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INTEGERS IN NUMBER SYSTEMS WITH POSITIVE AND NEGATIVE QUADRATIC PISOT BASE

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Abstract. We consider numeration systems with base β and $-\beta$, for quadratic Pisot numbers β and focus on comparing the combinatorial structure of the sets $\mathbb{Z}_\beta$ and $\mathbb{Z}_{-\beta}$ of numbers with integer expansion in base β, resp. $-\beta$. Our main result is the comparison of languages of infinite words u_β and $u_{-\beta}$ coding the ordering of distances between consecutive β- and $(-\beta)$-integers. It turns out that for a class of roots β of $x^2 - mx - m$, the languages coincide, while for other quadratic Pisot numbers the language of u_β can be identified only with the language of a morphic image of $u_{-\beta}$. We also study the group structure of $(-\beta)$-integers.

Mathematics Subject Classification. 11K16, 68R15.

Introduction

Numeration systems with negative non-integer base obtained a non-negligible attention since the paper [12] of Ito and Sadahiro in 2009. Some of the articles that followed point out that many properties of these systems are analogous to those of systems with non-integer positive base defined by Rényi in 1957 [22], and are easily derivable. On the other hand, there are interesting results showing that the analogy is sometimes very non-trivial, and that negative base systems have certain features very different from properties of Rényi systems. It is therefore appealing to focus on the comparison of these two types of numeration systems. The comparison of the dynamical aspects of these systems is the topic of [13]. The present paper is another attempt in this direction, concentrating on comparing the combinatorial structure of the sets $\mathbb{Z}_\beta$ and $\mathbb{Z}_{-\beta}$ of numbers with integer expansion

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in base β, resp. $-\beta$. We perform the study for the case where β is a quadratic Pisot number. For this purpose, we have to put β- and $(-\beta)$-integers into several different contexts. Thus the first part of the paper can be viewed as a review on properties of $\mathbb{Z}_\beta$ and $\mathbb{Z}_{-\beta}$ found in [2,6,8,11,16,18,25] and others. In particular, we use the cut-and-project scheme, that describes $\mathbb{Z}_\beta$ and $\mathbb{Z}_{-\beta}$ for β being an algebraic unit. We study the infinite words u_β and $u_{-\beta}$ coding the ordering of distances in $\mathbb{Z}_\beta$ and $\mathbb{Z}_{-\beta}$ and provide prescriptions for the morphisms/antimorphisms under which these infinite words are invariant.

The main result of the paper is the comparison of languages of u_β and $u_{-\beta}$. We show that languages of these infinite words coincide if β is a root of $x^2 - mx - m$, $m \geq 1$. For all other quadratic Pisot numbers β one has to first apply a morphism π to the infinite word coding $(-\beta)$-integers in order to obtain equality of the corresponding languages. We use the notion of conjugate morphisms.

In the last part of the paper we concentrate on the arithmetical properties of $\mathbb{Z}_\beta$ and $\mathbb{Z}_{-\beta}$. In particular, we generalize the result of [7], presenting a group operation $\oplus$ on $\mathbb{Z}_\beta$, resp. $\mathbb{Z}_{-\beta}$, with which these sets are isomorphic to $\mathbb{Z}$. The authors of [7] define such an operation on $\mathbb{Z}_\beta$ for quadratic Pisot units β and show that such an operation is compatible with ordinary addition, *i.e.* whenever the result of $x+y$ is an element of $\mathbb{Z}_\beta$, then $x \oplus y = x + y$. By an easy combinatorial argument we show that one can define an operation $\oplus$ compatible with addition on every discrete set with finitely many distances between consecutive points which are linearly independent over $\mathbb{Q}$. We use again the cut-and-project scheme for determining the possible outcomes of $x + y - (x \oplus y)$ for $x, y \in \mathbb{Z}_{-\beta}$ for quadratic Pisot units β.

The organization of the paper is as follows. In Section 1 we recall necessary notions from combinatorics on words, present the numeration systems with positive and negative base and define the β- and $(-\beta)$-integers. In Section 2 we show the simplest cut-and-project scheme that can be applied for studying β- and $(-\beta)$-integers in the quadratic Pisot case. We cite the results identifying $\mathbb{Z}_\beta$ and $\mathbb{Z}_{-\beta}$ as cut-and-project sets. Section 3 introduces the infinite words u_β and $u_{-\beta}$ coding the distances between consecutive β- and $(-\beta)$-integers. These words are invariant under morphisms, resp. antimorphisms. We list the values of the distances and provide the prescriptions of the corresponding morphisms, resp. antimorphisms, by citing the known results and giving proofs where such are nowhere explicitly given. The main results of this paper are in Sections 4 and 5. Section 4 demonstrates the geometric and combinatorial relation of $\mathbb{Z}_\beta$ and $\mathbb{Z}_{-\beta}$. We compare the languages of the infinite words u_β and $u_{-\beta}$. Section 5 concentrates on arithmetical aspects of $(-\beta)$-integers.

1. Preliminaries

An alphabet $\mathcal{A}$ is a finite set of symbols, called letters. By $\mathcal{A}^*$ we denote the monoid of all finite words over the alphabet $\mathcal{A}$ equipped with the empty word ϵ and the operation of concatenation. The concatenation of k copies of a word w is denoted by $w^k = w \dots w$, for $k \in \mathbb{N}$. The length of a finite word $w = w_1 \dots w_n$,

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$w_i \in \mathcal{A}$, is denoted by $|w| = n$. By $|w|_a$ we denote the number of occurrences of a letter a in the word w. We will consider one-directional infinite words $u = u_0u_1u_2\ldots$, resp. $u = \ldots u_{-2}u_{-1}$, but also pointed bidirectional infinite words $u = \ldots u_{-2}u_{-1}|u_0u_1u_2\ldots$. The delimiter $|$ marks the position of the origin. A word w is a factor of a word u, if $u = vwv'$ for some (finite or infinite) words v, v'. If $v = \epsilon$, then w is a prefix of u, if $v' = \epsilon$, then w is a suffix of u. The language of an infinite word u is the set $\mathcal{L}(u)$ of all factors of u. The factor complexity of the word u is the function $\mathcal{C} : \mathbb{N} \to \mathbb{N}$, where $\mathcal{C}(n)$ is defined as the number of factors of u of length n.

The infinite repetition of a finite word w is denoted by $w^\omega = www\ldots$ A one-directional infinite word u is said to be eventually periodic if $u = vw^\omega = vwww\ldots$, for some finite words v, w, otherwise it is called aperiodic. It is not difficult to show that an aperiodic one-directional word must satisfy $\mathcal{C}(n) \geq n + 1$ for $n \in \mathbb{N}$. Aperiodic one-directional infinite words of minimal complexity $\mathcal{C}(n) = n + 1$ for $n \in \mathbb{N}$ are called Sturmian words. Sturmian words are by definition binary words (for $\mathcal{C}(1) = 2$) and have many equivalent definitions, for a recent overview of them see [3]. We will be interested in the following one. A one-directional infinite word over an alphabet $\mathcal{A}$ is called balanced if $\big||w|_a - |v|_a\big| \leq 1$ for every pair of factors $w, v \in \mathcal{L}(u)$ of the same length $|w| = |v|$ and every $a \in \mathcal{A}$. A binary aperiodic word is Sturmian if and only if it is balanced, see [16].

The definition of Sturmian words can be extended to bidirectional infinite words, by requiring aperiodicity and balance, see [9]. The language of such a Sturmian bidirectional word u is the same as the language of any of its infinite suffix, which is a Sturmian word in the original sense.

For finite alphabets $\mathcal{A}$ and $\mathcal{B}$, a morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ satisfies $\sigma(vw) = \sigma(v)\sigma(w)$ for any $v, w \in \mathcal{A}^*$. Its action can be extended to both one-directional and pointed bidirectional infinite words by

$$\sigma(u_0u_1u_2\ldots) = \sigma(u_0)\sigma(u_1)\sigma(u_2)\ldots,$$
$$\sigma(\ldots u_{-2}u_{-1}|u_0u_1u_2\ldots) = \ldots\sigma(u_{-2})\sigma(u_{-1})|\sigma(u_0)\sigma(u_1)\sigma(u_2)\ldots$$

If the morphism $\sigma : \mathcal{A}^* \to \mathcal{A}^*$ is non-erasing, *i.e.* $\sigma(a) \neq \epsilon$ for all $a \in \mathcal{A}$, and there exists a letter $a \in \mathcal{A}$ such that $\sigma(a) = aw$ for some non-empty word w, then it is called a substitution. A substitution has always a fixed point, *i.e.* one-directional infinite word u such that $\sigma(u) = u$, namely $u = \lim_{j\to\infty} \sigma^j(a)$. Existence of a bidirectional fixed point is ensured requiring moreover a letter b such that $\sigma(b) = vb$ for some non-empty word v. Then the fixed point is $u = \lim_{n\to\infty} \sigma^n(b)|\sigma^n(a)$. The limit is taken over the product topology on $\mathcal{A}^{\mathbb{N}}$, resp. $\mathcal{A}^{\mathbb{Z}}$.

In a similar way, one defines antimorphisms on $\mathcal{A}^*$ by $\overleftarrow{\sigma}(vw) = \overleftarrow{\sigma}(w)\overleftarrow{\sigma}(v)$ for any $v, w \in \mathcal{A}^*$. A fixed point of an antimorphism $\overleftarrow{\sigma}$ is a bidirectional infinite word $\ldots u_{-2}u_{-1}|u_0u_1u_2\ldots$ such that

$$\overleftarrow{\sigma}(\ldots u_{-2}u_{-1}|u_0u_1u_2\ldots) = \ldots\overleftarrow{\sigma}(u_2)\overleftarrow{\sigma}(u_1)\overleftarrow{\sigma}(u_0)|\overleftarrow{\sigma}(-1)\overleftarrow{\sigma}(-2)\ldots$$

Note that a second iteration of an antimorphism is a morphism.

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Given a morphism σ over $\mathcal{A} = \{a_1, \ldots, a_d\}$, one defines its incidence matrix M_σ, $(M_\sigma)_{ij} = |\sigma(a_j)|_{a_i}$. A morphism is called primitive, if there is a constant $k \in \mathbb{N}$ such that $M_\sigma^k > 0$. The eigenvalues of the matrix M_σ predicate for example about balance properties of the fixed point of σ.

We will be interested in properties of infinite words associated with the set of numbers with integer expansion in an irrational base. Let $\alpha \in \mathbb{R}$, $|\alpha| > 1$, and let $\mathcal{A}$ be a finite subset of $\mathbb{R}$. An $(\alpha, \mathcal{A})$-representation of a real number x is the series

$$x = \sum_{i \le k} x_i \alpha^i, \qquad x_i \in \mathcal{A}.$$

Sometimes we may write x equal to the sequence of digits x_i, namely

$$x = x_k x_{k-1} \ldots x_1 x_0 \bullet x_{-1} x_{-2} \ldots$$

A specific representation with the base $\alpha \in \mathbb{R}$, $|\alpha| > 1$ is obtained as follows. Consider the transformation $T : I \to I$ on an interval $I = [l, l+1)$, given by $T(x) = \alpha x - \lfloor \alpha x - l \rfloor$. For an $x \in I$ set

$$x_{-i} = \lfloor \alpha T^{i-1}(x) - l \rfloor, \qquad \text{for } i \ge 1;$$

the digits x_{-i} take values in a finite set $\mathcal{A}$ of consetutive integers. Then

$$x = \frac{x_{-1}}{\alpha} + \frac{x_{-2}}{\alpha^2} + \frac{x_{-3}}{\alpha^3} + \ldots$$

The string of digits is then denoted by $d(x) = x_{-1} x_{-2} x_{-3} \ldots$

A string $y_1 y_2 y_3 \ldots$ of elements of the digit set $\mathcal{A}$ is called $(\alpha, \mathcal{A})$-admissible, if it is equal to the $(\alpha, \mathcal{A})$-representation $d(x)$ for some $x \in I$. One can derive for example from [10] that a string $y_1 y_2 y_3 \ldots$ of integers is $(\alpha, \mathcal{A})$-admissible if and only if every suffix $y_i y_{i+1} y_{i+2} \ldots$ satisfies

$$d(l) \preceq y_i y_{i+1} y_{i+2} \ldots \prec \lim_{\delta \to 0+} d(l + 1 - \delta), \tag{1.1}$$

where $\preceq$ stands for the lexicographical order $\preceq_{\text{lex}}$ when $\alpha > 0$ and for the alternate order $\preceq_{\text{alt}}$ when $\alpha < 0$. Recall that $x_1 x_2 x_3 \ldots \preceq_{\text{alt}} y_1 y_2 y_3 \ldots$ if $(-1)^k x_k < (-1)^k y_k$, where $k = \min\{i \mid x_i \neq y_i\}$.

It is not difficult to show that the corresponding order $\preceq$ reflects the natural order on reals in the interval I, namely that

$$d(x) \preceq d(y) \quad \iff \quad x \le y.$$

The two specific choices of numeration systems which we consider here are the β-expansions introduced by Rényi (*cf.* [22]) and $(-\beta)$-expansions considered by Ito and Sadahiro (*cf.* [12]) which share many properties but display also important differences.

First set $\alpha = \beta > 1$ and $I = [0, 1)$. The digits x_i then take values in the set $\mathcal{A} = \mathcal{A}_\beta = \{d \in \mathbb{Z} \mid 0 \le d < \beta\}$. For the sake of clarity, we denote $d(x) = d_\beta(x)$

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for $x \in [0,1)$. The admissibility condition (1.1) states that a string of integers $y_1y_2y_3\ldots$ is admissible in this system if and only if for every $i \geq 1$, we have

$$0^\omega \preceq_{\text{lex}} y_iy_{i+1}y_{i+2}\ldots \prec_{\text{lex}} \lim_{\delta\to 0+} d_\beta(1-\delta). \tag{1.2}$$

This condition was first derived by Parry [21].

In order to expand any positive number $x \in \mathbb{R}$, one uses the expansion of x/β^k, where $k \in \mathbb{Z}$ such that $x/\beta^k \in [0,1)$. Then we have

$$d_\beta(x/\beta^k) = x_{-1}x_{-2}x_{-3}\ldots \quad \Rightarrow \quad x = x_{-1}x_{-2}\ldots x_{-k} \bullet x_{-k-1}\ldots$$

This representation of x is called the β-expansion and is denoted $\langle x\rangle_\beta$. We focus on the set of numbers with integer β-expansion, namely the set

$$\mathbb{Z}_\beta = \mathbb{Z}_\beta^+ \cup -\mathbb{Z}_\beta^+, \qquad \text{where} \quad \mathbb{Z}_\beta^+ = \{x \geq 0 \mid \langle x\rangle_\beta = x_k\ldots x_1x_0 \bullet 0^\omega\}.$$

Negative base systems allow to expand every real number x (even negative) without the use of a sign. Ito and Sadahiro considered $\alpha = -\beta < -1$, $I = [l, l+1)$, where $l = -\beta/(\beta+1)$, *i.e.* $I = \big[-\beta/(\beta+1), 1/(\beta+1)\big)$. The digit set is now equal to $\mathcal{A} = \mathcal{A}_{-\beta} = \{d \in \mathbb{Z} \mid 0 \leq d \leq \beta\}$. Here we denote $d(x) = d_{-\beta}(x)$ for $x \in [l, l+1)$. A string of integers $y_1y_2y_3\ldots$ is admissible in this setting if and only if for every $i \geq 1$, we have

$$d_{-\beta}(l) \preceq_{\text{alt}} y_iy_{i+1}y_{i+2}\ldots \prec_{\text{alt}} \lim_{\delta\to 0+} d_{-\beta}(l+1-\delta). \tag{1.3}$$

In [12] it is shown that $\lim_{\delta\to 0+} d_{-\beta}(l+1-\delta)$ is strongly dependent on the string $d_{-\beta}(l)$, namely $\lim_{\delta\to 0+} d_{-\beta}(l+1-\delta) = \big(0d_1\ldots d_{q-1}(d_q-1)\big)^\omega$ if $d_{-\beta}(l) = (d_1\ldots d_{q-1}d_q)^\omega$ is purely periodic with odd period-length q, and $\lim_{\delta\to 0+} d_{-\beta}(l+1-\delta) = 0d_{-\beta}(l)$ otherwise.

In order to expand any real number $x \in \mathbb{R}$, take a suitable power of $(-\beta)$ such that $x/(-\beta)^k \in I^\circ = \big(-\beta/(\beta+1), 1/(\beta+1)\big)$. If $d_{-\beta}(x/(-\beta)^k) = x_{-1}x_{-2}x_{-3}\ldots$ then we put $\langle x\rangle_{-\beta} = x_{-1}\ldots x_{-k} \bullet x_{-k-1}\ldots$. The set of $(-\beta)$-integers is defined by

$$\mathbb{Z}_{-\beta} = \{x \in \mathbb{R} \mid \langle x\rangle_{-\beta} = x_k\ldots x_1x_0 \bullet 0^\omega\}.$$

One can determine the distances between consecutive elements of $\mathbb{Z}_\beta$ and $\mathbb{Z}_{-\beta}$ and, in case that the distances take only finitely many values, one can code their ordering by an infinite word over a finite alphabet. It is interesting to study combinatorial properties of these infinite words, such as invariance under morphism. We will describe these infinite words for quadratic Pisot numbers β and use them to relate $\mathbb{Z}_\beta$ and $\mathbb{Z}_{-\beta}$.

2. β- and $(-\beta)$-integers as cut-and-project sequences

A Pisot number β is the root > 1 of a monic irreducible polynomial with integer coefficients whose other roots (the algebraic conjugates of β) are in modulus smaller

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than 1. In this paper we focus on the quadratic case. It can be easily shown that quadratic Pisot numbers are precisely the larger roots of the polynomials

$$\begin{aligned} &x^2 - mx - n, && m \geq n \geq 1, \\ &x^2 - mx + n, && m - 2 \geq n \geq 1. \end{aligned} \tag{2.1}$$

A special role is played by quadratic Pisot units, *i.e.* roots of (2.1) with $n = 1$.

For any algebraic number β, the minimal subfield of $\mathbb{C}$ containing β is denoted by $\mathbb{Q}(\beta)$. For quadratic β we have $\mathbb{Q}(\beta) = \{a + b\beta \mid a, b \in \mathbb{Q}\}$. The only non-trivial automorphism over such a field is the Galois automorphism $' : x = a + b\beta \mapsto x' = a + b\beta'$, where β' is the conjugate of β.

In many cases, it is advantageous to apply the Galois conjugation in the study of numeration systems. In particular, we will find useful to study $\{x' \mid x \in \mathbb{Z}_\beta\}$, resp. $\{x' \mid x \in \mathbb{Z}_{-\beta}\}$. Obviously, if β is a Pisot number, this set is bounded as β-integers have in their β-expansion only non-negative powers of β. The same holds for $(-\beta)$-integers. However, if β is a Pisot unit, one can say more, using the so-called cut-and-project scheme.

For given irrational numbers ε, η, $\varepsilon \neq \eta$, consider vectors $\boldsymbol{x}_1 = \frac{1}{\varepsilon - \eta}(\varepsilon, -1)$, $\boldsymbol{x}_2 = \frac{1}{\eta - \varepsilon}(\eta, -1)$. Then, $(a + b\eta)\boldsymbol{x}_1 + (a + b\varepsilon)\boldsymbol{x}_2 = (a, b)$. Consequently, for a given point $(a, b) \in \mathbb{Z}^2$, the value $\pi_1(a, b) = a + b\eta$ is its projection to the line $V_1 = \mathbb{R}\boldsymbol{x}_1$; the value $\pi_2(a, b) = a + b\varepsilon$ is its projection to the line $V_2 = \mathbb{R}\boldsymbol{x}_2$. Since ε, η are irrational, the mapping π_1 restricted to $\mathbb{Z}^2$ is an injection, and we can define $^* : \mathbb{Z} + \mathbb{Z}\eta \to \mathbb{Z} + \mathbb{Z}\varepsilon$ by $^* = \pi_2 \circ \pi_1^{-1}$. We have thus the cut-and-project scheme,

$$\begin{array}{ccccc} V_1 & \xleftarrow{\pi_1} & \mathbb{R}^2 = V_1 \times V_2 & \xrightarrow{\pi_2} & V_2 \\ & & \cup & & \\ \mathbb{Z} + \mathbb{Z}\eta & \xleftarrow{\pi_1} & \mathbb{Z}^2 & \xrightarrow{\pi_2} & \mathbb{Z} + \mathbb{Z}\varepsilon. \end{array}$$

For a bounded set $\Omega \subset \mathbb{R}$ we define a cut-and-project set $\Sigma_{\varepsilon,\eta}(\Omega)$ by

$$\Sigma_{\varepsilon,\eta}(\Omega) = \{x \in \mathbb{Z} + \mathbb{Z}\eta \mid x^* \in \Omega\}.$$

For our purposes, it suffices to consider Ω to be an interval.

In general, a cut-and-project set can be defined by projection of certain lattice points $z \in L \subset \mathbb{R}^{r+s} = V_1 \times V_2$ to a suitably oriented r-dimensional subspace V_1, where the choice of points to be projected is directed by the projection to the subspace V_2. For details and a list of references, see [20]. For our purposes, it is sufficient to limit our considerations to projection of $\mathbb{Z}^2$ to one-dimensional subspaces. Such cut-and-project schemes are subject of [11]. One of the results shown there is that the distances between consecutive points of $\Sigma_{\varepsilon,\eta}(\Omega)$ take two or three values and that the corresponding infinite word is a coding of exchange of two or three intervals. In particular, we will use the following statement which can be derived from [11].

Theorem 2.1. *Let ε, η be distinct irrational numbers and let Ω be a non-degenerated interval containing 0. If $\Sigma_{\varepsilon,\eta}(\Omega) = \{t_j \mid j \in \mathbb{Z}\} \subset \mathbb{R}$, $t_j < t_{j+1}$,*

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$t_0 = 0$, then there exist positive values Δ_0, Δ_1 such that the distances between consecutive elements of $\Sigma_{\varepsilon,\eta}(\Omega)$ take values $t_{j+1} - t_j \in \{\Delta_0, \Delta_1, \Delta_0 + \Delta_1\}$. Moreover, if $t_{j+1} - t_j \in \{\Delta_0, \Delta_1\}$, then the pointed bidirectional infinite word $u_{\varepsilon,\eta}(\Omega) = \ldots u_{-2}u_{-1}|u_0u_1u_2\ldots$ over $\{0,1\}$ defined by $u_j = X$ if $t_{j+1} - t_j = \Delta_X$ is a bidirectional Sturmian word.

In connection to β- and $(-\beta)$-integers we will need a special case of the set $\Sigma_{\varepsilon,\eta}(\Omega)$, namely for $\varepsilon = \beta'$ and $\eta = \beta$ for a given quadratic Pisot unit β. Then $\mathbb{Z} + \mathbb{Z}\eta = \mathbb{Z}[\beta]$ and $\mathbb{Z} + \mathbb{Z}\varepsilon = \mathbb{Z}[\beta'] = \mathbb{Z}[\beta]$, and the mapping * coincides with the Galois conjugation. For a bounded set $\Omega \subset \mathbb{R}$ we thus have the cut-and-project set

$$\Sigma_\beta(\Omega) = \Sigma_{\beta',\beta}(\Omega) = \{x \in \mathbb{Z}[\beta] \mid x' \in \Omega\},$$

where we simplify the notation in indices. We will need the following properties of $\Sigma_\beta(\Omega)$. The first of the statements of Proposition 2.2 would be trivially valid for any cut-and-project set, the last two are specific for the algebraic choice of parameters ε and η.

Proposition 2.2. *Let $\alpha, x_0 \in \mathbb{Z}[\beta]$ and let α be an algebraic unit. Let $\Omega, \Omega_1, \Omega_2$ be intervals such that $\Omega = \Omega_1 \cup \Omega_2$. Then*

(1) $\Sigma_\beta(\Omega_1) \cup \Sigma_\beta(\Omega_2) = \Sigma_\beta(\Omega_1 \cup \Omega_2)$;
(2) $x_0 + \Sigma_\beta(\Omega) = \Sigma_\beta(x_0' + \Omega)$;
(3) $\Sigma_\beta(\alpha'\Omega) = \alpha\Sigma_\beta(\Omega)$.

Proof. The first statement follows directly from the definition. For the second statement, we have

$$x_0 + \Sigma_\beta(\Omega) = \{x_0 + x \in \mathbb{Z}[\beta] \mid x' \in \Omega\} = \{x \in \mathbb{Z}[\beta] \mid (x - x_0)' \in \Omega\} = \Sigma_\beta(x_0' + \Omega).$$

For the third one, we have

$$\begin{aligned}\Sigma_\beta(\alpha'\Omega) &= \{x \in \mathbb{Z}[\beta] \mid x' \in \alpha'\Omega\} = \{x \in \mathbb{Z}[\beta] \mid (\alpha^{-1}x)' \in \Omega\} \\ &= \{\alpha x \in \mathbb{Z}[\beta] \mid x' \in \Omega\} = \alpha\Sigma_\beta(\Omega),\end{aligned}$$

where we have used that α is an algebraic unit and hence $\alpha\mathbb{Z}[\beta] = \mathbb{Z}[\beta]$. This means that $\alpha x \in \mathbb{Z}[\beta]$ is equivalent to $x \in \mathbb{Z}[\beta]$. □

As shown in [6], for quadratic Pisot units β, the positive part $\mathbb{Z}_\beta^+$ of β-integers can be identified with the positive part of a cut-and-project set. The reason why one cannot write the equality in the following proposition for the entire set $\mathbb{Z}_\beta$ is that $\mathbb{Z}_\beta$ is in some sense 'artificially' defined so that it is symmetric with respect to zero.

Proposition 2.3. *Let $\beta > 1$ be the root of $x^2 - mx - 1$, $m \geq 1$. Then*

$$\mathbb{Z}_\beta^+ = \Sigma_\beta\big((-1,\beta)\big) \cap \mathbb{R}^+ = \big\{x \in \mathbb{Z}[\beta] \big| x' \in (-1,\beta)\big\} \cap \mathbb{R}^+.$$

Let $\beta > 1$ be the root of $x^2 - mx + 1$, $m \geq 3$. Then

$$\mathbb{Z}_\beta^+ = \Sigma_\beta\big([0,\beta)\big) \cap \mathbb{R}^+ = \big\{x \in \mathbb{Z}[\beta] \big| x' \in [0,\beta)\big\} \cap \mathbb{R}^+.$$

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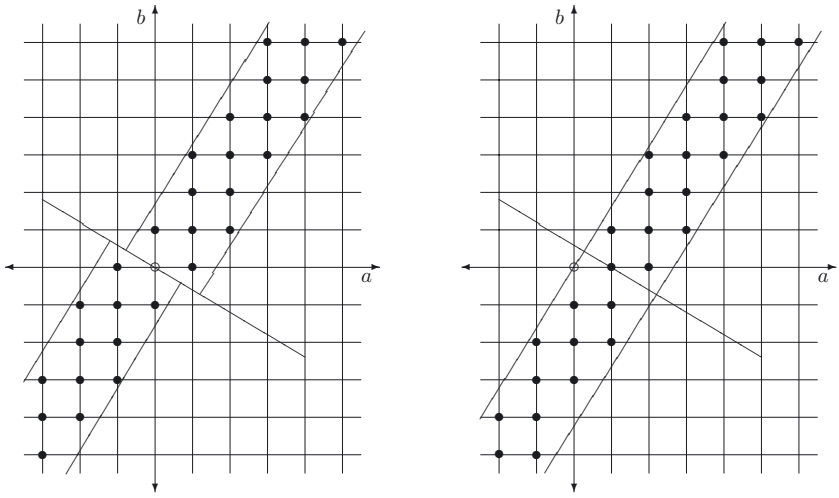


FIGURE 1. The cut-and-project scheme for the set of τ-integers (left) and $(-\tau)$-integers (right). Every element $x \in \mathbb{Z}_\tau$ and $y \in \mathbb{Z}_{-\tau}$ can be written in the form $x = a + b\tau$, $y = c + d\tau$, where $a, b, c, d \in \mathbb{Z}$. The picture represents x and y as pairs $(a, b) \in \mathbb{Z}^2$, resp. $(c, d) \in \mathbb{Z}^2$.

Unlike the case of positive base, with negative base one can represent negative numbers without the use of a sign. This allows to identify the entire set $\mathbb{Z}_{-\beta}$ as a cut-and-project set. The difference between integers in positive and negative base is illustrated in Figure 1 for the case $\beta = \tau = \frac{1}{2}(1 + \sqrt{5})$, the golden ratio. The following proposition appeared as Lemmas 5 and 9 in [17].

Proposition 2.4. *Let $\beta > 1$ be the root of $x^2 - mx - 1$, $m \geq 1$. Then*

$$\mathbb{Z}_{-\beta} = \begin{cases} \Sigma_\beta\big([0,\beta)\big) = \big\{z \in \mathbb{Z}[\beta] \mid z' \in [0,\beta)\big\} & \text{for } m \geq 2, \\ \Sigma_\beta\big([0,\beta^2)\big) = \big\{z \in \mathbb{Z}[\beta] \mid z' \in [0,\beta^2)\big\} & \text{for } m = 1. \end{cases}$$

Let $\beta > 1$ be the root of $x^2 - mx + 1$, $m \geq 3$. Then

$$\mathbb{Z}_{-\beta} = \Sigma_\beta\left(\left(-\tfrac{\beta-1}{\beta+1}, \beta\tfrac{\beta-1}{\beta+1}\right)\right) = \left\{z \in \mathbb{Z}[\beta] \middle| z' \in \left(-\tfrac{\beta-1}{\beta+1}, \beta\tfrac{\beta-1}{\beta+1}\right)\right\}.$$

3. Combinatorial properties of $(-\beta)$-integers

Before starting to compare $\mathbb{Z}_\beta$, resp. $\mathbb{Z}_{-\beta}$ we shall state the necessary properties of β- and $(-\beta)$-integers. We cite the known ones and provide proofs for the rest.

Recall that the distances between consecutive points of $\mathbb{Z}_\beta$ can be described using the infinite word $\lim_{\delta \to 0+} d_\beta(1-\delta)$ of (1.2), see [25]. They take finitely many values if and only if this sequence is eventually periodic. Bases β, for which this

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happens are called Parry numbers. Similarly, the distances between consecutive points of $\mathbb{Z}_{-\beta}$ can be described using the infinite word $d_{-\beta}(l)$ of (1.3), *cf.* [2]. They take finitely many values for the Ito–Sadahiro numbers β (in [24] called Yrrap numbers), *i.e.* such that $d_{-\beta}(l)$ is eventually periodic.

Although the class of Parry numbers and the class of Ito–Sadahiro numbers do not coincide, as shown in [14], Pisot numbers belong to both classes. In fact, assuming that β is a quadratic number, the notions of Parry numbers, Ito–Sadahiro numbers and Pisot numbers coincide [5,18]. If β is a Pisot number, the distances between consecutive points of $\mathbb{Z}_\beta$, resp. $\mathbb{Z}_{-\beta}$ take only finitely many values. In particular, for quadratic Pisot numbers, the distances are only two. From [25], one can easily derive the following statement.

Proposition 3.1. *Let $\beta > 1$ be a quadratic Pisot number. Let $\mathbb{Z}_\beta = \{t_j \mid j \in \mathbb{Z}\}$, where $t_0 = 0$ and $t_j < t_{j+1}$. Then the distances between consecutive β-integers take values $t_{j+1} - t_j \in \{\Delta_0^+, \Delta_1^+\}$, where*

$$\Delta_0^+ = 1 \quad \text{and} \quad \Delta_1^+ = \begin{cases} \frac{n}{\beta} & \text{if } \beta^2 = m\beta + n,\ m \geq n \geq 1, \\ 1 - \frac{n}{\beta} & \text{if } \beta^2 = m\beta - n,\ m - 2 \geq n \geq 1. \end{cases}$$

The ordering of distances in $\mathbb{Z}_\beta$ can be coded by an infinite word u_β. Since $\mathbb{Z}_\beta$ is symmetric with respect to 0, we define u_β as a one-sided infinite word

$$u_\beta = u_0 u_1 u_2 \ldots \in \{0,1\}^{\mathbb{N}}, \qquad \text{where } u_j = X \text{ if } t_{j+1} - t_j = \Delta_X^+. \tag{3.1}$$

It is known that the infinite word u_β is a fixed point of the canonical substitution [8].

Proposition 3.2. *Let $\beta > 1$ be a quadratic Pisot number. Let $\varphi_\beta : \{0,1\}^* \to \{0,1\}^*$ be the canonical morphism*

$$\begin{aligned} &\varphi_\beta(0) = 0^m 1, \quad \varphi_\beta(1) = 0^n, && \text{if } \beta^2 = m\beta + n,\ m \geq n \geq 1, \\ &\varphi_\beta(0) = 0^{m-1} 1, \varphi_\beta(1) = 0^{m-n-1} 1, && \text{if } \beta^2 = m\beta - n,\ m - 2 \geq n \geq 1. \end{aligned}$$

Then the one-sided infinite word u_β coding the sequence of non-negative β-integers $\mathbb{Z}_\beta^+$ is a fixed point of φ_β. In particular, we have $u_\beta = \lim_{j \to +\infty} \varphi_\beta^j(0)$.

An infinite word u_β coding positive β-integers and the corresponding canonical morphism φ_β fixing u_β can be defined for every Parry number β. Among all such words, exactly those corresponding to quadratic Pisot units are Sturmian.

Proposition 3.3. *Let u_β be the one-directional infinite word coding $\mathbb{Z}_\beta^+$. Then u_β is Sturmian if and only if β is a quadratic Pisot unit.*

Statements similar to Propositions 3.1, 3.2 and 3.3 can be stated for $(-\beta)$-integers. Before presenting these results, recall that the admissibility of digit strings as $(-\beta)$-expansions is decided by the alternate order condition (1.3) using the

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expansion $d_{-\beta}(l)$ of the left-end point $l = -\frac{\beta}{\beta+1}$ of the interval $I = [l, l+1)$. In case of quadratic Pisot numbers these strings are given as follows,

$$d_{-\beta}(l) = m(m-n)^\omega, \qquad \text{for } \beta^2 = m\beta + n,\ m \geq n \geq 1, \tag{3.2}$$

$$d_{-\beta}(l) = \big((m-1)n\big)^\omega, \qquad \text{for } \beta^2 = m\beta - n,\ m-2 \geq n \geq 1. \tag{3.3}$$

For the class (3.2), one can use results of [2] in order to describe the distances between consecutive $(-\beta)$-integers and substitution properties of the corresponding infinite word.

Proposition 3.4. *Let $\beta > 1$ be the root of $x^2 - mx - n$, $m \geq n \geq 1$. Let $\mathbb{Z}_{-\beta} = \{t_j \mid j \in \mathbb{Z}\}$, where $t_0 = 0$ and $t_j < t_{j+1}$. Then the distances between consecutive $(-\beta)$-integers take values $t_{j+1} - t_j \in \{\Delta_0^-, \Delta_1^-\}$, where*

$$\Delta_0^- = 1 \quad \text{and} \quad \Delta_1^- = \begin{cases} \frac{m}{\beta} & \text{if } m = n, \\ 1 + \frac{n}{\beta} & \text{if } m < n. \end{cases}$$

Proof. If $m > n$, then the string $d_{-\beta}(l) = m(m-n)^\omega$ satisfies conditions of Theorem 18 from [2] and the statement follows. For $m = n$, we have $d_{-\beta}(l) = m0^\omega$ and we use Theorem 21 from [2]. □

Similarly as in case of positive base, one can code the sequence of $(-\beta)$-integers by an infinite word, say $u_{-\beta}$, over the alphabet $\{0, 1\}$, where the letter 0 is used if distance Δ_0^- occurs, and 1 otherwise. Since negative numbers may have integer $(-\beta)$-expansion with non-negative digits, the infinite word is pointed bidirectional, where we denote the position of 0 by a delimiter |. Formally, we define

$$u_{-\beta} = \ldots u_{-2}u_{-1}|u_0u_1u_2\ldots \in \{0,1\}^{\mathbb{Z}}, \text{ where } u_i = X \text{ if } t_{i+1} - t_i = \Delta_X^-. \tag{3.4}$$

For further use, let us denote the suffix of $u_{-\beta}$ starting at the delimiter by $u_{-\beta}^+ = u_0u_1u_2\ldots$

The results of [2] can be used to derive the following statement about invariance of $u_{-\beta}$ under an antimorphism.

Proposition 3.5. *Let $\beta > 1$ be the root of $x^2 - mx - n$, $m \geq n \geq 1$. Then the infinite word $u_{-\beta}$ is a fixed point of the antimorphism $\overleftarrow{\varphi_{-\beta}} : \{0,1\}^* \to \{0,1\}^*$, given by*

$$\begin{aligned} &\overleftarrow{\varphi_{-\beta}}(0) = 0^{m-1}1, \\ &\overleftarrow{\varphi_{-\beta}}(1) = 0^{m+n-1}1, \end{aligned} \quad \text{for } m > n, \quad \text{and} \quad \begin{aligned} &\overleftarrow{\varphi_{-\beta}}(0) = 0^m1, \\ &\overleftarrow{\varphi_{-\beta}}(1) = 0^m, \end{aligned} \quad \text{for } m = n.$$

In particular, we have

$$u_{-\beta} = \lim_{n\to+\infty} \overleftarrow{\varphi_{-\beta}}^n(1)|\overleftarrow{\varphi_{-\beta}}^n(0).$$

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For the class of quadratic Pisot numbers β, which are not covered by Proposition 3.4, we have by (3.3) that $d_{-\beta}(l) = \big((m-1)n\big)^\omega$. For such a case, results of [2] cannot be used neither for deriving the distances, nor for determining invariance under an antimorphism, thus we provide our own demonstration. For that, we need to consider the admissibility condition (1.3) for this particular case, which reads as follows: A string of integer digits $y_1y_2y_3\ldots$ is a $(-\beta)$-expansion of a number $x \in [-\frac{\beta}{\beta+1}, \frac{1}{\beta+1})$ if and only if every suffix $y_iy_{i+1}y_{i+2}\ldots$ satisfies

$$\big((m-1)n\big)^\omega \preceq_{\text{alt}} y_iy_{i+1}y_{i+2}\ldots \prec_{\text{alt}} 0\big((m-1)n\big)^\omega.$$

In particular, for strings ending in 0^ω, this results in requiring that the infinite word $y_1y_2y_3\ldots$ does not contain a forbidden factor from the set $\{(m-1)A \mid 0 \le A < n\}$.

The following proposition is stated with more details than previous results about distances in $\mathbb{Z}_\beta$ and $\mathbb{Z}_{-\beta}$, in order to facilitate the proof of Proposition 3.7.

Proposition 3.6. *Let $\beta > 1$ be the root of $x^2 - mx + n$, $m - 2 \ge n \ge 1$. Let $\mathbb{Z}_{-\beta} = \{t_j \mid j \in \mathbb{Z}\}$, where $t_0 = 0$ and $t_j < t_{j+1}$. Then the distances between consecutive $(-\beta)$-integers take values $t_{j+1} - t_j \in \{\Delta_0^-, \Delta_1^-\}$, where*

$$\Delta_0^- = 1 \quad \text{and} \quad \Delta_1^- = 2 - \frac{n}{\beta}.$$

In particular, let $x < y$ be consecutive $(-\beta)$-integers.

(1) *If $\langle x\rangle_{-\beta} = x_k\ldots x_1A\bullet$, with $A \le m-3$, then $y - x = \Delta_0^- = 1$.*
(2) *If $\langle x\rangle_{-\beta} = x_k\ldots x_1(m-2)\bullet$, then $y - x = \Delta_1^- = 2 - \frac{n}{\beta}$.*

Proof. Since the natural order of reals corresponds to the alternate order of their $(-\beta)$-expansions, it suffices to find, for every $(-\beta)$-integer x with $\langle x\rangle_{-\beta} = x_k\ldots x_1A\bullet$, the $(-\beta)$-integer y with the smallest alternately greater expansion. We do not consider strings ending with the digit $(m-1)$, because they are not admissible.

(1) Let $A \le m-3$. Then $y = x + \Delta_0 = x + 1$, because

$$\begin{aligned} x &= x_k \ldots x_1 \quad A \quad \bullet \\ x+1 &= x_k \ldots x_1 \ (A+1) \bullet \end{aligned}$$

(2) Let $A = (m-2)$. Then $y = x + \Delta_1$, which can be justified as follows. The $(-\beta)$-expansion of x is now of the form

$$\langle x\rangle_{-\beta} = \ldots XY[(m-1)n]^k(m-2)\bullet$$

where $k \in \mathbb{N}_0$ and $XY \ne (m-1)n$. We distinguish two cases:

(a) Let $X \le m-2, Y \ge 1$ or $X = m-1, Y \ge n+1$. Then

$$\begin{aligned} x &= \ldots XY[(m-1)n]^k(m-2)\bullet \\ x + \Delta_1^- &= \ldots X(Y-1)0[(m-1)n]^k\bullet \end{aligned}$$

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(b) Let $X \leq m-2, Y = 0$. Then

$$x = \ldots X0[(m-1)n]^k(m-2)\bullet$$
$$x + \Delta_1^- = \ldots X+1m-1[n(m-1)]^k n\bullet \qquad \square$$

Since the distances between consecutive $(-\beta)$-integers take two values, denoted by Δ_0^- and Δ_1^-, we may define the pointed bidirectional infinite word $u_{-\beta}$ and one-directional infinite word $u_{-\beta}^+$ for the class of quadratic Pisot numbers β, roots of $x^2 - mx + n$, in the same way as before, namely by (3.4). In the following proposition we find the antimorphism fixing $u_{-\beta}$.

Proposition 3.7. *Let $\beta > 1$ be the root of $x^2 - mx + n$, $m - 2 \geq n \geq 1$. Then the infinite word $u_{-\beta}$ is a fixed point of the antimorphism $\overleftarrow{\varphi_{\text{-}\beta}} : \{0,1\}^* \to \{0,1\}^*$, given by*

$$\overleftarrow{\varphi_{\text{-}\beta}}(0) = 0^{m-2}1$$
$$\overleftarrow{\varphi_{\text{-}\beta}}(1) = 0^{m-2}10^{m-n-2}1$$

In particular, we have

$$u_{-\beta} = \lim_{n\to+\infty} \overleftarrow{\varphi_{\text{-}\beta}}^n(1)|\overleftarrow{\varphi_{\text{-}\beta}}^n(0).$$

Proof. From the definition, the set $\mathbb{Z}_{-\beta}$ is self-similar, *i.e.* $(-\beta)\mathbb{Z}_{-\beta} \subset \mathbb{Z}_{-\beta}$. We will determine the antimorphism $\overleftarrow{\varphi_{\text{-}\beta}}$ under which $u_{-\beta}$ is invariant by showing that between consecutive points of $(-\beta)\mathbb{Z}_{-\beta}$ (which lie also in $\mathbb{Z}_{-\beta}$) we find always the same configuration of elements of $\mathbb{Z}_{-\beta}$. More precisely, let $x < y$ be consecutive points in $\mathbb{Z}_{-\beta}$. Then $-\beta y < -\beta x$ are consecutive points in $(-\beta)\mathbb{Z}_{-\beta}$. We will show that there are only two types of the configurations $[-\beta y, -\beta x] \cap \mathbb{Z}_{-\beta}$, according to whether $y - x = \Delta_0^-$ or $y - x = \Delta_1^-$. The sequence of distances in the configuration obtained for $y - x = \Delta_0^-$ determines the word $\overleftarrow{\varphi_{\text{-}\beta}}(0)$; similarly, the sequence of distances in the configuration obtained for $y - x = \Delta_1^-$ determines the word $\overleftarrow{\varphi_{\text{-}\beta}}(1)$. Recall that $\Delta_0^- = 1$ is coded by the letter 0 and $\Delta_1^- = 2 - \frac{n}{\beta}$ by the letter 1.

Consider consecutive $(-\beta)$-integers x, y such that $y = x + 1 = x + \Delta_0^-$. By Proposition 3.6, the expansions of x, y are of the form

$$\langle x\rangle_{-\beta} = x_k \ldots\ x_1 \quad A \quad \bullet$$
$$\langle y\rangle_{-\beta} = x_k \ldots\ x_1\ (A+1)\ \bullet$$

Multiplying by $(-\beta)$ we have

$$\langle -\beta x\rangle_{-\beta} = x_k \ldots\ x_1 \quad A \quad 0\ \bullet$$
$$\langle -\beta y\rangle_{-\beta} = x_k \ldots\ x_1\ (A+1)\ 0\ \bullet$$

and we have $-\beta x > -\beta y$. From Proposition 3.6, we derive that $(m-2)$ right neighbours of the point $-\beta y$ are $-\beta y + 1, -\beta y + 2, \ldots, -\beta y + m - 2$. There are no

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other elements of $\mathbb{Z}_{-\beta}$ in the interval $(-\beta y, -\beta x)$. We have $-\beta y + (m-2)\Delta_0^- + \Delta_1^- = -\beta x$, and thus we may put $\overleftarrow{\varphi_{-\beta}}(0) = 0^{m-2}1$.

Let now $y = x + \Delta_1^-$. From the proof of Proposition 3.6, we can derive that the number $-\beta y$ may have two possible expansions, namely

$$\langle -\beta y\rangle_{-\beta} = \dots\ (m-1)\ n\ 0\ \bullet$$
$$\langle -\beta y\rangle_{-\beta} = \dots\quad X\quad 0\ 0\ \bullet$$

In both cases, the neighbouring $m-2$ elements of $\mathbb{Z}_{-\beta}$ are again $-\beta y+1$, $-\beta y+2$, $\dots$, $-\beta y + m - 2$, and we have

$$\langle -\beta y + (m-2)\Delta_0^-\rangle_{-\beta} = \dots\ (m-1)\ n\ (m-2)\ \bullet$$
$$\langle -\beta y + (m-2)\Delta_0^-\rangle_{-\beta} = \dots\quad X\quad 0\ (m-2)\ \bullet$$

where X denotes an arbitrary digit in $\{0, 1, \dots, m-2\}$. By Proposition 3.6, the neighbour is

$$\langle -\beta y + (m-2)\Delta_0^- + \Delta_1^-\rangle_{-\beta} = \dots\quad \dots\quad (m-1)\ n\ \bullet$$
$$\langle -\beta y + (m-2)\Delta_0^- + \Delta_1^-\rangle_{-\beta} = \dots\ (X+1)\ (m-1)\ n\ \bullet$$

Necessarily, the $(m-n-2)$ right neighbours are obtained by adding Δ_0^-. The last one is of the form

$$-\beta y + (m-2)\Delta_0^- + \Delta_1^- + (m-n-2)\Delta_0^- = -\beta x - \Delta_1^-.$$

Hence there are no other points of $\mathbb{Z}_{-\beta}$ in the interval $(-\beta y, -\beta x)$ and the prescription for the antimorphism is $\overleftarrow{\varphi_{-\beta}}(1) = 0^{m-2}10^{m-n-2}1$. □

As a consequence of the above descriptions of distances in $\mathbb{Z}_{-\beta}$ and antimorphisms fixing infinite words coding $\mathbb{Z}_{-\beta}$, we derive an analogue of Proposition 3.3, characterizing Sturmian words among $u_{-\beta}$, which is defined analogously to (3.4) for any Ito–Sadahiro number β.

Corollary 3.8. *Let $\beta > 1$ be an Ito–Sadahiro number and let $u_{-\beta}$ be the bidirectional infinite word coding $\mathbb{Z}_{-\beta}$. Then $u_{-\beta}$ is Sturmian if and only if β is a quadratic Pisot unit.*

Proof. From Proposition 16 of [2] it follows that the infinite word $u_{-\beta}$ is binary only if β is quadratic. As mentioned above, the only quadratic Ito–Sadahiro numbers are quadratic Pisot numbers. For β a quadratic Pisot unit, Proposition 2.4 identifies the set $\mathbb{Z}_{-\beta}$ as a cut-and-project set $\Sigma_\beta(\Omega)$ for certain interval Ω. Propositions 3.4 and 3.6 state that the distances between consecutive elements of $\mathbb{Z}_{-\beta}$ take two values. By Theorem 2.1, the infinite word $u_{-\beta}$ is a bidirectional Sturmian word. On the other hand, if β is a quadratic Pisot number, but not a unit, then we use the combinatorial characterization of Sturmian words as aperiodic balanced words to exclude that $u_{-\beta}$ is Sturmian.

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Recall that words $u_{-\beta}$ are fixed points

$$u_{-\beta} = \lim_{n\to\infty} \overleftarrow{\varphi_{\text{-}\beta}}^n(1)|\overleftarrow{\varphi_{\text{-}\beta}}^n(0)$$

of antimorphisms given in Propositions 3.5 and 3.7. We distinguish three cases according to the prescription of the antimorphism.

(1) Let $\beta^2 = m\beta + m$, $m \geq 2$. Then $\overleftarrow{\varphi_{\text{-}\beta}}(0) = 0^m1$, $\overleftarrow{\varphi_{\text{-}\beta}}(1) = 0^m$. Necessarily, $u_{-\beta}$ contains the factors 10^m1 and 0^{2m}, which contradicts the balance property of Sturmian words.
(2) Let $\beta^2 = m\beta + n$, $2 \leq n < m$. Then $\overleftarrow{\varphi_{\text{-}\beta}}(0) = 0^{m-1}1$, $\overleftarrow{\varphi_{\text{-}\beta}}(1) = 0^{m+n-1}1$. Then $u_{-\beta}$ contains the factors $10^{m-1}1$ and 0^{m+n-1}, which is a contradiction.
(3) Let $\beta^2 = m\beta - n$, $2 \leq n \leq m-2$. Then $\overleftarrow{\varphi_{\text{-}\beta}}(0) = 0^{m-2}1$, $\overleftarrow{\varphi_{\text{-}\beta}}(1) = 0^{m-2}10^{m-n-2}1$. Then $u_{-\beta}$ contains the factors $10^{m-n-2}1$ and 0^{m-2}, which is again a contradiction. □

Remark 3.9. In fact, when showing that $u_{-\beta}$ is a Sturmian word for a quadratic Pisot unit β one could avoid argumentation by cut-and-project scheme, providing thus an alternative proof of one implication in Corollary 3.8. For that, if would be sufficient to demonstrate that $u_{-\beta}$ is a fixed point of a primitive Sturmian morphism. A morphism σ is called Sturmian, if for every Sturmian word u, the infinite word $\sigma(u)$ is Sturmian. A morphism over a binary alphabet is Sturmian if and only if it is a composition of morphisms $E, \varphi, \widetilde{\varphi}$, where

$$E: \begin{array}{l} 0 \mapsto 1 \\ 1 \mapsto 0 \end{array} \qquad \varphi: \begin{array}{l} 0 \mapsto 01 \\ 1 \mapsto 0 \end{array} \qquad \widetilde{\varphi}: \begin{array}{l} 0 \mapsto 10 \\ 1 \mapsto 0. \end{array}$$

The infinite word $u_{-\beta}$ is a fixed point of the morphism $\overleftarrow{\varphi_{\text{-}\beta}}^2$, where $\overleftarrow{\varphi_{\text{-}\beta}}$ is the antimorphism from Propositions 3.5 and 3.7. We can justify that $\overleftarrow{\varphi_{\text{-}\beta}}^2$ is Sturmian by providing its decomposition into morphisms $E, \varphi, \widetilde{\varphi}$. We have

(1) Let $\beta^2 = \beta + 1$. Then $\beta = \tau = \frac{1}{2}(1+\sqrt{5})$, and $u_{-\beta}$ is a fixed point of the Sturmian morphism $\overleftarrow{\varphi_{\text{-}\beta}}^2 = \varphi \circ \widetilde{\varphi}$, hence it is a Sturmian word.
(2) Let $\beta^2 = m\beta + 1$, $1 < m$. Then the second iteration $\overleftarrow{\varphi_{\text{-}\beta}}^2$ is a morphism with prescription

$$0 \mapsto 0^m1(0^{m-1}1)^{m-1}, \qquad 1 \mapsto 0^m1(0^{m-1}1)^m.$$

One can verify that

$$\overleftarrow{\varphi_{\text{-}\beta}}^2 = (\varphi \circ E)^{m-1} \circ (E \circ \widetilde{\varphi})^m \circ \varphi \circ E.$$

(3) Let $\beta^2 = m\beta - 1$. Then $\overleftarrow{\varphi_{\text{-}\beta}}^2$ is given by the prescription

$$\begin{aligned} 0 &\mapsto 0^{m-2}10^{m-3}1\left(0^{m-2}1\right)^{m-2}, \\ 1 &\mapsto 0^{m-2}10^{m-3}1\left(0^{m-2}1\right)^{m-2}0^{m-3}1\left(0^{m-2}1\right)^{m-2}. \end{aligned}$$

This morphism is Sturmian since

$$\overleftarrow{\varphi_{\text{-}\beta}}^2 = (\varphi \circ E)^{m-3} \circ E \circ \widetilde{\varphi} \circ (\widetilde{\varphi} \circ E)^{m-2} \circ E \circ \widetilde{\varphi} \circ \varphi \circ E.$$

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4. Relation of $\mathbb{Z}_\beta$ and $\mathbb{Z}_{-\beta}$

One of the main purposes of this paper is demonstrating the relation of β- and $(-\beta)$-integers and thus continuing the study performed for the golden ratio τ in [19]. There we have shown that

$$\mathbb{Z}_{-\tau} \cap \mathbb{R}^+ = \mathbb{Z}^+_{\tau^2}. \tag{4.1}$$

In fact, such an equality relating integers in base $-\beta$ and β^2 is exceptional, ensured by the fact that $\mathbb{Z}[\tau] = \mathbb{Z}[\tau^2]$. This allows that

$$\mathbb{Z}_{\tau^2} \cap \mathbb{R}^+ = (\mathbb{Z}_\tau + 1) \cap \mathbb{R}^+. \tag{4.2}$$

This can be seen for example from the fact that τ is a root of $x^2 - x - 1$, and so by the first statement of Proposition 2.3,

$$\mathbb{Z}^+_\tau = \Sigma_\tau\big((-1,\tau)\big) \cap \mathbb{R}^+ = \big\{x \in \mathbb{Z}[\tau] \big| x' \in (-1,\tau)\big\} \cap \mathbb{R}^+,$$

and similarly, τ^2 is a root of $x^2 - 3x + 1$, and so by the second statement of Proposition 2.3,

$$\begin{aligned}\mathbb{Z}^+_{\tau^2} &= \Sigma_{\tau^2}\big([0,\tau^2)\big) \cap \mathbb{R}^+ = \big\{x \in \mathbb{Z}[\tau^2] = \mathbb{Z}[\tau] \big| x' \in [0,\tau^2)\big\} \cap \mathbb{R}^+ \\ &= \Sigma_\tau\big([0,\tau^2)\big) \cap \mathbb{R}^+ = \mathbb{Z}_{-\tau} \cap \mathbb{R}^+.\end{aligned}$$

We obviously have $(-1,\tau)+1 = (0,\tau^2)$, which according to statement 2 of Proposition 2.2 and Proposition 3.6 justifies (4.2) and (4.1).

In [19] we have used a combinatorial approach for proving (4.1). Argumentation using the cut-and-project scheme as above can be extended to clarify the relation of $\mathbb{Z}_{-\beta}$ and $\mathbb{Z}_\beta$ for all quadratic Pisot units β roots of $x^2 - mx - 1$, $m \geq 1$.

Theorem 4.1. *Let $\beta > 1$ be the root of $x^2 - mx - 1$, $m \geq 2$. Then*

$$\mathbb{Z}_\beta \cap \mathbb{R}^+ = (\mathbb{Z}_{-\beta} \cup \beta\mathbb{Z}_{-\beta}) \cap \mathbb{R}^+.$$

Proof. By Proposition 2.3 we have $\mathbb{Z}_\beta \cap \mathbb{R}^+ = \Sigma_\beta(-1,\beta) \cap \mathbb{R}^+$. Since $\beta' = -\frac{1}{\beta}$, we have from Proposition 2.2 that

$$\Sigma_\beta(-1,\beta) = \Sigma_\beta(-1,0] \cup \Sigma_\beta[0,\beta) = \Sigma_\beta\big(-\tfrac{1}{\beta}[0,\beta)\big) \cup \Sigma_\beta[0,\beta) = \beta\Sigma_\beta[0,\beta) \cup \Sigma_\beta[0,\beta).$$

By Proposition 2.4 we have $\mathbb{Z}_{-\beta} = \Sigma_\beta[0,\beta)$, and the statement of the theorem follows. □

Note that $\mathbb{Z}_{-\beta} \cap \beta\mathbb{Z}_{-\beta} = \{0\}$, as can be seen from the proof of Theorem 4.1.

If we try to obtain similar geometric relation for $\beta > 1$ root of $x^2 - mx + 1$, $m \geq 3$, we do not succeed, for, the closures of $\{x' \mid x \in \mathbb{Z}_\beta\}$, resp. $\{x' \mid x \in \mathbb{Z}_{-\beta}\}$ are shifted with respect to each other by a constant that does not belong to $\mathbb{Z}[\beta'] = \mathbb{Z}[\beta]$. Even if we consider for example the set

$$(\mathbb{Z}_\beta \setminus \beta\mathbb{Z}_\beta) \cap \mathbb{R}^+ = \Sigma_\beta[1,\beta) \cap \mathbb{R}^+,$$

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which has the same interval length as the cut and project set

$$\mathbb{Z}_{-\beta} = \Sigma_\beta\left(\left(-\tfrac{\beta-1}{\beta+1}, \beta\tfrac{\beta-1}{\beta+1}\right)\right),$$

we cannot identify them as a shift one of the other. Nevertheless, every finite piece in $\Sigma_\beta\left(\left(-\frac{\beta-1}{\beta+1}, \beta\frac{\beta-1}{\beta+1}\right)\right)$ can be found translated in $\Sigma_\beta[1,\beta)$ and *vice versa*. This suggests that we can still obtain an interesting combinatorial relation between $\mathbb{Z}_\beta$ and $\mathbb{Z}_{-\beta}$ when studying the corresponding infinite words u_β and $u_{-\beta}$. In particular, we will study the language of these infinite words, thus comparing the finite pieces of $\mathbb{Z}_\beta$ and $\mathbb{Z}_{-\beta}$. In fact, such an approach allows us to give a relation between the β- and $(-\beta)$-integers not only for quadratic Pisot units β, but also for other quadratic Pisot numbers.

Since the language $\mathcal{L}(u_{-\beta})$ of $u_{-\beta} = \ldots u_{-2}u_{-1}|u_0u_1u_2\ldots$ is the same as the one of its suffix $u^+_{-\beta} = u_0u_1u_2\ldots$, it is natural to compare $\mathcal{L}(u_\beta)$ and $\mathcal{L}(u^+_{-\beta})$. The tool is to compare the morphisms under which the words are invariant. For that, the notion of conjugation is useful. A morphism ψ is a right conjugate of a morphism σ, if there is a finite word w, such that

$$\sigma(a)w = w\psi(a) \quad \text{for all } \ a \in \mathcal{A}^*.$$

Incidence matrices M_σ and M_ψ of conjugate morphisms coincide. The opposite implication is obviously not satisfied. We will use the following lemma, which appears to be a folklore, thus we include it without proof.

Lemma 4.2. *Languages of fixed points of conjugate primitive morphisms coincide.*

In the terminology of dynamical systems, the fact that languages of two infinite words coincide means that they belong to the same subshift. However, it does not necessarily mean that one is a shift of the other.

Theorem 4.3. *Let $\beta > 1$ be the root of $x^2 - mx - m$, $m \geq 1$. Then $\mathcal{L}(u_\beta) = \mathcal{L}(u_{-\beta})$. Moreover, if $m = 1$, then $u^+_{-\beta} = 0u_\beta$. When $m \geq 2$, then neither $u^+_{-\beta}$ is a suffix of u_β, nor u_β is a suffix of $u^+_{-\beta}$.*

Proof. By Proposition 3.5, we have $\overleftarrow{\varphi_{-\beta}}(0) = 0^m1$, $\overleftarrow{\varphi_{-\beta}}(1) = 0^m$ and thus

$$\overleftarrow{\varphi_{-\beta}}^2(0) = 0^m(0^m1)^m, \qquad \overleftarrow{\varphi_{-\beta}}^2(1) = (0^m1)^m\,.$$

The second iteration of the canonical substitution φ_β, $\varphi_\beta(0) = 0^m1$, $\varphi_\beta(1) = 0^m$ is given by

$$\varphi^2(0) = (0^m1)^m0^m, \qquad \varphi^2(1) = (0^m1)^m.$$

Therefore $\overleftarrow{\varphi_{-\beta}}^2$ is a right conjugate of φ^2 and the conjugation factor w is equal to $w = (0^m1)^m$.

If $m = 1$, then $\beta = \tau$ and the statement $u^+_{-\tau} = 0u_\tau$ is found in [19]. On the other hand, if $m \geq 2$, we can exclude that u_β is a suffix of $u^+_{-\beta}$ or *vice versa*. Recall

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that letters $0, 1$ code the same distances $\Delta_0^+ = \Delta_0^- = 1$, $\Delta_1^+ = \Delta_1^- = \frac{m}{\beta}$ in u_β and $u_{-\beta}^+$, *cf.* Propositions 3.1 and 3.4. If $u_{-\beta}^+ = wu_\beta$ or $wu_{-\beta}^+ = u_\beta$ for some finite word $w \in \{0,1\}^*$, then

$$(\mathbb{Z}_{-\beta} \pm c) \cap \mathbb{R}^+ = \mathbb{Z}_\beta^+, \tag{4.3}$$

where $c = |w|_0\Delta_0 + |w|_1\Delta_1 \in \mathbb{Z} + \mathbb{Z}\frac{m}{\beta} = \mathbb{Z}[\beta]$. Taking the Galois images, one would have

$$\{x' \mid x \in \mathbb{Z}_{-\beta} \cap \mathbb{R}^+\} \pm c' = \{x' \mid x \in \mathbb{Z}_\beta^+\}.$$

However, this is not possible. We will show that by finding $\sup\{x' \mid x \in \mathbb{Z}_{-\beta} \cap \mathbb{R}^+\}$ and $\sup\{x' \mid x \in \mathbb{Z}_\beta^+\}$. It is easy to realize that supremum $\sup\{x' \mid x \in \mathbb{Z}_\beta^+\}$ is approached by β-integers $x = \sum_{i=0}^k a_i\beta^i \in \mathbb{Z}[\beta]$ with β-expansion $\langle x\rangle_\beta = m0m0\dots m0m\bullet$. We have

$$x' = \sum_{i=0}^k a_i\beta'^i = \sum_{i=0}^k a_i\left(-\frac{m}{\beta}\right)^i < m\sum_{i=0}^\infty \left(\frac{m}{\beta}\right)^{2i} = \frac{\beta^3}{\beta+m}.$$

Similarly, one can compute $\sup\{x' \mid x \in \mathbb{Z}_{-\beta} \cap \mathbb{R}^+\}$, realizing that it is obtained considering $(-\beta)$-integers of the form $\langle x\rangle_{-\beta} = mmm\dots m\bullet$,

$$x' = \sum_{i=0}^k a_i(-\beta')^i < m\sum_{i=0}^\infty \left(\frac{m}{\beta}\right)^i = \frac{m\beta}{\beta-m} = \beta^2.$$

Thus the difference $\sup\{x' \mid x \in \mathbb{Z}_\beta^+\} - \sup\{x' \mid x \in \mathbb{Z}_{-\beta} \cap \mathbb{R}^+\} = \frac{\beta^3}{\beta+m} - \beta^2 \notin \mathbb{Z}[\beta]$, and hence equality (4.3) cannot be valid for any $c \in \mathbb{Z}[\beta]$. □

One cannot expect that morphisms corresponding to u_β and $u_{-\beta}$ will be conjugated for quadratic Pisot β other than roots of $x^2 - mx - m$. This is because the incidence matrices of morphisms φ_β^2 and $\overleftarrow{\varphi_{\text{-}\beta}}^2$ do not coincide, compare Proposition 3.2 with Propositions 3.5 and 3.7. This is also in agreement with the fact that the distances in $\mathbb{Z}_\beta$ and $\mathbb{Z}_{-\beta}$ coincide only if β is a root of $x^2 - mx - m$. For other quadratic Pisot numbers β, we have interestingly

$$\Delta_0^+ = \Delta_0^- = 1, \quad \Delta_1^+ = \frac{n}{\beta} = \Delta_1^- - 1 \qquad \text{if } \beta^2 = mx + n,\ m > n \geq 1,$$

$$\Delta_0^+ = \Delta_0^- = 1, \quad \Delta_1^+ = 1 - \frac{n}{\beta} = \Delta_1^- - 1 \qquad \text{if } \beta^2 = mx - n,\ m - 2 \geq n \geq 1,$$

which means that $\Delta_1^- = \Delta_0^+ + \Delta_1^+$. It is therefore appealing to see what happens when adding to $\mathbb{Z}_{-\beta}$ certain points which would split every distance Δ_1^- into two distances of lengths $\Delta_0^+ = \Delta_0^- = 1$ and Δ_1^+. Theorem 4.1 shows that at least for one class of considered numbers, this splitting on the positive half-line gives the same set as $\mathbb{Z}_\beta^+$. Note, however, that we can choose the order in which we split the long distance into two.

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Realize that performing this procedure on the geometric representation of $\mathbb{Z}_{-\beta}$ corresponds to applying a suitable morphism on the infinite word $u_{-\beta}$. Define $\pi, \tilde{\pi} : \{0,1\}^* \to \{0,1\}^*$ by

$$\pi : \begin{array}{l} 0 \mapsto 0 \\ 1 \mapsto 10 \end{array} \quad \text{and} \quad \tilde{\pi} : \begin{array}{l} 0 \mapsto 0 \\ 1 \mapsto 01. \end{array} \tag{4.4}$$

As we shall prove, application of π, resp. $\tilde{\pi}$ on the infinite word $u_{-\beta}$ leads to an infinite word with the same language as u_β. For showing that, we will use the following easy statement.

Lemma 4.4. *Let σ, ψ, π be morphisms such that $\sigma : \mathcal{A}^* \to \mathcal{A}^*$, $\psi : \mathcal{B}^* \to \mathcal{B}^*$, $\pi : \mathcal{A}^* \to \mathcal{B}^*$ and $\pi \circ \sigma = \psi \circ \pi$. If u is a fixed point of σ, then $v := \pi(u)$ is a fixed point of ψ.*

Proof. We have $v = \pi(u) = \pi\big(\sigma(u)\big) = \psi\big(\pi(u)\big) = \psi(v)$. □

Theorem 4.5. *Let π and $\tilde{\pi}$ be morphisms as in (4.4).*

Let $\beta > 1$ be the root of $x^2 - mx - n$, $m > n \geq 1$. Then,

$$\mathcal{L}(u_\beta) = \mathcal{L}\big(\tilde{\pi}(u_{-\beta})\big).$$

Moreover, $u_\beta = \pi(u^+_{-\beta})$ for $n = 1$.

Let $\beta > 1$ be the root of $x^2 - mx + n$, $m - 2 \geq n \geq 1$.

$$\mathcal{L}(u_\beta) = \mathcal{L}\big(\pi(u_{-\beta})\big).$$

Proof. Let β be the quadratic Pisot number satisfying $\beta^2 = m\beta + n$, $m > n \geq 1$. Define a morphism $\psi : \{0,1\}^* \to \{0,1\}$ by

$$\psi : \begin{array}{l} 0 \mapsto 0^{m+n}1(0^m1)^{m-1} \\ 1 \mapsto (0^m1)^n. \end{array}$$

One can easily verify that the morphism ψ satisfies $\tilde{\pi} \circ \overleftarrow{\varphi_{-\beta}}^2 = \psi \circ \tilde{\pi}$. Therefore by Lemma 4.4, $\tilde{\pi}(u_{-\beta})$ is a fixed point of ψ. We can also verify that ψ is a right conjugate of φ^2_β, which is by Proposition 3.2 equal to

$$\varphi^2_\beta : \begin{array}{l} 0 \mapsto (0^m1)^m0^n \\ 1 \mapsto (0^m1)^n. \end{array}$$

The conjugacy factor w is equal to $w = (0^m1)^m$. By Lemma 4.2, the infinite words u^+_β and $\tilde{\pi}(u_{-\beta})$ have the same language.

Similarly, for the quadratic Pisot number satisfying $\beta^2 = m\beta - n$, $m - 2 \geq n \geq 1$, define $\psi : \{0,1\}^* \to \{0,1\}$ by

$$\psi : \begin{array}{l} 0 \mapsto (0^{m-2}1)0^{m-n-1}1(0^{m-1}1)^{m-2}0 \\ 1 \mapsto (0^{m-2}1)0^{m-n-1}1(0^{m-1}1)^{m-n-2}0. \end{array}$$

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The morphism ψ satisfies $\pi \circ \overleftarrow{\varphi_{-\beta}}^2 = \psi \circ \pi$ and therefore $\pi(u_{-\beta})$ is a fixed point of ψ. We verify that φ_β^2 is a right conjugate of ψ, which is by Proposition 3.2 equal to

$$\varphi_\beta^2 : \begin{array}{l} 0 \mapsto (0^{m-1}1)^{m-1}0^{m-n-1}1 \\ 1 \mapsto (0^{m-1}1)^{m-n-1}0^{m-n-1}1. \end{array}$$

The conjugacy factor w is equal to $w = 0^{m-2}10^{m-n-1}1$. Thus by Lemma 4.2, we derive $\mathcal{L}(u_\beta) = \mathcal{L}\big(\pi(u_{-\beta})\big)$.

It remains to justify that for $\beta^2 = m\beta + 1$ we have equality $u_\beta = \pi(u^+_{-\beta})$. In this case

$$\varphi_\beta^2 : \begin{array}{l} 0 \mapsto (0^m1)^m0 \\ 1 \mapsto 0^m1 \end{array} \quad \text{and} \quad \overleftarrow{\varphi_{-\beta}}^2 : \begin{array}{l} 0 \mapsto 0^m1(0^{m-1}1)^{m-1} \\ 1 \mapsto 0^m1(0^m1)^m \end{array} \tag{4.5}$$

and therefore one can verify that $\pi \circ \overleftarrow{\varphi_{-\beta}}^2 = \varphi_\beta^2 \circ \pi$. By Lemma 4.4, $\pi(u^+_{-\beta})$ is a fixed point of φ_β^2, but φ_β^2 has only one fixed point, namely $u_\beta = \lim_{j\to\infty} \varphi_\beta^j(0)$. □

5. Addition of $(-\beta)$-integers

In the following, let us focus on the arithmetical aspect of $(-\beta)$-integers. For non-integer base $-\beta$, the set $\mathbb{Z}_{-\beta}$ is not closed under addition and subtraction. One can nevertheless define the operation $\oplus$ of addition on $\mathbb{Z}_{-\beta}$ as $t_j \oplus t_k = t_{j+k}$, with the neutral element $t_0 = 0$ and opposite element $\ominus t_j = t_{-j}$, so that $\mathbb{Z}_{-\beta}$ be an additive group isomorphic to $\mathbb{Z}$. Such a definition is of course possible for any countable set. Here we show that for every quadratic Pisot number β, the 'sum' $t_j \oplus t_k$ yields always a result which is not far from the ordinary sum of real numbers t_j and t_k. More formally, we show that $t_j \oplus t_k - (t_j + t_k)$ is bounded independently on j and k, and we determine the possible values of the distance for quadratic Pisot units. We also show that the operation $\oplus$ is compatible with ordinary addition of real numbers, namely that whenever the result $t_j + t_k$ is a $(-\beta)$-integer, then $t_j \oplus t_k = t_j + t_k$. Note that for the case of positive base β, where β is a quadratic Pisot unit, this was done in [7] by a technical arithmetic study. Here we choose a combinatorial approach which allows us to derive a general result about compatibility. We will show that for any discrete set $\Sigma = \{t_j \mid j \in \mathbb{Z}\} \subset \mathbb{R}$, $t_j < t_{j+1}$, with finitely many distances between consecutive points, *i.e.* values $t_{j+1} - t_j$, one can define a binary operation $\oplus$ so that $(\Sigma, \oplus) \simeq (\mathbb{Z}, +)$, which is compatible with addition in $\mathbb{R}$ if the values of the distances are linearly independent over $\mathbb{Q}$.

Theorem 5.1. *Let $u = \ldots u_{-2}u_{-1}|u_0u_1u_2\ldots$ be a bidirectional infinite word over a finite alphabet $\mathcal{A} = \{0, 1, \ldots, d-1\}$. Let $\Delta_0, \ldots, \Delta_{d-1}$ be positive real values linearly independent over $\mathbb{Q}$. Set $t_0 = 0$. For $k \geq 1$, define $t_k = \sum_{i=0}^{d-1} |w|_i \Delta_i$, where $w = u_0u_1\ldots u_{k-1}$. For $k \leq -1$, define $t_k = -\sum_{i=0}^{d-1} |w|_i \Delta_i$, where $w = u_{-k}u_{-k+1}\ldots u_{-1}$. On the set $\Sigma = \{t_k \mid k \in \mathbb{Z}\}$ define a binary operation $\oplus$,*

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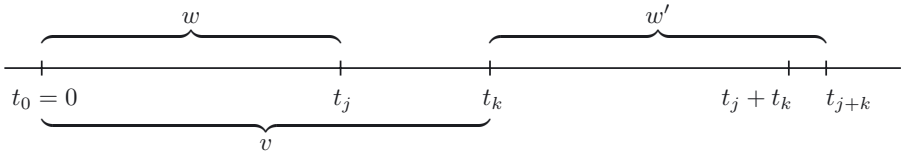


FIGURE 2. Addition $t_j + t_k$ in Σ.

by $t_j \oplus t_k = t_{j+k}$*, for* $j, k \in \mathbb{Z}$*. Then* $(\Sigma, \oplus)$ *is isomorphic to* $(\mathbb{Z}, +)$ *and* $\oplus$ *is compatible with addition in* $\mathbb{R}$*.*

Proof. We have to show that whenever $t_j + t_k = t_l$, then $l = j + k$. For the contradiction, let us assume that $t_j + t_k = t_l$, where $l \neq j + k$. Suppose first that $0 \leq j \leq k$ which obviously implies $l \geq 0$. Let t_j correspond to a prefix w, t_k to a prefix v and t_l to a prefix z of the word $u_0u_1 \dots$ We have

$$t_j + t_k - t_l = 0 \qquad \Longleftrightarrow \qquad \sum_{i=0}^{d-1}(|w|_i + |v|_i - |z|_i)\Delta_i = \sum_{i=0}^{d-1}(|wv|_i - |z|_i)\Delta_i = 0. \tag{5.1}$$

The last equality of (5.1) is a nontrivial combination of Δ_i's which follows from $|wv| = j + k \neq l = |z|$. Here we have a contradiction with the linear independence of $\Delta_0, \dots, \Delta_{d-1}$ over $\mathbb{Q}$.

The same argument can be applied when considering indices $j \leq 0 \leq k$ or $j \leq k \leq 0$. □

Let us derive the possible outcomes of $t_j + t_k - t_{j+k}$. For $0 \leq j \leq k$, let t_j correspond to the prefix $w = u_0 \dots u_{j-1}$, let t_k correspond to the prefix $v = u_0 \dots u_{k-1}$ and let t_{j+k} correspond to the prefix $z = vw'$, where $w' = u_k \dots u_{j+k-1}$, see Figure 2. Then we have

$$t_j + t_k - t_{j+k} = \sum_{i=0}^{d-1}(|w|_i + |v|_i - |vw'|_i)\Delta_i = \sum_{i=0}^{d-1}(|w|_i - |w'|_i)\Delta_i. \tag{5.2}$$

The same looking formula can be derived for other cases $j \leq k \leq 0$ or $j \leq 0 \leq k$, only the factors w and w' are dependent on the case, however, they are always of the same length.

Equality (5.2) states that the distance of $t_j + t_k$ from $t_j \oplus t_k = t_{j+k}$ depends on the difference of the number of occurrences of a specific letter in two factors w, w' of the same length. This is captured by the notion of generalized balance defined in [1]. An infinite word u over an alphabet $\mathcal{A}$ is called C-balanced for a $C \in \mathbb{N}$, if

$$\big||w|_a - |w'|_a\big| \leq C, \quad \text{for every } a \in \mathcal{A} \text{ and every pair } w, w' \in \mathcal{L}(u),\ |w| = |w'|.$$

We sometimes say that u has bounded balances, if there exists $C < +\infty$, such that u is C-balanced. Obviously, u is balanced if it is 1-balanced.

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Using (5.2), we derive that if the set Σ is defined by an infinite word with bounded balances, then the result of $t_j \oplus t_k$ is not far from the actual sum of t_j and t_k.

Corollary 5.2. *Let $u = \dots u_{-2}u_{-1}|u_0u_1u_2\dots$ be a bidirectional infinite word over a finite alphabet $\mathcal{A} = \{0, 1, \dots, d-1\}$ with bounded balances. Define $(\Sigma, \oplus)$ as in Theorem 5.1. Then $t_j + t_k - (t_j \oplus t_k)$ is bounded independently of $j, k \in \mathbb{Z}$.*

Note that when Σ is coded by a binary word over the alphabet $\mathcal{A} = \{0, 1\}$, we can write $|w|_1 = |w| - |w|_0$. In this case, (5.2) can be rewritten as

$$t_j + t_k - t_{j+k} = \big(|w|_0 - |w'|_0\big)\Delta_0 + \big(|w|_1 - |w'|_1\big)\Delta_1 = \big(|w|_0 - |w'|_0\big)(\Delta_0 - \Delta_1). \quad (5.3)$$

Similarly, we would get

$$-t_{-j} - t_j = \big(|w|_0 - |w'|_0\big)(\Delta_0 - \Delta_1), \quad (5.4)$$

where $w = u_{-j}u_{-j+1}\dots u_{-1}$, $w' = u_0u_1\dots u_{j-1}$ for $j \geq 0$ and *vice versa* otherwise.

Let us apply Theorem 5.1 and Corollary 5.2 to β- and $(-\beta)$-integers in case of quadratic Pisot β. For a positive base, we know that the infinite word u_β has bounded balances, they have been determined explicitly in [4,26]. It is not difficult to prove that also the infinite word $u_{-\beta}$ has bounded balances, if we use the result of Adamczewski [1] about balance properties of fixed points of morphisms. He shows that boundedness/unboundedness of balances is decided by the second (in modulus) eigenvalue of the incidence matrix of the morphism.

Lemma 5.3. *For every quadratic Pisot β there exist a constant C such that the infinite word $u_{-\beta}$ is C-balanced. If β is a unit, then $C = 1$.*

Proof. For any quadratic Pisot β, the infinite word $u_{-\beta}$ is a fixed point of a substitution, which is a second iteration of the antimorphism $\overleftarrow{\varphi}$ given in Propositions 3.5 and 3.7. The incidence matrix of the substitution has β^2 as its dominant eigenvalue. Necessarily, the second eigenvalue is β'^2, which is in modulus smaller than 1. Consequently, by [1], the fixed point of $\overleftarrow{\varphi}_\beta^2$ has bounded balances. If β is a unit, then Proposition 3.8 states that the infinite word $u_{-\beta}$ is a Sturmian word, and thus it is balanced, *i.e.* $C = 1$. □

Relations (5.3) and (5.4) describe the possible outcomes of $t_j \oplus t_k - (t_j + t_k)$ and $\ominus t_j - (-t_j)$. By Lemma 5.3, there exists a constant C such that for any pair of factors $w, v \in \mathcal{L}(u_{-\beta})$ we have $\big||v|_0 - |w|_0\big| \leq C$. This, together with (5.3) and (5.4) shows that $t_j \oplus t_k - (t_j + t_k)$ and $\ominus t_j - (-t_j)$ are bounded independently of j, k, and that when β is a unit, then $C = 1$ and thus

$$t_j \oplus t_k - (t_j + t_k), \ominus t_j - (-t_j) \in \big\{0, \pm(\Delta_0 - \Delta_1)\big\}. \quad (5.5)$$

Corollary 5.4. *Let $\beta > 1$ be the root of $x^2 - mx + 1$, $m \geq 3$. Then*

$$t_j \pm t_k = t_{j\pm k} + \{-\eta, 0, \eta\}, \qquad \text{where } \eta = 1 - \frac{1}{\beta}.$$

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For the second class of quadratic Pisot units, one can observe even stronger properties. This is given by the fact that the prefixes of $u^+_{-\beta} = u_0u_1u_3\dots$ are 'light', or in other words, contain at least as many letters 0 as any factor of $u_{-\beta}$ of the same length. In fact, by Corollary 3.8, $u_{-\beta}$ is a Sturmian word, *i.e.* it is balanced. Hence for any given length $k \in \mathbb{N}$, the number $|w|_0$ in factors $w \in \mathcal{L}(u_{-\beta})$, $|w| = k$, takes only two possible values. Factors having the larger number of 0's are called light, the other ones are called heavy.

Lemma 5.5. *Let $\beta > 1$ be the root of $x^2 - mx - 1$, $m \geq 1$. Let $u_{-\beta} = \dots u_{-2}u_{-1}|u_0u_1u_2\dots$ be the infinite word coding the set of $(-\beta)$-integers. Let $w = u_0u_1\dots u_{n-1}$ and let w' be any factor in $\mathcal{L}(u_{-\beta})$ of length $|w'| = |w| = n$. Then $|w|_0 - |w'|_0 \in \{0, 1\}$. In other words, every prefix of $u^+_{-\beta}$ is light.*

Proof. When $\beta > 1$ is the root of $x^2 - mx - 1$, $m \geq 1$, we use Proposition 2.4, which states that $\mathbb{Z}_{-\beta} = \Sigma_\beta\big([0, \beta^2)\big)$ for $m = 1$ and $\mathbb{Z}_{-\beta} = \Sigma_\beta\big([0, \beta)\big)$ otherwise. Consider first the case $m > 1$. The infinite word $u^+_{-\beta}$ codes the non-negative part of $\mathbb{Z}_{-\beta}$, and in fact, coincides with the Sturmian word arising as coding of 0 in the exchange of intervals $[0, \beta - 1)$, $[\beta - 1, 1)$. By rescaling, it can be viewed also as a coding of 0 under a rotation by the angle $\alpha = \frac{1}{\beta}$ on the unit interval, thus $u^+_{-\beta}$ is in fact what is called a Sturmian word with null intercept. As a consequence of the proof of Theorem 8 in [23], any prefix of such a Sturmian word is light. □

Proposition 5.6. *Let $\beta > 1$ be the root of $x^2 - mx - 1$, $m \geq 1$. Then*

$$t_j + t_k = t_{j+k} + \{0, \xi\}, \qquad t_j - t_k = t_{j-k} - \{0, \xi\} \qquad \text{for all } j, k \in \mathbb{Z}$$

and

$$-t_j = t_{-j} - \xi \quad \text{for } j \in \mathbb{Z} \setminus \{0\},$$

where

$$\xi = \Delta_0^- - \Delta_1^- = \begin{cases} -\frac{1}{\beta} & \text{for } m \geq 2, \\ \frac{1}{\beta^2} & \text{for } m = 1. \end{cases}$$

Moreover, if $m \geq 2$, then $t_j \oplus t_k$ is the closest $(-\beta)$-integer to $t_j + t_k$.

Proof. Let us first prove $-t_j = t_{-j} - \xi$ for all $j \in \mathbb{Z} \setminus \{0\}$. According to (5.5) we have that $-t_j = t_{-j} + \{0, \pm(\Delta_0^- - \Delta_1^-)\}$, where $\Delta_0^- - \Delta_1^- = \xi$. We take the Galois images of those three possible cases, namely

$$-t'_j = t'_{-j}, \qquad -t'_j = t'_{-j} + \xi' \qquad \text{or} \qquad -t'_j = t'_{-j} - \xi', \tag{5.6}$$

where $\xi' = \beta^2$ and $\xi' = \beta$ for $m = 1$ and $m \geq 2$, respectively. Proposition 2.4 states that $t_j \in \mathbb{Z}_{-\beta}$ implies $t'_j \in \Omega$, where $\Omega = [0, \beta^2)$ for $m = 1$ and $\Omega = [0, \beta)$ otherwise (note that 0 corresponds to $t_0 = 0$ which is not considered in this statement). Therefore $-t'_j \in -\Omega$ and substituting together with $t'_{-j} \in \Omega$ into (5.6) we get that the only possible option is $-t'_j = t'_{-j} - \xi'$.

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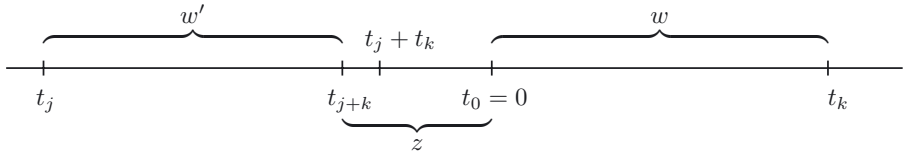


FIGURE 3. Addition $t_j + t_k$ in the case $j \le 0 \le k$ with $|j| \ge k$.

Let $0 \le j \le k$. Using (5.2) and the fact that the factor $w = u_0 \dots u_{j-1}$ is light, we obtain $t_j + t_k - t_{j+k} = (|w|_0 - |w'|_0)(\Delta_0^- - \Delta_1^-) \in \{0, \xi\}$. When $k \le j \le 0$ we can write

$$\begin{aligned} -(t_j + t_k - t_{j+k}) &= t_{-j} - \xi + t_{-k} - \xi - t_{-j-k} + \xi \\ &= \underbrace{t_{-j} + t_{-k} - t_{-j-k}}_{\in\{0,\xi\}} - \xi \in \{-\xi, 0\}. \end{aligned} \tag{5.7}$$

The first equality follows from $-t_j = t_{-j} + \xi$ and then we used already proven case $0 \le j \le k$. Thus is holds $t_j + t_k - t_{j+k} \in \{0, \xi\}$.

Consider the case $j \le 0 \le k$ with $|j| \ge k$. Let $w = u_0 u_1 \dots u_{k-1}$, $w' = u_j u_{-j+1} \dots u_{j+k-1}$, and $z = u_{j+k} u_{j+k+1} \dots u_{-1}$, *i.e.* t_k correspond to the prefix w, t_j correspond to the suffix $v = w'z$, and t_{j+k} to the suffix z, see Figure 5. Then we have

$$\begin{aligned} t_j + t_k - t_{j+k} &= \sum_{i=0}^{1} (-|w'z|_i + |w|_i + |z|_i)\Delta_i^- \\ &= (|w|_0 - |w'|_0)(\Delta_0^- - \Delta_1^-) = (|w|_0 - |w'|_0)\xi. \end{aligned}$$

Since w is light and factors w, w' are of the same length, we have $|w|_0 - |w'|_0 \in \{0, 1\}$.

When $j \le 0 \le k$ with $|j| < k$ we use the same approach as in (5.7) to get the statement.

Substituting $j = p,\ k = q - p$, into $t_j + t_k = t_{j+k} + \{0, \eta\}$, one gets $t_p + t_{q-p} = t_q + \{0, \eta\}$, and thus $t_q - t_p = t_{q-p} - \{0, \eta\}$.

If $\beta > 1$ is the root of $x^2 - mx - 1$, $m \ge 2$, then the distances between consecutive $(-\beta)$-integers take values $\Delta_0^- = 1$ and $\Delta_1^- = 1 + \frac{1}{\beta}$, and thus $\Delta_0^- - \Delta_1^- = -\frac{1}{\beta}$. Moreover, since $\lfloor \beta \rfloor = m \ge 2$, we have $\frac{1}{\beta} < \frac{1}{2}$. Therefore both $t_j \oplus t_k - (t_j + t_k)$ and $\ominus t_j - (-t_j)$ are in modulus smaller than $\frac{1}{2} < \min(\Delta_0^-/2, \Delta_1^-/2)$. □

In the above proposition, we show that $t_j \oplus t_k$ is the closest $(-\beta)$-integer to t_{j+k} only for β root of $x^2 - mx - 1$ with $m \ge 2$. We can show that if $m = 1$, *i.e.* when β is the golden ratio, then similar statement does not hold.

Example 5.7. Let β be the golden ratio, root of $x^2 - x - 1$. Then $t_j \oplus t_k$ is not always the closest $(-\beta)$-integer to $t_j + t_k$. The situation is illustrated in Figure 4.

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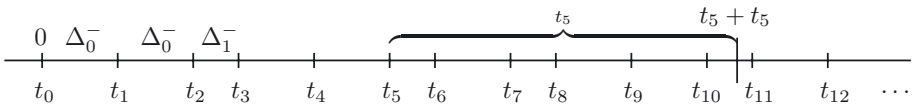


FIGURE 4. Addition in $(-\beta)$-integers for $\beta = \frac{1}{2}(1+\sqrt{5})$.

Since the distances between consecutive $(-\beta)$-integers are $\Delta_0^- = 1$ and $\Delta_1^- = \frac{1}{\beta}$, we can see in Figure 4 that $t_5 + t_5 = 2(4 + \frac{1}{\beta}) = 8 + \frac{2}{\beta}$, $t_{10} = 7 + \frac{3}{\beta}$ and $t_{11} = 7 + \frac{4}{\beta}$, so that

$$t_5 + t_5 - t_{10} = 1 - \frac{1}{\beta} \approx 0.38 \quad \text{but} \quad t_5 + t_5 - t_{11} = 1 - \frac{2}{\beta} \approx -0.24,$$

which means that $\left|t_5 + t_5 - (t_5 \oplus t_5)\right| > \left|t_5 + t_5 - t_{11}\right|$.

The following example shows that also for β root of $x^2 - mx + 1$, $m \geq 3$, the 'sum' $t_j \oplus t_k$ is not always the closest $(-\beta)$-integer to $t_j + t_k$.

Example 5.8. Using the morphism $\overleftarrow{\varphi}_{-\beta}$ from Proposition 3.7, one can list several letters of $u_{-\beta}$ around the delimiter marking the origin, namely

$$u_{-\beta} = \dots 1\big(0^{m-2}1\big)^{m-2}|0^{m-2}10^{m-3}1\big(0^{m-2}1\big)^{m-2}0^{m-2}10^{m-3}1\big(0^{m-2}1\big)^{m-2}\dots$$

Taking into account that $\Delta_0^- = 1$ and $\Delta_1^- = 2 - \frac{n}{\beta}$, we derive the following.

(1) For the case $m = 3$ we have $t_6 = 3\Delta_0^- + 3\Delta_1^- = 9 - \frac{3}{\beta}$ and $t_6 + t_6 = 18 - \frac{6}{\beta}$. The closest $(-\beta)$-integer to $t_6 + t_6$ is $t_{11} = 4\Delta_0^- + 7\Delta_1^- = 18 - \frac{7}{\beta}$ instead of $t_{12} = t_{11} + \Delta_1^- = 20 - \frac{7}{\beta}$.
(2) When $m \geq 4$ we have $t_2 = 2\Delta_0^- = 2$ and $t_{m-2} = (m-2)\Delta_0^- = m - 2$ and hence $t_2 + t_{m-2} = m$ while $t_m = (m-1)\Delta_0^- + \Delta_1^- = m + 1 - \frac{1}{\beta}$. The closest $(-\beta)$-integer to $t_2 + t_{m-2}$ is $t_{m-1} = m - \frac{1}{\beta}$.

There is an interesting consequence of Proposition 5.6. The relation $-t_j = t_{-j} - \xi$ implies

$$-\left(t_j - \frac{\xi}{2}\right) = t_{-j} - \frac{\xi}{2} \qquad \text{for all} \quad j \in \mathbb{Z} \setminus \{0\}.$$

The geometrical interpretation of this fact is following.

Corollary 5.9. *Let $\beta > 1$ be the root of $x^2 - mx - 1, m \geq 1$. Then the set $\mathbb{Z}_{-\beta} \setminus \{0\}$ is symmetrical with respect to $\xi = \Delta_0^- - \Delta_1^-$.*

The fact that the compatibility of the operations $\oplus$ and addition in $\mathbb{R}$ does not always hold is illustrated in the next example.

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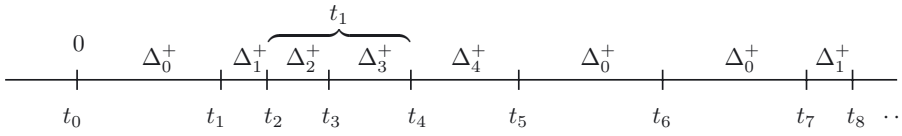


FIGURE 5. Operation $\oplus$ in $\mathbb{Z}_\beta$ is not compatible with addition in $\mathbb{R}$ for minimal Pisot number β.

Example 5.10. Let $\beta > 1$ be the minimal Pisot number, zero of the polynomial $x^3 - x + 1$. Let us show that in this case the group operation $\oplus$ defined on β-integers by $t_j \oplus t_k = t_{j+k}$ is not compatible with addition in $\mathbb{R}$.

The Rényi expansion of 1 is equal to $d_\beta(1) = 10001$, and thus the distances between consecutive β-integers take the values

$$\Delta_0^+ = 1, \quad \Delta_1^+ = \beta^{-4}, \quad \Delta_2^+ = \beta^{-3}, \quad \Delta_3^+ = \beta^{-2}, \quad \Delta_4^+ = \beta^{-1}.$$

These values are not linearly independent over $\mathbb{Q}$. In particular, we have $\Delta_0^+ = \Delta_2^+ + \Delta_3^+$.

The canonical substitution φ_β for β is of the form $0 \mapsto 01$, $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 4$, $4 \mapsto 0$. The infinite word u_β coding the set $\mathbb{Z}_\beta$ is the fixed point of φ_β, namely

$$\lim_{n\to\infty} \varphi^n(0) = 0123400101201230123401230\ldots$$

The first few β integers can be written as

$$t_0 = 0, \quad t_1 = \Delta_0^+ = 1, \quad t_2 = \Delta_0^+ + \Delta_1^+ = 1 + \frac{1}{\beta^4},$$
$$t_3 = \Delta_0^+ + \Delta_1^+ + \Delta_2^+ = 1 + \frac{1}{\beta^4} + \frac{1}{\beta^3},$$
$$t_4 = \Delta_0^+ + \Delta_1^+ + \Delta_2^+ + \Delta_3^+ = 1 + \frac{1}{\beta^4} + \frac{1}{\beta^3} + \frac{1}{\beta^2}\ldots$$

We have $t_1 \oplus t_2 = t_3$, while $t_1 + t_2 = 2\Delta_0^+ + \Delta_1^+ = \Delta_0^+ + \Delta_1^+ + \Delta_2^+ + \Delta_3^+ = t_4$.

The last example illustrates that compatibility does not enforce that the distance $|t_j + t_k - t_{j+k}|$ is bounded.

Example 5.11. Let $\beta > 1$ be the zero of the irreducible polynomial $x^6 - x^5 - 1$. Such β has conjugates outside of the unit circle, so it is not a Pisot number. However, it is a Parry number. The Rényi expansion of 1 is $d_\beta(1) = 100001$. The distances between consecutive β-integers take the values

$$\Delta_0^+ = 1, \quad \Delta_1^+ = \beta^{-5}, \quad \Delta_2^+ = \beta^{-4}, \quad \Delta_3^+ = \beta^{-3}, \quad \Delta_4^+ = \beta^{-2}, \quad \Delta_5^+ = \beta^{-1},$$

which are now linearly independent over $\mathbb{Q}$. By Theorem 5.1, the operation $\oplus$ defined on $\mathbb{Z}_\beta$ by $t_j \oplus t_k = t_{j+k}$ is compatible with addition in $\mathbb{R}$. However, the infinite

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word u_β is fixed by a non-Pisot substitution φ_β. The second largest eigenvalue λ_2 of its incidence matrix is in modulus greater than 1, and thus by [1], the infinite word u_β has unbounded balances. In particular, it means that $t_j + t_k$ can be arbitrarily far from the ordinary sum $t_j \oplus t_k$. Nevertheless, by Theorem 5.1, whenever $t_j + t_k$ is a β-integer, it coincides with $t_j \oplus t_k$. Moreover, since the word u_β is recurrent, *i.e.* each factor of u_β (and the prefix of u_β in particular) occurs infinitely many times, one can find arbitrarily large indices j, k such that $t_j + t_k = t_{j+k}$.

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