Ergod. Th. & Dynam. Sys. (2005), **25**, 1521–1538
doi:10.1017/S0143385705000052
© *2005 Cambridge University Press*
Printed in the United Kingdom

Spatial and non-spatial actions of Polish groups

E. GLASNER† and B. WEISS‡

† *Department of Mathematics, Tel Aviv University, Ramat Aviv, Israel*
(e-mail: glasner@math.tau.ac.il)
‡ *Institute of Mathematics, Hebrew University of Jerusalem, Jerusalem, Israel*
(e-mail: weiss@math.huji.ac.il)

(*Received* 1 *June* 2004 *and accepted in revised form* 21 *January* 2005)

Abstract. For locally compact groups all actions on a standard measure algebra have a spatial realization. For many Polish groups this is no longer the case. However, we show here that for non-archimedean Polish groups all measure algebra actions do have spatial realizations. In the other direction we show that an action of a Polish group is whirly ('ergodic at the identity') if and only if it admits no spatial factors and that all actions of a Lévy group are whirly. We also show that in the Polish group $\mathrm{Aut}(X, \mathcal{X}, \mu)$, for the generic automorphism T the action of the subgroup $\Lambda(T) = \mathrm{cls}\{T^n : n \in \mathbb{Z}\}$ on the Lebesgue space $(X, \mathcal{X}, \mu)$ is whirly.

0. *Introduction*

Classical theorems, due to Mackey and Ramsay, assert that all measure algebra actions of a locally compact second countable topological group admit spatial realizations (see [**10**] and [**14**, Theorem 3.3]). In [**7**], the authors have shown that this is no longer the case for a general Polish topological group and established, given a Boolean action of a Polish group G on a measure algebra $(\mathcal{X}, \mu)$, a necessary and sufficient condition for the action to admit a spatial model. This necessary and sufficient condition was formulated in terms of G-continuous functions. A function $f \in L^\infty(\mu)$ is G-continuous if $\|f \circ g_n - f\| \to 0$ in $L^\infty(\mu)$ whenever $g_n \to e$ in G. Another key notion introduced in [**7**] was that of a whirly G-action. Roughly speaking an action of G on a measure algebra $(\mathcal{X}, \mu)$ is whirly if it is ergodic at the identity, i.e. $\mu(UA) = 1$ for every positive set $A \in \mathcal{X}$ and every neighborhood U of the identity $e \in G$. It was shown in [**7**] that a whirly action admits no non-trivial spatial factors. In the present work we exploit these new notions and results in several ways.

In §1 we provide some background and recall the relevant facts from [**7**]. A topological group is non-archimedean if there is a basis for the topology at the identity consisting of open subgroups. The class of non-archimedean Polish groups coincides with that of closed subgroups of the group S_∞ of permutations of $\mathbb{N}$ (with the topology of pointwise convergence). In §2 we show that for a non-archimedean Polish group every Boolean

https://doi.org/10.1017/S0143385705000052 Published online by Cambridge University Press

action is spatial. In §3 we define stable sets and show that the sub-σ-algebra of $\mathcal{X}$ generated by the G-continuous functions coincides with the σ-algebra generated by the stable sets. As corollaries we deduce that: (i) a G-action is whirly if and only if it admits no non-trivial spatial factors; (ii) every ergodic action of a Polish Lévy group is whirly. In §4 we deduce some facts concerning the structure of whirly and spatial systems.

In §5 we show that in the Polish group $\mathrm{Aut}\,(X, \mathcal{X}, \mu)$ of automorphisms of a standard Lebesgue space, (topologically) almost every automorphism defines a whirly action on $(X, \mathcal{X}, \mu)$. Finally, in §6 we show that there are Polish groups which admit whirly actions and yet are not Lévy. First we find $T \in \mathrm{Aut}\,(X)$ for which the Polish group $\Lambda(T) = \mathrm{cls}\,\{T^n : n \in \mathbb{Z}\}$ has this property, and then we show that the abelian Polish group which underlies a separable infinite-dimensional Hilbert space H admits both spatial and whirly faithful actions. We thank Matt Foreman for a helpful conversation.

1. *Background*

In this introductory section we present our basic definitions and review some of the results of [**7**], sometimes in a slightly modified way, that will be used here. The reader is also referred to that paper for more motivation and background.

A *Borel action* of G on a Borel space $(X, \mathcal{X})$ is a Borel map $G \times X \to X$ satisfying the conditions, $ex = x$ and $g(hx) = (gh)x$ for all $g, h \in G$ and all $x \in X$ (e denotes the identity element of G). Such an object is also called a *Borel G-space*. When a G-space carries an invariant measure we define the notion of spatial G-action.

Definition 1.1. Let G be a Polish group. By a *spatial G-action* we mean a Borel action of G on a standard Lebesgue space $(X, \mathcal{X}, \mu)$ such that each $g \in G$ preserves the measure μ. We say that two spatial actions are isomorphic if there exists a measure-preserving one-to-one map between two G-invariant subsets of full measure in the corresponding spaces which intertwines the G-actions (the same two sets for all $g \in G$).

Often, however, it is the case that one has to deal with near-actions (see Definition 1.2 below and Zimmer [**18**, Definition 3.1]) or merely with an action of the group on a *measure algebra* (i.e. the Borel algebra modulo sets of measure zero) and it is then desirable to find a spatial model.

Definition 1.2. Let G be a Polish group and $(X, \mathcal{X}, \mu)$ a standard Borel space with a probability measure μ. By a *near-action* of G on $(X, \mathcal{X}, \mu)$ we mean a Borel map $G \times X \to X$, $(g, x) \mapsto gx$ with the following properties.

(i) With e the identity element of G, $ex = x$ for almost every x.

(ii) For each pair $g, h \in G$, $g(hx) = (gh)x$ for almost every x (where the set of points $x \in X$ of measure one where this equality holds may depend on the pair g, h).

(iii) Each $g \in G$ preserves the measure μ.

Let $\mathrm{Aut}\,(X) = \mathrm{Aut}\,(X, \mathcal{X}, \mu)$ be the Polish group of all equivalence classes of invertible measure-preserving transformations $X \to X$, with the neighborhood basis at the identity formed by sets of the form

$$N(A, \varepsilon) = \{T \in \mathrm{Aut}\,(X) : \mu(A \triangle TA) < \varepsilon\},$$

for $A \in \mathcal{X}$ and $\varepsilon > 0$. The following proposition is from the introduction to [**7**].

https://doi.org/10.1017/S0143385705000052 Published online by Cambridge University Press

PROPOSITION 1.3. *The following three notions are equivalent.*

(I) *A near-action of G on $(X, \mathcal{X}, \mu)$.*

(II) *A continuous homomorphism from G to* Aut (X).

(III) *A Boolean action of G on $(X, \mathcal{X}, \mu)$; that is, a continuous homomorphism from G to the automorphism group of the associated measure algebra.*

Every spatial action is also a near-action and in that case the spatial action will be called a *spatial model* of the near-action.

Recall that a Polish G-space is a Polish space X together with a continuous action $G \times X \to X$ of a Polish group G. Such an action will be called a *Polish action*. If, in addition, X is compact then it is a *compact* (Polish) G-space.

Every Polish action is also a Borel action. In that case the Polish action will be called a *Polish model* of the Borel action.

We have the following classical theorems, due to Mackey [**10**], Varadarajan [**16**, Theorem 3.2, **17**] and Ramsay [**14**, Theorem 3.3].

THEOREM 1.4. *Let G be a locally compact second countable topological group.*

(a) *Every near-action (or Boolean action) of G admits a spatial model.*

(b) *Every spatial action of G admits a Polish model.*

A powerful generalization to Polish groups of Theorem 1.4(b), given in [**1**], is the following.

THEOREM 1.5. (Becker and Kechris [**1**, Theorem 5.2.1]) *We have:*

(a) *every Borel action of a Polish group admits a Polish model;*

(b) *every Borel G-space is embedded (as a G-invariant Borel subset) into a compact Polish G-space.*

It was shown in [**7**] that many Polish groups have near-actions that do not admit spatial models. In fact, it is shown there that for a Polish Lévy group any spatial action is necessarily trivial [**7**, Theorem 1.1].

Lévy groups were introduced and studied by Gromov and Milman [**8**]. We briefly recall the definitions of Lévy families and Lévy groups; see [**9, 12**], for detailed studies of the phenomenon of concentration of measure and a plethora of examples. Concise proofs of some of the outstanding instances of this phenomenon can be found in the appendix of [**7**]. Refer to [**11, 13**] for further details on the history of the subject.

A family (X_n, d_n, μ_n), $n = 1, 2, 3 \ldots$ of metric spaces with probability measures μ_n is called a *Lévy family* if the following condition is satisfied. Whenever $A_n \subset X_n$ is a sequence of subsets such that $\liminf \mu_n(A_n) > 0$ then for any $\varepsilon > 0$, $\lim \mu_n(B_\varepsilon(A_n)) = 1$ (here $B_\varepsilon(A)$ is the ε neighborhood of A). A Polish group G is a *Lévy group* if there exits a family of compact subgroups $K_n \subset K_{n+1}$ such that the group $F = \bigcup_{n\in\mathbb{N}} K_n$ is dense in G and the corresponding family (K_n, d, m_n) is a Lévy family; here m_n is the normalized Haar measure on K_n and d is a right-invariant compatible metric on G.

Since the group Aut (X) is a Lévy group it follows, in particular, that the natural near-action of Aut (X) on $(X, \mathcal{X}, \mu)$ does not admit a spatial model.

https://doi.org/10.1017/S0143385705000052 Published online by Cambridge University Press

Another approach used in [**7**] to the problem of finding spatial models to Boolean actions is through the notions of G-continuous function and whirly action.

Definition 1.6. Having a near-action (or Boolean action) of G on $(X, \mathcal{X}, \mu)$ we say that $f \in L^\infty(\mu)$ is *G-continuous*, if $f \circ g_n$ converges to f in $L^\infty(\mu)$ norm whenever $g_n \to e$.

The following theorem is a reformulation of Theorem 2.2 and Remark 2.3 of [**7**].

THEOREM 1.7. *Let $(\mathcal{X}, \mu, G)$ be a Boolean G-system and let $\mathcal{D} \subset \mathcal{X}$ be the G-invariant sub-σ-algebra generated by the G-continuous functions in $L^\infty(\mu)$. Then the factor Boolean system $(\mathcal{D}, \mu, G)$ is the maximum factor of $(\mathcal{X}, \mu, G)$ which admits a spatial model. Thus, $A \in \mathcal{X}$ is in $\mathcal{D}$ if and only if for every $\varepsilon > 0$ there is a G-continuous function $f \in L^\infty$ such that $\|f - \mathbf{1}_A\|_2 < \varepsilon$. In particular, $(\mathcal{X}, \mu, G)$ admits a spatial model if and only if the collection of G-continuous functions is dense in $L^2(\mu)$.*

The following definition and proposition are from [**7**] (Proposition 1.9(c) follows directly from the definition).

Definition 1.8. A near-action of G on $(X, \mathcal{X}, \mu)$ is *whirly*, if for all sets $A, B \in \mathcal{X}$ of positive measure, every neighborhood of e (the unit of G) contains g such that $\mu(A \cap gB) > 0$.

PROPOSITION 1.9. *We have the following.*

(a) *If a near-action is whirly, then all G-continuous functions are constants.*

(b) *A whirly action has no spatial model; moreover, such an action cannot have non-trivial spatial factor.*

(c) *A factor of a whirly action is whirly.*

In Theorem 3.10 we show that, conversely, a Boolean action with no non-trivial spatial factor is whirly.

It was shown in [**7**] that the action of $\mathrm{Aut}(X)$ on $(X, \mathcal{X}, \mu)$ is whirly, thereby providing a second proof to the fact that this action does not admit a spatial model. Theorem 3.11 below asserts that, in fact, every near-action of a Polish Lévy group is whirly.

2. *Non-archimedean group actions are spatial*

Recall that a topological group is called *non-archimedean* if there is a basis for the topology at the identity consisting of open subgroups. In this section, we apply our criterion for an action to admit a spatial model to show that for Polish non-archimedean groups every near-action admits a spatial model.

Although we will not have an occasion to use it, we remind the reader of the following interesting characterization of Polish non-archimedean groups (see [**1**, Theorem 1.5.1]).

THEOREM 2.1. *A Polish topological group G is non-archimedean if and only if it is isomorphic to a closed subgroup of the group S_∞ of permutations of $\mathbb{N}$ (with the topology of pointwise convergence).*

A theorem that will be used in our proof is the celebrated Ryll-Nardzewski theorem [**15**] (see also [**4**, Theorem III.5.2]). Recall that a topological dynamical system (Q, T),

https://doi.org/10.1017/S0143385705000052 Published online by Cambridge University Press

where a group T acts continuously on a compact space Q, is called *affine* if Q is a convex subset of a topological linear space E and each $t \in T$ acts as an affine transformation (that is, $t(\alpha x + (1-\alpha)y) = \alpha tx + (1-\alpha)ty$ for every $x, y \in Q$ and $0 \leq \alpha \leq 1$). The action is called *distal* with respect to a norm $\|\cdot\|$ on E if for every $x \neq y$ in Q we have $\inf_{t\in T} \|tx - ty\| > 0$.

THEOREM 2.2. *Let E be a separable Banach space, Q a weakly compact convex subset of E. Let (Q, T) be an affine dynamical system such that the action of T is distal in the norm topology, then T has a fixed point in Q.*

THEOREM 2.3. *Every near-action of a Polish non-archimedean group admits a spatial model.*

Proof. Let $(X, \mathcal{X}, \mu, G)$ be a near-action of the Polish non-archimedean group G and let $g \mapsto U_g$ be the associated Koopman representation on $L^2(\mu)$ given by $U_g f(x) = f(g^{-1}x)$, $g \in G$, $f \in L^2(X, \mu)$. Given $f \in L^2(\mu)$ and $\varepsilon > 0$ we next show that the ball $B_\varepsilon(f) = \{k \in L^2(\mu) : \|k - f\| \leq \varepsilon\}$ contains a G-continuous $L^\infty(\mu)$ function.

By the continuity of the representation $g \mapsto U_g$, there exists an open subgroup $H \subset G$ such that $U_h f \in B_\varepsilon(f)$ for every $h \in H$. Being a norm closed bounded convex subset of $L^2(\mu)$, $B_\varepsilon(f)$ is weakly closed, hence, also weakly compact. We let

$$Q = w\text{-cls co}\,\{U_h f : h \in H\},$$

the w-closed convex hull of the H-orbit of f in $L^2(\mu)$. Clearly Q is a weakly-compact convex H-invariant subset of $B_\varepsilon(f)$. By the Ryll-Nardzewski fixed point theorem (Theorem 2.2) there exists a function $f \in Q$ which is H-fixed. If we let, for each $M > 0$,

$$f_M = \begin{cases} f(x) & \text{if } |f(x)| \leq M, \\ -M & \text{if } f(x) < -M, \\ M & \text{if } f(x) > M, \end{cases}$$

then clearly each f_M is H-fixed and for sufficiently large M we have $f_M \in B_\varepsilon(f)$. We then fix such an M and claim that f_M is G-continuous.

In fact, the group G being second countable, we see that the homogeneous space G/H is a countable space and that consequently the G-orbit of f_M in $L^\infty(\mu)$ is a countable set. By Proposition 2.6 of [**7**] we deduce that f_M is indeed a G-continuous function.

Note that what we have shown so far clearly implies that the G-continuous, essentially bounded functions are dense in $L^2(\mu)$. Thus, in order to complete the proof of the theorem it only remains to apply Theorem 1.7. □

Remark 2.4. One can use a weaker fixed point theorem here. In fact, since the action of H on Q is *weakly almost periodic*, it follows from the general theory of such systems that every minimal subsystem of Q is equicontinuous (see, e.g., [**6**, Ch. 1]). Now an affine dynamical system always contains a *strongly proximal* minimal subsystem (see [**4**, Ch. III]). Finally, a minimal system which is both proximal and equicontinuous is necessarily a fixed point.

https://doi.org/10.1017/S0143385705000052 Published online by Cambridge University Press

Remark 2.5. In [**2**], Danilenko showed that another large class of topological groups, namely the class CILLC (countable inductive limit of locally compact groups) has the property that for every member G of the class, all G-actions admit spatial realizations.

3. *Stable subsets*

Let G be a Polish group and fix a sequence $\{U_m\}_{m=1}^{\infty}$ of neighborhoods of the identity element $e \in G$ with the following properties.

(i) For every m, $U_{m+1}^2 \subset U_m$.
(ii) For every m, $U_m^{-1} = U_m$.
(iii) $\bigcap_{m=1}^{\infty} U_m = \{e\}$.

Let $(\mathcal{X}, \mu, G)$ be a Boolean G-dynamical system. We say that $A \in \mathcal{X}$ is *positive* when $\mu(A) > 0$. Given a positive set $A \in \mathcal{X}$ and an open set $U \subset G$, recall that the set $UA \in \mathcal{X}$ was defined in [**7**] as

$$UA = \bigcup \{\gamma A : \gamma \in U'\}.$$

Here $U' \subset U$ is a countable dense subset of U. This defines UA uniquely as an element of the measure algebra $\mathcal{X}$ and the definition does not depend on the choice of U'. Note that for all $u \in U$ we have $uA \subset UA$.

Definition 3.1. For any positive set $A \in \mathcal{X}$ set

$$\tilde{A} = \bigcap_{m=1}^{\infty} U_m A.$$

We say that $A \in \mathcal{X}$ is *stable* if it is either empty or $\tilde{A} = A$.

PROPOSITION 3.2. *For any A, B positive subsets in $\mathcal{X}$:*

(1) $A \subset \tilde{A}$;
(2) $\tilde{\tilde{A}} = \tilde{A}$;
(3) *for every $g \in G$,* $\widetilde{gA} = g\tilde{A}$;
(4) $\widetilde{A \cup B} = \tilde{A} \cup \tilde{B}$;
(5) $\widetilde{A \cap B} \subset \tilde{A} \cap \tilde{B}$, *hence for stable $A, B \in \mathcal{X}$,* $\widetilde{A \cap B} = \tilde{A} \cap \tilde{B}$.

Proof. (1) Clear.

(2) For each k,

$$\begin{aligned}\tilde{\tilde{A}} &= \bigcap_{m=1}^{\infty} U_m \tilde{A} \\ &= \bigcap_{m=1}^{\infty} U_m \Big(\bigcap_{n=1}^{\infty} U_n A \Big) \\ &\subset U_{k+1}^2 A \subset U_k A,\end{aligned}$$

hence $\tilde{\tilde{A}} \subset \bigcap_{k=1}^{\infty} U_k A = \tilde{A}$. By 1, also $\tilde{A} \subset \tilde{\tilde{A}}$, hence $\tilde{\tilde{A}} = \tilde{A}$.

(3) Given $g \in G$ and $n \in \mathbb{N}$ there exists an $m \in \mathbb{N}$ with $g^{-1} U_m g \subset U_n$. Hence,

$$\widetilde{gA} = \bigcap_{m=1}^{\infty} U_m g A = \bigcap_{m=1}^{\infty} g(g^{-1} U_m g) A \subset g \bigcap_{n=1}^{\infty} U_n A = g\tilde{A}.$$

https://doi.org/10.1017/S0143385705000052 Published online by Cambridge University Press

Now

$$\tilde{A} = \widetilde{g^{-1}(gA)} \subset g^{-1}\widetilde{(gA)},$$

hence also $g\tilde{A} \subset \widetilde{gA}$.

Claims (4) and (5) are easy to verify. □

Example 3.3. Take $G = X = \mathbb{T}$, the unit circle, where $\mathbb{T}$ acts on itself by translations. Let A be a dense open subset of measure $\frac{1}{2}$. Then, for $B = X \setminus A$ we have $A \cap B = \emptyset$ but $\tilde{A} = X$ and $\tilde{B} = B$, hence $\tilde{A} \cap \tilde{B} = B$.

PROPOSITION 3.4. *If $\{A_k\}_{k=1}^{\infty}$ is a sequence of stable sets then so is $A = \bigcap_{k=1}^{\infty} A_k$.*

Proof. By Proposition 3.2 we can assume that $A_{k+1} \subset A_k$ for every k. Fix k_0. Then for each n, $A \subset U_n A \subset U_n A_{k_0}$, hence

$$\tilde{A} = \bigcap_{n=1}^{\infty} U_n A \subset \bigcap_{n=1}^{\infty} U_n A_{k_0} = \tilde{A}_{k_0} = A_{k_0}.$$

Thus

$$\tilde{A} \subset \bigcap_{k=1}^{\infty} A_k = A.$$ □

LEMMA 3.5. *Let A be a positive stable set and $\varepsilon > 0$, then there exists a stable set $D \subset A^c$ with $\mu(A^c \setminus D) < \varepsilon$ and a k such that $\mu(U_k A \cap U_k D) = 0$.*

Proof. Let $B = A^c$ and choose an m with $\mu(U_m A \setminus A) < \varepsilon$. Set $E = (U_m A)^c$, so that $E \subset B$. If $\mu(U_{m+1}E \cap U_{m+1}A) > 0$, then for some $\gamma, \delta \in U_{m+1}$ we have $\mu(\gamma E \cap \delta A) > 0$, hence $\mu(E \cap \gamma^{-1}\delta A) > 0$. However, $\gamma^{-1}\delta \in U_{m+1}^2 \subset U_m$ and, therefore, $\mu(E \cap U_m A) > 0$, contradicting the fact that $E \subset B = (U_m A)^c$. Thus, $\mu(U_{m+1}E \cap U_{m+1}A) = 0$.

We now set $D = \tilde{E}$ so that D is stable and $E \subset D \subset U_{m+2}E$. Now $U_{m+2}D \subset U_{m+2}^2 E \subset U_{m+1}E$, hence $\mu(U_k A \cap U_k D) = 0$ with $k = m + 2$. □

PROPOSITION 3.6. *If $A = \tilde{A} \in \mathcal{X}$ is stable then A^c is a union of an increasing sequence of stable sets.*

Proof. If $A = \emptyset$ the assertion is clear. Otherwise, given ε, we use Lemma 3.5 to find a stable subset $D_\varepsilon \subset A^c$ with $\mu(A^c \setminus D_\varepsilon) < \varepsilon$. Next use an exhaustion argument to show that A^c is a union of a sequence of stable sets. Finally we can arrange for this sequence to be an increasing one by means of Proposition 3.2. □

Definition 3.7. Let $\mathcal{A} \subset \mathcal{X}$ be the σ-algebra generated by the stable sets.

By Lemma 3.2(3), $\mathcal{A}$ is G-invariant.

THEOREM 3.8. *Every $A \in \mathcal{A}$ is a union of an increasing sequence of stable sets.*

Proof. Let $\mathcal{A}_0$ be the collection of sets $A \in \mathcal{X}$ which are a union of an increasing sequence of stable sets. We will show that $\mathcal{A}_0 = \mathcal{A}$. Since $\mathcal{A}_0$ contains the stable sets it is enough to show that it is closed under the operations of taking countable union and taking complements. The closure under countable union is evident. Suppose that $A = \bigcup_{k=1}^{\infty} A_k$

https://doi.org/10.1017/S0143385705000052 Published online by Cambridge University Press

is the union of the increasing sequence of stable sets A_k. Set $B = A^c$ and $B_k = A_k^c$ so that $B = \bigcap_{k=1}^{\infty} B_k$.

By Proposition 3.6 we know that each B_k is an increasing union of stable sets. Thus, if $\varepsilon > 0$ is given we can choose, for each k, a stable subset $D_k \subset B_k$ with $\mu(B_k \setminus D_k) < \varepsilon/2^{k+1}$. By Proposition 3.4 the set $D_\varepsilon = \bigcap_{k=1}^{\infty} D_k \subset B$ is stable and we have

$$\mu(B \setminus D_\varepsilon) \leq \sum_{k=1}^{\infty} \mu(B_k \setminus D_k) < \varepsilon.$$

By an exhaustion argument and an application of Proposition 3.2 we conclude that B is an increasing union of stable sets. □

The proof of the next theorem is a variation on the classical Urysohn lemma.

THEOREM 3.9. *If $(\mathcal{X}, \mu, G)$ is a Boolean G-system then $\mathcal{D} = \mathcal{A}$, i.e. the sub-$\sigma$-algebra generated by the G-continuous factors coincides with the sub-σ-algebra generated by the stable sets. In particular, $(\mathcal{X}, \mu, G)$ admits a spatial model if and only if $\mathcal{X} = \mathcal{A}$.*

Proof. If f is a G-continuous function and $A_t = \{x : f(x) \leq t\}$ then for every $n \in \mathbb{N}$ there exists an $m = m(n) \in \mathbb{N}$ such that $\|f \circ u - f\|_\infty < 1/n$ for every $u \in U'_m$ (recall that U'_m is a countable dense subset of U_m). For almost every $x \in A_t$ and every $u \in U'_m$ we then have

$$f(ux) < f(x) + 1/n \leq t + 1/n.$$

Thus, $U_m A_t \subset A_{t+1/n} = \{x : f(x) \leq t + 1/n\}$ and it follows that $\tilde{A}_t = \bigcap_{m=1}^{\infty} U_m A_t \subset \bigcap_{n=1}^{\infty} A_{t+1/n} = A_t$, so that $\tilde{A}_t = A_t$. This shows that $\mathcal{D} \subset \mathcal{A}$.

In proving the converse inclusion we can now restrict our attention to the Boolean factor $(\mathcal{A}, \mu, G)$. In other words we now assume, with no loss of generality, that $\mathcal{X} = \mathcal{A}$.

Next observe that, in view of Theorem 1.7, it suffices to prove the following claim. Given $A \in \mathcal{X}$ and $\varepsilon > 0$ there exists a G-continuous function $f \in L^\infty(\mu)$ such that $\|f - \mathbf{1}_A\|_2 < \varepsilon$.

Of course we can assume that both A and A^c are positive.

Applying Theorem 3.8 and Lemma 3.5 we choose stable subsets $D_0 \subset A$, $D_1 \subset A^c$ and an m_0 such that

$$\mu(A \setminus D_0) < \frac{\varepsilon}{2}, \quad \mu(A^c \setminus D_1) < \frac{\varepsilon}{2} \quad \text{and} \quad \mu(U_{m_0} D_0 \cap U_{m_0} D_1) = 0.$$

(We say that $U_{m_0} D_0$ and $U_{m_0} D_1$ are disjoint.) Define f to have the values 0 on D_0 and 1 on D_1.

Next define, by induction on the binary tree, a system of disjoint stable subsets $\{D_r : r = j/2^n,\ n \in \mathbb{N},\ 0 \leq j < 2^n\}$ whose union will be X, with the following provision. Note that if $UD = D$, for a positive set D and a symmetric neighborhood U of e in G, then also $UD^c = D^c$ and both sets are stable. In the following construction we assume, for convenience, that at each stage the various sets of the form $U_m D_r \setminus D_r$ are positive sets. If this is not the case then, by the above remark D as well as D^c are stable and the construction will terminate at that point. The corresponding interval of dyadic rationals will be missing from the range of f.

https://doi.org/10.1017/S0143385705000052 Published online by Cambridge University Press

Step 1. Let $E_0 = (U_{m_0}(D_0 \cup D_1))^c$ and observe that the sets $U_{m_0+1}E_0$ and $U_{m_0+1}(D_0 \cup D_1)$ are disjoint. Therefore, the stable set

$$D_{1/2} = \tilde{E}_0 = ((U_{m_0}(D_0 \cup D_1))^c)^\sim$$

is disjoint from $D_0 \cup D_1$. Note that

$$U_{m_0+2}D_{1/2} \subset U_{m_0+2}U_{m_0+2}E_0 \subset U_{m_0+1}E_0,$$

hence also the sets $U_{m_0+2}D_0$, $U_{m_0+2}D_{1/2}$ and $U_{m_0+2}D_1$ are pairwise disjoint. We now choose $m_1 > m_0 + 2$ such that

$$\sum\{\mu(U_{m_1}D_r \setminus D_r) : r = 0, \tfrac{1}{2}, 1\} < \tfrac{1}{2},$$
$$\text{and } U_{m_1}D_0,\ U_{m_1}D_{1/2},\ U_{m_1}D_1 \text{ are pairwise disjoint.}$$

We define f to have the value $\frac{1}{2}$ on the stable set $D_{1/2}$ and declare that f will have values in the interval $[0, \frac{1}{2}]$ on $X_0^1 = U_{m_0}D_0 \cup D_{1/2}$ and values in the interval $[\frac{1}{2}, 1]$ on $X_1^1 = U_{m_0}D_1 \cup D_{1/2}$, so that $X_0^1 \cup X_1^1 = X$.

Step 2. Next consider the two disjoint stable sets D_0 and $D_{1/2}$ as subsets of the set $X_0^1 = U_{m_0}D_0 \cup D_{1/2}$. And, at the same time, the two disjoint stable sets D_1 and $D_{1/2}$ as subsets of the set $X_1^1 = U_{m_0}D_1 \cup D_{1/2}$. Repeating the procedure described in step one we set $E_1 = (U_{m_1}(D_0 \cup D_{1/2} \cup D_1))^c$ and note that the sets $U_{m_1+1}E_1$ and $U_{m_1+1}(D_0 \cup D_{1/2} \cup D_1)$ are disjoint. Let

$$D_{1/4} = \tilde{E}_1 \cap X_0^1 = (U_{m_0}D_0 \setminus U_{m_1}(D_0 \cup D_{1/2}))^\sim$$

and

$$D_{3/4} = \tilde{E}_1 \cap X_1^1 = (U_{m_0}D_1 \setminus U_{m_1}(D_1 \cup D_{1/2}))^\sim.$$

Note that, e.g.,

$$U_{m_1+2}D_{1/4} \subset U_{m_1+2}U_{m_1+2}E_1 \subset U_{m_1+1}E_1,$$

hence also the sets $U_{m_1+2}D_r$, $r \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ are pairwise disjoint.

For a suitable $m_2 > m_1 + 2$ we will have

$$\sum\left\{\mu(U_{m_2}D_r \setminus D_r) : j = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\} < \frac{1}{2^2},$$
$$\text{and the sets } U_{m_2}D_r,\ r \in \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\} \text{ are pairwise disjoint.}$$

We define f to have the value r on the stable set D_r, $r \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ and declare that f will have values in the interval $[0, \frac{1}{4}]$ on $X_{00}^2 = U_{m_1}D_0 \cup D_{1/4}$, values in the interval $[\frac{1}{4}, \frac{1}{2}]$ on $X_{01}^2 = (X_0^1 \cap U_{m_1}D_{1/2}) \cup D_{1/4}$, etc. Note that we have

$$U_{m_2}X_0^1 \text{ is disjoint from } X_{11}^2 \text{ and } U_{m_2}X_1^1 \text{ is disjoint from } X_{00}^2.$$

Step $n+1$. Consider, for each $r = r(j)$ in the set $\{j2^{-(n-1)},\ 0 \le j < 2^{(n-1)}\}$ the two disjoint stable sets D_r and $D_{r+2^{-n}}$ as subsets of the set $X_{j0}^n = (X_j^{n-1} \cap U_{m_{n-1}}D_r) \cup D_{r+2^{-n}}$, and the two disjoint stable sets $D_{r+2^{-(n-1)}}$ and $D_{r+2^{-n}}$ as subsets of the set $X_{j1}^n = (X_j^{n-1} \cap U_{m_{n-1}}D_{r+2^{n-1}}) \cup D_{r+2^{-n}}$ (here $j0$ denotes the first n bits in the expansion of

https://doi.org/10.1017/S0143385705000052 Published online by Cambridge University Press

$j/2^{n-1}$ in base 2 and $j1$ the first n bits in the expansion of $j/2^{n-1}+1/2^n$). Set $E_n = \left(\bigcup\{U_{m_n}D_s : s = i2^{-n},\ 0 \le i < 2^n\}\right)^c$, then for $r = r(j) = j2^{-(n-1)},\ 0 \le j < 2^{(n-1)}$ let

$$D_{r+2^{-(n+1)}} = \tilde{E}_n \cap X^n_{j0} = ((X^n_{j0} \cap U_{m_{n-1}}D_r) \setminus U_{m_n}(D_r \cup D_{r+2^{-n}}))^{\sim},$$

and

$$D_{r+2^{-(n-1)}-2^{-(n+1)}} = \tilde{E}_n \cap X^n_{j1} = ((X^n_{j1} \cap U_{m_{n-1}}D_{r+2^{-(n-1)}}) \setminus U_{m_n}(D_{r+2^{-(n-1)}} \cup D_{r+2^{-n}}))^{\sim}.$$

For a suitable $m_{n+1} > m_n + 2$ we will have

$$\sum\{\mu(U_{m_{n+1}}D_s \setminus D_s) : s \in \{i2^{-(n+1)} :\ 0 \le i < 2^{(n+1)}\}\} < \frac{1}{2^{n+1}},$$

and the sets $U_{m_{n+1}}D_s,\ s \in \{i2^{-(n+1)} :\ 0 \le i < 2^{(n+1)}\}$ are pairwise disjoint.

We define f to have the value s on the stable set $D_s, s \in \{i2^{-(n+1)} :\ 0 \le i < 2^{-(n+1)}\}$ and declare that for $r = r(j) = j2^{-(n-1)},\ 0 \le j < 2^{(n-1)}$, f will have values in the interval $[r, r+2^{-(n+1)}]$ on

$$X^{n+1}_{j00} = (X^n_{j0} \cap U_{m_n}D_r) \cup D_{r+2^{-(n+1)}},$$

and values in the interval $[r+2^{-(n+1)}, r+2^{-n}]$ on

$$X^{n+1}_{j01} = (X^n_{j0} \cap U_{m_n}D_{r+2^{-n}}) \cup D_{r+2^{-(n+1)}}.$$

Similarly, on the sets

$$X^{n+1}_{j10} = (X_{j1} \cap U_{m_n}D_{r+2^{-n}}) \cup D_{r+2^{-n}+2^{-(n+1)}}$$

and

$$X^{n+1}_{j11} = (X_{j1} \cap U_{m_n}D_{r+2^{-(n-1)}}) \cup D_{r+2^{-n}+2^{-(n+1)}},$$

f will have values in the intervals $[r+2^{-n}, r+2^{-n}+2^{-(n+1)}]$ and $[r+2^{-n}+2^{-(n+1)}, r+2^{-(n-1)}]$, respectively.

The key point (in proving the G-continuity of f) is to observe that, e.g.,

$$U_{m_{n+1}}X^n_{j0} \subset X^{n+1}_{(j-1)11} \cup X^n_{j0} \cup X^{n+1}_{j10}.$$

Once this construction is completed we note that

$$\mu\left(\bigcup\left\{D_r : r = \frac{j}{2^n},\ n \in \mathbb{N},\ 0 \le j < 2^n\right\}\right) = 1,$$

so that the function

$$f(x) = \sum_r r1_{D_r}$$

is defined on X. It is now easy to conclude that $\|f - \mathbf{1}_A\|_2 < \varepsilon$ and that f is G-continuous. □

The next two results are direct corollaries of Theorem 3.9.

THEOREM 3.10. *A Boolean G-system $(\mathcal{X}, \mu, G)$ is whirly if and only if it admits no non-trivial spatial factors.*

https://doi.org/10.1017/S0143385705000052 Published online by Cambridge University Press

Proof. If there are no spatial factors then Theorem 1.7 implies that there are no non-trivial G-continuous functions, hence, by Theorem 3.9, no non-trivial stable sets. It follows that $\mu(UA) = 1$ for every positive set $A \in \mathcal{X}$ and a neighborhood of the identity U in G. By [**7**] this condition is equivalent to $(\mathcal{X}, \mu, G)$ being whirly.

Conversely, if $(\mathcal{X}, \mu, G)$ is whirly then every stable set is either null or the whole space and by Theorem 3.9, $\mathcal{D} = \mathcal{A}$ is also trivial. Thus, $(\mathcal{X}, \mu, G)$ admits no non-constant G-continuous functions and by Theorem 1.7 there are no non-trivial spatial factors. □

THEOREM 3.11. *Every ergodic Boolean action of a Polish Lévy group is whirly.*

Proof. By Theorem 1.1 of [**7**] every ergodic spatial action of a Polish Lévy group is trivial; i.e. a one point system. Evidently our theorem is now a consequence of this fact together with Theorem 3.10. □

4. *Some structure theory*

Let G be a Polish group and $(Y, \mathcal{Y}, \nu, G)$ a spatial G-system. Denoting by $M(Y)$ the space of Borel measures on the Lebesgue space Y, we observe that the spatial action of G on Y induces an action of G on $M(Y)$. By [**1**] we can choose (Y, G) to be a compact model where the action $G \times Y \to Y$ is jointly continuous, so that the induced action $(M(Y), G)$ is also jointly continuous. This observation implies that every *quasifactor* (hence, in particular, every factor) of the system $(Y, \mathcal{Y}, \nu, G)$ has a spatial model (see [**6**]). Theorem 8.4 of [**6**] asserts that two ergodic systems $(X, \mathcal{X}, \mu, G)$ and $(Y, \mathcal{Y}, \nu, G)$ are *not disjoint* if and only if $(Y, \mathcal{Y}, \nu, G)$ admits a non-trivial quasifactor which is a factor of $(X, \mathcal{X}, \mu, G)$. Thus, part (2) of the following theorem is a direct consequence of Theorem 3.10 above. For completeness, and for those who are not familiar with the theory of quasifactors, we provide an alternative direct proof.

THEOREM 4.1. *Let G be a Polish group.*

(1) *Every factor of a spatial G-system is a spatial system.*

(2) *A G-system is whirly if and only if it is disjoint from every spatial G-system.*

Proof. (1) Let $(Z, \mathcal{Z}, \eta, G)$ be a Boolean factor of the spatial system $(Y, \mathcal{Y}, \nu, G)$. We consider $\mathcal{Z}$ as a G-invariant sub-σ-algebra of $\mathcal{Y}$ (so that $\eta = \nu \restriction \mathcal{Z}$) and let $E^{\mathcal{Z}} : L^2(\mathcal{Y}, \nu) \to L^2(\mathcal{Z}, \nu)$ denote the corresponding conditional expectation, i.e. the orthogonal projection of the Hilbert space $L^2(\mathcal{Y}, \nu)$ onto the closed subspace $L^2(\mathcal{Z}, \nu)$. Since $E^{\mathcal{Z}}$ is a contraction, both on L^2 and on L^∞, it follows easily that if $f \in L^\infty(\mathcal{Y}, \nu)$ is a G-continuous function then so is $E^{\mathcal{Z}} f$. An application of Theorem 1.7 yields an L^2-dense subset of $L^2(\mathcal{Y}, \nu)$ consisting of G-continuous functions. Its image under $E^{\mathcal{Z}}$ will provide an L^2-dense subset of $L^2(\mathcal{Z}, \nu)$. An application of the other direction of Theorem 1.7 finishes the proof.

(2) If a G-system is disjoint from every spatial system then, in particular, it admits no non-trivial spatial factors and by Theorem 3.10 it is whirly.

Conversely, assume that $(X, \mathcal{X}, \mu, G)$ is a whirly near-action and that $(Y, \mathcal{Y}, \nu, G)$ is spatial. By [**1**] we can assume that Y is a compact space and that the action $G \times Y \to Y$ is jointly continuous. Let λ be a joining of the two systems. That is, λ is a probability

https://doi.org/10.1017/S0143385705000052 Published online by Cambridge University Press

measure on $X \times Y$, with projections μ and ν on X and Y, respectively. Let

$$\lambda = \int_X (\delta_x \times \lambda_x)\, d\mu(x),$$

be the disintegration of λ over μ. Let $\Gamma \subset G$ be a countable dense subgroup. Then, by the G-invariance of λ, for μ-almost every x we have $\gamma\lambda_x = \lambda_{\gamma x}$ for all $\gamma \in \Gamma$. If $\lambda \neq \mu \times \nu$ then the map $x \mapsto \lambda_x$ is not a constant μ-almost everywhere and there exists a continuous function F on Y such that

$$f(x) = \int_Y F(y)\, d\lambda_x(y),$$

is not a constant μ-almost everywhere. Choose a value $c \in \mathbb{R}$ and a $\delta > 0$ such that

$$0 < \mu\{x : f(x) > c\} = a < 1 \quad \text{and} \quad 0 < \mu\{x : f(x) > c + \delta\} < a.$$

Let U be a neighborhood of e in G such that $\sup_{y \in Y} |F(\gamma y) - F(y)| < \delta/10$ for all $\gamma \in U$. Then for $\gamma \in U$

$$f(\gamma x) = \int_Y F(y)\, d\lambda_{\gamma x}(y) = \int_Y F(\gamma y)\, d\lambda_x(y),$$

for μ-almost every x. Thus,

$$|f(\gamma x) - f(x)| \leq \int_Y |F(\gamma y) - f(y)|\, d\lambda_x(y) \leq \frac{\delta}{10},$$

for μ-almost every x. Denote

$$A = \{x : f(x) > c + \delta\}, \quad B = \{x : f(x) > c\},$$

then we have for all $\gamma \in U$, $\gamma A \subset B$, i.e. $UA \subset B$. This implies that $A \subset \tilde{A} \subset B$, so that $\tilde{A}$ is a non-trivial stable set in $\mathcal{X}$. By Theorem 3.9 this contradicts our assumption that $(X, \mathcal{X}, \mu, G)$ is whirly. Thus, $\lambda = \mu \times \nu$ and we have shown that $(X, \mathcal{X}, \mu, G)$ and $(Y, \mathcal{Y}, \mu, G)$ are disjoint. □

Theorem 4.1 suggests an analogy—whirly is analogous to weak mixing while spatial is analogous to Kronecker (a system $(X, \mathcal{X}, \mu, G)$ is a *Kronecker system* if the finite-dimensional G-subrepresentations are dense in $L^2(\mu)$). The following corollary enhances this analogy.

COROLLARY 4.2. *Every whirly system is weakly mixing.*

Proof. Since every Kronecker action embeds in an action of a compact group we deduce that every Kronecker system admits a spatial model. Since weak mixing is characterized as the property of having no non-trivial Kronecker factor, the corollary follows from Theorem 3.10. □

5. *The generic automorphism of a Lebesgue space is whirly*

Let $(X, \mathcal{X}, \mu)$ be a standard Lebesgue space and denote by $G = \mathrm{Aut}(X)$ the Polish group of its automorphisms, i.e. the equivalence classes of invertible measure preserving transformations $X \to X$, equipped with the topology of convergence in measure.

https://doi.org/10.1017/S0143385705000052 Published online by Cambridge University Press

For elements T of G we have the following well-known dichotomy. Either the subgroup $\{T^n : n \in \mathbb{Z}\}$ is a discrete subset of G or $\Lambda(T) = \mathrm{cls}\{T^n : n \in \mathbb{Z}\}$ is a non-discrete monothetic Polish subgroup of G. In the latter case we say that T is a *rigid* transformation. Thus, T is rigid if and only if there exists a sequence $n_k \nearrow \infty$ with $\lim_{k\to\infty} T^{n_k} = \mathrm{Id}$. A rigid T is called whirly if the $\Lambda(T)$-action on $(X, \mathcal{X}, \mu)$ is whirly. This property of T can be characterized directly, with no reference to $\Lambda(T)$, as follows.

Definition 5.1. An automorphism $T \in G$ is called *whirly* if for every neighborhood U of the identity in G and any two sets $A, B \in \mathcal{X}$ of positive measure there is $n \in \mathbb{Z}$ such that $T^n \in U$ and $\mu(T^n A \cap B) > 0$.

The following theorem shows that whirly automorphisms are not at all rare.

THEOREM 5.2. *The collection of whirly transformations is a dense G_δ subset of G.*

Proof. For convenience we let $X = [0, 1]$ and μ be normalized Lebesgue measure on X. Choose a sequence of pairs $\{(A_k, B_k)\}_{k=1}^{\infty}$ which is dense in $\mathcal{X} \times \mathcal{X}$, the two-fold product of the measure algebra. For each $m \in \mathbb{N}$ and $0 \le j < 2^m$ let

$$J_j^{(m)} = \left(\frac{j}{2^m}, \frac{j+1}{2^m}\right)$$

and let

$$U_m = \{T \in G : \mu(T J_j^{(m)} \triangle J_j^{(m)}) < 2^{-2m},\ 0 \le j < 2^m\}.$$

For $k, m \in \mathbb{N}$ we now define

$$V_{k,m} = \{T \in G : \exists n \in \mathbb{Z},\ T^n \in U_m \quad \text{and} \quad \mu(T^n A_k \cap B_k) > \delta_m \mu(A_k)\mu(B_k)\}.$$

The sequence of positive constants δ_m will be determined later. Note, however, that δ_m is independent of k. Clearly each $V_{k,m}$ is an open subset of G and we set

$$R = \bigcap_{(k,m)\in\mathbb{N}\times\mathbb{N}} V_{k,m}.$$

CLAIM 1. *Each $S \in R$ is whirly.*

Proof. Given A, B positive sets in $\mathcal{X}$ and $m_0 \in \mathbb{N}$ we have to find an $n \in \mathbb{N}$ with $S^n \in U_{m_0}$ and $\mu(S^n A \cap B) > 0$. For an $\varepsilon > 0$, to be determined soon, we find a pair (A_{k_0}, B_{k_0}) with

$$\mu(A \triangle A_{k_0}) < \varepsilon, \quad \mu(B \triangle B_{k_0}) < \varepsilon.$$

By assumption $S \in V_{k_0,m_0}$, hence there exists n such that $S^n \in U_{m_0}$ and

$$\mu(S^n A_{k_0} \cap B_{k_0}) > \delta_{m_0} \mu(A_{k_0})\mu(B_{k_0}).$$

We now observe that for sufficiently small ε this will imply

$$\mu(S^n A \cap B) > \delta_{m_0} \mu(A)\mu(B). \qquad \square$$

By Baire's category theorem our proof will be complete once we show that each $V_{k,m}$ is dense in G.

https://doi.org/10.1017/S0143385705000052 Published online by Cambridge University Press

CLAIM 2. *For every $m \in \mathbb{N}$ there exists a number $\delta_m > 0$ such that for any positive sets $A, B \in \mathcal{X}$ the open subset*

$$V = \{T \in G : \exists n \in \mathbb{Z},\ T^n \in U_m \text{ and } \mu(T^n A \cap B) > \delta_m \mu(A)\mu(B)\}$$

is dense in G.

Proof. Since the set of weakly mixing automorphisms is dense in G it suffices to show that, given a weakly mixing $T_0 \in G$ and $\varepsilon > 0$, there exists an element $S \in V$ with $\mu\{x \in X : T_0 x \neq Sx\} < \varepsilon$.

By weak mixing of T_0 there exists n_0 such that $1/n_0 \ll \varepsilon$ and $\mu(T_0^{n_0} A \cap B) > \frac{1}{2}\mu(A)\mu(B)$. Denoting $D = T_0^{n_0} A \cap B$ we next apply the ergodic theorem to choose n sufficiently large so that the frequencies of appearances of D, as well as of each of the intervals $J_j^{(m)}$, $j = 0, \ldots, 2^m - 1$, in almost every sequence $\{x, T_0 x, \ldots, T_0^{n-1} x\}$, is a very good approximation of their respective measures.

Next construct a *Rohlin tower* with base H, height $N = nn_0$ and a remainder of measure less than ε. Thus, the sets $\{H, T_0 H, T_0^2 H, \ldots, T_0^{N-1} H\}$ are pairwise disjoint and the set $E = X \setminus \bigcup_{j=1}^{N-1} T_0^j B$ has measure $< \varepsilon$.

We split H into two parts H_γ and $H_{1-\gamma}$ of measures $\gamma\mu(H)$ and $(1-\gamma)\mu(H)$, respectively. The size of $0 < \gamma < 1$ will be determined later.

On each of the floors of the γ-part of the tower, except for the ceiling $T_0^{N-1} H_\gamma$, we set $S = T_0$. On the ceiling we set $S = T_0^{-N+1}$, so that S becomes periodic of period N on the γ-part of the tower.

On each of the n n_0-block of floors of the $(1-\gamma)$-part of the tower, we define S similarly to be periodic on that block with period n_0, so that again $S = T_0$ on all floors of the tower except for the n ceilings of the n_0-blocks. Finally, on the remainder E, S is defined as the identity. Clearly $\mu\{x \in X : T_0 x \neq Sx\} < \varepsilon$ and we will next show that S is in V.

We first observe that if $\gamma \leq 2^{-2m}$ then $S^{n_0} \in U_m$. Next note that, by the choice of n, on the γ-part of the tower, say τ_γ, we have

$$\mu(S^{n_0} A \cap B \cap \tau_\gamma) \approx \gamma\mu(D) > \frac{\gamma}{2}\mu(A)\mu(B),$$

hence

$$\mu(S^{n_0} A \cap B) > \frac{\gamma}{10}\mu(A)\mu(B).$$

We are finally in a position to determine the required size of the constants δ_m. If we set

$$\delta_m = \frac{1}{10} 2^{-2m},$$

then, choosing

$$\gamma = 2^{-2m},$$

we get

$$\mu(S^{n_0} A \cap B) > \frac{\gamma}{10}\mu(A)\mu(B) = \frac{2^{-2m}}{10}\mu(A)\mu(B) = \delta_m \mu(A)\mu(B),$$

so that S is indeed in V. □

https://doi.org/10.1017/S0143385705000052 Published online by Cambridge University Press

6. *Further examples*

As was shown in [**7**] a Lévy group admits no non-trivial spatial actions. On the other hand, every action of a locally compact group is spatial. In this section, we show that: (I) there exists a Polish monothetic group which is not Lévy but admits a whirly action; (II) the abelian Polish group which underlies an infinite-dimensional separable Hilbert space H admits both spatial and whirly faithful actions.

(I) In the following discussion we use some basic facts from the theory of *IP*-sequences and *IP*-convergence. We refer to Furstenberg's book [**3**] for the necessary background.

Definition 6.1. Let $T \in \mathrm{Aut}(X)$; we say that the action of T on X is *whirly of all orders* if for each $n \geq 1$ the action of $T \times T \times \cdots \times T$ (n-times) on X^n is whirly.

LEMMA 6.2. *If for $T \in \mathrm{Aut}(X)$ the Polish group $\Lambda(T)$ is Lévy then the action of T on X is whirly of all orders.*

Proof. By Corollary 4.2 the system (X, T) is weakly mixing and, therefore, for each $n \geq 1$ the system $(X^n, T \times T \times \cdots \times T)$ is ergodic. By Theorem 3.11 the latter system is also whirly. □

LEMMA 6.3. *Let $T \in \mathrm{Aut}(X)$ be such that the T-action on X is whirly of all orders. Let U be a neighborhood of the identity in $\Lambda(T)$. Given positive sets $A, B \in \mathcal{X}$ there exists an IP-sequence $\{p_\nu\}$ such that for every ν, $T^{p_\nu} \in U$ and $\mu(T^{p_\nu} A \cap B) > 0$.*

Proof. We denote by $\mathcal{N}_e$ the filter of neighborhoods of the identity in $\Lambda(T)$. Let $U_1 = U$ and choose p_1 such that $T^{p_1} \in U_1$ and $\mu(T^{p_1} A \cap B) > 0$. Let $D = T^{p_1} A \cap B$ and find $U_2 \in \mathcal{N}_e$ such that $U_2 \subset U_1$ and $T^{p_1} U_2 \subset U_1$. Since the action of T is whirly of all orders we can choose p_2 such that $T^{p_2} \in U_2$ and $\mu(T^{p_2} D \cap B) > 0$, $\mu(T^{p_2} A \cap B) > 0$. We now have, for $m = p_1$, p_2 and $p_1 + p_2$,

$$T^m \in U \quad \text{and} \quad \mu(T^m A \cap B) > 0. \tag{6.1}$$

Assume $p_1, p_2, \ldots, p_n$ have been found so that (6.1) is satisfied for all $m = p_{i_1} + p_{i_2} + \cdots + p_{i_k}$ with $i_1 < i_2 < \cdots < i_k \leq n$. For each such m set $D_m = T^m A \cap B$ and choose $U_{n+1} \in \mathcal{N}_e$ such that $U_{n+1} \subset U_n$ and $T^m U_{n+1} \subset U$ for every such m. Since the action of T is whirly of all orders we can choose p_{n+1} with $T^{p_{n+1}} \in U_{n+1}$ and with the property that

$$\mu(T^{p_{n+1}} D_m \cap B) > 0 \quad \text{for all } m = p_{i_1} + p_{i_2} + \cdots + p_{i_k}.$$

It now follows that (6.1) is satisfied for all $m = p_{i_1} + p_{i_2} + \cdots + p_{i_k}$ with $i_1 < i_2 < \cdots < i_k \leq n + 1$. This concludes the inductive step of the construction of the required *IP*-sequence. □

PROPOSITION 6.4. *Let $T \in \mathrm{Aut}(X)$ be such that the T-action on X is whirly of all orders and let α be an irrational number. Denote by R_α the rotation by α on $\mathbb{T}$: $R_\alpha y = y + \alpha$, $y \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$. Consider the system $(X \times Y, \mu \times m, T \times R_\alpha)$, where m is Lebesgue measure on $\mathbb{T}$. Then we have the following.*

(1) *This dynamical system is rigid and we denote by $L = \Lambda(T \times R_\alpha) \subset \mathrm{Aut}(X \times \mathbb{T})$ the corresponding perfect Polish monothetic topological group.*

(2) *The action of L on X via the projection on the first coordinate is whirly.*

https://doi.org/10.1017/S0143385705000052 Published online by Cambridge University Press

(3) *L admits a continuous character and a non-trivial spatial action. In particular, L is not Lévy.*

(4) *Let $K = \{\beta \in \mathbb{T} : (\mathrm{Id}, \beta) \in L\}$ then K is a closed subgroup of $\mathbb{T}$ and $K = \{0\}$ if and only if the natural projection $L \to \Lambda(T)$ is a topological isomorphism.*

Proof. (1) Since T is a rigid transformation there exists a sequence $n_i \nearrow \infty$ such that $\lim T^{n_i} = \mathrm{Id}$ in $\Lambda(T)$. It then follows that for some *IP* sequence n_ν we have *IP*-$\lim T^{n_\nu} = \mathrm{Id}$ (see [**3**]). This fact, in turn, implies that for some sub-*IP*-sequence $n_{\nu'}$ we have *IP*-$\lim n_{\nu'}\alpha = 0$ in $\mathbb{T}$ and, therefore, also

$$IP\text{-}\lim(T^{n_{\nu'}}, n_{\nu'}\alpha) = (\mathrm{Id}, 0)$$

in L. Thus, $T \times R_\alpha$ is a rigid transformation and L is a perfect Polish monothetic group.

(2) Note first that an arbitrary neighborhood of the identity in L contains a neighborhood of the form $V = L \cap (U \times J_\varepsilon)$, where U is a neighborhood of the identity in $\Lambda(T)$ and $J_\varepsilon = \{\beta \in \mathbb{T} : |\beta| < \varepsilon\}$ ($|\cdot|$ denotes the distance to the closest integer). Thus, in order to show that the action of L on X is whirly it suffices to show that for every pair of positive sets $A, B \in \mathcal{X}$ and every V as above there exists $g = (S, \beta) \in V$ with $\mu(SA \cap B) > 0$. An application of Lemma 6.1 yields an *IP*-sequence $\{p_\nu\}$ such that for every ν, $T^{p_\nu} \in U$ and $\mu(T^{p_\nu} A \cap B) > 0$. Since $\mathbb{T}$ is a compact topological group if follows that for some sub-*IP*-sequence $\{p_{\nu'}\}$, *IP*-$\lim n_{\nu'}\alpha = 0$, hence eventually

$$(T^{n_{\nu'}}, n_{\nu'}\alpha) \in V = L \cap (U \times J_\varepsilon).$$

This completes the proof of part (2).

(3) Clearly the projection $\chi : L \to \mathbb{T}$ to the second coordinate, $\pi(S, \beta) = \beta$, for $(S, \beta) \in L \subset \Lambda(T) \times \mathbb{T}$ is a continuous homomorphism, i.e. a continuous character. Via this character L acts spatially on $\mathbb{T}$. The image of χ contains the dense subgroup $\{n\alpha : n \in \mathbb{Z}\}$ and, in particular, χ is non-trivial. Since by Theorem 1.1 of [**7**] a Lévy group does not admit non-trivial spatial actions we conclude that L is not Lévy.

The proof of (4) is straightforward. □

Remark 6.5. It follows from Proposition 6.4(4) that if in addition to the conditions of that proposition T also satisfies the condition $K = \{0\}$, then the action of $\Lambda(T)$ on X is whirly yet $\Lambda(T)$ is not a Lévy group. Such a T will provide a negative answer to Problem 2 below (see also [**5**, Theorem 3.3]).

(II) We first define a spatial action of H represented as $H = \ell_2(\mathbb{N})$. As usual let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the 1-torus and let λ be normalized Lebesgue measure on $\mathbb{T}$. For $x \in \ell_2(\mathbb{N})$ with coordinates $x = (x_1, x_2, \dots)$ define a translation

$$A_1(x) : \mathbb{T}^{\mathbb{N}} \times \mathbb{T}^{\mathbb{N}} \to \mathbb{T}^{\mathbb{N}} \times \mathbb{T}^{\mathbb{N}}, \quad A_1(x)(t, s) = (t + x, s + \sqrt{2}x) \pmod 1.$$

Taking $Y = \mathbb{T}^{\mathbb{N}} \times \mathbb{T}^{\mathbb{N}}$ and $\mu = \lambda^{\mathbb{N}} \times \lambda^{\mathbb{N}}$, the map $x \mapsto A_1(x)$ clearly defines a spatial measure-preserving system $(Y, \mathcal{Y}, \mu, H)$. Moreover this action is a topological system on a compact space and the action is equicontinuous. In fact, the image of H under $A_1 : H \to \mathrm{Homeo}\,(Y)$ is dense in the compact subgroup of $\mathrm{Homeo}\,(Y)$ (isomorphic to $\mathbb{T}^{\mathbb{N}} \times \mathbb{T}^{\mathbb{N}}$) of translations of Y. Of course, the second component is there to make the action faithful.

https://doi.org/10.1017/S0143385705000052 Published online by Cambridge University Press

We next describe a faithful whirly near-action of H. This time we choose to view H as the Hilbert space $L^2([0,1],\lambda)$. Start from the Polish monothetic Lévy group $G = L_0([0,1], S^1)$ of all (equivalence classes of) measurable functions $[0,1] \to S^1$, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and the topology on G is given by L^2-convergence (see [**5**]). The map $\theta : H \to G$, $\theta(f) = e^{if}$ defines a continuous homomorphism of H onto G. Consider G as a subgroup of the unitary group $U(H)$ (the elements of G acting as multiplication operators) and let $(X, \mathcal{X}, \mu, G)$ be the associated Gaussian near-action. Define for $f \in L^2(\lambda)$,

$$A_2(f) : X \times X \to X \times X, \quad A_2(f)(x, x') = (\theta(f)x, \theta(\sqrt{2}f)x').$$

This defines a faithful near-action $(X \times X, \mu \times \mu, H)$.

Let $L = \mathrm{cls}\,(A_2(H))$, where the closure is taken in the Polish group $\mathrm{Aut}\,(X \times X)$. Of course, $L \subset G \times G \subset \mathrm{Aut}\,(X \times X)$, and we claim that actually $L = G \times G$. In fact, if for each n we denote by H_n the finite-dimensional subspace of H that consists of all the square integrable functions $f : [0,1] \to \mathbb{R}$ which are measurable with respect to the partition

$$\mathcal{J}_n = \left\{ \left[\frac{j}{2^n}, \frac{j+1}{2^n} \right] : 0 \le j < 2^n \right\},$$

and let $G_n = \theta(H_n)$, then it is easily checked that $\mathrm{cls}\, A_2(H_n) = G_n \times G_n$. Since the union of the increasing sequence of compact groups $G_n \times G_n$ is dense in $G \times G$ we conclude that indeed $L = G \times G$.

Next observe that the Polish groups G and $G \times G$ are clearly isomorphic (for example via the map which sends $g \in G$ to the pair (g_1, g_2) of its restrictions to $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, respectively). In particular, we deduce that $G \times G$ is a Lévy group. By Theorem 3.11 its action on $X \times X$ is whirly.

Now observe that whenever Λ is a dense subgroup of a Polish group Γ and $(X, \mathcal{X}, \mu, \Gamma)$ is a whirly action, then so is the restricted action $(X, \mathcal{X}, \mu, \Lambda)$. Applying this observation to $A_2(H) \subset \mathrm{cls}\,(A_2(H)) = G \times G$ we finally conclude that the faithful action $(X \times X, \mu \times \mu, H)$ via A_2 is whirly.

Problems

1. For a Polish group G, is the product of two whirly G-actions whirly?
2. Suppose that the natural action of a closed subgroup $G \subset \mathrm{Aut}\,(X)$ on $(X, \mathcal{X}, \mu)$ is whirly; is G necessarily a Lévy group? In particular, if for $T \in \mathrm{Aut}\,(X)$ the action of $\Lambda(T)$ on X is whirly, must $\Lambda(T)$ be a Lévy group?
3. Is the generic Polish group $\Lambda(T) = \mathrm{cls}\,\{T^n : n \in \mathbb{Z}\} \subset \mathrm{Aut}\,(X)$ Lévy?
4. For $G = L_0([0,1], S^1)$, observe that the map $A_2 : H \to G \times G$, which is a one-to-one continuous group homomorphism, is not a homeomorphism. Is there a topologically faithful whirly action of H?

References

[1] H. Becker and A. S. Kechris. *The Descriptive Set Theory of Polish Groups (London Mathematical Society Lecture Notes, 232)*. Cambridge University Press, Cambridge, 1996.

https://doi.org/10.1017/S0143385705000052 Published online by Cambridge University Press

[2] A. I. Danilenko. Point realization of Boolean actions of countable inductive limits of locally compact groups. *Mat. Fiz. Anal. Geom.*, **7** (2000), 35–48.
[3] H. Furstenberg. *Recurrence in Ergodic Theory and Combinatorial Number Theory*. Princeton University Press, Princeton, NJ, 1981.
[4] E. Glasner. *Proximal Flows (Lecture Notes in Mathematics, 517)*. Springer, Berlin, 1976.
[5] E. Glasner. On minimal actions of Polish groups. *Topology Appli.* **85** (1998), 119–125.
[6] E. Glasner. *Ergodic Theory via Joinings (Surveys Monogr., 101)*. American Mathematical Society, Providence, RI, 2003.
[7] E. Glasner, B. Tsirelson and B. Weiss. The automorphism group of the Gaussian measure cannot act pointwise. *Israel J. Math.*, to appear (math DS/0311450).
[8] M. Gromov and V. D. Milman. A topological application of the isoperimetric inequality. *Amer. J. Math.* **105** (1983), 843–854.
[9] M. Ledoux. *The Concentration of Measure Phenomenon*. American Mathematical Society, Providence, RI, 2001.
[10] G. W. Mackey. Point realizations of transformation groups. *Illinois J. Math.* **6** (1962), 327–335.
[11] V. D. Milman. The heritage of P. Lévy in geometrical functional analysis. *Colloque Paul Lévy sur les Processus Stochastiques (Palaiseau, 1987). Astérisque* **157–158** (1988), 273–301.
[12] V. D. Milman and G. Schechtman. *Asymptotic Theory of Finite-dimensional Normed Spaces with an Appendix by M. Gromov (Lecture Notes in Mathematics, 1200)*. Springer, Berlin, 1986.
[13] V. Pestov. Topological groups: where to from here? *Topology Proc.* **24** (1999), 421–502.
[14] A. Ramsay. Virtual groups and group actions. *Adv. Math.* **6** (1971), 253–322.
[15] C. Ryll-Nardzewski. On fixed points of semi-groups of endomorphisms. *Proc. 5th Berkeley Symp. on Mathematical Statistics and Probability*. Vol. II. Berkeley University Press, Berkeley, CA, 1966.
[16] V. S. Varadarajan. Groups of automorphisms of Borel spaces. *Trans. Amer. Math. Soc.* **109** (1963), 191–220.
[17] V. S. Varadarajan. *Geometry of Quantum Theory*. Vol. II. Van Nostrand, Princeton, NJ, 1970.
[18] R. J. Zimmer. Extensions of ergodic group actions. *Illinois J. Math.* **20** (1976), 373–409.

https://doi.org/10.1017/S0143385705000052 Published online by Cambridge University Press