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COMPLEXITY OF EQUIVALENCE RELATIONS AND PREORDERS FROM COMPUTABILITY THEORY

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Abstract. We study the relative complexity of equivalence relations and preorders from computability theory and complexity theory. Given binary relations R, S, a componentwise reducibility is defined by

$$R \leq S \iff \exists f \, \forall x, y \, [x \, R \, y \leftrightarrow f(x) \, S \, f(y)].$$

Here, f is taken from a suitable class of effective functions. For us the relations will be on natural numbers, and f must be computable. We show that there is a Π^0_1-complete equivalence relation, but no Π^0_k-complete for $k \geq 2$. We show that Σ^0_k preorders arising naturally in the above-mentioned areas are Σ^0_k-complete. This includes polynomial time m-reducibility on exponential time sets, which is Σ^0_2, almost inclusion on r.e. sets, which is Σ^0_3, and Turing reducibility on r.e. sets, which is Σ^0_4.

§1. Introduction. Mathematicians devote much of their energy to the classification of mathematical structures. In this endeavor, the tool most commonly used is the notion of a *reduction*: a map reducing a seemingly more complicated question to a simpler question. As a simple example, one classifies the countable rational vector spaces (under the equivalence relation of isomorphism) using the notion of dimension, by proving that two such vector spaces are isomorphic if and only if they have the same dimension over $\mathbb{Q}$. More generally, we have the following definition.

Definition 1.1. If E and F are equivalence relations on the domains D_E and D_F, a *reduction* from E to F is a function $f : D_E \to D_F$ with the property that, for all $x, y \in D_E$,

$$x \, E \, y \iff f(x) \, F \, f(y).$$

In our example, E is the isomorphism relation on the class D_E of countable rational vector spaces (with domain ω, say, to avoid set-theoretical issues), and F is the equality relation on the set $\{0, 1, 2, \ldots, \omega\}$ of possible dimensions of these spaces.

A reduction f of a complicated equivalence relation E to a simpler equivalence relation F is entirely possible if one allows f to be sufficiently complex. (For instance, every equivalence reduction E with exactly n classes can be reduced to the equality relation on $\{1, 2, \ldots, n\}$, but this reduction has the same complexity as E.) Normally, however, the goal is for f to be a readily understandable function,

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so that we can actually learn something from the reduction. In our example, one should ask how hard it is to compute the dimension of a rational vector space. It is natural to restrict the question to computable vector spaces over $\mathbb{Q}$ (i.e., those where the vector addition is given as a Turing-computable function on the domain ω of the space). Yet even when its domain D_E such vector spaces, computing the function which maps each one to its dimension requires a $\mathbf{0}'''$-oracle, hence is not as simple as one might have hoped. (The reasons why $\mathbf{0}'''$ is required can be gleaned from [7] or [8].)

1.1. Effective reductions. Reductions are normally ranked by the ease of computing them. In the context of Borel theory, for instance, a large body of research is devoted to the study of *Borel reductions* (the standard book reference is [17]). Here, the domains D_E and D_F are the set 2^ω or some other standard Borel space, and a Borel reduction f is a reduction (from E to F, these being equivalence relations on 2^ω) which, viewed as a function from 2^ω to 2^ω, is Borel. If such a reduction exists, one says that E is *Borel reducible to* F, and writes $E \leq_B F$. A stronger possible requirement is that f be continuous, in which case we have (of course) a *continuous reduction*. In case the reduction is given by a Turing functional from reals to reals, it is a (*type-2*) *computable reduction*.

A further body of research is devoted to the study of the same question for equivalence relations E and F on ω, and reductions $f : \omega \to \omega$ between them which are computable. If such a reduction from E to F exists, we say that E is *computably reducible to* F, and write $E \leq_c F$, or often just $E \leq F$. These reductions will be the focus of this paper. Computable reducibility on equivalence relations was perhaps first studied by Ershov [12] in a category theoretic setting.

The main purpose of this paper is to investigate the complexity of equivalence relations under these reducibilities. In certain cases we will generalize from equivalence relations to preorders on ω. We restrict most of our discussion to relatively low levels of the hierarchy, usually to Π^0_n and Σ^0_n with $n \leq 4$. One can focus more closely on very low levels: Such articles as [3, 5, 18], for instance, have dealt exclusively with Σ^0_1 equivalence relations. The work [15] considers arithmetical equivalence relations, in particular Σ^0_3 ones. At the other extreme, the papers [13, 14] consider equivalence relations outside the arithmetical or even hyperarithmetical, obtaining certain results about Σ^1_1-completeness of various equivalence relations on ω. The work [11] is focused on much the same levels of the arithmetical hierarchy as our work here, but through the prism of Borel equivalence relations on 2^ω, by restricting such relations to the class of computably enumerable subsets of ω. A few of their results will be extended here. Finally, the complexity of equivalence relations under this kind of reducibility has recently been studied by a number of authors from various disciplines. Within complexity theory it seems to have been first introduced in [16], and later studied in [6].

1.2. Completeness. Equivalence relations on ω are subsets of ω^2, and so those with arithmetical definitions can be ranked in the usual hierarchy as Σ^0_n and/or Π^0_n for various $n \in \omega$. The question of *completeness* then naturally arises. A Σ^0_n equivalence relation F is *complete* (among equivalence relations under $\leq_c$) if for every Σ^0_n equivalence relation E, we have $E \leq_c F$. Π^0_n-completeness is defined similarly. Much of this paper is devoted to the study of complete equivalence relations at

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various levels of the arithmetical hierarchy, and we will often speak of their Σ^0_n-completeness or Π^0_n-completeness without specifying the reducibility $\leq_c$. At certain times we will need to consider other reducibilities on subsets of ω, such as Turing reducibility $\leq_T$ or 1-reducibility $\leq_1$, and these reducibilities give rise to their own (analogous) notions of completeness, but when referring to those notions we will always name them specifically.

The standard notion of an m-reduction from one set to another is relevant here, since computable reducibility on equivalence relations can be seen as one of two natural ways to extend m-reducibility from sets to binary relations. However, when comparing the complexity between equivalence relations, it is more natural to use computable reducibility. Following Definition 1.1, we define:

$$A \leq_c B \iff \exists \text{ computable } g \, \forall x, y \, [\langle x, y\rangle \in A \leftrightarrow \langle g(x), g(y)\rangle \in B]. \quad (1)$$

The resulting reducibility has many interesting and, at times, surprising properties. It can be readily seen that (1) is a stronger property than m-reducibility: $A \leq_c B$ via g implies $A \leq_m B$ via the function f given by $f(x, y) = \langle g(x), g(y)\rangle$, so a relation complete for a level of the arithmetic hierarchy in the sense of relations as in (1) is also complete in the sense of sets. The converse fails: in Subsection 3.2 we shall see that some levels do not have complete equivalence relations at all.

It is easy to see that for every n, there exists an equivalence relation E that is Σ^0_n-complete under $\leq_c$: let $(V_e)_{e\in\omega}$ be an effective list of all symmetric Σ^0_n relations, let $(S_e)_{e\in\omega}$ be the effective list of their transitive closure, and let $E = \bigoplus_e S_e$ be their effective sum. In Section 3 we demonstrate the existence of Π^0_1-complete equivalence relations, which is not nearly as obvious. We give a natural example from complexity, namely equality of quadratic time computable functions. In contrast, we also show that for $n \geq 2$, no equivalence relation is Π^0_n-complete.

1.3. Preorders. Equivalence relations were the first context in which reductions arose, but the notion of a reduction can be applied equally well to any pair of binary relations: The E and F in Definition 1.1 need not be required to be equivalence relations. In fact, a reduction could be considered as a homomorphism of structures in the language containing a single binary relation symbol, whether or not that symbol defines an equivalence relation on either structure. Preorders (i.e., reflexive transitive binary relations $\preceq$, with $x \preceq y \preceq x$ allowed even when $x \neq y$) form a natural collection of binary relations on which to study reductions, and in Section 4 we will turn to the question of computable reducibility on preorders, with some interesting uses of effectively inseparable Boolean algebras. In this way we obtain some examples of natural Σ^0_n-complete preorders from computability theory. For instance, almost inclusion $\subseteq^*$ of recursively enumerable (r.e.) sets, and weak truth table reducibility on r.e. sets, are Σ^0_3 complete preorders. To obtain corresponding completeness results for equivalence relations, we use the easy Fact 2.1 that completeness of a preorder P at some level of the arithmetical hierarchy implies completeness of the corresponding symmetric fragment which is an equivalence relation. Using this we obtain, for instance, the Σ^0_3 completeness of several of the equivalence relations considered in [11]. In Section 5 we show that Turing reducibility on r.e. sets is a Σ^0_4-complete preorder, and hence Turing equivalence on r.e. sets is also Σ^0_4-complete as an equivalence relation. Using a direct construction, Fokina,

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Friedman, and Nies [15] have proved that 1-equivalence and m-equivalence on r.e. sets are Σ^0_3-complete.

Our results can be applied to obtain completeness results for preorders and equivalence relations from effective algebra. For instance, Ianovski [19] has proved the Σ^0_3-completeness of effective embeddability for computable subgroups of the additive group of rationals by reducing almost inclusion of r.e. sets. On the other hand, the completeness of several natural equivalence relations, such as isomorphism of finitely presented groups, remains open. Such questions are discussed in the final section.

The articles already cited here are among the principal references for this subject. For ordinary computability-theoretic questions, we suggest [30], a second edition of which is to appear soon.

§2. Preliminaries. Let $X^{[2]}$ denote the unordered pairs of elements of X. A set $R \subseteq X^{[2]}$ is *transitive* if $\{u, v\}, \{v, w\} \in R$ and $u \neq w$ implies $\{u, w\} \in R$. We view equivalence relations as transitive subsets of $\omega^{[2]}$.

FACT 2.1. *Let P be a preorder, $E = \{\{x, y\} \colon \langle x, y\rangle \in P \text{ and } \langle y, x\rangle \in P\}$, and $n \in \omega$. If P is complete among Σ^0_n $[\Pi^0_n]$ preorders under $\leq_c$, then E is complete among Σ^0_n $[\Pi^0_n]$ equivalence relations under $\leq_c$.*

PROOF. We give the proof for Σ^0_n. Note that E is clearly Σ^0_n. Let F be an arbitrary Σ^0_n equivalence relation. In particular, F can be seen as a preorder and so there is a computable function f such that for every x, y, $\{x, y\} \in F$ iff $\langle f(x), f(y)\rangle \in P$. Then, f witnesses that $F \leq_c E$. ⊣

In parts of the Subsections 3.1 and 4.3 we will assume familiarity with the basic notions of computational complexity theory; see for instance [4]. We consider languages $A \subseteq \Sigma^*$ for the alphabet $\Sigma = \{0, 1\}$. Recall that for a function $h \colon \omega \to \omega$, a language A is *computable in time h* if there is an $O(h)$ time-bounded multitape Turing machine (TM) deciding membership in A. The class of such languages is denoted $\mathrm{DTIME}(h)$. Languages in $\mathrm{DTIME}(n)$ are also called *linear time*.

Recall that a function h from numbers to strings with $h(n) \geq n$ is *time constructible* if $h(n)$ can be computed in $O(h(n))$ steps on a multitape TM. Suppose $c \in \omega$ is a constant. If a function $h \colon \omega \to \omega$ is time constructible, then we can equip any given Turing machine with a "clock" on an extra tape. This is a counter initialized at $c \cdot h(n)$ and decremented each step of the given machine. It stops the given machine on reaching 0. Since h is time constructible, the number of steps the clocked machine needs is still within $O(h)$. Numbering all Turing machines clocked in this way yields an effective listing of the class $\mathrm{DTIME}(h)$.

§3. Π^0_n-completeness for equivalence relations. In this section we show that there is a Π^0_1-complete equivalence relation, but no Π^0_n-complete equivalence relation for $n \geq 2$.

3.1. Π^0_1 equivalence relations. For each function $f \colon \omega \times \omega \to \omega$ and $x \in \omega$, let f_x denote the function $n \mapsto f(x, n)$. If f is computable,

$$E_f = \{\langle x, y\rangle \colon f_x = f_y\}$$

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appears to be a natural example of a Π^0_1 equivalence relation. We show that indeed every Π^0_1 equivalence relation is obtained in this way. A slight extension of the argument yields a function f such that the corresponding equivalence relation E_f is Π^0_1-complete. We show that f can in fact be chosen polynomial time computable.

For each Π^0_1 equivalence relation E, there is a computable sequence of cofinite relations $(E_t)_{t\in\omega}$ contained in $\omega^{[2]}$ with $E = \bigcap_t E_t$ such that from t we can compute a strong index for the complement of E_t, we have $E_t \supseteq E_{t+1}$ and

$$E_t \cap [0,t]^{[2]} \text{ is transitive.} \tag{2}$$

PROPOSITION 3.1. *For each Π^0_1 equivalence relation E, there is a computable binary function f such that $E = E_f$, namely, for each x, y, we have*

$$x\ E\ y \leftrightarrow f_x = f_y. \tag{3}$$

PROOF. Let $(E_t)_{t\in\omega}$ be as in (2). We define $f(x,n)$ by recursion on x. Let

$$f(x,n) = \begin{cases} f(r,n), & \text{if } r < x \text{ for the least } r \text{ such that } r\ E_{\max(x,n)}\ x, \\ x, & \text{otherwise.} \end{cases}$$

Note that for each z, for sufficiently large k, we have $f(z,k) = \min[z]_E$. Here, $[z]_E$ is the equivalence class of z with respect to E. We verify that (3) is satisfied.

For the implication "$\rightarrow$", we show the following by induction on x.

CLAIM. *For each $y < x$, for each n, if $y\ E_{\max(x,n)}\ x$ then $f_y(n) = f_x(n)$.*

To see this, let $r < x$ be as in the definition of $f(x,n)$. Then $r \le y < x$. By definition of f we have $f_x(n) = f_r(n)$. By transitivity in (2), we have $r\ E_{\max(x,n)}\ y$, and hence $r\ E_{\max(y,n)}\ y$. Then, by inductive hypothesis, $f_r(n) = f_y(n)$.

For the implication "$\leftarrow$", suppose that $y\ \neg E\ x$. Then, $f_y(k) \neq f_x(k)$ for sufficiently large k by the remark after the definition of f above. ⊣

As a corollary we obtain a presentation of Π^0_1 equivalence relations as the uniform intersection of recursive ones.

COROLLARY 3.2. *For each Π^0_1 equivalence relation E, there is a uniformly recursive sequence $(F_n)_{n\in\omega}$ of equivalence relations such that $F_n \supseteq F_{n+1}$ and $E = \bigcap_n F_n$.*

PROOF. Let $E = E_f$ for a computable function f. Let

$$F_n = \{\langle x,y\rangle :\ f_x\restriction_n = f_y\restriction_n\},$$

where $g\restriction_n$ denotes the tuple $\langle g(0),\dots,g(n-1)\rangle$. ⊣

We obtain a Π^0_1-complete equivalence relation by exposing a uniformity in the proof of Proposition 3.1.

THEOREM 3.3. *There is a computable binary function g such that the equivalence relation $E_g = \{\langle x,y\rangle :\ g_x = g_y\}$ is Π^0_1-complete.*

PROOF. Let $E^i = \omega^{[2]} - W_i$, where the i-th r.e. set W_i is viewed as a subset of $\omega^{[2]}$. Uniformly in i we may obtain a computable sequence of strong indices $(E^i_t)_{t\in\omega}$ as the one used in the proof of Proposition 3.1 such that $E^i = \bigcap_t E^i_t$ whenever E^i is transitive. Now define $g(\langle i,x\rangle, n)$ to have the value $f(x,n)$ as above where $E = E^i$. By the argument in that proof, if E^i is transitive, then $E^i \le_c E_g$ via the function $x \mapsto \langle i,x\rangle$. ⊣

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We can in fact obtain in Theorem 3.3 a function on binary strings that is quadratic time computable (see Section 2). We thank Moritz Müller at the Kurt Gödel Research Institute in Vienna for suggesting a simplified proof that we provide at the end of this subsection.

LEMMA 3.4. *For each computable binary function g, there is a quadratic time computable binary function G defined on strings over $\{0,1\}$, and a computable unary function p mapping numbers to strings, such that $g(x,n) = G(p(x),p(n))$, and $G(w,v) = 0$ for any string v not of the form $p(n)$.*

Unary quadratic time functions over the alphabet $\{0,1\}$ are given by indices for multitape Turing machines (TM) that are equipped with a counter forcing them to stop in time $O(n^2)$, where n is the input length.

THEOREM 3.5. *(i) There is a quadratic time computable binary function G such that E_G is Π^0_1-complete. (ii) Equality of unary quadratic time computable functions is a Π^0_1-complete equivalence relation.*

PROOF OF THEOREM. (i) Let G be the function obtained from the function g of Theorem 3.3 according to the lemma. Let p be as in the lemma. Then $x\ E_g\ y \leftrightarrow G_{p(x)} = G_{p(y)}$. Since E_g is complete, so is E_G.
(ii) Since all the Turing machines considered stop on every input in a quadratic time bound, the equivalence relations is Π^0_1. The completeness follows from (i) since from a string w we can compute an index for a quadratic time TM computing G_w. ⊣

PROOF OF LEMMA 3.4. Let $g(x,y)$ (with x,y represented in unary) be computable in time $t(x,y)$ where t is a computable function that is increasing in both x and y. We want a time constructible h satisfying $t(n,k) \leq h(n) + h(k)$. This can be achieved by having $h(n)$ run $g(n,n)$ and counting the number of steps. This satisfies $t(n,k) \leq h(n) + h(k)$ as for $n \leq k$, $t(n,k) \leq t(k,k) = h(k)$, and $h(n) = m$ can be calculated in time quadratic in m: Place a binary counter at the start of the tape and simulate g to the right of it. In the worst case scenario for every step of f the machine would iterate over $m + \log m$ cells, and whenever the counter grows in size the entire tape would need to be shifted right which in total would take $m \log m$, coming to $m \log m + m(m + \log m)$ or $O(m^2)$.

To compute $G(a,b)$, verify whether $a = 1^x01^{h(x)}$ and $b = 1^n01^{h(n)}$. If so output $g(x,n)$, else output 0. Observe that G is quadratic time: We can verify whether $a = 1^x01^z$ is of the form $1^x01^{h(x)}$ by beginning to compute $g(x,x)$, stopping whenever the number of steps exceeds z. If the input is of the right form we can compute $g(x,n)$ in time $t(x,n) \leq h(x) + h(n) = O(|a| + |b|)$.

Note that for every x,n,

$$G(1^x01^{h(x)}, 1^n01^{h(n)}) = g(x,n),$$

and for all other values the function is 0. Thus the function $p(y) = 1^y01^{h(y)}$ is as required. ⊣

In work independent of ours [10, Proposition 3.1] the authors offer a characterisation of Π^0_1 preorders: Every such preorder is computably isomorphic to the inclusion relation on a uniformly computable family of sets. As an equivalence relation is merely a symmetric preorder, a Π^0_1 equivalence relation is then isomorphic to equality on such a family. Since sets can be viewed as $0/1$ valued functions, this

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yields another proof of Proposition 3.1. Methods similar to the ones employed in the proof of Theorem 3.5 above now yield a version of that Theorem for the inclusion of quadratic time computable languages.

THEOREM 3.6 ([19] based on [10]). *Inclusion of quadratic time computable languages is a complete* Π^0_1 *preorder.*

For a proof see [19, Theorem 5.5]. Using that result one can also obtain a stronger form of Corollary. 3.2 where the F_n have at most 2^n equivalence classes.

3.2. For $n \geq 2$ there is no Π^0_n-complete equivalence relation. Π^0_n equivalence relations often occur naturally. For instance, equality of r.e. sets as a relation on indices is Π^0_2. See [11] for further examples along these lines. Computable model theory is a further source of interesting examples. Consider the class $\mathcal{P}$ of computable permutations on ω with all cycles finite. Isomorphism as a relation on computable indices for structures in this class is Π^0_2. This equivalence relation is easily seen to be computably equivalent to equality on r.e. sets. Melnikov and Nies [23] have studied isometry of compact computable metric spaces, They showed that it is properly Π^0_3 in general, and properly Π^0_2 for effectively compact metric spaces.

Here we show that there is no Π^0_n-complete equivalence relation for $n \geq 2$. Intuitively speaking, one would then expect that the computable reducibility degrees of Π^0_n equivalence relations are complicated.

THEOREM 3.7. *For each* Π^0_2 *equivalence relation* E *there is a* Δ^0_2 *equivalence relation* L *such that* $L \not\leq_c E$.

Let α be a computable ordinal. Relativizing to $\emptyset^{(\alpha)}$ yields an immediate corollary. We say that $E \leq_{\mathbf{0}^{(\alpha)}} F$ if Definition 1.1 holds for E and F for some $f \leq_T \emptyset^{(\alpha)}$.

COROLLARY 3.8. *Let* α *be a computable ordinal. No* $\Pi^0_{\alpha+2}$ *equivalence relation can be hard for all the* $\Delta^0_{\alpha+2}$ *equivalence relations under the reducibility* $\leq_{\mathbf{0}^{(\alpha)}}$, *let alone under computable reducibility* $\leq_c$. ⊣

In contrast, there is a Π^1_1-complete equivalence relation E. Similar to the case of Σ^0_n relations, the class of symmetric Π^1_1 relations is uniformly closed under taking the transitive closure. Let $(V_e)_{e\in\omega}$ be an effective list of all symmetric Π^1_1 relations, let $(S_e)_{e\in\omega}$ be the effective list of their transitive closure, and let $E = \bigoplus_e S_e$.

PROOF OF THEOREM 3.7. E be any Π^0_2 equivalence relation on ω. Since E is $\Pi^{0,\emptyset'}_1$ we may fix a $\emptyset'$-computable function f such that $\langle x, y\rangle \in E$ iff $f(x, y, t) = 1$ for every t. Now define L as follows.

For $e_0 \neq e_1$, $\{\langle e_0, s_0\rangle, \langle e_1, s_1\rangle\} \notin L$ for any s_0, s_1. For each e, we distinguish two cases.

CASE 1. There is a number t, chosen least, such that

$$\varphi_e(\langle e, 0\rangle)\downarrow = x, \varphi_e(\langle e, 1\rangle)\downarrow = y, \varphi_e(\langle e, t+2\rangle)\downarrow = z, \text{ and } f(x, y, t) = 0.$$

Let v be least such that $f(x, z, v) = 0$ or $f(y, z, v) = 0$.

- If $f(x, z, v) = 0$ we set $\{\langle e, 0\rangle, \langle e, t+2\rangle\} \in L$.
- If $f(y, z, v) = 0$ we set $\{\langle e, 1\rangle, \langle e, t+2\rangle\} \in L$.

Declare $\{\langle e, s_0\rangle, \langle e, s_1\rangle\} \notin L$ for every other pair $s_0 \neq s_1$.

CASE 2. There is no such number t. Then we set $\{\langle e, s_0\rangle, \langle e, s_1\rangle\} \notin L$ for every $s_0 \neq s_1$.

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To see that L is Δ^0_2, it suffices to note that for each e, if Case 1 applies then by the transitivity of E the number v exists. Transitivity of L follows from the fact that we relate each element to at most one other element.

Now suppose that $L \leq_c E$ via the total function φ_e. Since $\{\langle e,0\rangle, \langle e,1\rangle\} \notin L$ it follows that the least number t described above exists. By construction we ensure that either $\{\langle e,0\rangle, \langle e,t+2\rangle\} \in L$ and $\{x,z\} \notin E$, or we have $\{\langle e,1\rangle, \langle e,t+2\rangle\} \in L$ and $\{y,z\} \notin E$, a contradiction. ⊣

REMARK 3.9. Observe that given any two Π^0_{n+2} equivalence relations R, S, the disjoint union $R \sqcup S = \{\{2n,2m\} \mid n\ R\ m\} \cup \{\{2n+1,2m+1\} \mid n\ S\ m\}$ is also a Π^0_{n+2} equivalence relation and we have $R, S \leq_c R \sqcup S$. Hence Theorem 3.7 shows that amongst Π^0_{n+2} equivalence relations there can be no maximal element under $\leq_c$.

The following is an immediate consequence of Theorem 3.7; for preorders we also apply Fact 2.1.

COROLLARY 3.10. *For each $n \geq 2$, there is no Π^0_n-complete preorder, and no Δ^0_n-complete equivalence relation, or preorder.*

There also is no Π^0_2 analog of Corollary. 3.2. Let us say a Π^0_2 equivalence relation E is *effectively* Π^0_2 if there is a uniformly r.e. sequence $(F_n)_{n\in\omega}$ of equivalence relations such that $E = \bigcap_n F_n$. (We may also require $F_n \supseteq F_{n+1}$ without loss of generality.) If S is an r.e. nonrecursive set then the Δ^0_2 equivalence relation with the two classes S and $\omega - S$ is not effectively Π^0_2. Note that we can effectively list the effectively Π^0_2 equivalence relations. By putting together all the effectively Π^0_2 equivalence relations in the obvious way we see that there is a complete one under computable reducibility.

§4. Σ^0_k-complete preorders. The concept of an effectively inseparable r.e. Boolean algebra was introduced by Pour-El and Kripke [28, Lemmas 1,2] when they studied the complexity of logical derivability for recursively axiomatizable theories. The method was developed in more generality by Montagna and Sorbi [24]. For instance, in their Theorem 3.1 they showed that the preorder associated with any effectively inseparable Boolean algebra is Σ^0_1-complete. They developed their method further in order to show that for a sufficiently strong arithmetical theory T (such as a consistent axiomatizable extensions of PA), for any r.e. preorder $\preceq$ there is a Σ_1 formula G with $i \preceq j$ iff $T \vdash G(\underline{i}) \to G(\underline{j})$.

We are interested mostly in semantic preorders such as reducibilities, which are usually at higher levels of the arithmetical hierarchy. So we extend the methods discussed above in order to show the completeness of a Σ^0_k preorder, using effectively inseparable Σ^0_k Boolean algebras. Thereafter we apply it to preorders in the subrecursive setting, which are Σ^0_2, and to preorders on the r.e. sets arising from computability theory, which are Σ^0_3. Each time, we naturally embed an effectively inseparable Σ^0_k Boolean algebra for the appropriate k into the preorder.

For sets X, Y, we write $X \subseteq^* Y$ (X is almost contained in Y) if $X \setminus Y$ is finite. We write $X =^* Y$ if $X \subseteq^* Y \subseteq^* X$. The preorders we will consider in this section for the cases $k = 2, 3$ are given on certain natural classes of sets, either by almost containment, or by reducibilities.

4.1. The method of effectively inseparable Σ^0_k Boolean algebras. Our proofs rest on the following notion, which is an analog of creativity for disjoint pairs of sets.

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For details see [30, Exercise II.4.13]. We relativize r.e. sets to $\emptyset^{(k-1)}$, thereby obtaining Σ^0_k sets. Note, however, that the analog g of the productive function remains computable.

DEFINITION 4.1 (see [30, Exercise II.4.13]). We say that disjoint Σ^0_k sets $A, B \subseteq \omega$ are *effectively inseparable* (e.i.) if there is a computable binary function g (called a productive function) such that for each $p, q \in \omega$,

$$\text{if } A \subseteq W_p^{\emptyset^{(k-1)}}, B \subseteq W_q^{\emptyset^{(k-1)}} \text{ and } W_p^{\emptyset^{(k-1)}} \cap W_q^{\emptyset^{(k-1)}} = \emptyset,$$
$$\text{then } g(p,q) \notin W_p^{\emptyset^{(k-1)}} \cup W_q^{\emptyset^{(k-1)}}.$$

REMARK 4.2. (a) If we enlarge both components of the disjoint pair to disjoint Σ^0_k sets, the function g is still a productive function.
(b) We say that a disjoint pair A, B of Σ^0_k sets is *m-complete for disjoint pairs* if for each disjoint pair U, V of Σ^0_k sets, there is a computable map α such that

$$x \in U \leftrightarrow \alpha(x) \in A, \quad \text{and} \quad x \in V \leftrightarrow \alpha(x) \in B.$$

It is a well-known result going back to Smullyan [29] that each e.i. pair of Σ^0_k sets is m-complete in this sense. See [30, Exercise II.4.15] for a more recent reference. Note that the m-reduction in [30, Exercise II.4.15] can be determined from indices of the given Σ^0_k sets U, V, A, B, and the productive function g for A, B. Even If $U \cap V \neq \emptyset$ or $A \cap B \neq \emptyset$, as long as g is total the construction still yields a (total) computable function.

We consider Boolean algebras in the language with partial order $\leq$, meet $\wedge$, join $\vee$, and complementation $'$. A Σ^0_k Boolean algebra $\mathcal{B}$ is represented by a model $(\omega, \preceq, \vee, \wedge, ')$ such that $\preceq$ is a Σ^0_k preorder, $\vee, \wedge$ are computable binary functions, $'$ is a computable unary function, and the quotient structure $(\omega, \preceq, \vee, \wedge, ')/_{\approx}$ is isomorphic to $\mathcal{B}$. Here, $\approx$ is the equivalence relation corresponding to $\preceq$, and we assume the functions are compatible with $\approx$. We may also assume that $0 \in \omega$ denotes the least element of the Boolean algebra, and $1 \in \omega$ denotes the greatest element. If necessary, we write $\preceq_{\mathcal{B}}$ etc. to indicate the Boolean algebra a relation on ω belongs to.

The usual effectiveness notions defined for sets or functions on the natural numbers can be transferred to $\mathcal{B}$. For instance, we say that an ideal I of $\mathcal{B}$ is Σ^0_k if I is Σ^0_k when viewed as a subset of ω.

Let $\mathcal{F}$ be a computable Boolean algebra that is freely generated by a computable sequence $(p_n)_{n\in\omega}$. For instance, we can take as $\mathcal{F}$ the finite unions of intervals $[x, y)$ in $[0, 1)_{\mathbb{Q}}$. We fix an effective encoding of $\mathcal{F}$ by natural numbers. Then, equivalent to the definition above, a Σ^0_k Boolean algebra is given in the form $\mathcal{F}/\mathcal{I}$, where $\mathcal{I}$ is a Σ^0_k ideal of $\mathcal{F}$. We rely on this view for coding a Σ^0_k preorder into a Σ^0_k Boolean algebra. We slightly extend concepts and results of [24] where $k = 1$.

LEMMA 4.3. *For any Σ^0_k preorder $\preceq$, there is a Σ^0_k ideal $\mathcal{I}$ of $\mathcal{F}$ such that $n \preceq k \leftrightarrow p_n - p_k \in \mathcal{I}$.*

PROOF. Let $\mathcal{I}$ be the ideal of $\mathcal{F}$ generated by $\{p_n - p_k : n \preceq k\}$. The implication "$\rightarrow$" follows from the definition. For the implication "$\leftarrow$", let $\mathcal{B}_{\preceq}$ be the Boolean algebra generated by the subsets of ω of the form $\hat{i} = \{r : r \preceq i\}$. The map $p_i \mapsto \hat{i}$ extends to a Boolean algebra homomorphism $g : \mathcal{F} \rightarrow \mathcal{B}_{\preceq}$ that sends $\mathcal{I}$ to 0. If $n \not\preceq k$, then $\hat{n} \not\subseteq \hat{k}$, and hence $p_n - p_k \notin \mathcal{I}$. ⊣

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DEFINITION 4.4. We say that a Σ^0_k Boolean algebra $\mathcal{B}$ is Σ^0_k *effectively inseparable* (e.i.) if the sets $[0]_\approx$ and $[1]_\approx$ (that is, the names for $0 \in \mathcal{B}$ and for $1 \in \mathcal{B}$, respectively) are effectively inseparable.

We will frequently use the following criterion to show that a Σ^0_k Boolean algebra $\mathcal{B}$ is effectively inseparable.

FACT 4.5. *Suppose that U, V is a pair of e.i. Σ^0_k sets. Let $\mathcal{B}$ be a Σ^0_k Boolean algebra. Suppose there is a computable function g such that*

$$x \in U \to g(x) \approx 0 \quad \textit{and} \quad x \in V \to g(x) \approx 1.$$

Then, $\mathcal{B}$ is effectively inseparable.

PROOF. Let $A = \{x\colon\ g(x) \approx 0\}$ and $B = \{x\colon\ g(x) \approx 1\}$. Then A, B are Σ^0_k sets, $A \cap B = \emptyset$, $U \subseteq A$, and $V \subseteq B$. By Remark 4.2 this implies that A, B are effectively inseparable as Σ^0_k sets, whence $\mathcal{B}$ is effectively inseparable as a Σ^0_k Boolean algebra. ⊣

We slightly extend a result [24, Proposition. 3.1]. As mentioned already, that work goes back to results of Pour-El and Kripke [28, Lemmas 1,2] who only worked in the setting of axiomatizable theories. It is worth including a proof of the extension, because we only partially relativize the setting of [24]: The preorder of the Boolean algebra is Σ^0_k, while all of its algebraic operations, as well as the reduction function in the definition of completeness, are still computable.

THEOREM 4.6. *Suppose a Boolean algebra $\mathcal{B}$ is Σ^0_k-effectively inseparable. Then, $\preceq_\mathcal{B}$ is a Σ^0_k-complete preorder.*

PROOF. We say that a Σ^0_k Boolean algebra is Σ^0_k-*complete* if each Σ^0_k Boolean algebra $\mathcal{C}$ is computably embedded into it. We first show that the given Boolean algebra $\mathcal{B}$ is Σ^0_k-complete. For an element z of a Boolean algebra define $z^{(0)} = z$ and $z^{(1)} = z'$.

Given a Σ^0_k Boolean algebra $\mathcal{C}$, we construct the desired computable embedding h of $\mathcal{C}$ into $\mathcal{B}$ recursively. Recall that the domain of a presentation of $\mathcal{C}$ is ω by our definition. Suppose $y_i = h(i)$ has been defined for $i < n$. For each n-bit string τ let

$$p_\tau = \bigwedge_{i<n} i^{(\tau_i)} \quad \text{and} \quad y_\tau = \bigwedge_{i<n} y_i^{(\tau_i)}.$$

In particular, if $n = 0$, we let $p_\emptyset = 1_\mathcal{C}$ and $y_\emptyset = 1_\mathcal{B}$.

We define h in such a way that for each n and each string τ of length n, we have

$$p_\tau \approx_\mathcal{C} 0 \leftrightarrow y_\tau \approx_\mathcal{B} 0. \tag{$*$}$$

This is clearly the case for $n = 0$. Inductively assume $(*)$ holds for n. To define $h(n)$ we use the following.

CLAIM. *From a string τ of length n we can effectively determine an element $z_\tau \preceq_\mathcal{B} y_\tau$ such that*

$$n \wedge p_\tau \approx_\mathcal{C} 0 \leftrightarrow z_\tau \approx_\mathcal{B} 0, \quad \text{and} \quad n \succeq_\mathcal{C} p_\tau \leftrightarrow z_\tau \approx_\mathcal{B} y_\tau.$$

To see this, first suppose $y_\tau \not\approx_\mathcal{B} 0$. Then, the pair of Σ^0_k sets

$$A = \{r\colon\ r \wedge y_\tau \approx_\mathcal{B} 0\}, B = \{r\colon\ r \succeq_\mathcal{B} y_\tau\}$$

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is disjoint. By Remark 4.2(a) this pair is e.i. with the same productive function g as the one for the e.i. pair of sets $\{r\colon\ r \approx_{\mathcal{B}} 0\}$, $\{r\colon\ r \approx_{\mathcal{B}} 1_{\mathcal{B}}\}$.

Since $y_\tau \not\approx_{\mathcal{B}} 0$, by $(*)$ for n we have $p_\tau \not\approx_{\mathcal{C}} 0$. Hence, the pair of Σ^0_k sets

$$U = \{k\colon\ k \wedge p_\tau \approx_{\mathcal{C}} 0\},\ V = \{k\colon\ k \succeq_{\mathcal{C}} p_\tau\}$$

is also disjoint. By Remark 4.2(b), we are uniformly given an m-reduction α from U, V to A, B. We let $z_\tau = y_\tau \wedge \alpha(n)$. Then, the claim is satisfied in case $y_\tau \not\approx_{\mathcal{B}} 0$.

If $y_\tau \approx_{\mathcal{B}} 0$, then, by $(*)$ for n, we have $p_\tau \approx_{\mathcal{C}} 0$. None of the pairs of sets is disjoint now, but (based on the productive function g) we still have a computable index for a function α, and hence a definition of z_τ such that $z_\tau \approx_{\mathcal{B}} 0$, which vacuously satisfies the claim.

Now let $h(n) = \bigvee_{|\tau|=n} z_\tau$. It is clear that $(*)$ holds for $n+1$. Then, by induction, the map h is a computable embedding of $\mathcal{C}$ into $\mathcal{B}$, as required.

To conclude the proof of the theorem, let $\preceq_P$ be any Σ^0_k preorder. By Lemma 4.3 there is a Σ^0_k ideal $\mathcal{I}$ of the free Boolean algebra $\mathcal{F}$ such that $n \preceq k \leftrightarrow p_n - p_k \in \mathcal{I}$. Since $\mathcal{B}$ is Σ^0_k-complete, there is a computable embedding g of $\mathcal{F}/\mathcal{I}$ into $\mathcal{B}$. Thus, $n \preceq_P k \leftrightarrow g(p_n) \preceq_{\mathcal{B}} g(p_k)$. ⊣

A forth-and-back version of the argument above shows that any two e.i. Σ^0_k Boolean algebras are effectively isomorphic. This fact for $k = 1$ was already noted in [24, 28]. It is interesting in view of the applications below: for a fixed k, all the preorders considered are complete "for the same reason".

4.2. Derivability in first-order logic. Let T be a recursively axiomatized sufficiently strong theory in the language of arithmetic, such as Robinson arithmetic Q. Note that such a theory T can be finitely axiomatizable. The following was first observed in [24, Section 4], building on Pour-El and Kripke [28].

THEOREM 4.7. *For sentences ϕ, ψ in the language of T, let $\phi \preceq \psi$ if $\vdash_T \phi \to \psi$. Then, $\preceq$ is a Σ^0_1-complete preorder.*

PROOF. Clearly $\preceq$ is Σ^0_1. It was first observed by Smullyan [29] that the Lindenbaum algebra of T is Σ^0_1-effectively inseparable (or see [24, Section 4]). Now Theorem 4.6 implies that $\preceq$ is complete. ⊣

In [19] the first author has given a direct proof that logical implication for the first order language in a full signature is Σ^0_1-complete. His proof relies on a coding of Turing machine computations.

4.3. Σ^0_2 preorders in the resource-bounded setting. For relevant notation from complexity theory see Section 2. In particular, there is an effective listing of the class $\mathrm{DTIME}(h)$, which we fix below without further mention. We use a technical tool:

LEMMA 4.8. *Given a disjoint pair of Σ^0_2 sets U, V, we can effectively in x determine a linear time computable language L_x such that*

$$x \in U \to L_x \text{ is finite, and } x \in V \to L_x \text{ is cofinite.}$$

PROOF. It follows from [30, p. 66] that there is a uniformly r.e. pair of sequences $(S_x)_{x\in\omega}, (T_x)_{x\in\omega}$ such that

$$x \in U \to S_x \text{ is finite, and } x \in V \to T_x \text{ is finite.}$$

To define L_x, at stage n, in linear time we determine whether $w \in L_x$ for each string w of length n. Let $t \leq n$ be largest such that $t > 0$ and in n steps one can verify that

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(a) $S_{x,t} \neq S_{x,t-1}$ or (b) $T_{x,t} \neq T_{x,t-1}$; if there is no such t, let $t = 0$. If (a) applies or $t = 0$, declare that w is in L_x. Otherwise, declare that w is not in L_x.

Clearly L_x is computable in linear time, uniformly in x. If $x \in U$, then S_x is finite, so T_x is infinite. Then, for almost all n we are in case (b), so L_x is finite. If $x \in V$, then T_x is finite, so S_x is infinite, so for almost all n we are in case (a) and L_x is cofinite. ⊣

We obtain a variation on Theorem 3.6.

THEOREM 4.9. *The preorder of almost inclusion $\subseteq^*$ among quadratic time computable languages is Σ^0_2-complete.*

PROOF. Let $(G_i)_{i\in\omega}$ be an effective listing of all the linear time computable languages. Fix a set A computable in quadratic time but not in linear time. We claim that the Σ^0_2 Boolean algebra

$$\mathcal{B} = (\{A \cap L\colon\ L \text{ is linear time}\}, \subseteq^*)/ =^*$$

with the canonical representation given by $(G_i)_{i\in\omega}$ is Σ^0_2 effectively inseparable. To see this, in Lemma 4.8 let U, V be any pair of Σ^0_2 effectively inseparable sets. If $x \in U$, then $A \cap L_x =^* \emptyset$. If $x \in V$, then $A \cap L_x =^* A$. Now we apply Fact 4.5.

The preorder of $\mathcal{B}$ on indices for linear time computable languages is Σ^0_2-complete by Theorem 4.6. From i we can compute an index for $A \cap G_i$ within our effective listing of the quadratic time sets. Hence, we can effectively reduce the preorder on $\mathcal{B}$ to almost inclusion $\subseteq^*$ among quadratic time computable languages. ⊣

In the following, $\leq^p_r$ will denote a polynomial time reducibility in between polynomial time many-one (m) and Turing (T) reducibility. Clearly this reducibility is Σ^0_2 on $\mathrm{DTIME}(h)$.

THEOREM 4.10. *Suppose the function h is time constructible and dominates all polynomials. Then $\leq^p_r$ on sets in $\mathrm{DTIME}(h)$ is Σ^0_2-complete preorder.*

For instance, we can let $h(n) = 2^n$, or $h(n) = n^{\log\log n}$. The proof relies on the following notion of Ambos-Spies [2].

DEFINITION 4.11. Let $f\colon \omega \to \omega$ be a strictly increasing, time constructible function. We say that a language $A \subseteq \{0\}^*$ is *super sparse* via f if $A \subseteq \{0^{f(k)} : k \in \omega\}$ and "$0^{f(k)} \in A$?" can be determined in time $O(f(k+1))$.

Super sparse sets exist in the time classes we are interested in.

LEMMA 4.12 ([2]). *Suppose that $h : \omega \to \omega$ is an increasing time constructible function with $\mathrm{P} \subset \mathrm{DTIME}(h)$, so that $h(n) \geq n+1$ and h eventually dominates all polynomials. Then, there is a super sparse language $A \in \mathrm{DTIME}(h) - \mathrm{P}$.*

SKETCH OF PROOF. Let $f(n) = h^{(n)}(0)$. Since h eventually dominates all polynomials, we can construct $A \subseteq \{0^{f(k)} : k \in \omega\}$ such that $A \in \mathrm{DTIME}(h)$, but still diagonalize against all polynomial time machines. ⊣

We will keep this set A fixed in what follows. Given a reducibility $\leq^p_r$, we write $\mathbf{a} = \deg^p_r(A)$ for the degree of A. We write $[\mathbf{0}, \mathbf{a}]$ for the initial segment of the degrees below $\mathbf{a}$, viewed as a partial order.

PROOF OF THEOREM 4.10. Recall the e.i. Σ^0_2 Boolean algebra $\mathcal{B}$ from the proof of Theorem 4.9. It is represented via an effective listing (G_i) of linear time computable languages. We will embed a Σ^0_2 quotient of $\mathcal{B}$ into $[\mathbf{0}, \mathbf{a}]$.

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Since A is super sparse, by [26, Lemma 6.2.9] for polynomial time sets U, V we have

$$A \cap U \leq_r^p A \cap V \leftrightarrow A \cap (U - V) \in \mathrm{P}.$$

So we may well-define a map $\Phi\colon \mathcal{B} \to [\mathbf{0}, \mathbf{a}]$ as follows. For a linear time set $L = G_i$, let

$$\Phi(A \cap L/=^*) = \deg_r^p(A \cap L).$$

Let $\mathcal{I}$ be the Σ_2^0 ideal of $\mathcal{B}$ consisting of all the equivalence classes $A \cap L/=^*$ such that $A \cap L \in \mathrm{P}$. Then, $\mathcal{C} = \mathcal{B}/\mathcal{I}$ is in a natural way a Σ_2^0 Boolean algebra, and $\mathcal{C}$ is effectively embedded into $[\mathbf{0}, \mathbf{a}]$. Since $\mathcal{B}$ is e.i. and the map Φ is effective, it follows that $\mathcal{C}$ is e.i. Hence, by Theorem 4.6, the Σ_2^0 preorder $\{\langle e, i\rangle\colon A \cap G_e \leq_r^p A \cap G_i\}$ is complete. Via a computable transformation of indices, this preorder is effectively reducible to $\leq_r^p$ on $\mathrm{DTIME}(h)$. ⊣

4.4. Almost inclusion of r.e. sets.

THEOREM 4.13. *The preordering $\{\langle e, i\rangle\colon W_e \subseteq^* W_i\}$ of almost inclusion among r.e. sets is Σ_3^0-complete.*

PROOF. All sets in this proof will be r.e. Fix a nonrecursive r.e. set A. By $X \sqcup Y = A$ we denote that r.e. sets X, Y are a splitting of A. That is, $X \cap Y = \emptyset$ and $X \cup Y = A$. We also write $X \sqsubseteq A$ to denote that X is part of a splitting of A.

The following technical tool parallels Lemma 4.8.

LEMMA 4.14. *Given a disjoint pair of Σ_3^0 sets U, V, we can effectively in x determine an r.e. splitting $A = E_x \sqcup F_x$ such that*

$$x \in U \to E_x \text{ is computable, and } x \in V \to F_x \text{ is computable.}$$

PROOF OF LEMMA. By [30, p. 66] we may choose a uniformly r.e. double sequence $(P_{x,n})_{x,n\in\omega}$ of initial segments of ω such that

$$x \in U \leftrightarrow \exists n\,[P_{x,2n} = \omega] \quad \text{and} \quad x \in V \leftrightarrow \exists n\,[P_{x,2n+1} = \omega].$$

We fix x. Let $g(k, s)$ be the greatest $t \leq s$ such that $t = 0$, or $P_{x,k,t-1} \neq P_{x,k,t}$. We enumerate sets E_x, F_x with $E_x \sqcup F_x = A$ as follows. At stage $s > 0$, if $a \in A_s - A_{s-1}$, see if there is a $k \leq s$ be least such that $a < g(k, s)$. If so, choose k least. If k is even, enumerate a into F_x; otherwise, or if there is no such k, enumerate a into E_x.

First suppose that $x \in U$. Then, the least k such that $P_{x,k} = \omega$ is even. We have $\lim_s g(k, s) = \infty$. If $k > 0$, then also $r = \max_{i<k} \lim_s g(i, s) < \infty$. For any $v \geq r$, we have $v \in E_x \leftrightarrow v \in E_{x,s}$, where s is least such that $g(k, s) > v$. Therefore, E_x is computable.

If $x \in V$, then the least k such that $P_{x,k} = \omega$ is odd. A similar argument shows that F_x is computable. ⊣

Since A is nonrecursive, there is a major subset $D \subset_m A$ (see [30, Section X.4]). Thus, if $R \subseteq A$ is computable, then $R \subseteq^* D$. We now define an e.i. Σ_3^0 Boolean algebra $\mathcal{B}$, which is a subalgebra of the complemented elements in $[D, A]/=^*$. Suppose we are given a pair of e.i. Σ_3^0 sets U, V. Let $(E_x)_{x\in\omega}$, $(F_x)_{x\in\omega}$ be the u.r.e. sequences obtained through the foregoing lemma. We let $\mathcal{B}$ be the Boolean algebra generated by all the elements of the form $(E_x \cup D)^*$ for $x \in \omega$. Since uniformly in x we can recursively enumerate both E_x and its complement $F_x = A - E_x$, it is clear that $\mathcal{B}$ is a Σ_3^0 Boolean algebra. Furthermore, by Lemma 4.14 and

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Fact 4.5, $\mathcal{B}$ is effectively inseparable. So by Theorem 4.6 the preorder of $\mathcal{B}$, namely, $\{\langle x, y\rangle : E_x \subseteq^* E_y \cup D\}$, is Σ^0_3-complete. Since the sequence $(E_x)_{x\in\omega}$ is uniformly r.e., this establishes the theorem. ⊣

Note that we have in fact shown that almost inclusion of sets of the form $(W_e \cup D) \cap A$ is Σ^0_3-complete. It is worth noting that by a result of Maass and Stob [22], the lattice $[D, A]/=^*$ is unique up to effective isomorphism whenever D is a major subset of A . Thus, only the complemented elements of the Maass–Stob lattice are needed to establish the completeness of $\subseteq^*$ as a Σ^0_3 preorder; this lattice is in itself merely a "tiny" part of the lattice of r.e. sets under almost inclusion.

COROLLARY 4.15. *The equivalence relation* $\{\langle e, i\rangle : W_e =^* W_i\}$ *is* Σ^0_3*-complete.*

PROOF. This follows immediately from Fact 2.1. ⊣

The relation $=^*$ on r.e. indices is the same relation as that called E^{ce}_0 in [11, Theorem 3.4]. In the spirit of that article, we consider this and several other equivalence relations carried over from the Borel theory. The relations E_0, E_1, E_2, and E_3 are all standard Borel equivalence relations, whose definitions can be found in [17] (and may also be gleaned from the definitions given shortly). If $A \subseteq \omega$, we write $(A)_n = \{x : \langle n, x\rangle \in A\}$ for the n-th column of A. The analogues of these for r.e. indices are relations on the natural numbers, derived by letting each natural number e represent the set W_e. Thus, for an equivalence relation E on reals, we let $i\ E^{ce}\ j$ iff $W_i\ E\ W_j$, yielding the following specific relations.

$$
\begin{aligned}
i\ E^{ce}_0\ j \quad &\iff \quad |W_i \triangle W_j| < \infty \qquad (\iff \quad W_i =^* W_j)\\
i\ E^{ce}_1\ j \quad &\iff \quad \forall^\infty n\ \left[(W_i)_n = (W_j)_n\right]\\
i\ E^{ce}_2\ j \quad &\iff \quad \sum_{i\in W_i \triangle W_j} \frac{1}{i} < \infty\\
i\ E^{ce}_3\ j \quad &\iff \quad \forall n\ \left[\ |(W_i)_n \triangle (W_j)_n| < \infty\right].
\end{aligned}
$$

Now it is not hard to see that E^{ce}_3 is Π^0_4-complete as a set (since $=^*$ is Σ^0_3-complete as a set), and, therefore, not computably reducible to any of the others, since the others are all Σ^0_3. In the Borel theory, $E_0 <_B E_1$, $E_0 <_B E_2$, and $E_0 <_B E_3$, with no other reductions holding among these relations. The proofs of the Borel reducibilities can be adapted to give proofs that $E^{ce}_0 \leq_c E^{ce}_1$, $E^{ce}_0 \leq_c E^{ce}_2$, and $E^{ce}_0 \leq_c E^{ce}_3$. Therefore, Corollary 4.15 gives a further result, providing the relation left open in [11, Figure. 3].

COROLLARY 4.16. E^{ce}_0, E^{ce}_1, *and* E^{ce}_2 *are all* Σ^0_3*-complete equivalence relations, and hence are bireducible with each other under* $\leq_c$.

In contrast, Corollary 3.8 showed that the relation E^{ce}_3 cannot be complete under $\leq_c$ at any level of the arithmetical hierarchy, as it is Π^0_4 but not Σ^0_4 by virtue of its Π^0_4-completeness as a set of pairs.

4.5. Weak truth-table reducibility on r.e. sets. It is well-known that weak truth-table reducibility $\leq_{\mathrm{wtt}}$ on r.e. sets is a Σ^0_3 preorder with computable supremum operation. It determines the distributive upper semilattice of r.e. weak truth-table degrees; see, e.g., [27].

THEOREM 4.17. *The preorder* $\{\langle e, i\rangle : W_e \leq_{\mathrm{wtt}} W_i\}$ *is* Σ^0_3*-complete.*

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PROOF. Ambos-Spies [1] called an r.e. set A *antimitotic* if for any r.e. splitting $A = X \sqcup Y$, the sets X and Y induce a minimal pair in the Turing degrees. He built an incomputable antimitotic set A, which we will fix in the following. Let $\mathbf{a} = \deg_{wtt}(A)$.

Note that the degrees $\{\deg_{wtt}(X)\colon X \sqsubseteq A\}$ form a Boolean algebra of complemented elements in $[\mathbf{0}, \mathbf{a}]$ with the usual degree ordering. We establish an e.i. Σ^0_3 subalgebra $\mathcal{C}$ of this Boolean algebra. We are given a pair of e.i. Σ^0_3 sets U, V. Let $(E_x), (F_x)$ be uniformly r.e. sequences as in Lemma 4.14. Thus, $E_x \sqsubseteq A$ for each x. Let $\mathcal{C}$ be the Boolean algebra generated by the $\deg_{wtt}(E_x)$, $x \in \omega$. Since A is antimitotic, intersection, union, and complementation for splits of A correspond to the operations of meet, join, and complementation in $[\mathbf{0}, \mathbf{a}]$. Since the former operations are effective, $\mathcal{C}$ is indeed a Σ^0_3 Boolean algebra.

If $x \in U$, then E_x is computable, so $\deg(E_x) = \mathbf{0}$. If $x \in V$, then F_x is computable, so $\deg(E_x) = \mathbf{a}$. This implies that $\mathcal{C}$ is effectively inseparable. So the preordering of $\mathcal{C}$, namely, $\{\langle x, y\rangle\colon E_x \leq_{\text{wtt}} E_y\}$, is Σ^0_3-complete by Theorem 4.6. Since the sequence $(E_x)_{x\in\omega}$ is uniformly r.e., this establishes the theorem. ⊣

§5. Turing reducibility on r.e. sets is a Σ^0_4-complete preorder.

THEOREM 5.1. *The preorder* $\{\langle i, j\rangle \mid W_i \leq_T W_j\}$ *is* Σ^0_4*-complete. Hence, the equivalence relation* $\{\langle i, j\rangle \mid W_i \equiv_T W_j\}$ *is also* Σ^0_4*-complete.*

PROOF. We fix a Σ^0_4 preorder R. We give a construction with the overall aim to build a uniform sequence of r.e. sets $(V_i)_{i\in\omega}$ such that $i\,R\,j$ iff $V_i \leq_T V_j$. Following Nies [25] we fix a uniformly r.e. sequence $X^{i,j}_{e,p}$ of initial segments of ω, such that

$$i\,R\,j \to \text{For almost all } e \text{ and } p,\ |X^{i,j}_{e,p}| < \infty,$$
$$\neg\, i\,R\,j \to \forall e \exists p\ X^{i,j}_{e,p} = \omega.$$

Here, $|X|$ denotes the cardinality of the set X. The facts "$X^{i,j}_{e,p} = \omega$" and "$X^{i,j}_{e,p} < \infty$" are (uniformly) Π^0_2 and Σ^0_2, respectively, so each such statement can be measured at a single node on a priority tree measuring infinitary or finitary behavior. That is, $\neg\, i\,R\,j$ would be equivalent to the fact that "$\forall e \exists p$ such that the node measuring $|X^{i,j}_{e,p}|$ has true infinitary outcome". Similarly, if $i\,R\,j$ then "for almost all e, p, the node measuring $|X^{i,j}_{e,p}|$ has true finitary outcome". Hence, the ordered pairs in R will determine the true path of the construction, i.e., which nodes are visited infinitely often and which are not. We will arrange the strategies on the construction tree to align our actions with the true path.

In this proof we use $Y \upharpoonright k$ to mean the first $k+1$ bits of Y. We say "Y changes below k" to mean a change in $Y \upharpoonright k$.

5.1. Requirements and a high-level description. The requirements to be met are $S_{i,j}$ for $i, j \in \omega$:

$$S_{i,j}\ :\ i\,R\,j \leftrightarrow V_i \leq_T V_j.$$

The way in which each $S_{i,j}$ is satisfied can only be answered by a $0''''$ oracle, and so it is difficult to split $S_{i,j}$ explicitly into subrequirements in a meaningful way. We will instead describe the local action of each node on the priority tree and then describe how the priority tree ensures that each requirement $S_{i,j}$ is met globally.

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The construction tree is a labeled binary tree. Each node of length $\langle i, j, e, p\rangle$ is assigned the set $X^{i,j}_{e,p}$ and has two outcomes labeled $\infty, 0$. As usual ∞ is to the left of 0. If α is assigned the set $X^{i,j}_{e,p}$ we write

$$i_\alpha, j_\alpha, e_\alpha, p_\alpha, X_\alpha \quad \text{for} \quad i, j, e, p, X^{i,j}_{e,p}, \text{respectively.}$$

Outcome ∞ stands for $X_\alpha = \omega$, while outcome 0 stands for X_α finite.

We now describe the local action of each node α. During the construction α has a dual role. During the nonexpansionary stages (i.e., $\alpha * 0$ stages) α builds a Turing reduction aiming to make $V_i \leq_T V_j$. Loosely speaking, this strategy monitors for changes in V_i and responds with a change in V_j each time it sees an element entering V_i. The other (conflicting) strategy of α acts at each expansionary stage ($\alpha * \infty$ stage). At each expansionary stage α checks to see if it can make $V_i \neq \Phi^{V_j}_{e_\alpha}$ by enumerating some number in V_i and restraining V_j (henceforth, we will write Φ_α instead of Φ_{e_α} for convenience). These two strategies are clearly conflicting, and we always initialize the Turing reduction whenever an expansionary stage occurs for α.

We now describe how $S_{i,j}$ can be met globally. Suppose $\neg\, i\, R\, j$. Then, for each e there is some node α along the true path with $i_\alpha = i, j_\alpha = j, e_\alpha = e$ and having true outcome ∞. This α will ensure that $\Phi^{V_j}_e \neq V_i$, and so overall we get $V_i \not\leq_T V_j$. In this case there could be infinitely many nodes β along the true path also working for i, j, but which have true outcome 0. The local strategy for β will attempt to demonstrate $V_i \leq_T V_j$, which cannot possibly be achieved. Indeed each such node β along the true path will be injured infinitely often by another node α further along the true path aiming to diagonalize $\Phi^{V_j}_e \neq V_i$. This is the key difference between this construction and a typical $\emptyset'''$ tree argument. In the latter case we usually only allow a finite amount of injury to each requirement along the true path, whereas in our case it is necessary that some node gets injured infinitely often along the true path.

Now suppose that $i\, R\, j$. Then, almost every node along the true path working for the pair i, j must have true outcome 0. In particular, there is a shortest node α along the true path with true outcome 0 (we will later call all possible candidates for α a *top node*). In this case there could be finitely many top nodes $\alpha' \prec \alpha$ with true outcome 0. However, the Turing reducibilities built by these α' will all be destroyed (by nodes between α' and α which are attempting to diagonalize), and $V_i \leq_T V_j$ will be demonstrated by the final top node α along the true path. It is important to note that each top node α' cannot possibly know anything about the nodes on the true path which are longer than α', and so the situation described above cannot be avoided.

We conclude this section with some technical definitions. We say that α is a top node if there is no $\beta \prec \alpha$ such that $i_\alpha = i_\beta$ and $j_\alpha = j_\beta$, or if $\alpha(|\beta|) = \infty$ for the maximal such β. Given a node α we define the top of α to be the maximal top node $\tau \preceq \alpha$ such that $i_\alpha = i_\tau$ and $j_\alpha = j_\tau$. We say that α is a child of τ if α has top τ.

We let $Z^\infty(\alpha) = \{\beta \prec \alpha \mid \beta * \infty \preceq \alpha$ and β is not a top node$\}$, which are all the nodes extended by α running the ∞ strategy. Similarly, we let $Z^0(\alpha) = \{\beta \prec \alpha \mid \beta * 0 \preceq \alpha$ and β is a top node$\}$.

5.2. Ensuring $V_i \leq_T V_j$. Each top node α will build a Turing reduction Δ_α to ensure that $V_{i_\alpha} \leq_T V_{j_\alpha}$ in the case that $\alpha * 0$ is along the true path. The Turing

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reduction is built indirectly by specifying the coding markers $\{\delta_\alpha(x)[s]\}_{x,s\in\omega}$. Each $\delta_\alpha(x)[s]$ is not in $V_{j_\alpha}[s]$ and obeys the following standard marker rules: For all x and s, we ensure

(i) $\delta_\alpha(x)[s] < \delta_\alpha(x+1)[s]$.
(ii) If $\delta_\alpha(x)[s] \neq \delta_\alpha(x)[s+1]$ or if $\delta_\alpha(x)[s]\downarrow$ and $\delta_\alpha(x)[s+1]\uparrow$, then $V_{j_\alpha,s-1} \upharpoonright (\delta_\alpha(x)[s]) \neq V_{j_\alpha,s} \upharpoonright (\delta_\alpha(x)[s])$.
(iii) If x is enumerated in V_{i_α} at stage s, and if $\delta_\alpha(x)[s]$ is defined then V_{j_α} must change below $\delta_\alpha(x)[s]$ at the same stage or later.
(iv) $\lim_s \delta_\alpha(x)[s]$ exists (if $\alpha * 0$ is along the true path).

It is easy to see that if these rules are ensured, and if $\alpha * 0$ is visited infinitely often, then $V_{i_\alpha} \leq_T V_{j_\alpha}$: To compute if x is in V_{i_α} we find recursively in V_{j_α} a stage s where $V_{j_\alpha}[s]$ is correct up to $\delta_\alpha(x)[s]$. This stage exists because of (iv). At all stages $t > s$ we have $\delta_\alpha(x)[s] = \delta_\alpha(x)[t]$ because of (ii). By (iii) we must have $x \in V_{i_\alpha}$ iff $x \in V_{i_\alpha}[s]$.

5.3. Informal description of the strategy. We now describe the strategy and the method of overcoming the key difficulties. Each top node α will have a dual role. At $\alpha * 0$ stages it must extend the Turing reducibility $\Delta_\alpha^{V_j}$ by picking a fresh value for $\delta_\alpha(x)$. It must also ensure the correctness of $\Delta_\alpha^{V_j}(x)$ by enumerating $\delta_\alpha(x)$ if x had been enumerated into V_i recently. At each $\alpha * \infty$ stage α will attempt to diagonalize by enumerating (if possible) some number l into V_i and preserving V_j. Clearly these two strategies are conflicting and we give the diagonalization strategy higher priority; Hence at each $\alpha * \infty$ stage we reset Δ_α and all δ_α markers. If $\alpha * 0$ is along the true path, then we ensure $V_i = \Delta_\alpha^{V_j}$, and if $\alpha * \infty$ is along the true path, then we ensure $\Phi_\alpha^{V_j} \neq V_i$. In the latter case, the reduction Δ_α fails. Analyzing the overall outcome of the construction we note that if $\neg\, i\, R\, j$ then for every e we ensure $\Phi_e^{V_j} \neq V_i$ at some node along the true path, and if $i\, R\, j$ then there is a final top node τ along the true path where $V_i = \Delta_\tau^{V_j}$ is successfully maintained. In this case there can be finitely many top nodes α where $\alpha * \infty \preceq \tau$ for which $\Delta_\alpha^{V_j}$ is initialized infinitely often.

We now describe the main issue which will arise in implementing the above strategy. Suppose $\alpha \prec \beta$ are both working for the same i, j, and α has true outcome 0, while β has true outcome ∞. We may as well assume that α is the top of β. Now α and β will have conflicting actions. At each $\beta * \infty$ stage, we may want β to diagonalize by putting some number l in V_i. If $\delta_\alpha(l)$ has already been defined, then α will later want to correct Δ_α by putting $\delta_\alpha(l)$ into V_j, which can destroy the diagonalization previously obtained by β. We overcome this by delaying β from diagonalizing. In the meantime, at each $\beta * \infty$ stage we lift $\delta_\alpha(k)$ for some fixed number k by enumerating $\delta_\alpha(k)$ into V_j and later picking a fresh value for δ_α. Since there are infinitely many $\beta * \infty$ stages, $\delta_\alpha(k)$ goes to infinity. Hence, β destroys Δ_α but will now be able to meet $\Phi_\beta^{V_j} \neq V_i$ (via diagonalization or an undefined computation). $\Delta_\alpha^{V_j}$ is now destroyed. However, this is fine, because β witnesses the fact that α is not the final top node along the true path. In this case the reduction $\Delta^{V_j} = V_i$ will be built further down the true path (if $i\, R\, j$) or may not be built at all (if $\neg\, i\, R\, j$).

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In the general situation (when considering nodes devoted to other pairs i', j') we consider the nodes $\beta_0, \beta_1 \prec \alpha$ where β_0, β_1 have true outcome 0, α has true outcome ∞, $i = i_\alpha = i_{\beta_1}$, $j_{\beta_1} = i_{\beta_0}$, and $j_{\beta_0} = j_\alpha = j$. Again if α wants to diagonalize at some $\alpha * \infty$ stage it must be careful; A number l enumerated in V_i might cause β_1 to later correct $\Delta_{\beta_1}^{V_{j_{\beta_1}}}$ by changing $V_{j_{\beta_1}}$, which might in turn cause β_0 to correct $\Delta_{\beta_0}^{V_j}$ by changing V_j, destroying the previous diagonalization attempt of α. In this case we observe that along the true path, we must have one of the following three situations hold:

(i) There is some child γ_0 of β_0 where $\gamma_0 * \infty$ is along the true path.
(ii) There is some child γ_1 of β_1 where $\gamma_1 * \infty$ is along the true path.
(iii) Almost every γ working for the same pair $\langle i, j\rangle$ as α along the true path has true 0 outcome.

One of these three situations must hold since R is transitive. If (iii) holds, then it is fine for α to fail in diagonalizing even though it has true outcome ∞. If (i) or (ii) holds, then γ_v (for some $v < 2$) witnesses that β_v is not the final top node along the true path; In this case γ_v will destroy Δ_{β_v} by pushing some $\delta_{\beta_v}(k)$ to infinity. In this case we delay α from diagonalizing until we find that $\Phi_\alpha^{V_j}(l)$ has converged on a number l with use u safely. That is, the combined response of β_0 and β_1 to the entry of l into V_i will not change V_j below u.

5.4. Markers for capricious destruction. Each node α, which is not a top node, is given a parameter k_α. At each α-expansionary stage, α will enumerate $\delta_\tau(k_\alpha)$ into V_{j_τ} and lift this marker. Hence, if $\alpha * \infty$ is along the true path, then $\delta_\tau(k_\alpha)$ goes to infinity. This is fine because τ is not the maximal top node along the true path; in this case there may be a top node further down the true path, or there may be none along the true path, and so the Turing reducibility witnessing $V_{i_\tau} \leq_T V_{j_\tau}$ need not be built at τ. The important thing here is to ensure that for each top node τ where $\tau * 0$ is along the true path and τ is not the maximal top node along the true path, we have to destroy the τ markers and create an interval (i, ∞) where every point is eventually cleared of τ markers. This allows a conflicting diagonalization strategy of lower priority to succeed.

If τ is the final top node along the true path, then each τ marker is lifted finitely often this way, and so the Turing reducibility witnessing $V_{i_\tau} \leq_T V_{j_\tau}$ will be built at τ.

5.5. Believable computations. A computation $\Phi_\alpha^{V_{j_\alpha}}(l)[s]$ with use u is said to be α-believable at stage s if the following two conditions are met.

- For every $\beta \in Z^0(\alpha)$ there is no β marker below u which is pending. A β marker $\delta_\beta(x)$ is said to be pending at t if x was enumerated into V_{i_β} at some $s < t$ where $\delta_\beta(x)[s]$ is defined and V_{j_β} has not changed below $\delta_\beta(x)[s]$.
- There does not exist a sequence of distinct nodes $\beta_0, \ldots, \beta_n \in Z^0(\alpha)$ such that $i_{\beta_n} = i_\alpha$, $j_{\beta_0} = j_\alpha$, and for every $m < n$, $j_{\beta_{m+1}} = i_{\beta_m}$, and $\delta_{\beta_m}(l)[s] \downarrow \leq u$.

The first item says that a computation is α-believable if there is no δ marker below the use that will be enumerated by some $\beta \prec \alpha$ due to coding. The second item ensures that if l were to be enumerated by α into V_{i_α} for the sake of diagonalization then the resulting sequence of correction actions will not cause the computation to be later destroyed.

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5.6. Formal construction. At stage 0 we initialize every node. This means to reset δ_α and k_α to be undefined. At stage $s > 0$ we define TP_s of length s, where TP_s is the stage s approximation to the true path of construction. Assume that $\alpha \prec TP_s$ has been defined. If $|X_\alpha|$ has increased since the last α-stage, then we play outcome ∞ for α, otherwise play outcome 0 for α. Initialize every node to the right of $\alpha * o$ where o is the outcome played. We now act for α. There are four cases.

(i) α is not a top node and outcome 0 is played. Do nothing.

(ii) α is a top node and outcome ∞ is played. Initialize α. Check if it is possible to diagonalize, and if so, do it. This means to check if there exists some $l < s$ such that $\Phi_\alpha^{V_{j_\alpha}}(l)[s]\downarrow$ is α-believable, $l \in \omega^{[\alpha]} - V_{i_\alpha}[s]$, and for every $l' \leq l$ we have $V_{i_\alpha}(l') = \Phi_\alpha^{V_{j_\alpha}}(l')[s]$. We also require l to be larger than the previous stage where α was initialized by another node. If such l is found enumerate l into V_{i_α} and initialize every node extending α.

(iii) α is not a top node and outcome ∞ is played. Let τ be the top of α. If $k_\alpha \uparrow$ we pick a fresh value for it. Otherwise if $\delta_\tau(k_\alpha)\downarrow$ enumerate it into V_{j_τ} and make it (and all larger τ marker) undefined. Check if it is possible to diagonalize, and if so, do it.

(iv) α is a top node and outcome 0 is played. We will correct and extend the Turing reduction Δ_α. For correction, we check to see if there is any number l enumerated into V_{i_α} since the last visit to α. Let l be the least. If $\delta_\alpha(l)\downarrow$ we enumerate it in V_{j_α} and make $\delta_\alpha(l)\uparrow$. To extend Δ_α we let $k < s$ be the least such that $\delta_\alpha(k)\uparrow$. We pick a fresh value $x \in \omega^{[\alpha]}$ and set $\delta_\alpha(k)\downarrow = x$.

5.7. Verification. Let $TP = \liminf_s TP_s$ be the true path of the construction, which clearly exists as the construction tree is finitely branching. We first prove the following key lemma.

LEMMA 5.2. *Suppose α is a node on the true path. Then, α makes finitely many diagonalization attempts.*

PROOF. Assume we are at a stage of the construction where each $\beta \prec \alpha$ makes no more diagonalization attempt, and we never move left of α. Suppose α makes infinitely many diagonalization attempts after this. Let l be the smallest number put in by α due to a diagonalization attempt, at some stage t. At t we have $\Phi_\alpha^{V_{j_\alpha}}(l)[t]\downarrow = 0$ with use u, and is an α-believable computation. We claim that at t, V_{j_α} is stable below u, which contradicts the assumption that infinitely many attempts are made.

By convention $u < t$, and so the only nodes which can enumerate a number less than u into any set after stage t, are nodes of the form $\beta \preceq \alpha$. In fact we cannot have $\beta = \alpha$, because if α is a top node then it is initialized at t and any further diagonalization action by α involves a number $l' > l$, and if α is not a top node then k_α or $\delta_\tau(k_\alpha)$ is undefined before the diagonalization at t (here τ is the top of α). Similarly, we cannot have $\beta * \infty \preceq \alpha$, because if β is a top node then it is initialized at t and never again performs diagonalization, and if β is not a top node then k_β or $\delta_\tau(k_\beta)$ is undefined when α was visited at t. Hence, if β is a node which enumerates a number smaller than u into any set after t, it must satisfy $\beta_0 \in Z^0(\alpha)$.

Let $\beta_0 \in Z^0(\alpha)$ be the first node after t to change V_{j_α} below u, say at $t_0 > t$. At t_0, β_0 is correcting Δ_{β_0} and responding to the action of β_1 at t_1, $t_1 < t_0$. Let l_0 be the number which was enumerated by β_1 into $V_{i_{\beta_0}}$ at t_1. We cannot have $t_1 < t$

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because of α-believability at t. Suppose $t_1 > t$. Since $l < \delta(l)$ for every l, δ, we have $\beta_1 \in Z^0(\alpha)$. Then, we have correction at t_1 and so $j_{\beta_1} = i_{\beta_0}$. We can then repeat to get a sequence $\beta_0, \ldots, \beta_n \in Z^0(\alpha)$ and $t_0 > \cdots > t_n > t_{n+1} = t$, where $\beta_{n+1} = \alpha$, $j_{\beta_0} = j_\alpha$, $i_{\beta_n} = i_\alpha$, and for every $m < n$, $j_{\beta_{m+1}} = i_{\beta_m}$. Furthermore for every $m \leq n$, $l_m < \delta_{\beta_m}(l_m) = l_{m-1}$, and $l_n = l$. Since $\delta_{\beta_m}(l_m)[t_m] \downarrow < u$ we have $\delta_{\beta_m}(l_m)[t] \downarrow < u$ (since δ markers are picked fresh), and so $\delta_{\beta_m}(l)[t] \downarrow < u$. By removing any cycles in the sequence, we may assume that the nodes $\beta_0, \ldots, \beta_n$ are distinct. This contradicts the second requirement for α-believability at t. $\dashv$

We devote the remainder of the proof to showing that each requirement $S_{i,j}$ is met. Fix i and j such that $i R j$. Let α be the shortest node along TP such that $i_\alpha = i$, $j_\alpha = j$ and for every node β such that $i_\beta = i$ and $j_\beta = j$ where $\alpha \preceq \beta \prec TP$, we have $\beta * 0 \preceq TP$. Clearly α is a top node. We argue that $V_i = \Delta_\alpha^{V_j}$ (by a symmetric argument we can then conclude $V_i \equiv_T V_j$). Since α has true 0 outcome, there are finitely many self-initializations, and by Lemma 5.2, there are only finitely many initializations to α. The marker rules (i)–(iii) are clearly met for δ_α. We argue (iv) holds. Since α is visited infinitely often it suffices to check that each marker is made undefined finitely often. Each $\delta_\alpha(k)$ can only be made undefined by some child of α, and only when $k = k_\alpha$. A child of α to the left of the true path only does this finitely often, while a child to the right of the true path has $k_\alpha = k$ at finitely many stages. A child on the true path must have true 0 outcome and acts at only finitely many stages.

Now fix i and j such that $\neg\, i\, R\, j$. Fix an e, and let α be the node on the true path such that $i_\alpha = i, j_\alpha = j, e_\alpha = e$ and $\alpha * \infty \prec TP$. We argue that $\Phi_\alpha^{V_j} \neq V_i$. Suppose that $\Phi_\alpha^{V_j} = V_i$.

LEMMA 5.3. *For almost every l, the true computation $\Phi_\alpha^{V_j}(l)$ is eventually α-believable.*

PROOF. Call a sequence of distinct nodes $\beta_0, \ldots, \beta_n \in Z^0(\alpha)$ such that $i_{\beta_n} = i_\alpha$, $j_{\beta_0} = j_\alpha$, and for every $m < n$, $j_{\beta_{m+1}} = i_{\beta_m}$, a bad sequence. There are only finitely many bad sequences, and for each bad sequence there exists a node γ which is a child of some β_m such that $\gamma * \infty \prec TP$. This follows by the transitivity of R. Since γ is not a top node it is initialized finitely often and so achieves a stable value for k_γ. We consider l larger than every k_γ and argue that $\Phi_\alpha^{V_j}(l)$ is eventually α-believable.

Fix l and fix a large stage s such that α is visited and $\Phi_\alpha^{V_j}(l)[s] \downarrow$ with the correct use u. The first requirement for α-believability must be met because any pending marker from any node in $Z^0(\alpha)$ must be enumerated in V_j below u before the next visit to α. For the second requirement, we can assume that s is large enough such that for each bad sequence and associated γ, we have $\delta_{\beta_m}(k_\gamma)[s] > u$. This is possible, because at each visit to $\gamma * \infty$ we make $\delta_{\beta_m}(k_\gamma)$ undefined.

Let us assume the second requirement for α-believability fails. Fix a bad sequence witnessing this, and let γ be the associated node. By the failure of believability we have $\delta_{\beta_m}(l)[s] \leq u$; On the other hand, by the choice of s we have $\delta_{\beta_m}(l)[s] > \delta_{\beta_m}(k_\gamma)[s] > u$, a contradiction. $\dashv$

By Lemma 5.2, there are only finitely many initializations to α initiated by a different node; Let s be large enough so that there are no more initializations to α of this type. By Lemma 5.3 pick l large enough, and wait for $\Phi_\alpha^{V_j}(l)$ to

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become α-believable, and for $\Phi_\alpha^{V_j}$ to agree with V_i below l. Hence, α will make another diagonalization attempt after s. Since this holds for any large s, we have a contradiction to Lemma 5.2. This ends the proof of Theorem 5.1. ⊣

We also obtain as a corollary, examples of complete equivalence relations at the Σ^0_{n+4} level for each $n \in \omega$:

COROLLARY 5.4. *For each $n \in \omega$, the preorder $\{\langle i, j\rangle \mid W_i^{\emptyset^{(n)}} \leq_T W_j^{\emptyset^{(n)}}\}$ is Σ^0_{n+4}-complete.*

PROOF. The proof of Theorem 5.1 produces a computable function f such that for each i, j, we have $i\,R\,j$ iff $W_{f(i)} \leq_T W_{f(j)}$. Relativizing this to $\emptyset^{(n)}$, for each Σ^0_{n+4} preorder R, we get a function $f \leq \emptyset^{(n)}$ such that $i\,R\,j$ iff $W_{f(i)}^{\emptyset^{(n)}} \leq_T W_{f(j)}^{\emptyset^{(n)}} \oplus \emptyset^{(n)}$.

For each oracle X and each function $g \leq_T X$ there is a computable function h such that $W_{g(i)}^X = W_{h(i)}^X$ for every i. To see this, observe that the set $\{(n, i) : n \in W_{g(i)}^X\}$ is $\Sigma_1^{0,X}$ and is hence equal to the domain of Ψ^X for some functional Ψ. The function h is obtained by applying the s-m-n Theorem.

Now applying this fact we may assume that f is computable. Let $\hat{f}$ be a computable function such that $W_{f(i)}^X \oplus X = W_{\hat{f}(i)}^X$ for all X and i. Then, $\hat{f}$ witnesses that R is reducible to the preorder $\{\langle i, j\rangle \mid W_i^{\emptyset^{(n)}} \leq_T W_j^{\emptyset^{(n)}}\}$. ⊣

COROLLARY 5.5. *For each $n \in \omega$, the preorder $\{\langle i, j\rangle \mid W_i^{(n)} \leq_T W_j^{(n)}\}$ is Σ^0_{n+4}-complete.*

PROOF. The proof of Corollary 5.4 shows that the preorder $\{\langle i, j\rangle \mid W_i^{\emptyset^{(n)}} \oplus \emptyset^{(n)} \leq_T W_j^{\emptyset^{(n)}} \oplus \emptyset^{(n)}\}$ is complete among the Σ^0_{n+4} preorders. By the uniformity of Sacks' Jump Inversion Theorem (see [30, Corollary VIII.3.6.]), for each n there is a computable function q_n such that for all x, we have $W_{q_n(x)}^{(n)} \equiv_T W_x^{\emptyset^{(n)}} \oplus \emptyset^{(n)}$. ⊣

COROLLARY 5.6. *For each $n \in \omega$ the relation*

$$\{\langle i, j\rangle : W_i^{\emptyset^{(n+1)}} \equiv_1 W_j^{\emptyset^{(n+1)}}\}$$

is Σ^0_{n+4}-complete.

§6. Discussion and open questions. Many effective equivalence relations from algebra can be considered under the aspect of relative complexity. For instance:

QUESTION 6.1. *Is the equivalence relation E of isomorphism between finite presentations of groups Σ^0_1-complete?*

Adyan and Rabin independently showed in 1958 that the triviality problem, whether a finite presentation describes the trivial group, is m-complete. See Lyndon and Schupp [21, Theorem IV.4.1], letting the given group H there have a word problem m-equivalent to the halting problem. In fact, by the discussion following the proof there, every single equivalence class of E is m-complete: Being isomorphic to a particular finitely presented group P is incompatible with free products in their sense, because $P * A$ has higher rank than P for any nontrivial group A.

QUESTION 6.2. *Is the equivalence relation E of isomorphism between automatic equivalence relations Π^0_1-complete?*

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Kuske, Liu, and Lohrey [20] showed that this is Π^0_1-complete as a set of pairs. A similar question can be asked about the Π^0_1 relation of elementary equivalence of automatic structures for the same finite signature.

Computable isomorphism of computable Boolean algebras is Σ^0_3 complete by [15]. By [13], isomorphism on many natural classes of computable structures, such as graphs, is a Σ^1_1-complete equivalence relation. The fourth author has observed that isomorphism of computable Boolean algebras also is a Σ^1_1-complete equivalence relation. This uses the fact that the coding of graphs into countable Boolean algebras from [9] is effective. Thus, uniformly in a computable graph it produces a computable Boolean algebra.

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