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The restricted Burnside problem for multilinear commutators

BY PAVEL SHUMYATSKY† AND JHONE CALDEIRA SILVA‡
Department of Mathematics, University of Brasilia, Brasilia-DF, 70910-900 *Brazil.*

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Abstract

The following theorem is proved. For any positive integer n there exists t depending only on n such that if the word w is a multilinear commutator, then the class of all groups G in which $w(G)$ is locally finite and the product of any t values of w has order dividing n is a variety.

1. *Introduction*

The celebrated solution by Zelmanov of the Restricted Burnside Problem [**19**, **20**] had a profound impact on development of group theory. The methods used in the solution happened to be very effective in treatment of other problems (see for example [**12**, **22**]). It is well known that the solution of the Restricted Burnside Problem is equivalent to each of the following statements:

1·1. *The class of locally finite groups of exponent n is a variety.*

1·2. *The class of locally nilpotent groups of exponent n is a variety.*

The equivalence of 1·1 and 1·2 follows from the famous Hall–Higman Reduction Theorem [**3**].

A variety is a class of groups defined by equations. More precisely, if W is a set of words in $x_1, x_2, \ldots$, the class of all groups G such that $W(G) = 1$ is called the *variety* determined by W. By a well-known theorem of Birkhoff varieties are precisely classes of groups closed with respect to taking quotients, subgroups and cartesian products of their members. If w is a word in variables $x_1, x_2, \ldots, x_m$ we think of it as a function of m variables defined on any given group G. We denote by $w(G)$ the verbal subgroup of G generated by the values of w. In [**15**] the following questions were raised.

PROBLEM 1·3. *Let $n \geqslant 1$ and w a group-word. Consider the class of all groups G satisfying the identity $w^n \equiv 1$ and having $w(G)$ locally finite. Is that a variety?*

PROBLEM 1·4. *Let $n \geqslant 1$ and w a group-word. Consider the class of all groups G satisfying the identity $w^n \equiv 1$ and having $w(G)$ locally nilpotent. Is that a variety?*

The word w is *commutator* if the sum of the exponents of any variable involved in w is zero. According to the solution of the Restricted Burnside Problem the answer to the above

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questions is positive if $w(x) = x$. In fact it is easy to see that the answer is positive whenever w is any non-commutator word. Indeed, suppose $w(x_1, x_2, \ldots, x_m)$ is such a word and that the sum of the exponents of x_i is $r \neq 0$. Now, given any group G satisfying the identity $w^n \equiv 1$, substitute the unit for all the variables except x_i and an arbitrary element $g \in G$ for x_i. We see that g^r is a w-value for all $g \in G$. Hence G satisfies the identity $x^{nr} \equiv 1$, that is, G has finite exponent. It is now not difficult to see that Statements 1·1 and 1·2 imply positive answers to Problems 1·3 and 1·4.

Hence, Problems 1·3 and 1·4 are essentially about commutator words. It is not known whether they are equivalent. In [**14**] Problem 1·4 was solved positively in the case that w is a lower central word $[x_1, x_2, \ldots, x_k]$. In [**15**, **16**] positive solutions of Problem 1·4 for some other words w were presented. In [**17**] Problem 1·3 was solved positively in the case that w is a product of several commutators of the form $[x_1, x_2, \ldots, x_k]$. More precisely, in [**17**] the following result was proved.

THEOREM 1·5. *For any positive integers k and n there exists t depending only on k and n such that the class of all groups G having $\gamma_k(G)$ locally finite and satisfying the condition that the product of any t commutators of the form $[x_1, x_2, \ldots, x_k]$ is of order dividing n is a variety.*

The present paper deals with Problem 1·3 in the case that w is a product of several multilinear commutators. A word w is called *multilinear commutator of weight k* if it has form of a multilinear Lie monomial in precisely k independent variables. Particular examples of multilinear commutators are the derived words, defined by the equations

$$\delta_0(x) = x$$
$$\delta_k(x_1, \ldots, x_{2^k}) = [\delta_{k-1}(x_1, \ldots, x_{2^{k-1}}), \delta_{k-1}(x_{2^{k-1}+1}, \ldots, x_{2^k})],$$

and the lower central words, defined by

$$\gamma_1(x) = x,$$
$$\gamma_{k+1} = [\gamma_k(x_1, x_2, \ldots, x_k), x_{k+1}].$$

In [**15**] Problem 1·4 was solved positively in the case that w is a multilinear commutator.

The main result of the present paper is the following theorem.

THEOREM 1·6. *For any positive integer n there exists t depending only on n such that if w is a multilinear commutator, then the class of all groups G in which $w(G)$ is locally finite and the product of any t values of w has order dividing n is a variety.*

This improves Theorem 1·5 in two ways:

(i) the result is proved for any multilinear commutator (Theorem 1·5 was only about $[x_1, x_2, \ldots, x_k]$); and
(ii) the function t depends only on n (in Theorem 1·5 it also depends on the word w).

We record one immediate corollary of Theorem 1·6 that is related to the results obtained in [**13**].

COROLLARY 1·7. *Let n be a positive integer and let t be as in Theorem* 1·6. *If w is a multilinear commutator and G is a residually finite group satisfying the condition that the product of any t values of w has order dividing n, then $w(G)$ is locally finite.*

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2. *Lie algebras associated with p-groups*

We use the expression "$\{a, b, c, \ldots\}$-bounded" to mean "bounded from above by some function depending only on $a, b, c, \ldots$".

In this section we briefly describe some Lie-theoretic machinery necessary for the proof of Theorem 1·6. We use Lie methods in the spirit of Zelmanov's solution of the Restricted Burnside Problem. For the reader's convenience we collect some definitions and facts on Lie algebras associated with p-groups. For reference see [**22**].

Let L be a Lie algebra over a field $\mathbb{F}$. We use the left normed notation

$$[x_1, x_2, x_3, \ldots, x_n] = [\ldots[[x_1, x_2], x_3], \ldots, x_n],$$

for all $x_1, x_2, x_3, \ldots, x_n \in L$. An element $y \in L$ is called *ad-nilpotent* if there exists a positive integer n such that $[x, \underbrace{y, \ldots, y}_{n}] = 0$ for all $x \in L$. If n is the least positive integer with the above property, then we say that y is *ad-nilpotent of index* n.

If $X \subseteq G$ is any subset of a group G, by a *commutator in elements of* X we mean any element of G that could be obtained from elements of X by means of repeated operation of commutation with an arbitrary system of brackets including the elements of X. In particular, the elements of X are viewed as commutators with the empty system of brackets.

Let G be a finite p-group. The terms of the lower central series of G will be denoted by $\gamma_j(G)$. Write $D_i = D_i(G) = \prod_{j \cdot p^k \geqslant i} \gamma_j(G)^{p^k}$. The subgroups D_i form a central series of G known as the Zassenhaus–Jennings–Lazard series (see [**5**, chapter 8]). Let $L(G) = \bigoplus D_i/D_{i+1}$. Then $L(G)$ can naturally be viewed as a Lie algebra over the field $\mathbb{F}_p$ with p elements. In fact $L(G)$ even has the structure of a restricted Lie algebra (Lie p-algebra) but we will treat it as just a Lie algebra. We will denote by $L_p(G)$ the subalgebra of $L(G)$ generated by D_1/D_2. Fix a positive number c and assume that G is generated by $a_1, a_2, \ldots, a_m$. Let $\rho_1, \rho_2, \ldots, \rho_s$ be the list of all commutators in $a_1, a_2, \ldots, a_m$ of weight at most c. Here s obviously is $\{c, m\}$-bounded. The following lemma is implicit in Zelmanov [**21**]. A detailed proof can be found in [**12**].

LEMMA 2·1. *If $L_p(G)$ is nilpotent of class c then the group G can be written as a product $G = \langle\rho_1\rangle\langle\rho_2\rangle \cdots \langle\rho_s\rangle$ of cyclic subgroups generated by the ρ_i.*

Let $x \in G$ and let $i = i(x)$ be the largest integer such that $x \in D_i$. We denote by $\widetilde{x}$ the element $xD_{i+1} \in L(G)$.

LEMMA 2·2 (Lazard). *For any $x \in G$ we have $(ad\widetilde{x})^p = ad(\widetilde{x^p})$. In particular, if $x^n = 1$ then $\widetilde{x}$ is ad-nilpotent of index at most n.*

Any group law that holds in G implies certain polynomial identity in the algebra $L_p(G)$. In Wilson and Zelmanov [**18**] one can find a description of an effective algorithm allowing to write explicitly the polynomial identity when the group law is given. Thus, we have

LEMMA 2·3. *Let G be a group satisfying an identity $w \equiv 1$. Then there exists a non-zero Lie polynomial f whose form depends only on p and w such that the algebra $L_p(G)$ satisfies the identity $f \equiv 0$.*

In view of Lemma 2·1, it is important to have criteria for a Lie algebra to be nilpotent of bounded class. The following result was proved in [**7**, corollary of Theorem 4] using a profound theorem of Zelmanov [**21**, III(0·4)].

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THEOREM 2·4. *Let L be a Lie algebra over $\mathbb{F}_p$ generated by m elements $a_1, a_2, \ldots, a_m$. Assume that L satisfies the identity $f \equiv 0$ and that each monomial in the generators $a_1, a_2, \ldots, a_m$ is ad-nilpotent of index at most n. Then L is nilpotent of $\{f, n, m, \mathbb{F}_p\}$-bounded class.*

3. *The case of δ_k-commutators*

In this section we will prove Theorem 1·6 in the particular case that w is a δ_k-commutator. Given a group G, an element $x \in G$ will be called a *δ_k-commutator* if there exist $x_1, \ldots, x_{2^k}$ such that $x = \delta_k(x_1, \ldots, x_{2^k})$. We will use freely the fact that inverses and conjugates of δ_k-commutators are δ_k-commutators.

The following important result is due to Segal [**11**].

THEOREM 3·1. *If G is a finite soluble group generated by m elements $y_1, y_2, \ldots, y_m$, then every element of the derived group of G is a product of an m-bounded number of commutators of the form $[a, b]$, where $a \in G$ and $b \in \{y_1, y_2, \ldots, y_m\}$.*

In [**9**] Nikolov and Segal generalized the above theorem to the case where G is not required to be soluble but this will not be used in the present paper. From Segal's result we deduce the following lemma.

LEMMA 3·2. *Let G be a finite soluble group generated by m elements $g_1, g_2, \ldots, g_m$ such that the order of each g_i divides a positive integer l. Then every element of G is a product of an $\{m, l\}$-bounded number $f = f(m, l)$ of conjugates of g_i.*

Proof. By Theorem 3·1, any element of G' can be written as a product of r commutators $[a_i, b_i]$, where r is an m-bounded number, $a_i \in G$ and $b_i \in \{g_1, g_2, \ldots, g_m\}$. If $x \in G'$, we write

$$\begin{aligned} x = [a_1, b_1][a_2, b_2] \cdots [a_r, b_r] &= b_1^{-a_1} b_1 b_2^{-a_2} b_2 \cdots b_r^{-a_r} b_r \\ &= b_1^{(l-1)a_1} b_1 b_2^{(l-1)a_2} b_2 \cdots b_r^{(l-1)a_r} b_r. \end{aligned}$$

So x is a product of at most lr of conjugates of g_i. Since the order of G/G' is $\{m, l\}$-bounded, if follows that modulo G' each element of G is a product of an $\{m, l\}$-bounded number of the generators g_i. Hence, every element of G is a product of an $\{m, l\}$-bounded number $f = f(m, l)$ of conjugates of g_i.

Throughout the rest of the paper $f(m, l)$ will stand for the $\{m, l\}$-bounded number as in the previous lemma.

LEMMA 3·3. *Let G be a group, d a δ_k-commutator and x any element of G. Then $[x, \underbrace{d, \ldots, d}_{k}]$ is a δ_k-commutator.*

Proof. Note that d can be viewed as a δ_i-commutator for each $i \leqslant k$. It is clear that $[x, d]$ is a δ_1-commutator. Arguing by induction on k assume that $k \geqslant 1$ and $[x, \underbrace{d, \ldots, d}_{k-1}]$ is a δ_{k-1}-commutator. Then $[x, \underbrace{d, \ldots, d}_{k}] = [[x, \underbrace{d, \ldots, d}_{k-1}], d]$ is a δ_k-commutator.

Recall that the *Fitting subgroup* $F(G)$ of a group G is the product of all normal nilpotent subgroups of G. The *Fitting series* of G can be defined by the rules: $F_0(G) = 1$, $F_1(G) = F(G)$, $F_{i+1}(G)/F_i(G) = F(G/F_i(G))$, for $i = 1, 2, \ldots$. If G is a finite soluble group, then

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the minimal number $h = h(G)$ such that $F_h(G) = G$ is called the *Fitting height* of G. If G is finite, then $F(G)$ contains all subnormal nilpotent subgroups of G. It is well-known that every Engel element of a finite group is contained in $F(G)$ [**4**, Theorem III·6·15]. We need a few results on the Fitting height of a finite soluble group.

LEMMA 3·4. *Let G be a finite soluble group such that $h(G) = 2$. Let $x \in G$ and assume that the subgroup $\langle x, F(G)\rangle$ is nilpotent. Then $x \in F(G)$.*

Proof. Set $F = F(G)$. Since G/F is nilpotent, it follows that $\langle x, F\rangle$ is subnormal. Thus, $\langle x, F\rangle$ is a subnormal nilpotent subgroup. It follows that $x \in F$.

As usual, $G^{(k)}$ stands for the kth term of the derived series of G.

LEMMA 3·5. *Let G be a finite soluble group. Assume that the Fitting height of $G^{(k)}$ is h. Then one can choose h δ_k-commutators in G, $d_1, d_2, \ldots, d_h$, such that the subgroup $\langle d_1, d_2, \ldots, d_h\rangle$ has Fitting height precisely h.*

Proof. We denote $F_i(G^{(k)})$ by F_i, $1 \leqslant i \leqslant h$. If $h = 1$, the lemma is obvious so we assume that $h \geqslant 2$. Since $F_h = G^{(k)}$, there exists a δ_k-commutator d_1 such that $d_1 \notin F_{h-1}$. Since the Fitting height of $G^{(k)}/F_{h-2}$ is 2, by Lemma 3·4 the subgroup $\langle d_1, F_{h-1}\rangle/F_{h-2}$ is not nilpotent. Therefore the image of d_1 in $\langle d_1, F_{h-1}\rangle/F_{h-2}$ is not Engel. Thus, there exists $x_1 \in F_{h-1}$ such that $[x_1, \underbrace{d_1, \ldots, d_1}_{i}] \notin F_{h-2}$ for any positive integer i. Put $d_2 = [x_1, \underbrace{d_1, \ldots, d_1}_{k}]$. By Lemma 3·3, d_2 is a δ_k-commutator. Moreover, $d_2 \in F_{h-1}\backslash F_{h-2}$.

Similarly, since $\langle d_2, F_{h-2}\rangle/F_{h-3}$ is not nilpotent, there exists $x_2 \in F_{h-2}$ such that $[x_2, \underbrace{d_2, \ldots, d_2}_{i}] \notin F_{h-3}$ for any positive integer i. Put $d_3 = [x_2, \underbrace{d_2, \ldots, d_2}_{k}]$. Note that d_3 is a δ_k-commutator and that $d_3 \in F_{h-2}\backslash F_{h-3}$.

Continuing this process, there exists $x_{r-1} \in F_{h-(r-1)}$ such that we can choose a δ_k-commutator d_r defined by $d_r = [x_{r-1}, \underbrace{d_{r-1}, \ldots, d_{r-1}}_{k}]$, where $1 \leqslant r \leqslant h$ and $d_r \in F_{h-(r-1)}\backslash F_{h-r}$.

Now let $H = \langle d_1, d_2, \ldots, d_h\rangle$. It remains only to show that the subgroup H has Fitting height precisely h. We have $d_h \in F(H)$. Note that $d_{h-1} \in F_2(H)$ but $d_{h-1} \notin F(H)$ (otherwise, there would exist j such that $[d_h, \underbrace{d_{h-1}, \ldots, d_{h-1}}_{j}] = 1$). More generally, $d_{h-r} \in F_{r+1}(H)$ but $d_{h-r} \notin F_r(H)$, where $0 \leqslant r \leqslant h-1$. We conclude that H has Fitting height h, as required.

Having fixed a positive integer k, the symbol w_j will denote the word that is the product of j δ_k-commutators, all of whose entries are independent. Thus, for example, the symbol w_2 will denote the word $\delta_k(x_1, \ldots, x_{2^k})\delta_k(y_1, \ldots, y_{2^k})$. It is clear that if an element of a group G is a w_i-value, then it is also a w_j-value for any $i \leqslant j$. Therefore, for any $i \leqslant j$, the identity $w_j^n \equiv 1$ implies the identity $w_i^n \equiv 1$ in G.

A well-known corollary of the Hall–Higman theory [**3**] says that the Fitting height of any finite soluble group of exponent n is bounded by the number of prime divisors of n, counting multiplicities. We will denote the number by $h(n)$.

LEMMA 3·6. *Let j and n be positive integers with the property that $j \geqslant f(h(n) + 1, n)$. Let G be a finite soluble group satisfying the identity $w_j^n \equiv 1$. Then $h(G^{(k)}) \leqslant h(n)$.*

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Proof. Assume the lemma is false. Set $h = h(n)$. By Lemma 3·5 we can choose $h+1$ δ_k-commutators $d_1, d_2, \ldots, d_{h+1}$ in G such that the subgroup $H = \langle d_1, d_2, \ldots, d_{h+1}\rangle$ has Fitting height at least $h+1$. By Lemma 3·2 any element of H can be written as a product of at most $f(h(n)+1, n)$ conjugates of d_i. By the hypothesis any such product has order dividing n. So H is of exponent n, whence $h(H) \leqslant h$, a contradiction.

COROLLARY 3·7. *Under the hypothesis of Lemma* 3·6, *we have* $h(G) \leqslant h(n) + k$.

Following [**3**] we call a group G *monolithic* if it has a unique minimal normal subgroup which is non-abelian simple.

PROPOSITION 3·8. *Let j, j_1 and n be positive integers such that $j \geqslant j_1 n + 1$ and let G be a finite group satisfying the identity $w_j^n \equiv 1$. Assume that G has no non-trivial normal soluble subgroups. Then G possesses a normal subgroup L such that L is residually monolithic and G/L residually belongs to the class of finite groups satisfying an identity $w_{j_1}^{n/p} \equiv 1$ for some prime divisor p of n.*

Proof. Let M be a minimal normal subgroup of G. We know that $M \cong S_1 \times S_2 \times \cdots \times S_r$, where $S_1, S_2, \ldots, S_r$ are isomorphic simple groups. The group G acts on M by permuting the simple factors so we obtain a representation of G by permutations of the set $\{S_1, S_2, \ldots, S_r\}$. Let L_M be the kernel of the representation. Choose in S_1 a non-trivial element b of the form w_1, and let p be a prime divisor of the order of b. We want to show that G/L_M satisfies the identity $w_{j_1}^{n/p} \equiv 1$. Suppose this is not true. Let $q = p^\alpha$ be the largest power of p dividing n. Since the identity $w_{j_1}^{n/p} \equiv 1$ does not hold in G/L_M, there exists an element $a \in G$ of the form w_{j_1} such that q divides the order of a modulo L_M. Write $n = n_1 q$ and $a_1 = a^{n_1}$. Then a_1 is an element of the form $w_{j_1 n_1}$ that has order q and permute regularly some q factors in $\{S_1, S_2, \ldots, S_r\}$. Without any loss of generality we will assume that S_1 is one of those factors. Write

$$(ba_1)^q = bb^{a_1^{-1}} b^{a_1^{-2}} \cdots b^{a_1}.$$

Since each of the elements $b^{a_1^{-i}}$ belongs to a different subgroup S_i, the product $bb^{a_1^{-1}} b^{a_1^{-2}} \cdots b^{a_1}$ has the same order as b. Thus, $(ba_1)^q$ has order divisible by p so the order of ba_1 is divisible by $p^{\alpha+1}$. However ba_1 is of the form w_j so its order must divide n. This contradiction shows that indeed G/L_M satisfies the identity $w_{j_1}^{n/p} \equiv 1$.

Let now L be the intersection of all the subgroups L_M, where M ranges through the minimal normal subgroups of G. The previous paragraph implies that the proof of the proposition will be completed once it is shown that L is residually monolithic. If R is the product of the minimal normal subgroups of G, it is clear that R is the product of pairwise commuting simple groups $S_1, S_2, \ldots, S_t$ and that L is the intersection of the normalizers of S_i. Since G has no non-trivial normal soluble subgroups, it follows that $C_G(R) = 1$ and therefore any element of L induces a non-trivial automorphism of some the S_i. Let ρ_i be the natural homomorphism of L into the group of automorphisms of S_i. It is easy to see that the image of ρ_i is monolithic and that the intersection of the kernels of all ρ_i is trivial. Hence L is residually monolithic.

Let us now choose a function $t(x)$ defined for any positive integer x with the following two properties:

(1) $t(x) \geqslant f(h(x)+1, x)$ for all x;
(2) $t(x) \geqslant x \cdot t(y) + 1$ for all x, y such that y is a proper divisor of x.

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Such a function can be constructed using induction on x. Indeed, define $t(1) = f(1, 1)$. Now suppose that $t(x)$ is defined for all $x \leqslant n-1$. Let M be the maximum of $f(h(n)+1, n)$ and $n \cdot t(y) + 1$, where y ranges through the set of all proper divisors of n. Put $t(n) = M$. Thus we established the existence of a function with the desired properties. Eventually, it will be shown that the chosen function satisfies the hypothesis of Theorem 1·6.

PROPOSITION 3·9. *Let m, n, l be positive integers and $t = t(n)$. Let G be a finite nilpotent group satisfying the identity $w_t^n \equiv 1$. Assume that G can be generated by m elements $g_1, g_2, \ldots, g_m$ such that each g_i and each commutator of the form $[g, x]$, where $g \in \{g_1, g_2, \ldots, g_m\}$, $x \in G$, have order dividing l. Then the order of G is bounded by a function depending only on k, m, n, l.*

Proof. Since G is a finite nilpotent group, it is clear that any prime divisor of $|G|$ is a divisor of l. Hence, it is sufficient to bound the order of the Sylow p-subgroup of G for any prime p. We can pass to the quotient $G/O_{p'}(G)$. Thus, G can be assumed to be a p-group and n a p-power.

Suppose first that G is soluble with derived length $j \leqslant k$. If G is abelian, it is easy to see that $|G|$ is $\{k, m, n, l\}$-bounded. Arguing by induction on j, we assume that $|G : G^{(j-1)}|$ is $\{k, m, n, l\}$-bounded.

Suppose that $G^{(j-1)}$ is central. In this case, $|G : Z(G)|$ is $\{k, m, n, l\}$-bounded and Schur's Theorem [**10**, p. 102] guarantees that so is $|G'|$. Since G can be generated by m elements of order dividing l, it follows that G has $\{k, m, n, l\}$-bounded order.

Let us see what happens if $G^{(j-1)}$ is not central. Consider the subgroup

$$A = \left\langle \left[g_1, G^{(j-1)}\right], \left[g_2, G^{(j-1)}\right], \ldots, \left[g_m, G^{(j-1)}\right] \right\rangle.$$

Clearly, A is normal in G. Applying the results of the previous paragraph to the quotient G/A, it follows that A has $\{k, m, n, l\}$-bounded index in G. Now Schreier's Theorem says that A can be generated by a $\{k, m, n, l\}$-bounded number of elements. Since A is abelian and, by the hypothesis, the order of each commutator of the form $[g, x]$ divides l, it follows that A has exponent l. Hence, $|A|$ and therefore $|G|$ is $\{k, m, n, l\}$-bounded.

Now consider the general case, that is, we do not assume anymore that G is soluble with derived length at most k. Applying the results of the previous paragraph to the quotient $G/G^{(k)}$, it follows that $|G : G^{(k)}|$ is $\{k, m, n, l\}$-bounded. It remains to show that $|G^{(k)}|$ is $\{k, m, n, l\}$-bounded. Let r be the minimal number of generators of $G^{(k)}$. Note that r is $\{k, m, n, l\}$-bounded. A well-known corollary of the Burnside Basis Theorem [**4**] says that if a p-group is r-generated, then any generating set contains a generating set of precisely r elements. Thus, $G^{(k)}$ can be generated by r δ_k-commutators $d_1, d_2, \ldots, d_r$.

Clearly, the order of any d_i divides n. Moreover, since $[d_i, d_j]$ is again a δ_k-commutator, it follows that any commutator in $d_1, d_2, \ldots, d_r$ has order dividing n. Let $L = L_p(G^{(k)})$. We know that the identity $\delta_k^n \equiv 1$ holds in $G^{(k)}$ so it follows by Lemma 2·3 that there exists a non-zero Lie polynomial f over $\mathbb{F}_p$, whose form depends only on k and n, such that the algebra L satisfies the identity $f \equiv 0$.

Consider an arbitrary Lie monomial σ in the generators $\widetilde{d}_1, \widetilde{d}_2, \ldots, \widetilde{d}_r$ of L and let ρ be the group commutator in $d_1, d_2, \ldots, d_r$ having the same arrangement of brackets as σ. The definition of L yields that either $\sigma = 0$ or $\sigma = \widetilde{\rho}$. Since $\rho^n = 1$, Lemma 2·2 implies that σ is ad-nilpotent of index at most n. Theorem 2·4 now says that L is nilpotent of class depending only on k, m and n. Combining this with Lemma 2·1 we conclude that there exists

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a $\{k, m, n\}$-bounded number s such that $G^{(k)}$ can be written as a product of at most s cyclic subgroups, each of order at most n. Therefore $G^{(k)}$ is of order at most n^s, as required.

In what follows, we denote by $\pi(G)$ the set of prime divisors of $|G|$.

PROPOSITION 3·10. *Let m, n, l be positive integers and $t = t(n)$. Let G be a finite group satisfying the identity $w_t^n \equiv 1$. Assume that G can be generated by m elements $g_1, g_2, \ldots, g_m$ such that each g_i and all commutators of the forms $[g, x]$ and $[g, x, y]$, where $g \in \{g_1, g_2, \ldots, g_m\}$, $x, y \in G$, have orders dividing l. Then the order of G is bounded by a function depending only on k, m, n, l.*

Proof. If $n = 1$, the group G is soluble of derived length at most k. If G is abelian, it is easy to see that $|G|$ is $\{k, m, n, l\}$-bounded. Arguing by induction on the derived length $j \leqslant k$ of G, we assume that $|G : G^{(j-1)}|$ is $\{k, m, n, l\}$-bounded.

Suppose first that $G^{(j-1)}$ is central. In this case, $G/Z(G)$ has $\{k, m, n, l\}$-bounded order and, by Schur's Theorem, G' has $\{k, m, n, l\}$-bounded order, too. Since G can be generated by m elements of order dividing l, it follows that $|G|$ is $\{k, m, n, l\}$-bounded.

Now suppose that $G^{(j-1)}$ is not central. Consider the subgroup

$$A = \left\langle \left[g_1, G^{(j-1)}\right], \left[g_2, G^{(j-1)}\right], \ldots, \left[g_m, G^{(j-1)}\right] \right\rangle.$$

Clearly, A is normal in G. Applying the results of the previous paragraph to the quotient G/A, it follows that A has $\{k, m, n, l\}$-bounded index in G. Now Schreier's Theorem says that A can be generated by a $\{k, m, n, l\}$-bounded number of elements. Since A is abelian and by the hypothesis, the order of each commutator of the form $[g, x]$ divides l, it follows that A has exponent l. Hence, $|A|$ and therefore $|G|$ is $\{k, m, n, l\}$-bounded.

We will now use induction on n, the case $n = 1$ being covered in the previous paragraph. Suppose that $n \geqslant 2$ and that the proposition is true for groups satisfying an identity $w_{t(n/p)}^{n/p} \equiv 1$ for a prime divisor p of n. In other words, the induction hypothesis is that there exists a $\{k, m, n, l\}$-bounded number N_0 such that if G is a finite group satisfying the identity $w_{t(n/p)}^{n/p} \equiv 1$ that can be generated by m elements $g_1, g_2, \ldots, g_m$ such that each g_i and all commutators of the forms $[g, x]$ and $[g, x, y]$ have orders dividing l, then $|G| \leqslant N_0$.

Suppose for a moment that G has no non-trivial normal soluble subgroups. Since $t \geqslant n \cdot t(n/p) + 1$, Proposition 3·8 tells us that G possesses a normal subgroup L such that L is residually monolithic and G/L residually belongs to the class of finite groups satisfying an identity $w_{t(n/p)}^{n/p} \equiv 1$ for some prime divisor p of n. It follows that G/L is residually of order at most N_0. Since G/L is m-generated, by [**2**, theorem 7·2·9], the number of distinct normal subgroups of index at most N_0 in G/L is $\{m, N_0\}$-bounded. Therefore $|G/L|$ is $\{k, m, n, l\}$-bounded. In particular, it follows that L can be generated by r elements for some $\{k, m, n, l\}$-bounded number r.

A result of Jones [**6**] says that any infinite family of finite simple groups generates the variety of all groups. It follows that up to isomorphism there exist only finitely many monolithic groups satisfying the identity $w_t^n \equiv 1$. Let $N_1 = N_1(n, t)$ be the maximum of their orders. Then L is residually of order at most N_1. Since L is r-generated, the number of distinct normal subgroups of index at most N_1 in L is $\{r, N_1\}$-bounded. Therefore L has $\{k, m, n, l\}$-bounded order. We conclude that $|G|$ is $\{k, m, n, l\}$-bounded.

Now let us drop the assumption that G has no non-trivial normal soluble subgroups. Let S be the product of all normal soluble subgroups of G. The above paragraph shows that G/S

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has $\{k, m, n, l\}$-bounded order. Since $t \geqslant f(h(n)+1, n)$, by Corollary 3·7 the Fitting height of S is $\{k, m, n, l\}$-bounded. Let $F = F(G)$ be the Fitting subgroup of G. Using induction on the Fitting height of S, we assume that F has $\{k, m, n, l\}$-bounded index in G.

Suppose first that F is central. In this case, $|G : Z(G)|$ is $\{k, m, n, l\}$-bounded and Schur's Theorem guarantees that so is $|G'|$. Since G can be generated by m elements of order dividing l, it follows that $|G|$ is $\{k, m, n, l\}$-bounded.

If F is not central, consider the subgroup

$$N = \langle [g_1, F], [g_2, F], \ldots, [g_m, F] \rangle.$$

It is easy to see that N is normal in G. Applying the results of the previous paragraph to the quotient G/N, it follows that $|G : N|$ is $\{k, m, n, l\}$-bounded. We will show that $|N|$, and therefore $|G|$, is $\{k, m, n, l\}$-bounded.

We know that N can be generated by a $\{k, m, n, l\}$-bounded number of elements. Let s be the minimal number of generators of N. Since N is nilpotent, $\pi(N)$ consists of prime divisors of l. Thus, it is sufficient to bound the order of the Sylow p-subgroup of N for every prime $p \in \pi(N)$. Let P be the Sylow p-subgroup of N and write $N = P \times O_{p'}(N)$. If $y_1, y_2, \ldots$ is the list of all elements of the form $[g_i, y]$, where $1 \leqslant i \leqslant m$ and $y \in F$, we write $b_1, b_2, \ldots$ for the corresponding projections of y_j in P. Then $P = \langle b_1, b_2, \ldots \rangle$. Since P is a finite s-generated p-group, the Burnside Basis Theorem shows that P is actually generated by s elements in the list $b_1, b_2, \ldots$. By the hypothesis, the order of each of them divides l. Each commutator of the form $[b_i, z]$ also has order dividing l. By Proposition 3·9 we conclude that P has $\{k, m, n, l\}$-bounded order. The proof is complete.

We are now ready to prove Theorem 1·6 in the case that w is a δ_k-commutator.

THEOREM 3·11. *Let n be a positive integer and $t = t(n)$. Let $\mathfrak{X}$ denote the class of all groups G having $G^{(k)}$ locally finite and any w_t-value of order dividing n. Then $\mathfrak{X}$ is a variety.*

Proof. It is easy to see that the class $\mathfrak{X}$ is closed to taking subgroups and quotients of its members. Hence, we only need to show that if D is a cartesian product of groups from $\mathfrak{X}$ then $D \in \mathfrak{X}$. Obviously, the identity $w_t^n \equiv 1$ holds in D so it remains only to show that $D^{(k)}$ is locally finite. Let T be any finite subset of $D^{(k)}$. Clearly one can find finitely many δ_k-commutators $h_1, h_2, \ldots, h_m \in D$ such that $T \leqslant \langle h_1, h_2, \ldots, h_m \rangle = H$. Thus it is sufficient to prove that the subgroup H is finite. The order of each h_i divides n. Moreover, if $h \in \{h_1, h_2, \ldots, h_m\}$ and $x, y \in H$, then each commutator of the form $[h, x]$ is a product of two δ_k-commutators and each commutator of the form $[h, x, y]$ is a product of four δ_k-commutators. It is clear from the choice of t that $t(n) \geqslant 4$ for any $n \geqslant 2$. So the order of each of the commutators divides n. Note that $D^{(k)}$ is residually locally finite. If Q is any locally finite quotient of $D^{(k)}$, by Proposition 3·10 the order of the image of H in Q is finite and $\{k, m, n\}$-bounded, so it follows that this order actually does not depend on Q. We conclude that H is finite, as required.

4. *The case of multilinear commutators*

Let k be a positive integer. In this section we denote by w a multilinear commutator of weight k. The following two lemmas are taken from [**13**] and their proofs are provided for the reader's convenience.

LEMMA 4·1. *Let G be a group. Then every δ_k-commutator in G is a w-value.*

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Proof. The case $k = 1$ is quite obvious so we assume that $k \geqslant 2$ and use induction on k. Write $w = [w_1, w_2]$, where w_1 and w_2 are multilinear commutators of weight k_1 and k_2, respectively, $k_1 + k_2 = k$, and the variables involved in one of the words w_1, w_2 do not occur in the other. Let k_0 be the maximum of k_1, k_2. By the induction hypothesis any δ_{k_0}-commutator in G is a w_1-value as well as a w_2-value. Since $w = [w_1, w_2]$, it follows that the set of w-values contains the set of elements of the form $[x, y]$, where x, y range independently through the set of δ_{k_0}-commutators. Hence any commutator of δ_{k_0}-commutators represents a w-value. It remains to remark that $k_0 + 1 \leqslant k$ so the lemma follows.

In the next lemma we will require the concept of a subcommutator of weight $s \leqslant k$ of w. This can be defined by backward induction on s in the following way. The only subcommutator of weight k of w is w itself. If $s \leqslant k - 1$, a multilinear commutator v of weight s is a subcommutator of w if and only if there exists a subcommutator u of weight $> s$ of w and a multilinear commutator v_1 such that either $u = [v, v_1]$ or $u = [v_1, v]$. It is quite obvious that if v is a subcommutator of w then $w(G) \leqslant v(G)$ for any group G.

LEMMA 4·2. *Let w be a multilinear commutator and G a soluble group in which all w-values have finite order. Then the verbal subgroup $w(G)$ is locally finite.*

Proof. Let G be a counter-example whose derived length is as small as possible, and let T be the last non-trivial term of the derived series of G. Passing to the quotient over the subgroup generated by all normal locally finite subgroups of G we can assume that G has no non-trivial normal locally finite subgroups.

Since T is abelian, it follows that no w-values lies in $T \setminus \{1\}$. Let $s = s(w, G)$ be the smallest number such that any subcommutator of weight s of w has no values in $T \setminus \{1\}$. Obviously, $s \geqslant 2$ since $T \neq 1$. We can choose a subcommutator $v = [v_1, v_2]$ of weight $\geqslant s$ of w such that both v_1 and v_2 are subcommutators of weight $< s$, at least one of which having non-trivial values in T. Let H_i be the subgroup of T generated by the v_i-values contained in T, $i = 1, 2$. By the choice of v at least one of these subgroups is non-trivial. Since v has no values in $T \setminus \{1\}$, it follows that $H_1 \leqslant C_G(v_2(G))$ and $H_2 \leqslant C_G(v_1(G))$. Taking into account that $w(G) \leqslant u(G)$ for any subcommutator u of w we conclude that H_1 and H_2 centralize the verbal subgroup $w(G)$. Hence both subcommutators v_1 and v_2 have no non-trivial value in the image of T in $G/C_G(w(G))$. This shows that $s(w, G/C_G(w(G))) \leqslant s-1$. The induction on $s(w, G)$ now shows that $w(G)/Z(w(G))$, the image of $w(G)$ in $G/C_G(w(G))$, is locally finite. Then, by Schur's Theorem, the derived group of $w(G)$ is locally finite [**10**, part 1, corollary to Theorem 4·12]. But then $w(G)$, being a group generated by elements of finite order, must be locally finite.

We observe that by Lemma 4·1 any word that is a product of t δ_k-commutators is also a product of t w-values. We are now able to conclude the proof of Theorem 1·6. In what follows the function $t(n)$ is the same as in Theorem 3·11.

THEOREM 4·3. *Let w be a multilinear commutator and n a positive integer. Let $\mathfrak{X}$ be the class of all groups G in which $w(G)$ is locally finite and any product of $t(n)$ values of w has order dividing n. Then $\mathfrak{X}$ is a variety.*

Proof. It is easy to see that the class $\mathfrak{X}$ is closed to taking subgroups and quotients of its members. Hence, we only need to show that if D is a cartesian product of groups from $\mathfrak{X}$ then $D \in \mathfrak{X}$. Obviously, D satisfies the property that any product of $t(n)$ values of w has order dividing n so it remains only to show that $w(D)$ is locally finite. Let w be of weight k.

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If $G \in \mathfrak{X}$, we have $G^{(k)} \leqslant w(G)$ so $G^{(k)}$ is locally finite. Theorem 3·11 tells us that $D^{(k)}$ is locally finite. Now, applying Lemma 4·2 with $D/D^{(k)}$ in place of G, we conclude that $w(D)$ is locally finite, as required.

Theorem 1·6 is an immediate consequence of Theorem 4·3. Let us prove Corollary 1·7. Suppose G is a residually finite group with the property that any product of $t(n)$ values of w in G has order dividing n. Let $\mathfrak{X}$ have the same meaning as in the above theorem. Then any finite quotient of G belongs to the variety $\mathfrak{X}$. However, it is clear that if a group residually belongs to a certain variety, then it actually belongs to the variety. Thus, it follows that $w(G)$ is locally finite.

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