A NOTE ON MAGNETIC FIELDS IN CONDUCTING MATERIALS

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1. *Introduction.* In a paper on Alfven's theory of sunspots Cowling (1) points out that the problems of electromagnetism in highly conducting fluids can be discussed by means of two known laws. The first concerns the magnetic forces, and the second states that the magnetic lines of force are 'frozen into' highly conducting material. In a previous paper (2) Cowling showed that solar material may be treated as perfectly conducting in the theory of sunspots. Hoyle (3) also has given an account of electromagnetism in the sun and agrees with this result.

In the present note, Cowling's approach is adopted and his second law is put into mathematical form and proved for a perfectly conducting fluid. It is assumed that the velocity of the fluid is a single-valued and differentiable function of position, so that the methods of classical hydrodynamics can be applied. It is important to notice that this assumption would not apply to electrical machines using sliding contacts and that a dynamo must always use a sliding contact. Thus, in astrophysical applications, the fact that this assumption is true limits in an important degree the way in which the field can change. The proof does not require any other approximations such as the neglect of space charge or of the electric field. It is shown how to estimate the error due to finite conductivity in any particular application provided the conductivity and $|\nabla^2\mathbf{H}|/|\mathbf{H}|$ can be estimated. The mathematical form of the law also suggests a method of picturing the motion of the field by means of tubes of force, which, when combined with Cowling's first law, is convenient for a qualitative discussion.

2. *The law of induction.* We now consider a perfectly conducting fluid in a state of single-stream motion. We also assume that the velocity is differentiable with respect to position, but we do not impose any other restrictions on the way in which the velocity varies from point to point. Walen (4) in his development of Alfven's theory of sunspots derives the equations of electromagnetic-hydrodynamic motion from the energy balance, but his treatment is non-relativistic and assumes the absence of space-charge. But we only require here one of his equations, the law of induction, and this is exact and can be derived from the principles of relativity. Ohm's law applies at a given point in the fluid in the particular frame in which the fluid at that point is at rest. Thus in a perfectly conducting fluid, an observer moving with the fluid measures zero electric field. Now, since we want to use the same frame for all points in the fluid, we must express this by an equation which is true in any particular given frame. In this frame let $\mathbf{E}$, $\mathbf{H}$ be the electric and magnetic fields and $\mathbf{v}$ be the velocity of the fluid. We know that $\mathbf{E} = 0$ in the frame in which $\mathbf{v} = 0$ and a Lorentz transformation to the given frame then gives

$$\mathbf{E} = -\mathbf{v} \wedge \mathbf{H}/c, \tag{1}$$

which is the law of induction.

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3. *To show that the flux linked with a closed curve moving with the fluid is constant.* We now discuss the problem of how the magnetic field varies in the general motion of a perfectly conducting fluid. The motion is of course affected by electromagnetic forces; Walen (4) gives all the equations of motion. Here we confine attention to half of the problem—how the field varies in a given motion. This is determined by the law of induction (1) and the Maxwell equation

$$\frac{\partial \mathbf{H}}{\partial t} = -c \operatorname{curl} \mathbf{E},$$

for then $\partial \mathbf{H}/\partial t$ is determined in terms of $\mathbf{H}$, $\mathbf{v}$ and their spatial derivatives.

We introduce potentials ϕ, $\mathbf{A}$, where

$$\mathbf{E} = -\operatorname{grad} \phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \quad \text{and} \quad \mathbf{H} = \operatorname{curl} \mathbf{A},$$

and rewrite (1)

$$\frac{\partial \mathbf{A}}{\partial t} = -c \operatorname{grad} \phi + \mathbf{v} \wedge (\operatorname{curl} \mathbf{A}), \tag{2}$$

in which $\partial \mathbf{A}/\partial t$ is the only time derivative.

To obtain an integral of the motion we consider a closed curve, σ, which moves with the fluid. The curve is regarded as being made up of particles of the fluid which remain adjacent and so continue to form a closed curve. Now the flux linked by σ is $F = \int_S \mathbf{H}.\mathbf{dS}$, where $\mathbf{dS}$ is the surface element, and S is any surface bounded by σ. We put $\mathbf{H} = \operatorname{curl} \mathbf{A}$ and apply Stokes's theorem giving $F = \oint_\sigma \mathbf{A}.\mathbf{ds}$, where $\mathbf{ds}$ is the line element with sign corresponding to the direction of $\mathbf{dS}$. We use d/dt to denote differentiation with respect to time following the motion of the fluid—$d/dt = \partial/\partial t + (\mathbf{v}.\nabla)$. The proof that $dF/dt = 0$ is similar to the proof of Thomson's theorem in hydrodynamics.

Let $\mathbf{r}$ be the position vector of a particle on σ at time t. Then $\mathbf{r} + \mathbf{ds}$ is the position vector of a neighbouring particle on σ. At time $t + \delta t$ the particle which was at $\mathbf{r}$ at time t will be at $\mathbf{r} + \mathbf{v}(\mathbf{r})\,\delta t$, and the particle which was at $\mathbf{r} + \mathbf{ds}$ will be at $\mathbf{r} + \mathbf{ds} + \mathbf{v}(\mathbf{r})\,\delta t + (\mathbf{ds}.\nabla)\,\mathbf{v}\,\delta t$.

Therefore $\mathbf{ds}(t+\delta t) = \mathbf{ds}(t) + (\mathbf{ds}.\nabla)\,\mathbf{v}\,\delta t$ and

$$F(t+\delta t) = \oint_{\sigma(t+\delta t)} \mathbf{A}(t+\delta t).\mathbf{ds} = \oint_{\sigma(t)} \left(\mathbf{A}(t) + \frac{\partial \mathbf{A}}{\partial t}\delta t + (\mathbf{v}.\nabla)\,\mathbf{A}\,\delta t\right)(\mathbf{ds} + (\mathbf{ds}.\nabla)\,\mathbf{v}\,\delta t).$$

So

$$\frac{dF}{dt} = \oint_\sigma \left(\left[\frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v}.\nabla)\,\mathbf{A}\right].\mathbf{ds} + \mathbf{A}.(\mathbf{ds}.\nabla)\,\mathbf{v}\right).$$

By partially integrating the last term and using the fact that σ is a closed curve this can be reduced to

$$\oint_\sigma \left(\left[\frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v}.\nabla)\mathbf{A}\right].\mathbf{ds} - \mathbf{v}.(\mathbf{ds}.\nabla)\,\mathbf{A}\right),$$

which can be expressed as

$$\oint_\sigma \left[\frac{\partial \mathbf{A}}{\partial t} - \mathbf{v} \wedge (\operatorname{curl} \mathbf{A})\right].\mathbf{ds} = -c \oint_\sigma (\operatorname{grad} \phi).\mathbf{ds},$$

by (2), and this is zero since σ is a closed curve. This is true for any closed curve lying wholly inside the fluid. It is Cowling's second law in mathematical form.

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4. *Conditions of validity.* We now estimate the error in the above treatment for a fluid of finite conductivity σ. Ohm's law gives $\mathbf{E} = \mathbf{j}/\sigma$ in the frame in which $\mathbf{v} = 0$ instead of $\mathbf{E} = 0$ in the above treatment. Provided that the relative velocities of different parts of the material are small compared with the velocity of light we can find a frame in which $|\mathbf{v}|/c$ is small everywhere. Then the correction to $\mathbf{E}$ in this frame is approximately $\mathbf{j}/\sigma$, and the correction to $\partial\mathbf{H}/\partial t$ is $-c\,\mathrm{curl}\,\mathbf{j}/\sigma$. Now $\mathbf{j} = (c/4\pi)\,\mathrm{curl}\,\mathbf{H}$, if we neglect $\partial\mathbf{E}/\partial t$, so that this correction is $-(c^2/4\pi\sigma)\,\nabla^2\mathbf{H}$. In general this represents the decay of the field due to Joule losses. If $|\nabla^2\mathbf{H}| = |\mathbf{H}|/a^2$, it will have an appreciable effect in a time $4\pi\sigma a^2/c^2$. This time can be estimated in particular cases, and for most solar phenomena is large compared with their observed time of duration, so that our result is valid. The effect of the magnetic field on the conductivity must be taken into account; it has been calculated by Cowling (5). He finds that the effect is negligible if $|\mathbf{H}| \ll 7{\cdot}9 \cdot 10^9 P_e T^{-\frac{1}{2}}$ gauss, where P_e is the electron pressure in dynes/cm.^{2}, and T is the temperature. He finds that this is satisfied in the main body of a sunspot.

The other assumptions required in the proof are that the velocity is continuous and single-valued. It will always be continuous in astrophysical applications. In the case of multiple-stream motion our result will apply to any of the streams, if the material of which it consists can be treated as perfectly conducting. If there are two such streams, since equation (1) applies to both, their relative velocity can only be along the lines of force.

5. *Application to tubes of force.* Our result may be pictured by means of tubes of force. A tube of force is the surface generated by the lines of force passing through a closed curve and so the component of the field normal to the surface is zero. Also the flux through any closed curve lying on the surface and linked once with the tube of force is the same for all such curves. Since our result applies to any closed curve, we see that, if we take a surface which is at one instant a tube of force and let it move with the fluid, it will remain a tube of force and the flux associated with it will be constant. That is, we may regard the tubes of force as moving with the fluid. They will therefore experience the same strain as the fluid.

Suppose we have a section of a tube of force of mean cross-section S and length l, closed by surfaces which are assumed to remain roughly perpendicular to the lines of force, and let the mean density of the material in this section be ρ. Then the mass contained in it, ρSl, is constant, since the surface moves with the fluid. If the average magnetic field in it is H, the flux through any cross-section is HS and this is also constant. So $H \propto 1/S \propto \rho l$. Also the magnetic energy $\propto \rho l^2$. We see that a contraction of the cross-section of the tube of force will increase the field and we can distinguish two types of strain which will do so. One is uniform compression and the other is stretching of the material parallel to the lines of force. In the first we have $\rho \propto l^{-3}$ so that $H \propto \rho^{\frac{1}{2}}$. The second can occur without a change of density. The tubes of force are stretched along their length and contract perpendicular to it. If the density is constant $H \propto l$.

6. *Irregularity of the field in the convection zone.* It can be seen from these considerations that the magnetic field in the convection zone of the sun can have no very regular form. The lines of force will be distorted by the convective motion and they will tend

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to become more and more irregular. This process is limited by one of two factors. The field may be so irregular that the Joule decay is important. The effect of this is to smooth out the irregularities, since it depends on $|\nabla^2\mathbf{H}|/|\mathbf{H}|$. Alternatively, if $|\mathbf{H}|$ is large enough, the magnetic forces will be appreciable. They act in such a way as to reduce the magnetic energy and so to pull the lines of force straight. In an irregular field the magnetic forces will be appreciable if the magnetic energy density is comparable with the kinetic energy density, i.e. $|\mathbf{H}|^2 \simeq 4\pi\rho|\mathbf{v}|^2$. Putting $\rho \simeq 10^{-8}$ g./c.c., $|\mathbf{v}| \simeq 1$ km./sec. for the photosphere, we require $|\mathbf{H}| \simeq 30$ gauss. The lack of reproducibility in measurements of the field at the photosphere suggest that it is in fact irregular. Fields of the order of 30 gauss have been observed, so that the magnetic forces probably have an important effect on the convective motion.

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