A TRIGONOMETRICAL INEQUALITY

By E. H. NEVILLE

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ABSTRACT. The inequality is

$$\operatorname{cosec}^2 x - \frac{1}{2m+1} < \sum_{-m}^{m} \frac{1}{(x-r\pi)^2} < \operatorname{cosec}^2 x,$$

which is established by an elementary argument and is shown to lead directly to the evaluation of the integral

$$\int_0^\infty \frac{\sin x}{x}\,dx,$$

and to the expression of $\sin x$ as an infinite product.

1. Since the roots of the equation

$$S\left\{\binom{2m+1}{1}(1-S)^m - \binom{2m+1}{3}S(1-S)^{m-1} + \ldots\right\}^2 = \sin^2(2m+1)\theta$$

are $\sin^2\{\theta - r\pi/(2m+1)\}$, for $-m \leqslant r \leqslant m$, we have

$$(2m+1)^2 \operatorname{cosec}^2(2m+1)\theta = \sum_{-m}^{m} \operatorname{cosec}^2\left\{\theta - \frac{r\pi}{2m+1}\right\},$$

that is,

$$(2m+1)^2 \operatorname{cosec}^2 x = \sum_{-m}^{m} \operatorname{cosec}^2 \frac{x-r\pi}{2m+1},$$

whence also

$$(2m+1)^2 \operatorname{cosec}^2 x - (2m+1) = \sum_{-m}^{m} \cot^2 \frac{x-r\pi}{2m+1}.$$

If $0 < |y| \leqslant \frac{1}{2}\pi$, $\cot^2 y < 1/y^2 < \operatorname{cosec}^2 y$, and if $0 < |x| \leqslant \frac{1}{2}\pi$ and $-m \leqslant r \leqslant m$, then also $0 < |(x-r\pi)/(2m+1)| \leqslant \frac{1}{2}\pi$. Hence, for any whole number m, if $0 < |x| \leqslant \frac{1}{2}\pi$, we have

$$\operatorname{cosec}^2 x - \frac{1}{2m+1} < \sum_{-m}^{m} \frac{1}{(x-r\pi)^2} < \operatorname{cosec}^2 x. \tag{1·1}$$

2. Since the convergence of $\Sigma 1/(x-r\pi)^2$ is absolute, the two ends can be approached independently, and therefore the sum of the series has the period π, and the restriction on x can be lifted:

For all real values of x, except integral multiples of π,

$$\operatorname{cosec}^2 x = \sum_{-\infty}^{\infty} \frac{1}{(x-r\pi)^2}.$$

3. Multiplying by $\sin^2 x$ and integrating from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$, we have, from (1·1),

$$\pi - \frac{\pi}{4m+2} < 2\int_0^{(m+\frac{1}{2})\pi} \frac{\sin^2 x}{x^2}\,dx < \pi.$$

If $(m+\frac{1}{2})\pi \leqslant X < (m'+\frac{1}{2})\pi$, where the integers m, m' need not be consecutive, then because the integrand is positive, the integral to X is greater than the integrand to $(m+\frac{1}{2})\pi$ and less than the integral to $(m'+\frac{1}{2})\pi$. Thus

$$\frac{\pi}{2} - \frac{\pi}{8m+4} < \int_0^X \frac{\sin^2 x}{x^2}\,dx < \frac{\pi}{2},$$

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implying $$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2},$$

however the upper limit is approached. If $X > \pi$, we may suppose $(m+\frac{1}{2})\pi > X - \pi$, and we have

$$\frac{\pi}{2} - \frac{\pi}{8(X-\pi)} < \int_0^X \frac{\sin^2 x}{x^2}\,dx < \frac{\pi}{2}.$$

4. Since $$\frac{d}{dx}\left(\frac{\sin^2 x}{x}\right) = \frac{\sin 2x}{x} - \frac{\sin^2 x}{x^2},$$

we have $$\int_0^X \frac{\sin^2 x}{x^2}\,dx = \int_0^{2X} \frac{\sin x}{x}\,dx - \frac{\sin^2 X}{X},$$

and since $(\sin^2 X)/X$ is between 0 and $1/X$, we have, for $X > 2\pi$,

$$\frac{\pi}{2} - \frac{\pi}{4(X-2\pi)} < \int_0^X \frac{\sin x}{x}\,dx < \frac{\pi}{2} + \frac{2}{X}, \tag{4·1}$$

whence $$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

5. From (1·1), $$\operatorname{cosec}^2 x - \frac{1}{x^2} - \frac{1}{2m+1} < \sum_{-m}^{m}{}' \frac{1}{(x-r\pi)^2} < \operatorname{cosec}^2 x - \frac{1}{x^2},$$

the dash indicating that the term for which $r = 0$ is excluded from the sum. Since $\cot x - 1/x \to 0$ as $x \to 0$, integration from 0 gives us, for $0 < x \leqslant \frac{1}{2}\pi$,

$$\cot x - \frac{1}{x} < \sum_{-m}^{m}{}' \left\{\frac{1}{x-r\pi} + \frac{1}{r\pi}\right\} < \cot x - \frac{1}{x} + \frac{x}{2m+1},$$

and by a second integration

$$\log\frac{\sin x}{x} < \sum_{-m}^{m}{}' \left\{\log\frac{r\pi - x}{r\pi} + \frac{x}{r\pi}\right\} < \log\frac{\sin x}{x} + \frac{x^2}{4m+2},$$

implying $$\frac{\sin x}{x} < \prod_{-m}^{m}{}' \left\{\left(1 - \frac{x}{r\pi}\right)e^{x/r\pi}\right\} < \frac{\sin x}{x}\, e^{x^2/(4m+2)}.$$

Hence
$$\left.\begin{aligned} \cot x &= \frac{1}{x} + \sum_{-\infty}^{\infty}{}' \left\{\frac{1}{x-r\pi} + \frac{1}{r\pi}\right\} = \frac{1}{x} + \sum_{1}^{\infty} \frac{2x}{x^2 - r^2\pi^2}, \\ \sin x &= x \prod_{-\infty}^{\infty}{}' \left\{\left(1 - \frac{x}{r\pi}\right)e^{x/r\pi}\right\} = x \prod_{1}^{\infty} \left(1 - \frac{x^2}{r^2\pi^2}\right). \end{aligned}\right\} \tag{5·1}$$

As before, the restriction on x has become superfluous.

6. In the form $$\cot x = \frac{1}{x} + \sum_{1}^{\infty} \left\{\frac{1}{x-r\pi} + \frac{1}{x+r\pi}\right\},$$

the expansion for the cotangent is related to the elementary identity

$$(2m+1)\cot m\theta = \sum_{-m}^{m} \cot\left(\theta - \frac{r\pi}{2m+1}\right). \tag{6·1}$$

It is possible to proceed directly to the limit from the identity, but the process has none of the simplicity of one that is controlled throughout by bracketing inequalities.

A similar identity to (6·1) is

$$(2m+1)\operatorname{cosec} m\theta = \sum_{-m}^{m} (-)^r \operatorname{cosec}\left(\theta - \frac{r\pi}{2m+1}\right),$$

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and this also is related to an infinite series, which, to avoid the difficulties of a direct passage to the limit, we may derive either from (5·1) or by integration. Deduction from (5·1) is classical, and our alternative again uses explicit inequalities.

Since $(1+\cos x)/\sin^2 x = 1/2\sin^2 \frac{1}{2}x$, we have

$$\frac{\cos x}{\sin^2 x} = \tfrac{1}{2}\operatorname{cosec}^2 \tfrac{1}{2}x - \operatorname{cosec}^2 x.$$

But if $0 < |x| \leqslant \frac{1}{2}\pi$

$$\tfrac{1}{4}\operatorname{cosec}^2 \tfrac{1}{2}x - \frac{1}{8m+4} < \sum_{-m}^{m} \frac{1}{(x-2r\pi)^2} < \tfrac{1}{4}\operatorname{cosec}^2 \tfrac{1}{2}x,$$

$$-\operatorname{cosec}^2 x < -\sum_{-2m}^{2m} \frac{1}{(x-r\pi)^2} < -\operatorname{cosec}^2 x + \frac{1}{4m+1};$$

hence

$$\frac{\cos x}{\sin^2 x} - \frac{1}{4m+2} < \sum_{-2m}^{2m} \frac{(-1)^x}{(x-r\pi)^2} < \frac{\cos x}{\sin^2 x} + \frac{1}{4m+1},$$

that is,

$$\frac{\cos x}{\sin^2 x} - \frac{1}{x^2} - \frac{1}{4m+2} < \sum_{-2m}^{2m}{}' \frac{(-1)^r}{(x-r\pi)^2} < \frac{\cos x}{\sin^2 x} - \frac{1}{x^2} + \frac{1}{4m+1}.$$

Hence, if $0 < x \leqslant \frac{1}{2}\pi$,

$$\operatorname{cosec} x - \frac{1}{x} - \frac{x}{4m+1} < \sum_{-2m}^{2m}{}' (-)^r \left\{\frac{1}{x-r\pi} + \frac{1}{r\pi}\right\} < \operatorname{cosec} x - \frac{1}{x} + \frac{x}{4m+2}, \tag{6·2}$$

implying

$$\operatorname{cosec} x = \frac{1}{x} + \sum_{-\infty}^{\infty}{}' (-)^r \left\{\frac{1}{x-r\pi} + \frac{1}{r\pi}\right\} = \frac{1}{x} + \sum_{1}^{\infty} (-)^r \frac{2x}{x^2 - r^2\pi^2}.$$

7. Writing (6·2) in the form

$$\operatorname{cosec} x - \frac{x}{4m+1} < \sum_{-2m}^{2m} \frac{(-1)^r}{x-r\pi} < \operatorname{cosec} x + \frac{x}{4m+2},$$

multiplying by $\sin x$, and integrating from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$, we have

$$\frac{\pi}{2} - \frac{1}{4m+1} < \int_0^{(2m+\frac{1}{2})\pi} \frac{\sin x}{x}\,dx < \frac{\pi}{2} + \frac{1}{4m+2}.$$

If $X > (2m+\frac{1}{2})\pi$, the integral from $(2m+\frac{1}{2})\pi$ to X is easily shown, on account of the changes of sign of $\sin x$, to have its greatest numerical value when $X = (2m+1)\pi$, and this value is less than $1/(2m+\frac{1}{2})\pi$. Hence, for $X > (2m+\frac{1}{2})\pi$,

$$\frac{\pi}{2} - \frac{1}{4m+1} - \frac{2}{(4m+1)\pi} < \int_0^X \frac{\sin x}{x}\,dx < \frac{\pi}{2} + \frac{1}{4m+2} + \frac{2}{(4m+1)\pi},$$

or, with a trivial relaxation,

$$\frac{\pi}{2} - \frac{\pi+2}{(4m+1)\pi} < \int_0^X \frac{\sin x}{x}\,dx < \frac{\pi}{2} + \frac{\pi+2}{(4m+1)\pi}.$$

Since, for a given X, we may take $(2m+\frac{1}{2})\pi > X - 2\pi$, we have explicitly

$$\frac{\pi}{2} - \frac{\pi+2}{2(X-2\pi)} < \int_0^X \frac{\sin x}{x}\,dx < \frac{\pi}{2} + \frac{\pi+2}{2(X-2)\pi}, \tag{7·1}$$

on the sole condition that $X > 2\pi$.

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The differences of detail between the two pairs of inequalities (4·1), (7·1) are unimportant; for theoretical purposes what matters usually is the qualitative formula

$$\int_0^X \frac{\sin x}{x}\,dx = \frac{\pi}{2} + O\left(\frac{1}{X}\right),$$

and not any numerical estimate of the constants implicit in such a formula.

8. The origin of this note was a casual inquiry on the integral

$$\int_0^\infty \frac{\sin x}{x}\,dx,$$

which prompted me to read again two contributions* to the *Mathematical Gazette* in which Hardy compared several methods of evaluation. Hardy's verdict was that the simplest of these methods used the infinite series for $\operatorname{cosec} x$, but he notes that the convergence of this series is not of the simplest kind, and he might have added that the series itself is not one that Macaulay's Taylor-minded schoolboy necessarily knows. The identification of the required integral with the integral

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$$

was familiar, but the application of the series for $\operatorname{cosec}^2 x$ had still to be made, and the possibility of establishing this series by an elementary argument repaid investigation.

The University
Reading

* 5 (1909), 98; 8 (1216), 30. The integral was the subject also of notes by A. C. Dixon, 6 (1912), 223, and M. F. Egan, 8 (1916), 299.

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