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Abstract

We introduce a new kind of action of a relatively hyperbolic group on a CAT(0) cube complex, called a relatively geometric action. We provide an application to characterize finite-volume Kleinian groups in terms of actions on cube complexes, analogous to the results of Markovic and Haïssinsky in the closed case.

1. Introduction

The Cannon Conjecture (see [Can91, Conjecture 11.34], [CS98, Conjecture 5.1]) is one of the central problems in geometric group theory. Using the work of Agol [Ago13], Markovic [Mar13, Theorem 1.1] gave an approach to proving the Cannon Conjecture using CAT(0) cube complexes and quasi-convex codimension-1 surface subgroups. This was slightly generalized by Haïssinsky [Haï15, Theorem 1.10], who proved that a Gromov hyperbolic group whose boundary is a 2-sphere is virtually Kleinian if and only if it acts properly and cocompactly on a CAT(0) cube complex.[1]

A relative version of the Cannon Conjecture states that a relatively hyperbolic group (with abelian parabolic subgroups) whose (Bowditch) boundary is a 2-sphere is a Kleinian group (see [Kap07, Problem 57], for example). In [GMS19], it was proved that the Relative Cannon Conjecture is implied by the Cannon Conjecture.

In this paper we introduce a new kind of action of a relatively hyperbolic group on a CAT(0) cube complex, called a *relatively geometric* action (see Definition 2.1 below). Contrary to proper and cocompact actions, whose coarse geometry is that of the Cayley graph, relatively geometric actions have the coarse geometry of the coned Cayley graph (see Proposition 2.2 below), and hence their geometry can be expected to more faithfully exhibit the relatively hyperbolic geometry of groups.

It follows from the work of Cooper and Futer [CF19, Theorem 1.1] and the Sageev construction [Sag97] that if M is a finite-volume hyperbolic 3-manifold then $\pi_1(M)$ admits a relatively geometric action on a CAT(0) cube complex (see Theorem 3.1 below).

Applying the results from [GMS19, GM18, CF19], we prove the following relative version of Haïssinsky's result.

Theorem 1.1. *Suppose that $(G,\mathcal{P})$ is relatively hyperbolic, that the elements of $\mathcal{P}$ are free abelian, and that the (Bowditch) boundary of $(G,\mathcal{P})$ is a 2-sphere. Then G is Kleinian if and only if G admits a relatively geometric action on a* CAT(0) *cube complex.*

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[1] In fact, Haïssinsky proved this result more generally for hyperbolic groups with planar boundary.

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2. Relative cubulations

The theory of hyperbolic groups acting properly and cocompactly on CAT(0) cube complexes is by now well developed (see [Wis, HW12, HW15, AGM16], etc.). In the relatively hyperbolic situation, two generalizations have been previously studied: proper and cocompact actions (as in [SW15, Wis]) and proper and 'cosparse' actions (see [HW14, SW15]).

The following definition provides another condition for relatively hyperbolic groups which restricts to being proper and cocompact in case G is hyperbolic and $\mathcal{P} = \emptyset$.

Definition 2.1. Suppose that $(G, \mathcal{P})$ is a group pair. A (cellular) action of G on a cell complex X is *relatively geometric (with respect to $\mathcal{P}$)* if:

(i) ${}_G\backslash^X$ is compact;
(ii) each element of $\mathcal{P}$ acts elliptically on X; and
(iii) each stabilizer in G of a cell in X is either finite or else conjugate to a finite-index subgroup of $\mathcal{P}$.

In § 3 below we give natural examples of relatively geometric actions on CAT(0) cube complexes, provided by the work of Cooper and Futer [CF19].

The authors will investigate relatively geometric actions of relatively hyperbolic groups on CAT(0) cube complexes in future work. For the remainder of this section, we record some basic features of relatively geometric actions.

The following proposition is an immediate consequence of [CC07, Theorem 5.1].

Proposition 2.2. *Suppose that $(G, \mathcal{P})$ is relatively hyperbolic, and that G admits a relatively geometric action on a* CAT(0) *cube complex X. Then X is quasi-isometric to the coned-off Cayley graph of $(G, \mathcal{P})$, and consequently is δ-hyperbolic for some δ.*

The following proposition is an immediate consequence of [GM18, Corollary 6.5]. See [GM18, § 6] for the definition of $\mathcal{Q}$-fillings, and more context.

Proposition 2.3. *Suppose that $(G, \mathcal{P})$ is relatively hyperbolic and that G admits a relatively geometric action on a* CAT(0) *cube complex X. Let $\mathcal{Q}$ be a collection of finite-index subgroups of elements of $\mathcal{P}$ so that any infinite cell stabilizer contains a conjugate of an element of $\mathcal{Q}$. For sufficiently long $\mathcal{Q}$-fillings*

$$G \to \overline{G} = {}^G\!/_K$$

of $(G, \mathcal{P})$, the quotient ${}_K\backslash^X$ is a CAT(0) *cube complex.*

Proposition 2.3 provides one major benefit of relatively geometric actions over proper and cocompact (or proper and cosparse) actions. Namely, if the images of the elements of $\mathcal{P}$ in ${}^G\!/_K$ are hyperbolic and virtually special (for example, finite or virtually cyclic) then [GM18, Theorem D] implies that ${}^G\!/_K$ is virtually special. This allows one to prove properties of G by taking virtually special hyperbolic Dehn fillings ${}^G\!/_K$ and applying the properties of virtually special hyperbolic groups. This technique is used in the proof of Theorem 1.1 below, and will be crucial in our future work.

Relatively quasi-convex subgroups of relatively hyperbolic groups were investigated in [Hru10]. See that paper for many equivalent definitions, or [AGM09, GM17] for yet more equivalent definitions. A relatively quasi-convex subgroup H of a relatively hyperbolic group $(G, \mathcal{P})$ is *full* if, for any $g \in G$ and $P \in \mathcal{P}$, the subgroup $H^g \cap P$ is either finite or of finite-index in P.

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Proposition 2.4. *Suppose that $(G, \mathcal{P})$ is relatively hyperbolic and that $\mathcal{H}$ is a finite collection of full relatively quasi-convex codimension-1 subgroups. Then each one-ended element of $\mathcal{P}$ acts elliptically on the cube complex dual to $\mathcal{H}$.*

Proof. Let $\mathcal{H}$ be a finite collection of full relatively quasi-convex codimension-1 subgroups of G and let X be a CAT(0) cube complex dual to $\mathcal{H}$ obtained by the Sageev construction. In order to obtain a contradiction, suppose that $P \in \mathcal{P}$ is one-ended and that P does not act elliptically on X. Then any orbit $P \cdot x$ of P in X is unbounded, and for any such orbit there is a hyperplane W in X so that there are elements of $P \cdot x$ on either side of W, arbitrarily far from W.

It is straightforward to see that $\mathrm{Stab}(W) \cap P$ is a codimension-1 subgroup of P. Since P is one-ended, every codimension-1 subgroup of P is infinite. But $\mathrm{Stab}(W)$ is full, so $\mathrm{Stab}(W) \cap P$ is finite-index in P. It follows that any orbit $P \cdot x$ is contained in a bounded neighborhood of W, contradicting our choice of W. □

For a hyperbolic group G, Sageev [Sag97, Theorem 3.1] proved that the cube complex associated to a finite collection of quasi-convex codimension-1 subgroups is G-cocompact. The following is the appropriate relatively geometric version, and follows quickly from results of Hruska and Wise [HW14].

Proposition 2.5. *Suppose that $(G, \mathcal{P})$ is relatively hyperbolic, that each element of $\mathcal{P}$ is one-ended, and that $\mathcal{H}$ is a finite collection of full relatively quasi-convex codimension-1 subgroups. Then the action of G on the cube complex dual to $\mathcal{H}$ is G-cocompact.*

Proof. The condition that elements of $\mathcal{H}$ are full implies that the cubulation of each element of $\mathcal{P}$ induced by $\mathcal{H}$ (in any variation; see [HW14]) is a finite cube complex. The result now follows immediately from [HW14, Theorem 7.12]. □

The following result is a slight variation of [BW12, Theorem 5.1], and provides a useful criterion for actions to be relatively geometric.

Theorem 2.6. *Let $(G, \mathcal{P})$ be relatively hyperbolic and suppose that, for every pair of distinct points $u, v \in \partial(G, \mathcal{P})$, there is a full relatively quasi-convex codimension-1 subgroup H of G so that u, v lie in H-distinct components of $\partial G \smallsetminus \Lambda H$. Then there exist finitely many full relatively quasi-convex codimension-1 subgroups of G so that the action of G on the dual cube complex is relatively geometric.*

Proof. For *any* finite collection of full quasi-convex codimension-1 subgroups, the action on the dual cube complex is G-cocompact by Proposition 2.5, and elements of $\mathcal{P}$ act elliptically by Proposition 2.4. Therefore, it remains to prove that there is a finite collection of full relatively quasi-convex codimension-1 subgroups with respect to which the stabilizers for the dual cube complex are finite or parabolic. Thus, we need to show that there is a finite collection of such subgroups which 'cut' each loxodromic element of G. This can be achieved by applying the proof of [BW12, Theorem 5.1] directly. □

3. Finite-volume hyperbolic 3-manifolds and relatively geometric actions

In this section we explain how the works of Cooper and Futer [CF19] and Bergeron and Wise [BW12] together imply the following result. Note that the fundamental group of a finite-volume hyperbolic 3-manifold admits a canonical relatively hyperbolic structure, where the peripheral subgroups are the cusp subgroups. We always (implicitly) assume this structure.

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Theorem 3.1. *Suppose that M is a finite-volume hyperbolic 3-manifold. Then $\pi_1(M)$ admits a relatively geometric action on a cube complex.*

In the case where M is closed, $\pi_1(M)$ is hyperbolic and (with empty peripheral structure) a relatively geometric action is just a proper and cocompact action. Therefore, in the closed case, Theorem 3.1 is due to Bergeron and Wise [BW12], using work by Kahn and Markovic [KM12]. In the finite-volume non-compact case, Wise [Wis, Theorem 14.29] first proved that $\pi_1(M)$ admits a proper and cocompact action on a CAT(0) cube complex. Using [CF19, Theorem 1.2] and the results in [BW12], Cooper and Futer provided an alternative proof. It is perhaps worth remarking that our proof of Theorem 3.1 relies on the work of Cooper and Futer, but not on [Wis].

Definition 3.2. [CF19] A collection of immersed surfaces in a hyperbolic 3-manifold M is *ubiquitous* if, for any pair of hyperbolic planes $\Pi, \Pi' \subset \mathbb{H}^3$ whose distance $d(\Pi, \Pi')$ is greater than 0, there is some surface S in the collection with an embedded preimage $\widetilde{S} \subset \mathbb{H}^3$ that separates Π from Π'.

Theorem 3.3 [CF19, Theorem 1.1]. *Let M be a complete, finite-volume hyperbolic 3-manifold. Then the set of closed immersed quasi-Fuchsian surfaces in M is ubiquitous.*

Noting that the closed surfaces in Theorem 3.3 contain no parabolics, and so the corresponding subgroups of $\pi_1(M)$ are full, Theorem 3.1 is an immediate consequence of Theorems 3.3 and 2.6.

4. Criterion for Relative Cannon

For the convenience of the reader, we recall the statement of Theorem 1.1.

Theorem 1.1. *Suppose that $(G, \mathcal{P})$ is relatively hyperbolic, that the elements of $\mathcal{P}$ are free abelian, and that the (Bowditch) boundary of $(G, \mathcal{P})$ is a 2-sphere. Then G is Kleinian if and only if G admits a relatively geometric action on a* CAT(0) *cube complex.*

Proof. Suppose that G is Kleinian. Then G admits a relatively geometric action by Theorem 3.1 above.

Conversely, suppose that G admits a relatively geometric action on a CAT(0) cube complex X. Let $\mathcal{Q}$ be as in the statement of Proposition 2.3. According to Proposition 2.3, for sufficiently long $\mathcal{Q}$-fillings $G \to {}^{G}\!/_{K}$ the quotient space ${}_{K}\backslash X$ is a CAT(0) cube complex.

We consider sufficiently long co-(virtually cyclic) fillings, obtained by choosing cyclic subgroups as filling kernels.[2] According to [GMS19, Theorem 1.2], for sufficiently long such fillings the quotient ${}^{G}\!/_{K}$ is a word-hyperbolic group whose (Gromov) boundary is a 2-sphere. On the other hand, such a ${}^{G}\!/_{K}$ acts cocompactly on the CAT(0) cube complex ${}_{K}\backslash X$, with virtually cyclic cell stabilizers. Since virtually cyclic groups are virtually special, and virtually cyclic subgroups of hyperbolic groups are quasi-convex, it follows from [GM18, Theorem D] that such a ${}^{G}\!/_{K}$ is a virtually special group (and, in particular, it is cubulable). By [Haï15, Theorem 1.10], any such ${}^{G}\!/_{K}$ is virtually Kleinian.In fact, since the parabolic subgroups of G are free abelian, G cannot have a finite normal subgroup. It now follows from [GM08, Theorem 7.2]

[2] Note that an element $P \in \mathcal{P}$ fixes a point $\xi_P \in \partial(G, \mathcal{P})$, and, by the dynamical characterization of relatively hyperbolic groups, [Yam04] the group P acts properly and cocompactly on $\partial(G, \mathcal{P}) \smallsetminus \xi_P \cong \mathbb{R}^2$. It follows that each element of $\mathcal{P}$ is free abelian of rank 2.

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that such a G/K has no finite normal subgroup, and hence it acts faithfully on its boundary. Therefore, G/K is Kleinian.

We now take a longer and longer sequence of fillings of this form, obtaining a collection of hyperbolic quotients $G \twoheadrightarrow G_i = G/K_i$ so that each G_i is Kleinian. As in the proof of [GMS19, Corollary 1.4], we get a sequence of representations $\rho_i \colon G \to \mathrm{Isom}(\mathbb{H}^3)$, and exactly as in [GMS19] this sequence must converge to a discrete faithful representation of G into $\mathrm{Isom}(\mathbb{H}^3)$, which shows that G is Kleinian, as required. □

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