cdot |f' x_a| d(x_a, c_a) \leq \Lambda(f)^{l(a)-1} \cdot |f' x_a| d(x_a, c_a) \\ &\leq \Lambda(f)^{-\log |f' x_a|/\lambda(f)} |f' x_a| d(x_a, c_a) \leq d(x_a, c_a) < \varepsilon. \end{aligned}$$

The second inequality follows from the fact that $d(x_a, c_a) < \varepsilon < \varepsilon_1$ and $|f'|$ is monotone decreasing from x_a to c_a; the last inequality follows from our choice of $\delta < \delta_1'$, which guarantees $\Lambda(f)^{-\log |f' x_a|/\lambda(f)} |f' x_a| \leq 1$. □

In the following sections, since the parameters $\varepsilon > 0$, $\delta > 0$, and $x \in X$ are fixed, we omit them from the notation.

We define $A(x) := \{a \in \mathbb{N} : |f' x_a| < \delta\}$ and

$$F(n) = \begin{cases} \Lambda(f) & \text{if } |f' x| \geq \delta, \\ 2^{K+1} \cdot |f' x| & \text{otherwise.} \end{cases}$$

When the base point x is clear from the context, we abbreviate $A := A(x)$.

Definition 4. An (ε, δ)-*piecewise shadowing interval for* $x \in X$, abbreviated as a (ε, δ)-PSI, or just a PSI, is a sequence of consecutive ε-shadowing intervals

$$I := \{[a_1, a_2[, [a_2, a_3[, \ldots, [a_n, a_{n+1}[\},$$

such that:

(1) $a_1 < a_2 < \cdots < a_{n+1}$;
(2) for all $1 \leq i \leq n$, $a_i \in A$;
(3) for all $a_1 \leq k < a_{n+1}$,

$$G^I(k) := \prod_{a_1 \leq j \leq k} F(j) < 1.$$

The *support of* I, supp I, is the union $[a_1, a_{n+1}[$. We also set

$$G^I(a_{n+1}) := \prod_{a_1 \leq j \leq a_{n+1}} F(j).$$

Note that a PSI is not an interval but a collection of intervals.

Note that if $\{[a_1, a_2[, \ldots, [a_n, a_{n+1}[\}$ is ε-piecewise shadowing, so will be $\{[a_1, a_2[, \ldots, [a_s, b[\}$ for all $s \leq n$ and $b \in [a_s, a_{s+1}]$.

Definition 5. A piecewise shadowing interval $I = \{[a_1, a_2[, \ldots, [a_n, a_{n+1}[\}$ is right maximal if $G^I(a_{n+1}) \geq 1$.

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LEMMA 2.3. *Let* $I = \{[a_1, a_2[, \ldots, [a_n, a_{n+1}[\}$ *be a right maximal PSI. Then,* $a_{n+1} \notin A$.

Proof. Since I is a PSI, $G^I(a_{n+1} - 1) < 1$. If $a_{n+1} \in A$, then $F(a_{n+1}) = 2^{K+1} \cdot |f' x_{a_{n+1}}| < 1$ and

$$G^I(a_{n+1}) = G^I(a_{n+1} - 1) \cdot F(a_{n+1}) < G^I(a_{n+1}) = G^I(a_{n+1} - 1) < 1,$$

which contradicts our assumption that I is right maximal. □

LEMMA 2.4. *Let* $I_1 = \{[a_1, a_2[, \ldots, [a_n, a_{n+1}[\}$, $I_2 = \{[b_1, b_2[, \ldots, [b_m, b_{m+1}[\}$ *be two PSIs such that* $[a_1, a_{n+1}[\cap [b_1, b_{m+1}[\neq \emptyset$. *Denote by*

$$\{s_1 < s_2 < \cdots < s_{l+1}\} = \{a_1, \ldots a_n\} \cup \{b_1, \ldots b_m\} \cup \max\{a_{n+1}, b_{m+1}\},$$

then $J := \{[s_1, s_2[, \ldots, [s_l, s_{l+1}[\}$ *is again a PSI. We denote it by* $I_1 \cap I_2$.

Proof. Up to changing notation, we assume $a_1 \leq b_1$.

Note that for $j \leq l$, $s_j = a_i$ with $i \leq n$ or b_j with $j \leq m$. Thus:

(1) all $s_j \in A$;

(2) $[s_j, s_{j+1}[$ is a segment of some $[a_i, a_{i+1}[$ or $[b_i, b_{i+1}[$, with $s_i = a_i$ or b_i; hence, a shadowing interval.

It remains to check that $G^J(k) < 1$ for all $k \in [s_1, s_{l+1}[$. Now, if $k \leq a_{n+1}$, then $G^J(k) = G^{I_1}(k) < 1$. If $k > a_{n+1}$, then $b_1 < k < b_{m+1}$ and

$$G^J(k) = G^{I_1}(b_1) \cdot G^{I_2}(k) < 1,$$

since $\mathrm{I}_1 \cap \mathrm{I}_2 \neq \emptyset$ implies $b_1 < a_{n+1}$. □

LEMMA 2.5. *Let* $I_1 = \{[a_1, a_2[, \ldots, [a_n, a_{n+1}[\}$, $I_2 = \{[b_1, b_2[, \ldots, [b_m, b_{m+1}[\}$ *be two right maximal PSIs. Then, their supports are either disjoint or nested.*

Proof. We proceed by contradiction. Let I_1, I_2 be as above. We can assume that $a_1 < b_1$, $\operatorname{supp} I_1 \cap \operatorname{supp} I_2 \neq \emptyset$, and I_2 is not contained in I_1. Thus, $b_1 < a_{n+1}$ and $a_{n+1} < b_{m+1}$.

By Lemma 2.4, $I_1 \cup I_2$ should be a piecewise shadowing interval. However, this implies $G^{I_1}(a_{n+1}) = G^{I_1}(b_1) G^{I_2}(a_{n+1}) < 1$, which contradicts I_1 being maximal. □

Recall that we defined $K = K(f) := \max_{c \in C(f)}\{d_c\}$.

LEMMA 2.6. *If* $I = \{[a_1, a_2[, \ldots, [a_n, a_{n+1}[\}$ *is a right maximal PSI, then*

$$a_{n+1} - a_1 \geq \sum_{a \in A \cap [a_1, a_{n+1}[} \left(l(a) - \frac{K+1}{\lambda} \cdot \log 2 \right).$$

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Proof. By definition of right maximality,

$$\begin{aligned} 1 \le G(a_{n+1}) &= \prod_{a_1 \le k < a_{n+1}} F(k) \\ &= \prod_{a \in A \cap [a_1, a_{n+1}[}^{n} (|f' x_a| 2^{K+1}) \cdot \prod_{k \notin A} \Lambda \\ &\le \left(\prod_{a \in A \cap [a_1, a_{n+1}[}^{n} (|f' x_a| 2^{K+1}) \right) \cdot \Lambda^{a_{n+1} - a_1}. \end{aligned}$$

Taking the logarithm, we get

$$0 \le \sum_{a \in A \cap [a_1, a_{n+1}[}^{n} (-\lambda \cdot l(a) + (K+1) \log 2) + (a_{n+1} - a_1) \cdot \lambda.$$

Thus,

$$a_{n+1} - a_1 \ge \sum_{a \in A \cap [a_1, a_{n+1}[} \left(l(a) - \frac{K+1}{\lambda} \cdot \log 2 \right). \qquad \square$$

2.2. *Inductive construction of $\mathcal{S}(x)$.* In this subsection, we produce the sequence $\mathcal{S}(x)$ of intervals announced in Theorem 2.1 by an inductive scheme. We can assume that $C(f) \cap \operatorname{supp} \nu \neq \emptyset$, otherwise Theorem 2.1 is trivial.

We fix $\varepsilon > 0$, $L \in \mathbb{N}_+$, and let $\nu \in \mathbb{P}_{\text{erg}}(f)$, with $h(\nu) > 0$. Note that for all $\delta > 0$, $\nu(\{x : |f' x| < \delta\}) > 0$. Otherwise, $\alpha = 0$ and Theorem 1.3 is trivial.

Definition 6. Let $x \in X$ and $I = [a, b[$ be a shadowing interval for $x \in X$ with $b \in A$. We say that *I is non-switching* if

$$d(x_b, c_b) \ge d(x_b, f^{b-a} c_a).$$

Otherwise, we say that it is *switching*.

The terminology is explained by the following proposition.

PROPOSITION 2.7. *Let $I = [a, b[$ be a shadowing interval with $b \in A$.*

If I is non-switching, then

$$d(f x_b, f^{b-a+1} c_a) \le 2^{K+1} |f' x_b| \cdot d(x_b, f^{b-a} c_a);$$

if I is switching, then

$$d(f x_b, f c_b) \le 2^{K+1} |f' x_b| \cdot d(x_b, f^{b-a} c_a).$$

Proof. First, note that as f has only non-flat critical points and ε is small, $|f'|$ is monotone from x_b to c_b and from $f^{b-a} c_a$ to c_b. Thus, by the mean value theorem,

$$d(x_{b+1}, f^{b-a+1} c_a) \le \max\{|f' x_b|, |f' f^{b-a} c_a|\} \cdot d(x_b, f^{b-a} c_a).$$

We consider three cases.

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Case A.1: When I is non-switching and $|f'x_b| \geq |f'f^{b-a}c_a|$. By the previous discussion,

$$d(x_{b+1}, f^{b-a+1}c_a) \leq |f'x_b| \cdot d(x_b, f^{b-a}c_a) \leq 2^{K+1}|f'x_b| \cdot d(x_b, f^{b-a}c_a).$$

Case A.2: When I is non-switching and $|f'x_b| \leq |f'f^{b-a}c_a|$. Then,

$$d(x_{b+1}, f^{b-a+1}c_a) \leq |f'f^{b-a}c_a| \cdot d(x_b, f^{b-a}c_a).$$

Then, we compare $|f'f^{b-a}c_a|$ with $|f'x_b|$. Since I is non-switching, $d(f^{b-a}c_a, c_b) < d(f^{b-a}c_a, x_b) + d(x_b, c_b) \leq 2d(x_b, c_b)$. Thus,

$$\begin{aligned}\frac{|f'f^{b-a}c_a|}{|f'x_b|} &= \frac{|f^{(d_b)}(c_b)(f^{b-a}c_a - c_b)^{d_b} \cdot g_{c_b}(f^{b-a}c_a)|}{|f^{(d_b)}(c_b)(x_b - c_b)^{d_b} \cdot g_{c_b}(x_b)|} \\ &\leq \frac{|(2(x_b - c_b))^{d_b}|}{|(x_b - c_b)^{d_b}|} \cdot \frac{|g_{c_b}(f^{b-a}c_a)|}{|g_{c_b}(x_b)|} \\ &\leq 2^{d_b} \cdot 2 \leq 2^{K+1},\end{aligned}$$

where the last inequality follows from (1.1) and (1.2).

Case B: When I is switching. Since $|f'|$ is decreasing from x_b to c_b, we have

$$d(x_{b+1}, fc_b) \leq |f'x_b|\, d(x_b, c_b).$$

We conclude

$$\begin{aligned}d(x_{b+1}, fc_b) &< |f'x_b|\, d(x_b, c_b) \\ &< |f'x_b|\, d(x_b, f^{b-a}c_b) < 2^{K+1}|f'x_b|\, d(x_b, f^{b-a}c_b).\end{aligned}$$

□

Definition 7. Given $x \in X$ and a pair (I, a^*), where $I := \{[a_1, a_2[, \ldots [a_s, a_{s+1}[\}$ is a PSI and $a^* \in A \cap [a_s, a_{s+1}[$, define

$$G^I_{a^*} : [a_1, \infty[\to \mathbb{R}_+$$

by

$$G^I_{a^*}(k) := \prod_{a_1 \leq a \leq k, a \in A_{a^*}} (|f'x_a| 2^{K+1}) \prod_{a_1 \leq l \leq k, l \notin A_{a^*}} \Lambda,$$

where $A_{a^*} := A \cap [a_1, a^*[$.

This pair (I, a^*) is *an intermediate piecewise shadowing interval* (IPSI) if we have:

(1) for all $k \in \operatorname{supp} I := [a_1, a_{s+1}[$,

$$G^I_{a^*}(k) < 1;$$

(2) if $k \in [a_j, a_{j+1}[$ for some $1 \leq j \leq s$, then

$$d(x_k, f^{k-a_j}c_{a_j}) < G^I_{a^*}(k)\varepsilon.$$

The terminology is explained in the following proposition.

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PROPOSITION 2.8. *Given $\varepsilon > 0$, there exists $\delta_1 > 0$ such that for all $0 < \delta < \delta_1$, every $x \in X$ such that $|(f^n)'x| \to \infty$, for each $a_1 \in A$, there exists I, a right maximal PSI starting at a_1.*

Proof. Fix $\varepsilon > 0$ small, then Lemma 2.2 gives $\delta_1 > 0$. Then, choose a $0 < \delta < \delta_1$. Let $x \in X$ and $a_1 \in A$.

CLAIM 1. *There is a finite sequence of pairs (I_s, a_s^*), $s = 1, \dots S$, with I_s a finite collection of integer intervals and $a_s^* \in \mathbb{N}$ such that:*

(1) *(I_s, a_s^*) is an IPSI, $s = 1, \dots, S-1$;*
(2) *$\min I_s = a_1$;*
(3) *for $1 < s \le S$, $\operatorname{supp} I_{s-1} \subseteq \operatorname{supp} I_s$, $a_{s-1}^* < a_s^*$;*
(4) *$S > 1$ is the minimal $s \ge 1$ such that $a_s^* = \max I_s + 1$;*
(5) *I_S is a PSI and it is right maximal.*

Proof. We build by induction a sequence of IPSIs satisfying conditions (1), (2), and (3) until condition (4) holds. Then, we check condition (5).

For $s = 1$, we define (I_1, a_1^*) by setting $I_1 := \{[a_1, b_1[\}$, with $b_1 := a_1 + l(a_1) - ((K+1)/\lambda) \cdot \log 2$ and $a_1^* = a_1$.

Note that $[a_1, b_1[$ is a shadowing interval by Lemma 2.2, so (I_1, a_1^*) is an IPSI.

Now, assume by induction that for some $s \ge 1$, there are some IPSIs $(I_1, a_1^*), \dots, (I_s, a_s^*)$ satisfying conditions (2), (3), and $a_t^* \le \max I_t$ for $t \le s$. The induction ends when $s + 1 = \max I_{s+1} + 1$.

Now, we build a new IPSI according to the following two cases.

Case A: $A \cap]a_s^*, \max I_s] \neq \emptyset$. Write $I_s := \{[a_1, a_2[, \dots, [a_r, b[\}$. Notice that since I_s is an IPSI, $[a_r, b[$ is a shadowing interval with $a_s^* \in [a_r, b[$.

Fix $a_{s+1}^* := \min\{a \in A \cap]a_s^*, \max I_s]\} \in [a_r, b[$. Note that $[a_r, a_{s+1}^*[$ is either switching or non-switching.

Case A.1: $[a_r, a_{s+1}^*[$ *is non-switching.* Define

$$b' := \min\{i > b, i \in \mathbb{N} : G_{a_{s+1}^*}^{I_s}(i) \ge 1\};$$
$$I_{s+1} := \{[a_1, a_2[, \dots, [a_r, b'[\}.$$

Note that these two are well defined, since $G_{a_{s+1}^*}^{I_s}$ is a map defined on $[a_1, \infty[$. We now verify that (I_{s+1}, a_{s+1}^*) is indeed an IPSI.

First, note that by our choice of b', automatically, $G_{a_{s+1}^*}^{I_{s+1}}(k) < 1$ for all $k \in \operatorname{supp} I_s$.

Note that by our choice, $a_{s+1}^* > a_s^*$, so $A \cap]a_s^*, a_{s+1}^*[= \emptyset$. Now, since $G_-^J(k)$ depends on $A \cap [a_1, -[$, we see that for $a_1 \le k \le a_{s+1}^*$,

$$\begin{aligned} G_{a_{s+1}^*}^{I_{s+1}}(k) &= G_{a_s^*}^{I_{s+1}}(k) \\ &= G_{a_s^*}^{I_s}(k); \end{aligned}$$

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and for $a^*_{s+1} < k < b'$,

$$\begin{aligned} G^{I_{s+1}}_{a^*_{s+1}}(k) &= \prod_{a_1 \le a_j < k, a_j \in A_{a^*_{s+1}}} (|f' x_{a_j}|2^{K+1}) \prod_{a \le j < k, j \notin A_{a^*_{s+1}}} \Lambda \\ &= G^{I_s}_{a^*_s}(a^*_{s+1})(|f' x_{a^*_{s+1}}|2^{K+1})\Lambda^{k-a^*_{s+1}-1} \end{aligned}$$

for all $1 \le k \le b'$. So, for $a_1 \le a_i \le k \le a_{i+1} \le a^*_{s+1}$, by inductive hypothesis, $d(x_k, f^{k-a_i} c_{a_i}) < G^{I_{s+1}}_{a^*_{s+1}}(k)\varepsilon$. For $a_r < a^*_{s+1} < k < b'$,

$$\begin{aligned} d(x_k, f^{k-a_i} c_{a_r}) &\le \Lambda^{k-a^*_{s+1}-1} d(x_{a^*_{s+1}+1}, f^{a^*_{s+1}-a_r+1} c_{a_r}) \\ &\le \Lambda^{k-a^*_{s+1}-1}(|f' x^*_{a_{s+1}}|2^{K+1}) d(x_{a^*_{s+1}}, f^{a^*_{s+1}-a_r} c_{a_r}) \\ &\le G^{I_s}_{a^*_s}(a_{s+1})(|f' x_{a^*_{s+1}}|2^{K+1})\Lambda^{k-a^*_{s+1}-1}\varepsilon \qquad (2.1) \\ &= G^{I_s}_{a^*_{s+1}}(k)\varepsilon, \end{aligned}$$

where the second inequality follows from $a^*_{s+1} \in [a_r, b'[$ being non-switching and Proposition 2.7; the third inequality follows from $a^*_{s+1} \in [a_r, b[$ and inductive assumption; the fourth equality follows from our definition of $G^{I_{s+1}}_{a^*_{s+1}}$. From (2.1) and the fact that $G^{I_{s+1}}_{a^*_{s+1}}(k) < 1$, it is easy to check that $[a_r, b'[$ is shadowing.

Case A.2: $[a_r, b[$ *is switching.* Define

$$\begin{aligned} &b' := \min\{i > b, i \in \mathbb{N} : G^{I_s}_{a^*_{s+1}}(i) \ge 1\}; \\ &a_{r'} := a^*_{s+1}; \\ &I_{s+1} := \{[a_1, a_2[, \ldots, [a_r, a^*_{s+1}[, [a^*_{s+1}, b'[\}; \\ &a_{r+1} := a^*_{s+1}. \end{aligned}$$

We verify that I_{s+1} is an IPSI.

For $a_1 \le k \le a^*_{s+1}$, since by our choice $a^*_{s+1} > a^*_s$,

$$G^{I_{s+1}}_{a^*_{s+1}}(k) = G^{I_s}_{a^*_s}(k);$$

and for $a^*_{s+1} < k < b'$,

$$G^{I_{s+1}}_{a^*_{s+1}}(k) = G^{I_s}_{a^*_s}(a^*_{s+1})(|f' x_{a^*_{s+1}}|2^{K+1})\Lambda^{k-a^*_{s+1}-1}.$$

Thus, for $a_1 \le k \le a^*_{s+1}$, naturally $d(x_k, f^{k-a_i} c_{a_i}) < G^{I_{s+1}}_{a^*_{s+1}}(k)\varepsilon$. For $a^*_{s+1} < k < b'$, as $k \in [a_{r'}, b'[$ and $a_{r'} = a^*_{s+1}$ by our construction,

$$\begin{aligned} d(x_k, f^{k-a_{s+1}} c_{a_{s+1}}) &\le \Lambda^{k-a_{s+1}-1}\, d(x_{a_{s+1}+1}, f c_{a_{s+1}}) \\ &\le \Lambda^{k-a_{s+1}-1}(|f' x_{a_{s+1}}|2^{K+1})\, d(x_{a_{s+1}}, f^{a_{s+1}-a_{l_s}} c_{a_{l_s}}) \\ &\le G^{I_s}_{a^*_s}(a_{s+1})(|f' x_{a_{s+1}}|2^{K+1})\Lambda^{k-a_{s+1}-1}\varepsilon \\ &= G^{J_{n+1}}_{a^*_{s+1}}(k)\varepsilon \le \varepsilon, \end{aligned}$$

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where the second inequality follows from Lemma 2.7, applied to $a^*_{s+1} \in [a_r, b[$, the switching interval.

The integer intervals $[a_1, a_2[, \ldots [a_{r-1}, a_r[$ are all shadowing by inductive assumption. The interval $[a_r, a^*_{s+1}[$ is also shadowing, since $[a_r, a^*_{s+1}[\subseteq [a_r, b[$, and the latter is shadowing by inductive assumption. Additionally, $[a^*_{s+1}, b'[$ is shadowing by the arguments above as well as the definition of b'.

Case B: $A \cap]a^*_s, \max I_s] = \emptyset$. In this case, we define

$$\begin{aligned} I_{s+1} &:= I_s := \{[a_1, a_2[, \ldots, [a_r, b[\}; \\ a^*_{s+1} &:= \max I_s + 1; \\ S &:= s + 1. \end{aligned}$$

By our assumption of this case, $a_{s+1} > b$, for I_s,

$$A_{a^*} = A \cap [a_1, a_s] = A \cap [a_1, b[.$$

In this case, for all $a_1 < k < b$,

$$G^{I_S}(k) = G^{I_S}_{a^*_s}(k) < 1,$$

so I_S is a PSI by the definition of b.

Note that in Case B, the constructed sequence (I_s, a^*_s) satisfies the first three conditions (1), (2), and (3) by construction. It remains to check that (I_s, a^*_s) also satisfies conditions (4) and (5).

Condition (4) follows by induction hypothesis.

Then, we check condition (5). By construction, I is a PSI, so it remains to check that it is maximal. Note that by our construction in Case B, $I_S = I_{S-1}$. By our definition of b' for I_{S-1}, we know that I_S is right maximal.

Now, it remains to show that for all $x \in X$ with $|(f^n)'x \to \infty|$, the inductions end up with Case B within a finite amount of steps.

We prove by contradiction. Note that for each $x \in X$, the collection of visits near the critical set A is a monotone non-decreasing sequence in integers. Thus, by induction, when s goes to infinity, a^*_s is also monotonely non-decreasing. Thus, $a^*_s \geq s$. Hence, for all $k \geq a_1$, $k \leq a^*_k < b$, and $k \in [a_1, b[$.

On the one had, note that by definition of IPSI, for this k,

$$1 > G^{I_k}_{a^*_k}(k) \geq |(f^{k-a_1})'x_{a_1}|.$$

On the other hand, we have assumed $|(f^{k-a_1})'x_{a_1}| \to \infty$. For k large enough, this yields a contradiction.

This finishes the proof of the claim. □

Just take $I = I_S$ and we have proved the proposition. □

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2.3. *Proof of Theorem 2.1.* We start with one basic lemma from abstract ergodic theory.

LEMMA 2.9. *Let $(X, \mathcal{B}, \nu)$ be a probability space and $f : X \to X$ be a ν-ergodic map. Let $\Psi : X \to [-\infty, \infty)$ be a $L^1(\nu)$-function such that*

$$\int \Psi \, d\nu \neq 0.$$

If for all $x \in X$ there are two sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ such that:

(1) $a_n \leq n \leq b_n$, $a_n \uparrow \infty$;

(2) *for all $x \in X$,*

$$\inf_{n\geq 1} \sum_{i=a_n}^{b_n-1} \Psi(x_i) \geq 0;$$
$$\eta(x) := \sup_{n\geq 1} \sum_{i=a_n}^{b_n-1} \Psi(x_i) < \infty,$$

then for ν-a.e. $x \in X$, $b_n - n = o(n)$, $n - a_n = o(n)$.

Proof. Since $b_n \geq n$, $b_n \to \infty$ for all $x \in X$.

Since $\nu \in \mathbb{P}_{\text{erg}}(f)$, by Birkhoff's ergodic theorem, for ν-a.e. $x \in X$,

$$\sum_{i=0}^{b_n-1} \Psi(x_i) = b_n \cdot \int \Psi \, d\nu + o(b_n),$$
$$\sum_{i=0}^{a_n-1} \Psi(x_i) = a_n \cdot \int \Psi \, d\nu + o(a_n).$$

Subtracting the second equation from the first, we have

$$\sum_{i=a_n}^{b_n-1} \Psi(x_i) = (b_n - a_n) \cdot \int \Psi \, d\nu + o(b_n) + o(a_n). \tag{2.2}$$

By the first assumption, $a_n \leq b_n$. Hence, an $o(a_n)$ is also an $o(b_n)$. Moreover, by the second assumption, $0 \leq \sum_{i=a_n}^{b_n-1} \Psi x_i \leq \eta$, $\sum_{i=a_n}^{b_n-1} \Psi x_i = o(b_n)$. Replacing corresponding terms in (2.2), we have

$$0 = (b_n - a_n) \cdot \int \Psi \, d\nu + o(b_n).$$

Now that $\int \Psi \, d\nu \neq 0$ as assumed, we have $b_n - a_n = o(b_n)$.

Since $b_n - a_n = |b_n - n| + |a_n - n|$,

$$b_n - n = o(n) = a_n - n.$$

□

Now, we prove Theorem 2.1 using Proposition 2.8 and Lemma 2.9.

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Proof. Fix $\varepsilon > 0$, $L \in \mathbb{N}^+$.

Fix δ_1 as in Proposition 2.8. Then, take $0 < \delta < \delta_1$ such that

$$-\log \delta > L \cdot \left(L + \frac{K+1}{\lambda} \right).$$

Let $\nu \in \mathbb{P}_{\text{erg}}(f)$ with $\lambda(\nu) > 0$.

Step 1: The construction of $\mathcal{S}(x)$. This implies $\nu(X_0) = 1$, where $X_0 := \{x \in X : |(f^n)'x| \to \infty\}$. By Proposition 2.8, there exists $X_1 \subseteq X_0$, $\nu(X_1) = 1$, such that for all $x \in X_1$, there exists a right maximal PSI $J_1(x)$, with $\min J_1(x) = \min A(x)$. Consecutively, given $l \in \mathbb{N}^+$, applying Proposition 2.8 again to X_l with $a_{l+1} := \min\{a \in A(x) : a > \max J_l(x)\}$, we get X_{l+1} with $\nu(X_{l+1}) = 1$ and $J_{l+1}(x)$, PSI with $\min J_{l+1}(x) = a_{l+1}$. Take $X' := \bigcap_{l \geq 0} X_l$. Then, $\nu(X') = 1$ and for all $x \in X'$, there exists a collection of finite right maximal PSIs $\mathcal{E}(x) := \{J_l = J_l(x)\}_{l \in \mathbb{N}^+}$.

Fix $x \in X'$. We choose the initial position a_{l+1} to be $\min\{a \in A : a > \max J_l\}$. So, by Lemma 2.5, each J, $J' \in \mathcal{E}(x)$ are disjoint. Thus, we may assume $\max J_l < \min J_{l+1}$ for all $l \in \mathbb{N}$. In particular, given $x \in X'$, and J, $J' \in \mathcal{E}(x)$, J and J' cannot be nested. Moreover, for all $J \in \mathcal{E}(x)$, by Lemma 2.6,

$$\begin{aligned} |\text{supp } J| &\geq \sum_{a_i \in A \cap J} \left(l(a_i) - \frac{K+1}{\lambda} \log 2 \right) \\ &\geq \sum_{a_i \in A \cap J} \left(l(a_i) - \frac{K+1}{\lambda} \right). \end{aligned}$$

Remember that each $J \in \mathcal{E}(x)$ is a PSI, which is a collection of shadowing intervals. For each $x \in X'$, define

$$\mathcal{S}(x) := \{I = [a, b[\subseteq \mathbb{N} : \text{ there exists } J(x) \in \mathcal{E}(x) \text{ such that } I \in J(x), |I| \geq L\},$$

that is, $\mathcal{S}(x)$ consists of shadowing intervals in some $J \in \mathcal{E}(x)$ with length greater than L.

Step 2: The 'tail' interval. For the sake of simplicity, we first introduce some notation. Abbreviate $\overline{J} := \text{supp } J$ for $J \in \mathcal{E}(x)$. For all $n \in \mathbb{N}^+$, let l_n be the maximal integer such that $\min \overline{J}_{l_n} < n$.

Claim that if we define

$$a_n = \begin{cases} \min \overline{J}_{l_n} & \text{if } n \in \overline{J}_{l_n}, \\ n & \text{otherwise,} \end{cases} \tag{1}$$

$$b_n = \begin{cases} \max \overline{J}_{l_n} & \text{if } n \in \overline{J}_{l_n}, \\ n & \text{otherwise,} \end{cases} \tag{2}$$

then $b_n - a_n = o(n)$. To prove the claim, we use Lemma 2.9.

Define

$$\Psi(x) := \begin{cases} -\lambda(f) & \text{if } |f'x| \geq \delta, \\ -(\log |f'x| + (K+1)\log 2) & \text{if } |f'x| < \delta. \end{cases} \tag{3}$$

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We check that a_n, b_n, and Ψ satisfy all the hypotheses of Lemma 2.9. First, we check that $\int \Psi \, d\nu \neq 0$. By definition,

$$\begin{aligned}\int \Psi \, d\nu &= \int_{\{|f'|<\delta\}} -(\log|f'x| + (K+1)\log 2)\, d\nu + \int_{\{|f'|\geq\delta\}} -\lambda(f)\, d\nu \\ &\leq -\int \log|f'x|\, d\nu - \int_{\{|f'|<\delta\}} (K+1)\log 2\, d\nu < 0.\end{aligned}$$

The first inequality follows from the fact that $\lambda(f) \geq \log|f'x|$ for all $x \in X$. The last inequality follows from our assumption that $\lambda(\nu) = \int \log|f'|\, d\nu > 0$ and our choice of δ being small.

When $n \notin \overline{J}_{l_n}$, then $a_n = b_n = n$ and $\sum_{i=a_n}^{b_n-1} \Psi(x_i) = 0$. Assume $n \in \overline{J}_{l_n}$. Remember that by definition, J_{l_n} is a right maximal PSI. Thus, $G^{J_{l_n}}(\max \overline{J}_{l_n} + 1) \geq 1$, which implies

$$\begin{aligned}0 &\leq \log G^{J_{l_n}}(\max \overline{J}_{l_n} + 1) \\ &= \sum_{j \in \overline{J}_{l_n} \cap A^c} \lambda(f) + \sum_{a \in \overline{J}_{l_n} \cap A} (\log|f'| + (K+1)\log 2) \\ &= -\sum_{i=a_n}^{b_n} \Psi(x_i) \\ &= -\sum_{i=a_n}^{b_n-1} \Psi(x_i) + \lambda(f),\end{aligned}$$

that is, $\sum_{j=a_n}^{b_n-1} \Psi(x_j) \leq \lambda(f)$. The last equality follows from Lemma 2.3: $b_n = \max \overline{J}_{l_n} + 1 \notin A$, so $\log \Psi(x_{b_n}) = \lambda(f)$.

Meanwhile, since J_{l_n} is a PSI, $G^{J_{l_n}}(\max \overline{J}_{l_n}) \leq 1$. So, $\sum_{j=a_n}^{b_n-1} \Psi(x_j) \geq 0$. So, a_n, b_n, and Ψ satisfy all the requirements of Lemma 2.9.

Step 3: Verifying the properties in Theorem 2.1.

(1) Since elements in $\mathcal{S}(x)$ are constructed in Proposition 2.8 by visits to critical points, the map $x \mapsto \mathcal{S}(x)$ is naturally measurable.

(2) Note that since elements in $\mathcal{E}(x)$ are finite, pairwise disjoint, so will elements be in $\mathcal{S}(x)$. Moreover, by construction, elements in $\mathcal{S}(x)$ are naturally shadowing intervals.

(3) Let

$$\psi(x) := 1_{\{|f'|<\delta\}} \left(\frac{-\log|f'| + K + 1}{\lambda(f)} \right).$$

Note that $\psi(x) \geq 0$ and $\psi(x_j) > 0$ if and only if $j \in \bigcup \mathcal{E}(x)$. In particular,

$$\sum_{i=1}^{l_n-1} |\overline{J}_i| \leq \sum_{i=0}^{n-1} \psi(x_i) \leq \sum_{i=1}^{l_n} |\overline{J}_i|.$$

Thus, $\sum_{i=1}^{l_n-1} |\overline{J}_i| = \sum_{i=0}^{n-1} \psi(x_i) + o(n)$, by Step 2.

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Now,

$$\sum_{I\in\mathcal{S}(x),I^-<n} |I| \le \sum_{i=0}^{n-1}(L\cdot 1_{\{|f'|<\delta\}}).$$

Thus, by our choice of δ,

$$\begin{aligned}|\mathcal{S}(x)\cap[0,n[| &= \sum_{I\in\mathcal{S}(x),I^-<n} |I| \ge \sum_{i=1}^{l_n-1}|\overline{J}_i| - \sum_{i=0}^{n}(L\cdot 1_{\{|f'|<\delta\}})\\ &\ge \sum_{i=0}^{n-1}\psi(x_i) - \sum_{i=0}^{n}(L\cdot 1_{\{|f'|<\delta\}}) + o(n).\end{aligned}$$

The proof is concluded. □

3. *The entropy gap*

3.1. *Conditions for entropy gap.* In this section, we summarize our previous construction and deduce an entropy gap.

Let X be a compact metric space and $f : X \to X$ be a continuous map, with $h_{\text{top}}(f) > 0$. Fix $\varepsilon > 0$.

A system of coverings is a family $\{X_I\}$ where $I = [a, b[\subseteq \mathbb{N}$ ranges over integer intervals and each $X_I \subseteq X$ is a finite set. The entropy of this system $\{X_I\}$ is defined as

$$h(\{X_I\}) := \limsup_{n\to\infty} \max_{|I|=n} \frac{1}{n}\log|X_I|.$$

Take a measurable function $\varphi = \varphi_L : X \to \mathbb{N}\cup\{\infty\}$. We say that a subset $X' \subseteq X$ admits a partial shadowing by the system of covering $\{X_I\}$, with minimum length L, controlled by φ, if for all $x \in X'$, there is a collection of disjoint integer intervals, denoted by $\mathcal{E}(x) = \mathcal{E}_L(x)$, such that:

(1) each $I \in \mathcal{E}(x)$ is finite, with $|I| \ge L$;
(2) for each $I = [a, b[\in \mathcal{E}(x)$, there exists $z \in X_I$ such that $x_a \in B(z, \varepsilon, |I|)$;
(3) the map $x \mapsto \mathcal{E}(x)$ is measurable on X: for each collection of subsets J, $\mathcal{E}_n^J(x) := \{x \in X' : \{x \in X' : \{I \cap [0, n[: I \in \mathcal{E}(x)\} = J\}\}$ is measurable;
(4) for all $x \in X'$,

$$\liminf_{n\to\infty} \frac{1}{n}\left(\sum_{I\in\mathcal{E}_n(x)} |I| - \left(1-\frac{1}{L}\right)\sum_{i=0}^{n-1}\varphi(x_i)\right) \ge 0,$$

where $\mathcal{E}_n(x) := \{I \cap [0, n[: I \in \mathcal{E}(x)\}$.

We now state and prove an entropy gap given by partial shadowing. Before that, we recall some notation.

To begin with, we state and prove a lemma concerning combinatorics of subintervals inside $[0, n[$. This is the reason we take subintervals of length greater than L in the definition of partial shadowings.

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Definition 8. Given $L \in \mathbb{N}^+$, for all $n \in \mathbb{N}$, denote by

$$\mathcal{T}_L(n) := \{\{I_j\} : I_j \subseteq [0, n[,\ I_{j_1} \cap I_{j_2} = \emptyset,\ |I_j| \geq L\}.$$

Call the elements in $\mathcal{T}_L(n)$ the *combinatorical types in* $[0, n[$.

LEMMA 3.1. *Denote* $H(t) := t \cdot \log(1/t) + (1-t) \cdot \log(1/(1-t))$ *for* $t \in (0, 1)$. *Then, for all* $L \geq 4$,

$$|\mathcal{T}_L(n)| \leq \exp\left(n \cdot H\left(\frac{2}{L}\right) + o(n)\right).$$

Proof. Note that $H(t)$ is a function defined on $(0, 1)$, increasing from 0 to $\frac{1}{2}$ and decreasing from $\frac{1}{2}$ to 1.

By definition, every finite collection in $\mathcal{T}_L(n)$ consists of disjoint intervals in $[0, n[$ of lengths at least L. Thus, there can be at most $\llcorner n/L \lrcorner$ such intervals maximal in the union. These give

$$\sum_{i=0}^{\llcorner n/L \lrcorner} \binom{n}{2i}$$

many possible types. By assumption, $(1/L) < \frac{1}{4}$. Thus, each $\binom{n}{2i} \leq \binom{n}{2\llcorner n/L \lrcorner}$ and

$$\begin{aligned}\sum_{i=0}^{\llcorner n/L \lrcorner} \binom{n}{i} &\leq \frac{1}{2} \cdot 2^{\llcorner \frac{n}{L} \lrcorner} \cdot \binom{n}{2^{\llcorner \frac{n}{L} \lrcorner}} \\ &\leq \frac{1}{2} \exp\left(n \cdot H\left(\frac{2}{L}\right) + o(n)\right)\end{aligned}$$

using de Moivre's approximation. □

From the definition of topological entropy, we see the following lemma.

LEMMA 3.2. *For all* $\gamma > 0$, $\varepsilon > 0$, *there exists* $C_{\gamma,\varepsilon} > 0$ *such that for all* $n \in \mathbb{N}^+$,

$$r\left(f, n, \frac{\varepsilon}{2}, X\right) \leq C_{\gamma,\varepsilon} \cdot \exp(n \cdot (h_{\text{top}}(f) + \gamma)).$$

LEMMA 3.3. *If* $N = \sum n_i$, *where* $N, n_i \in \mathbb{N}^+$, *then*

$$r(f, N, \varepsilon, X) \leq \prod r\left(f, n_i, \frac{\varepsilon}{2}, f^{n_1+\cdots+n_{i-1}} X\right).$$

Proof. Simply note that given a $((\varepsilon/2), n)$-Bowen ball $B(z, n, (\varepsilon/2))$ of $f^m X$ and a covering $\mathcal{B}$ of X by $((\varepsilon/2), m)$-Bowen balls, each intersection $f^{-m} B(z, n, (\varepsilon/2)) \cap B$ with $B \in \mathcal{B}$ is contained in a $(\varepsilon, m+n)$-Bowen ball. □

THEOREM 3.4. *Let* $f : X \to X$ *be a continuous map over a compact metric space.*

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Given $\varepsilon > 0$, $0 < \gamma < \frac{1}{2}$, $L \in \mathbb{N}_+$, assume there exists a partial shadowing $(X', \varepsilon, L, \varphi, \{X_I\}, \mathcal{E}(x))$. For all $\nu \in \mathbb{P}_{\mathrm{erg}}(f)$, if $\nu(X') > 0$, then

$$h(\nu, \varepsilon) \le h_{\mathrm{top}}(f) - \left(1 - \frac{1}{L}\right) \cdot \int \varphi \, d\nu \cdot h_{\mathrm{top}}(f) - h(\{X_I\}) + K + 2\gamma,$$

where $K = K(\varepsilon, \gamma, L)$, and for all $\varepsilon, \gamma > 0$,

$$\lim_{L \to \infty} K(\varepsilon, \gamma, L) = 0.$$

Proof. Given ε, γ, let $(X', \varepsilon, L, \varphi, \{X_I\}, \mathcal{E}(x))$ be a partial shadowing on (X, f). Take $C := C_{\gamma,\varepsilon}$ as in Lemma 3.2 such that

$$r\left(f, n, \frac{\varepsilon}{2}, X\right) \le C \cdot \exp(n \cdot (h_{\mathrm{top}}(f) + \gamma)).$$

Since we have defined $h(\{X_I\}) := \limsup_{n\to\infty} \max_{|I|=n} (1/n) \log |X_I^\varepsilon|$, for $\gamma > 0$ given, there exists $D = D_\gamma$ such that for all $n \in \mathbb{N}^+$,

$$\max_{|I|=n} |X_I| \le D \cdot \exp(n \cdot (h(\{X_I\}) + \gamma)).$$

Take one $\nu \in \mathbb{P}_{\mathrm{erg}}(f)$.

Step 1: Finding a suitable Borel subset. By definition of a partial shadowing, the function φ is strictly positive on X. So, there exists a Borel subset $Z \subseteq X'$ such that $\nu(Z) = \nu(X')$ and for all $x \in Z$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(x_i) = \int \varphi \, d\nu.$$

We then subdivide Z into smaller pieces. Define the n-combinatorical types of $\mathcal{E}(x)$ to be

$$T(\mathcal{E}(x), n) := \bigcup_{I \in \mathcal{E}(x)} I \cap [0, n[.$$

Note that $T(\mathcal{E}(x), n) \in \mathcal{T}_L(n)$ by the zeroth assumption on partial shadowing. For a given type $T \in \mathcal{T}_L(n)$, let

$$Z^T := \{x \in Z : T(\mathcal{E}(x), n) = T\}.$$

By the second assumption on partial shadowing, Z^T is a measurable subset of X. By Lemma 3.1,

$$|\mathcal{T}_L(n)| \le \exp\left(n \cdot H\left(\frac{2}{L}\right) + o(n)\right).$$

Thus, for all $n \in \mathbb{N}^+$,

$$r(f, n, \varepsilon, Z) \le \exp\left(n \cdot H\left(\frac{2}{L}\right) + o(n)\right) \cdot \max_{T \in \mathcal{T}(x,n)} \{r(f, n, \varepsilon, Z^T)\}.$$

Step 2: Entropy estimate. By construction, given each Z^T, the combinatorical type for $x \in Z^T$ is a fixed union of disjoint intervals $\mathcal{E}_n(x) = T$. Denote by T^c the collection of integer intervals which are complements of those in T.

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Note that

$$h(f,\nu,\varepsilon)=\limsup_{n\to\infty}\frac{1}{n}\log r(f,n,\varepsilon,Z)$$
$$\le\limsup_{n\to\infty}\frac{1}{n}\log\left(\exp\left(n\cdot H\left(\frac{2}{L}\right)+o(n)\right)\cdot\max_{T\in\mathcal{T}}\{r(f,n,\varepsilon_0,Z^T)\}\right).$$

Meanwhile, by our definition of Z and Lemma 3.3, for all $n\in\mathbb{N}^+$,

$$\max_{T\in\mathcal{T}}\{r(f,n,\varepsilon_0,Z^T)\}$$
$$\le\prod_{[a,a'[\in T}r\left(f,a'-a,\frac{\varepsilon}{2},f^aZ^T\right)\prod_{[b,b'[\in T^c}r\left(f,b'-b,\frac{\varepsilon}{2},f^bZ^T\right).$$

Since each $[a,a'[$ is part of the system of covering, $r(f,a-a',\varepsilon_0/2,f^aZ_k^T)\le\max_{|I|=n}|X_I|$, as f^aZ^T could be covered by a union of $(a'-a,\varepsilon)$-Bowen balls. Thus, we may apply Lemma 3.2 to see that

$$\max_{T\in\mathcal{T}}\{r(f,n,\varepsilon_0,Z^T)\}$$
$$\le\left(\prod_{[a,a'[\in T}\max_{|I|=n}|X_I|\right)\cdot\left(\prod_{[b,b'[\in T^c}r\left(f,b'-b,\frac{\varepsilon}{2},f^bZ^T\right)\right),$$

where

$$\prod_{[a,a'[\in T}\max_{|I|=n}|X_I|\le\exp\left((h(\{X_I\})+\gamma)\cdot\left(\sum_{I\in T}|I\cap[0,n[|\right)+|T|\log D\right),$$

$$\prod_{[b,b'[\in T^c}r\left(f,b'-b,\frac{\varepsilon}{2},f^bZ^T\right)\le\exp\left((h_{\mathrm{top}}(f)+\gamma)\cdot\left(\sum_{I\in T^c}|I\cap[0,n[|\right)+|T^c|\log C\right)$$

for all $x\in Z^T$.

Note that, by our assumption (4) and assumption (1), for all $x\in Z^T$, $T=\mathcal{E}_n(x)$ and

$$\liminf_{n\to\infty}\frac{1}{n}\left(\sum_{I\in\mathcal{E}_n(x)}|I|-\left(1-\frac{1}{L}\right)\sum_{i=0}^{n-1}\varphi(x_i)\right)\ge0,$$
$$\limsup_{n\to\infty}\frac{|\mathcal{E}_n(x)|}{n}\le\frac{1}{L}.$$

Thus,

$$h(f,\nu,\varepsilon)\le H\left(\frac{2}{L}\right)+\frac{\log D}{L}+\frac{\log C}{L}+h_{\mathrm{top}}(f)$$
$$-\left(\left(1-\frac{1}{L}\right)\int\varphi\,d\nu\right)(h_{\mathrm{top}}(f)-h(\{X_I\}))+2\gamma.$$

Just let

$$K(\varepsilon,\gamma,L):=\lim_{n\to\infty}H\left(\frac{2}{L}\right)+\frac{\log C}{L}+\frac{\log D}{L}.$$

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With Lemma 3.1, one verifies easily that $K(\varepsilon, \gamma, L) \to 0$ when ε and γ are fixed, and $L \to \infty$. □

3.2. *Proof of the main theorem.* Now, we use Theorem 3.4 to prove the main theorem, Theorem 1.3.

Proof. First, fix $\gamma_0 > 0$.

Then, take ε_0 small, so that $h(f, \nu, \varepsilon_0) < h(f, \nu) + \gamma_0$ for all $\nu \in \mathbb{P}_{\text{erg}}(f)$. This is possible since $C(f)$ is a finite set and f is C^∞. Third, note that by Theorem 3.4, with ε_0, γ_0 fixed,

$$\lim_{L\to\infty} K(\varepsilon_0, \gamma_0, L) = 0.$$

Thus, we may take L_0 large so that $K(\varepsilon_0, \gamma_0, L_0) < \gamma_0$ and $(1/L_0) < \gamma_0$.

By assumption, $\alpha := \lim_{\delta\to 0} \limsup_{k\to\infty} \int_{\{|f'|<\delta\}} -\log|f'|\, d\mu_k > 0$. So, we may take $\delta_0 > 0$ and take a subsequence if necessary, so that for all $k \in \mathbb{N}_+$,

$$\frac{1}{\lambda} \int_{\{|f'|<\delta_0\}} -\log|f'|\, d\mu_k > \frac{\alpha}{\lambda} - \gamma_0.$$

Finally, shrink δ_0, so that we get the sequence $\mathcal{S}(x)$ as in Theorem 2.1 for $([0, 1], f, \varepsilon_0, L_0)$. Then, define

$$\varphi_0(x) := -1_{\{x:|f'x|<\delta_0\}} \frac{\log|f'(x)|}{\lambda(f)}.$$

Take one μ_k. We check that the $\mathcal{S}(x)$ satisfies all the conditions for being partially shadowing. Take X' to be the subset such that:

(1) $\mu_k(X') > \frac{1}{2}$;

(2) $|\mathcal{S}(x) \cap [0, n[| \geq (1 - (1/L_0)) \sum_{i=0}^{n-1} \varphi_0(x_i) - g_n(x)$.

The existence of X' is guaranteed by Theorem 2.1.

For each I, integer interval, define X_I to be $C(f)$. In this case, $h(\{X_I\}) = 0$. We now verify that $(X', \varepsilon_0, L_0, \varphi_0, C(f), \mathcal{S}(x))$ really gives a partial shadowing for (f, μ_k).

(1) By construction, for all $I \in \mathcal{S}(x)$, $|I| \geq L_0$.

(2) By construction, $I = [a, b[\in \mathcal{S}(x)$ is ε-shadowing, so $x_a \in B(c, \varepsilon, |I|)$ for some $c \in C(f)$.

(3) This is obvious, as I is characterized by visiting to $\{|f'x| < \delta_0\}$.

(4) This is property (3) of Theorem 2.1.

Now, apply Theorem 3.4 to $(X', f, \varepsilon_0.L_0, \varphi_0, \mu_k)$. We get

$$h(\mu_k, \varepsilon_0) \leq \left(1 - (1 - \gamma_0)\left(\frac{\alpha}{\lambda} - \gamma_0\right)\right) h_{\text{top}}(f) + 2\gamma_0.$$

So,

$$\begin{aligned} h_{\text{top}}(f) - h(\mu_k) &\geq h_{\text{top}}(f) - h(\mu_k, \varepsilon_0) - \gamma_0 \\ &\geq h_{\text{top}}(f) - \left(1 - (1 - \gamma_0)\left(\frac{\alpha}{\lambda} - \gamma_0\right)\right) h_{\text{top}}(f) - 3\gamma_0 \\ &\geq \frac{\alpha}{\lambda} h_{\text{top}}(f) - 4\gamma_0 - \gamma_0\left(\frac{\alpha}{\lambda} - \gamma_0\right) h_{\text{top}}(f). \end{aligned}$$

As the choice of γ_0 is arbitrary, we get the main theorem. □

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