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Singularity, weighted uniform approximation, intersections and rates

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Abstract

A classical argument was introduced by Khintchine in 1926 in order to exhibit the existence of totally irrational singular linear forms in two variables. This argument was subsequently revisited and extended by many authors. For instance, in 1959 Jarník used it to show that for $n \geqslant 2$ and for any non-increasing positive f there are totally irrational matrices $A \in M_{m,n}(\mathbb{R})$ such that for all large enough t there are $\mathbf{p} \in \mathbb{Z}^m$, $\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}$ with $\|\mathbf{q}\| \leqslant t$ and $\|A\mathbf{q} - \mathbf{p}\| \leqslant f(t)$. We denote the collection of such matrices by $\mathrm{UA}^*_{m,n}(f)$. We adapt Khintchine's argument to show that the sets $\mathrm{UA}^*_{m,n}(f)$, and their weighted analogues $\mathrm{UA}^*_{m,n}(f, \boldsymbol{\omega})$, intersect many manifolds and fractals, and have strong intersection properties. For example, we show that: (i) when $n \geqslant 2$, the set $\bigcap_{\boldsymbol{\omega}} \mathrm{UA}^*(f, \boldsymbol{\omega})$, where the intersection is over all weights $\boldsymbol{\omega}$, is non-empty, and moreover intersects many manifolds and fractals; (ii) for $n \geqslant 2$, there are vectors in $\mathbb{R}^n$ which are simultaneously k-singular for every k, in the sense of Yu; and (iii) when $n \geqslant 3$, $\mathrm{UA}^*_{1,n}(f) + \mathrm{UA}^*_{1,n}(f) = \mathbb{R}^n$. We also obtain new bounds on the rate of singularity which can be attained by column vectors in analytic submanifolds of dimension at least 2 in $\mathbb{R}^n$.

1. Introduction

1.1 Background and history

In 1926, Khintchine [Khi26] introduced the property of the singularity of vectors, which was later extended to systems of linear forms by Jarník [Jar59]. A matrix $A \in M_{m,n}(\mathbb{R})$ (viewed as a system of m linear forms in n variables) is *singular* (notation: $A \in \mathrm{Sing}_{m,n}$) if for any $\varepsilon > 0$ there is $t_0 > 0$ such that for all $t \geqslant t_0$ there exist $\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}$ and $\mathbf{p} \in \mathbb{Z}^m$ such that

$$\|\mathbf{q}\| \leqslant t \quad \text{and} \quad \|A\mathbf{q} - \mathbf{p}\| \leqslant \frac{\varepsilon}{t^{n/m}}. \tag{1.1}$$

Here $M_{m,n}(\mathbb{R})$ is the collection of $m \times n$ real matrices, and $\|\cdot\|$ stands for the supremum norm on $\mathbb{R}^m$ and $\mathbb{R}^n$, although the choice of the norm will be immaterial for this work.

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Note that (1.1) always has non-trivial integer solutions if $\varepsilon = 1$ (Dirichlet's Theorem). Khintchine, and then Jarník with greater generality, showed that while, for $m = n = 1$, the only singular numbers are rationals, for $\max(m, n) > 1$ there exist uncountably many singular matrices which are *totally irrational*, that is, not contained in any rational affine subspace of $M_{m,n}(\mathbb{R})$. We will denote by $\mathrm{Sing}^*_{m,n}$ the set of totally irrational singular $m \times n$ matrices. Since the work of Khintchine, a large body of work in Diophantine approximation has been devoted to the understanding of $\mathrm{Sing}^*_{m,n}$ and related sets. See [Mos10] for a detailed survey and an extensive bibliography, and [KKLM17, DFSU24, BCC19] for the computation of the Hausdorff dimension of $\mathrm{Sing}_{m,n}$ and for a discussion of related work.

A natural way to generalize (1.1) is to replace $\varepsilon/t^{n/m}$ in the right-hand side by an arbitrary approximating function.

DEFINITION 1.1. Given a function $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, we say that $A \in M_{m,n}(\mathbb{R})$ is *uniformly f-approximable*[1], or *f-uniform* for brevity, if for every sufficiently large $t > 0$ one can find $\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}$ and $\mathbf{p} \in \mathbb{Z}^m$ with

$$\|A\mathbf{q} - \mathbf{p}\| \leqslant f(t) \quad \text{and} \quad \|\mathbf{q}\| \leqslant t. \tag{1.2}$$

Denote by $\mathrm{UA}_{m,n}(f)$ the set of f-uniform $m \times n$ matrices, and by $\mathrm{UA}^*_{m,n}(f)$ the set of totally irrational $A \in \mathrm{UA}_{m,n}(f)$.

We remark that, in general, the field of *uniform Diophantine approximation* deals with the solvability of systems of type (1.1) or (1.2) for *all large enough* values of t, as opposed to the (much better understood) theory of *asymptotic approximation*. There one can define the sets $\mathcal{A}_{m,n}(f)$ of f-approximable $m \times n$ matrices, where those systems are required to have integer solutions for *an unbounded set* of $t > 0$.

Dirichlet's Theorem asserts that $\mathrm{UA}_{m,n}(\phi_{n/m}) = M_{m,n}(\mathbb{R})$, where we use the notation $\phi_a(t) \stackrel{\text{def}}{=} t^{-a}$. Also one clearly has

$$\mathrm{Sing}_{m,n} = \bigcap_{\varepsilon > 0} \mathrm{UA}_{m,n}(\varepsilon\phi_{n/m})$$

and, more generally, $\mathrm{UA}_{m,n}(f) \subset \mathrm{Sing}_{m,n}$ as long as f satisfies

$$t^{n/m} f(t) \to_{t\to\infty} 0. \tag{1.3}$$

It is well known (see [Cas57, Chapter V, § 7]) that the set of singular matrices is a nullset in $M_{m,n}(\mathbb{R})$, and hence so is the set of f-uniform matrices for f satisfying (1.3). It is also clear that matrices A such that $A\mathbf{q} = \mathbf{p}$ for some $\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}$ and $\mathbf{p} \in \mathbb{Z}^m$ are in $\mathrm{UA}_{m,n}(f)$ for any f. On the other hand we have the following theorem.

THEOREM 1.2 (Jarník [Jar59]). *Let $m, n \in \mathbb{N}$.*

(a) *If $n > 1$, then for any non-increasing $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ the set of totally irrational f-uniform $m \times n$ matrices is uncountable and dense.*
(b) *If, in addition, one assumes that*

$$\lim_{t\to\infty} t f(t) = \infty, \tag{1.4}$$

[1]This property, with a slightly different parametrization, has also been called 'f-Dirichlet' and 'f-singular' in the literature, see e.g. [KW18] and [DFSU24].

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then for any $m>1$ the set of totally irrational f-uniform $m\times 1$ matrices (column vectors) is uncountable and dense.

Remark 1.3. The above theorem is often stated in a simplified form by introducing the notion of the *uniform Diophantine exponent* $\hat\omega(A)$, which is the supremum of a such that A is ϕ_a-uniform. In other words,

$$\hat\omega(A)\stackrel{\text{def}}{=}\sup\left\{a>0\;\middle|\;\begin{array}{c}\text{for all large enough } t>0\\ \exists\,\mathbf{q}\in\mathbb{Z}^n\smallsetminus\{0\},\ \mathbf{p}\in\mathbb{Z}^m\\ \text{such that (1.2) holds for } f=\phi_a\end{array}\right\}. \tag{1.5}$$

Then Theorem 1.2 asserts that there exist uncountably many totally irrational $m\times n$ matrices A with $\hat\omega(A)=\infty$ whenever $n>1$, and uncountably many totally irrational $m\times 1$ matrices (column vectors) $\mathbf{x}$ with $\hat\omega(\mathbf{x})=1$ as long as $m>1$.

Note that in the column vector case the extra condition (1.4) cannot be removed: indeed, if $\mathbf{x}$ is such that $\hat\omega(\mathbf{x})>1$, then one has $\hat\omega(x_i)>1$ for every component x_i of $\mathbf{x}$, which necessarily implies that $x_i\in\mathbb{Q}$ for all i. See also the discussion in [DFSU24, § 3.2].

The argument of Khintchine and Jarník has been utilized and extended by many people, including [Dan85, Wei02, KW05]. Most recently, in [KMW21] three of the authors of the present paper showed that, in the case $m=1$, there exist uncountably many f-uniform row vectors on certain submanifolds and fractals. As a special case, they showed that if Y is either a connected analytic submanifold of $\mathbb{R}^n$ of dimension at least 2 not contained in a rational affine hyperplane, or a product $\mathcal{C}\times\cdots\times\mathcal{C}$ of $n\geqslant 2$ copies of the middle-third Cantor set $\mathcal{C}$, then Y contains uncountably many f-uniform vectors for any approximating function f.

1.2 Diophantine systems

The goal of this paper is to strengthen the aforementioned results, namely to provide conditions on countably many maps $\{\varphi_k:k\in\mathbb{N}\}$ from some metric space Y to $M_{m_k,n_k}(\mathbb{R})$ under which there is an uncountable and dense set of $y\in Y$ for which $\varphi_k(y)$ is f-uniform for every k. All this will follow from an abstract result (Theorem 1.5) which will generalize many constructions of singular/uniform objects existing in the literature. The proof of Theorem 1.5 once again utilizes the original strategy introduced by Khintchine.

In order to state our construction in full generality we introduce the notion of a *Diophantine system*, which is a triple $\mathcal{X}=(X,\mathcal{D},\mathcal{H})$, where:

- X is a topological space;
- $\mathcal{D}=\{d_s:s\in\mathcal{I}\}$ is a sequence of continuous functions on X taking nonnegative values (*generalized distance functions*); and
- $\mathcal{H}=\{h_s:s\in\mathcal{I}\}$ is a sequence of positive real numbers; we will say that h_s is the *height* associated with d_s.

Here $\mathcal{I}$ is a fixed countable set[2] of indices. This generalizes the definition of a *Diophantine space* originally introduced in [FSU14]; namely, by a Diophantine space one means a triple $(X,\mathcal{R},H)$, where X is a complete metric space, $\mathcal{R}$ is a countable dense subset of X, and $H:\mathcal{R}\to(0,\infty)$ gives the heights of points in $\mathcal{R}$. To view it as a Diophantine system as in our definition one can simply enumerate the elements of $\mathcal{R}=\{r_s:s\in\mathcal{I}\}$ and let $d_s(x)\stackrel{\text{def}}{=}\operatorname{dist}(x,r_s)$ and $h_s\stackrel{\text{def}}{=}H(r_s)$.

[2]For our exposition it will be convenient to have a freedom of choice of $\mathcal{I}$ instead of always using $\mathcal{I}=\mathbb{N}$.

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Similarly one can treat a more general situation when $\mathcal{R}$ is a countable collection of closed subsets of X (*resonant sets*), which in our notation are simply the zero sets of the generalized distance functions d_s. Our setup, however, makes it possible to define the distance from each individual set in a different way, as well as to count the same set several (perhaps infinitely many) times.

Now suppose we are given a Diophantine system $\mathcal{X} = (X, \mathcal{D}, \mathcal{H})$. Then we can introduce the notion of f-uniform points for any function $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ (the *approximating function*).

Definition 1.4. A point $x \in X$ is $(\mathcal{X}, f)$-*uniform* if for every sufficiently large t one can find $s \in \mathcal{I}$ such that

$$d_s(x) \leqslant f(t) \quad \text{and} \quad h_s \leqslant t.$$

We will denote by $\mathrm{UA}_{\mathcal{X}}(f)$ the set of $(\mathcal{X}, f)$-uniform points of X.

As the reader can easily check, the following choices define a Diophantine system, which we refer to as the *standard Diophantine system* $\mathcal{X}_{m,n}$:

$$\begin{gathered} X = M_{m,n}(\mathbb{R}), \quad \mathcal{I} = \mathbb{Z}^m \times (\mathbb{Z}^n \smallsetminus \{0\}), \\ h_{\mathbf{p},\mathbf{q}} = \|\mathbf{q}\|, \quad d_{\mathbf{p},\mathbf{q}}(A) = \|A\mathbf{q} - \mathbf{p}\|. \end{gathered} \tag{1.6}$$

It is clear that a matrix is $(\mathcal{X}_{m,n}, f)$-uniform if and only if it is f-uniform in the sense of Definition 1.1.

Trivially, every point in the union of resonant sets $d_s^{-1}(\{0\})$ is $(\mathcal{X}, f)$-uniform for any choice of $\mathcal{H}$ and f. In the next section we will state our main theorem, which provides a set of conditions guaranteeing the existence of uncountably many non-trivial $(\mathcal{X}, f)$-uniform points. More generally, it involves an auxiliary metric space Y and countably many maps from Y to possibly different metric spaces X_i, each endowed with its own Diophantine system.

1.3 The main result

To state our main abstract theorem, it will be convenient to introduce some more terminology. Let Y be a metric space and let $\mathcal{L}$, $\mathcal{R}$ be two collections of proper closed subsets of Y. We say that $\mathcal{L}$ is *totally dense relative to* $\mathcal{R}$ if

$$\bigcup_{L \in \mathcal{L}} L \text{ is dense in } Y, \tag{1.7}$$

and

$$\begin{gathered} \forall \text{ open } W \subset Y, \, \forall \, L \in \mathcal{L} \text{ such that } L \cap W \neq \varnothing, \text{ and } \forall \, R \in \mathcal{R} \\ \exists \, L' \in \mathcal{L} \text{ such that } L' \cap L \cap W \neq \varnothing \text{ and } L' \not\subset R. \end{gathered} \tag{1.8}$$

We will say that $\mathcal{L}$ is *totally dense* if it is totally dense relative to the collection of all closed nowhere dense subsets of Y. In this case (1.8) is equivalent to

$$\begin{gathered} \forall \text{ open } W \subset Y \text{ and } \forall \, L \in \mathcal{L} \text{ such that } L \cap W \neq, \varnothing \\ \text{the closure of } \bigcup_{L' \in \mathcal{L} \,:\, L' \cap L \cap W \neq \varnothing} L' \text{ has a non-empty interior.} \end{gathered} \tag{1.9}$$

For example, for any $1 \leqslant d < n$ the collection $\mathcal{L}$ of d-dimensional rational affine subspaces of $Y = \mathbb{R}^n$ is totally dense, with the union in (1.9) being dense in $\mathbb{R}^n$. More examples will be described later in the paper. On the other hand, the collection of straight lines in $\mathbb{R}^3$ that are parallel to one of the coordinate axes satisfies properties (1.7) and (1.8) with $\mathcal{R} =$ all planes in $\mathbb{R}^3$, but not (1.9).

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Here is another definition which will be important for us throughout the paper. Let us say that $\mathcal{L}$ *respects* a subset R of Y if whenever $L \in \mathcal{L}$ is such that $L \cap R$ has non-empty interior in L (in the topology induced from Y), it follows that $L \subset R$. In other words, elements L of $\mathcal{L}$ are not allowed to behave disrespectfully by intersecting R in an open set and then wandering off. We will say that $\mathcal{L}$ *respects* $\mathcal{R}$ if it respects every $R \in \mathcal{R}$. As an example, if each element of $\mathcal{L}$ is either connected or has no isolated points, it respects any singleton $\{y\} \subset Y$. Another example is given by collections $\mathcal{L}$ and $\mathcal{R}$ consisting of analytic submanifolds of $\mathbb{R}^n$ (where we equip them with the metric inherited from the ambient space $\mathbb{R}^n$) such that every $L \in \mathcal{L}$ is connected, and every $R \in \mathcal{R}$ is closed; see Lemma 3.2 for more details.

Our final definition involves a collection $\mathcal{L}$ of proper closed subsets of Y, a Diophantine system $\mathcal{X} = (X, \mathcal{D}, \mathcal{H})$, and a map $\varphi : Y \to X$. Let us say that $\mathcal{L}$ *is aligned with* $\mathcal{X}$ *via* φ if for any $L \in \mathcal{L}$ there exists $s \in \mathcal{I}$ such that $d_s|_{\varphi(L)} \equiv 0$; in other words, if $\varphi(L)$ is contained in one of the resonant sets $d_s^{-1}(\{0\})$. In the special case when $Y = X$ and $\varphi = \mathrm{Id}$ we will simply say that $\mathcal{L}$ is aligned with $\mathcal{X}$. Note that this property is independent both of the choice of the height functions h_s and of a reparametrization of generalized distance functions as long as their zero sets are fixed.

Now we are ready to state the main theorem of the paper.

Theorem 1.5. *Let Y be a locally compact metric space. For any $k \in \mathbb{N}$ suppose we are given a Diophantine system*

$$\mathcal{X}_k = (X_k, \mathcal{D}_k = \{d_{k,s} : s \in \mathcal{I}\}, \mathcal{H}_k = \{h_{k,s} : s \in \mathcal{I}\}) \tag{1.10}$$

and a non-increasing function $f_k : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, and let φ_k be a continuous map from Y to X_k. Let $\mathcal{L}$ and $\mathcal{R}$ be countable collections of closed subsets of Y such that $\mathcal{L}$ is totally dense relative to $\mathcal{R}$, respects $\mathcal{R}$, and is aligned with $\mathcal{X}_k$ via φ_k for every $k \in \mathbb{N}$. Then:

(a) *the set*

$$\bigcap_k \varphi_k^{-1}(\mathrm{UA}_{\mathcal{X}_k}(f_k)) \smallsetminus \bigcup_{R \in \mathcal{R}} R \tag{1.11}$$

is dense in Y; and

(b) *in addition, if every $L \in \mathcal{L}$ is either connected or has no isolated points, the set (1.11) is uncountable.*

Remark 1.6. A few comments are in order.

(1) In the above statement we are working with an *indexed collection* $\mathcal{X}_k$, which is shorthand for a function $k \mapsto \mathcal{X}_k$; we do not assume that this function is injective. It might also happen that the underlying set X_k of $\mathcal{X}_k = (X_k, \mathcal{D}_k, \mathcal{H}_k)$ arises several times, with distinct generalized distance functions or height functions.

(2) Neither the approximation functions f_k nor the heights $\mathcal{H}_k$ appear in the conditions of the theorem. Moreover, the collections $\mathcal{D}_k$ of generalized distance functions appear only through their zero sets $d_{k,s}^{-1}(\{0\})$. Thus if the assumptions of the above theorem are satisfied, then they are satisfied for all positive non-increasing functions f_k, all choices of the heights, and all choices of generalized distance functions as long as their zero sets are fixed.

(3) It is easy to see that Theorem 1.5 holds for finite intersections, that is, when the set of indices k is finite; this follows immediately from the statement of the theorem, taking $f_k, \varphi_k, \mathcal{X}_k$ to be the same for all sufficiently large k.

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Theorem 1.5 is proved in § 2, and in § 3 we show that it implies a strengthening of Theorem 1.2(a), making it possible to avoid countably many proper analytic submanifolds in $M_{m,n}(\mathbb{R})$ (see Theorem 3.3). In the remainder of the introduction we present a list of additional applications of Theorem 1.5.

1.4 Matrices uniformly approximable with different weights

Many results in Diophantine approximation extend to *approximation with weights*, an approach in which one gives different treatment to different linear forms A_i (the rows of A), as well as different components of $\mathbf{q}$.

For an $(m+n)$-tuple of positive weights[3] $\boldsymbol{\omega} = (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbb{R}^{m+n}_{>0}$, where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m_{>0}$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n_{>0}$, let us introduce *quasi-norms*

$$\|\mathbf{x}\|_{\boldsymbol{\alpha}} \stackrel{\text{def}}{=} \max_i |x_i|^{1/\alpha_i} \quad \text{and} \quad \|\mathbf{y}\|_{\boldsymbol{\beta}} \stackrel{\text{def}}{=} \max_j |y_j|^{1/\beta_j}$$

on $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. Then, for f as above, one says that $A \in M_{m,n}(\mathbb{R})$ is $(f, \boldsymbol{\omega})$-*uniform*, denoted by $A \in \mathrm{UA}_{m,n}(f, \boldsymbol{\omega})$, if for every sufficiently large t one can find $\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}$ and $\mathbf{p} \in \mathbb{Z}^m$ with

$$\|A\mathbf{q} - \mathbf{p}\|_{\boldsymbol{\alpha}} \leqslant f(t) \quad \text{and} \quad \|\mathbf{q}\|_{\boldsymbol{\beta}} \leqslant t. \tag{1.12}$$

In other words, we are considering the solvability of the system

$$\begin{cases} |A_i\mathbf{q} - p_i| \leqslant f(t)^{\alpha_i}, & i = 1, \dots, m; \\ |q_j| \leqslant t^{\beta_j}, & j = 1, \dots, n. \end{cases}$$

See [KR22] for a discussion. Clearly the unweighted case corresponds to the choice $\boldsymbol{\omega} = (1, \dots, 1)$. As in Definition 1.1, let us denote by $\mathrm{UA}^*_{m,n}(f, \boldsymbol{\omega})$ the set of totally irrational $A \in \mathrm{UA}_{m,n}(f, \boldsymbol{\omega})$.

Clearly for any $\boldsymbol{\omega} = (\boldsymbol{\alpha}, \boldsymbol{\beta})$ and f as above one has

$$\mathrm{UA}_{m,n}(f, \boldsymbol{\omega}) \supset \mathrm{UA}_{m,n}(\tilde{f}) \quad \text{where } \tilde{f}(t) \stackrel{\text{def}}{=} f(t^{1/\min_j \beta_j})^{\max_i \alpha_i}. \tag{1.13}$$

Hence it immediately follows from Theorem 1.2(a) that $\mathrm{UA}^*_{m,n}(f, \boldsymbol{\omega})$ is uncountable and dense. Using Theorem 1.5 one can extend this to matrices which are simultaneously $(f_k, \boldsymbol{\omega}_k)$-uniform for countably many k.

Theorem 1.7. *Let $n \geqslant 2$ and $m \in \mathbb{N}$.*

(a) *For any sequence $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \dots \in \mathbb{R}^{m+n}_{>0}$ of weight vectors and any sequence $f_1, f_2, \dots$ of positive non-increasing functions, the intersection $\bigcap_k \mathrm{UA}^*_{m,n}(f_k, \boldsymbol{\omega}_k)$ is dense and uncountable.*

(b) *The intersection $\bigcap_{\boldsymbol{\omega} \in \mathbb{R}^{m+n}_{>0}} \mathrm{UA}^*_{m,n}(f, \boldsymbol{\omega})$ is dense and uncountable for any positive non-increasing f.*

In § 3 we prove an even stronger statement, with extra uniformity in choosing approximating vectors and making it possible to avoid countably many proper analytic submanifolds (see Theorem 3.4).

[3] In the literature it is often assumed that the weight vectors are normalized so that $\sum_i \alpha_i = \sum_j \beta_j = 1$, but for this work it makes no difference, since our results are valid for arbitrary choice of approximating functions.

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1.5 Uniform approximation of higher order on submanifolds

In § 4 we show that, in the row vector case ($m=1$), vectors satisfying the conclusions of Theorem 1.7 exist on many manifolds and fractals. For example the following theorem, which is a special case of Theorem 4.2, was already proved in [KMW21].

Theorem 1.8. *Let Y be a connected analytic submanifold of $\mathbb{R}^n$ of dimension at least 2 that is not contained in a rational affine hyperplane, and let f be any positive non-increasing function. Then the intersection of $\mathrm{UA}^*_{1,n}(f)$ with Y is dense and uncountable.*

Our proof, based on Theorem 1.5, makes it possible to streamline the argument and derive other related results. In particular, one benefit of our level of abstraction is that we can similarly study uniform approximation *of higher order*. Let us introduce the following

Definition 1.9. Given a function $f:\mathbb{R}_{>0}\to\mathbb{R}_{>0}$ and $1\leqslant g\leqslant n$, we say that $\mathbf{x}\in\mathbb{R}^n$ is *f-uniform of order g* (notation: $\mathbf{x}\in\mathrm{UA}_{1,n}(f;g)$, or $\mathbf{x}\in\mathrm{UA}^*_{1,n}(f;g)$ if $\mathbf{x}$ is totally irrational) if for every sufficiently large $t>0$ one can find linearly independent $\mathbf{q}_1,\dots,\mathbf{q}_g$ in $\mathbb{Z}^n$ and $p_1,\dots,p_g\in\mathbb{Z}$ such that

$$\text{for all } i\in\{1,\dots,g\},\quad \|\mathbf{q}_i\|\leqslant t \quad\text{and}\quad |\mathbf{q}_i\cdot\mathbf{x}-p_i|\leqslant f(t).$$

Note that any rational affine subspace of codimension g is an intersection of g rational affine hyperplanes with linearly independent normal vectors, and hence every point of it is f-uniform of order g for any f. The following generalization of Theorem 1.8 is a special case of Theorem 4.2.

Theorem 1.10. *Let Y be a connected analytic submanifold of $\mathbb{R}^n$ of dimension $d\geqslant 2$ not contained in a rational affine hyperplane, let f be any positive non-increasing function, and let $1\leqslant g\leqslant d-1$. Then the intersection of Y with $\mathrm{UA}^*_{1,n}(f;g)$ is dense and uncountable.*

1.6 Vectors which are k-singular for all k

A vector $\mathbf{x}\in\mathbb{R}^n$ is called *k-singular* [KPW16] if for any $\varepsilon>0$ there is $Q_0>0$ such that for all $Q>Q_0$ there exists a polynomial $P\in\mathbb{Z}[X_1,\dots,X_n]$ such that

$$\deg(P)\leqslant k,\quad H(P)\leqslant Q,\quad\text{and}\quad |P(\mathbf{x})|\leqslant\varepsilon Q^{-N(k,n)}.$$

Here $H(P)$ is the *height* of P, defined as the maximum of the absolute value of the coefficients of P, $\deg(P)$ is the total degree of P, and

$$N(k,n)=\binom{k+n}{n}-1. \tag{1.14}$$

Note that when $k=1$, k-singularity is the same as the singularity of $\mathbf{x}$ considered as a row vector in $M_{1,n}(\mathbb{R})$, and the existence of singular vectors in certain manifolds and fractals is one of the main results of [KMW21].

The study of k-singular vectors is related to the approximation of $\mathbf{x}$ by vectors with algebraic coefficients of degree at most k; see e.g. [Yu87]. Following [KPW16], let us say that $\mathbf{x}$ is *k-algebraic* if there is a nonzero polynomial $P\in\mathbb{Z}[X_1,\dots,X_n]$ such that $P(\mathbf{x})=0$ and $\deg P=k$, and we say it is *algebraic* if it is k-algebraic for some k. It is easy to see that a k-algebraic vector is k'-singular for every $k'\geqslant k$, and it was shown in [KPW16] that for $n\geqslant 2$ and for each k there are k-singular vectors which are not k-algebraic. In § 5 we show the following theorem.

Theorem 1.11. *If $n\geqslant 2$, then there is a dense uncountable set of $\mathbf{x}\in\mathbb{R}^n$ such that $\mathbf{x}$ is k-singular for all $k\in\mathbb{N}$, but $\mathbf{x}$ is not algebraic.*

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Theorem 1.11 answers a question raised in [KPW16, § 2.3] regarding the existence of vectors which are (k, ε)-Dirichlet improvable for some fixed ε, for all positive integers k. Note that we are going to prove a statement much more general than Theorem 1.11; namely, we will discuss vectors that are uniformly f_k-approximable of degree k for each k, where f_k is a sequence of arbitrary positive non-increasing functions. Moreover, those vectors will be found in many manifolds and fractals.

1.7 $\mathbf{Sing}^*_{1,n} + \mathbf{Sing}^*_{1,n} = \mathbb{R}^n$ and generalizations

Here we consider the case $m = 1$. It is known [DFSU24] that for rapidly decaying f the sets $\mathrm{UA}_{1,n}(f)$ have very small Hausdorff dimension. The next theorem shows that when $n \geqslant 3$ the sum of two such sets is the whole space.

Theorem 1.12. *If $n \geqslant 3$, then for any two non-increasing functions $f_1, f_2 : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ one has*

$$\mathrm{UA}^*_{1,n}(f_1) + \mathrm{UA}^*_{1,n}(f_2) = \mathbb{R}^n. \tag{1.15}$$

See Schleischitz [Sch22, § 3] for related work. We prove this theorem in § 6 and discuss several extensions.

1.8 Improved rates of singularity for column vectors

Recall from Theorem 1.2 that for $n > 1$, any $m \in \mathbb{N}$ and any positive non-increasing f one can find an $m \times n$ matrix which is totally irrational and f-uniform. For $n = 1$ and $m > 1$, that is for the case of column vectors, this is no longer the case. Indeed, as was mentioned in Remark 1.3, for all totally irrational $\mathbf{x} \in \mathbb{R}^m \cong M_{m,1}(\mathbb{R})$ one has $\hat{\omega}(\mathbf{x}) \leqslant 1$, and, according to Theorem 1.2(b), this maximum value is attained on an uncountable dense set.

It is natural to inquire about the value of $\hat{\omega}$ that vectors in certain fractals and submanifolds may attain. One can approach this question by the standard transference argument as in [Cas57, Chapter V, Theorem II] and [Jar38] and show that any column vector $\mathbf{x} \in \mathbb{R}^n \cong M_{n,1}(\mathbb{R})$ with $\hat{\omega}(\mathbf{x}^T) = \infty$ satisfies

$$\hat{\omega}(\mathbf{x}) \geqslant \frac{1}{n-1}. \tag{1.16}$$

Therefore an application of Theorem 1.8 coupled with transference will produce, for manifolds Y as in that theorem, a dense uncountable set of $\mathbf{x} \in Y$ satisfying (1.16).

It is natural to seek an improvement of the above bound; however, replacing $1/(n-1)$ with 1 on an arbitrary analytic submanifold of $\mathbb{R}^n$ is an impossible task. Indeed, in [KM19] it was shown that certain submanifolds do not contain column vectors $\mathbf{x}$ with $\hat{\omega}(\mathbf{x}) = 1$. Namely, letting $H_n \in (\frac{1}{2}, 1)$ be the unique positive root of the equation $x + \cdots + x^{n+1} = 1$, it was proved that any $\mathbf{x}$ in the sphere $\{(x_1, \ldots, x_n) : \sum x_i^2 = 1\}$ satisfies $\hat{\omega}(\mathbf{x}) \leqslant H_{n-1}$, and any $\mathbf{x}$ in the paraboloid $\{(x_1, \ldots, x_n) : x_n = \sum_{i<n} x_i^2\}$ satisfies $\hat{\omega}(\mathbf{x}) \leqslant H_n$. Furthermore, in [KMW21] it was shown that there are d-dimensional affine subspaces $L \subset \mathbb{R}^n$ such that any $\mathbf{x} \in L$ satisfies $\hat{\omega}(\mathbf{x}) \leqslant (d+1)/(n-d)$, which is strictly smaller than 1 for $d < (n-1)/2$. In § 7, using uniform approximation of higher order, i.e. Theorem 1.10, together with a transference argument, we are able to improve the bound in (1.16) and prove the following theorem.

Theorem 1.13. *Let Y be a connected analytic submanifold of $\mathbb{R}^n \cong M_{n,1}(\mathbb{R})$ of dimension $d \geqslant 2$ not contained in a rational affine hyperplane. Suppose that a continuous non-decreasing function f satisfies*

$$t^{1/(n-d+1)} \cdot f(t) \to \infty \quad \textit{monotonically as } t \to \infty. \tag{1.17}$$

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Then the intersection of $\mathrm{UA}^*_{n,1}(f)$ *with* Y *is dense and uncountable. In particular, there exists a dense and uncountable set of totally irrational* $\mathbf{x} \in Y$ *satisfying* $\hat{\omega}(\mathbf{x}) \geqslant 1/(n-d+1)$.

Remark 1.14. Using methods from [Wei02] and [KMW21] and a dynamical interpretation of uniform approximation through divergence of trajectories in the space of lattices, Datta and Tamam [DT24] recently proved a weighted version of Theorem 1.13 independently of the present paper and not using transference. More precisely,

- [DT24, Theorem 1.1] produces totally irrational vectors on analytic submanifolds that are singular with respect to multiple weights in the spirit of Theorem 1.7; and
- [DT24, Theorem 1.2] constructs vectors in $\mathrm{UA}^*_{n,1}(f, \boldsymbol{\omega}) \cap Y$ for any weight $\boldsymbol{\omega}$ and any affine subspace Y of $\mathbb{R}^n$ not contained in a rational affine hyperplane, with optimal decay conditions on approximating functions f, which, in particular, proves Theorem 1.13 for Y being an affine subspace of $\mathbb{R}^n$.

It is likely that the above results can be derived from the main theorem of the present paper by combining Theorem 1.10 with a weighted transference argument as in [CGGM+20] and [Ger20].

1.9 Other applications

In the last section of the paper we briefly survey a few other settings where our main theorem can be applied. These include: a relationship between the conclusion of our main theorem and the absence of the Kan–Moshchevitin phenomenon as exhibited in [KM10]; Diophantine approximation with restrictions on $\mathbf{p}$ and $\mathbf{q}$; inhomogeneous approximation; and approximating subspaces of $\mathbb{R}^d$ by rational subspaces. Proofs will appear in a sequel to this paper.

2. Proof of the abstract result

Proof of Theorem 1.5. Recall that the statement of the theorem involves two countable collections $\mathcal{L}$ and $\mathcal{R}$ of subsets of Y. In the course of the proof we will *index* those collections by $\mathbb{N}$, writing $\mathcal{L} = \{L_i : i \in \mathbb{N}\}$ and $\mathcal{R} = \{R_\ell : \ell \in \mathbb{N}\}$. That is, we will essentially work with functions $i \mapsto L_i$ and $\ell \mapsto R_\ell$, and will not assume these functions to be injective.

Take a non-empty open $W \subset Y$, which, since Y is locally compact, we can assume to be relatively compact. We will inductively construct a nested sequence of open sets $U_\ell \subseteq U_0 \stackrel{\text{def}}{=} W$, an increasing sequence $T_\ell \to \infty$, and a sequence of indices i_ℓ so that for all $\ell \in \mathbb{N}$:

(i) $\varnothing \neq \overline{U_\ell} \subseteq U_{\ell-1}$;
(ii) $L_{i_\ell} \cap U_\ell \neq \varnothing$ and $U_\ell \cap R_\ell = \varnothing$;
(iii) for all $1 \leqslant k \leqslant \ell$ there exists $s = s(k,\ell) \in \mathcal{I}$ such that $d_{k,s(k,\ell)} \equiv 0$ on $\varphi_k(L_{i_\ell})$ and $h_{k,s(k,\ell)} \leqslant T_\ell$; and
(iv) for all $1 \leqslant k \leqslant \ell - 1$ and all $x \in U_\ell$, $d_{k,s(k,\ell-1)}(\varphi_k(x)) < f_k(T_\ell)$, where $s(k,\ell-1)$ is as in (iii) (for $\ell - 1$).

Step 1: Sufficiency. Let us first check that this is sufficient for part (a) of the theorem. First observe that (i) and the relative compactness of W imply that $\bigcap_{\ell\in\mathbb{N}} \overline{U_\ell}$ is non-empty. We claim that

$$x \in \bigcap_{\ell\in\mathbb{N}} \overline{U_\ell} \implies \forall\, k \in \mathbb{N}, \quad \varphi_k(x) \text{ is } (\mathcal{X}_k, f_k)\text{-uniform.}$$

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Indeed, for any k take $T \geqslant T_k$, and let ℓ be such that $T_\ell \leqslant T < T_{\ell+1}$; then clearly $\ell \geqslant k$. Take $s = s(k,\ell)$. Then $h_{k,s} \leqslant T_\ell \leqslant T$ by (iii), and by (iv) and since f_k is non-increasing and $x \in U_{\ell+1}$, we have that

$$d_{k,s}(\varphi_k(x)) < f_k(T_{\ell+1}) \leqslant f_k(T).$$

Therefore $\varphi_k(x)$ is $(\mathcal{X}_k, f_k)$-uniform. Also from (ii) it follows that $x \notin \bigcup_\ell R_\ell$. Thus x belongs to the set (1.11), which implies its density and finishes the proof of (a).

Step 2: Base case of induction. By the total density of $\mathcal{L}$ relative to $\mathcal{R}$ there exists i_1 so that

$$L_{i_1} \cap W \neq \varnothing \quad \text{and} \quad L_{i_1} \not\subset R_1.$$

Since $\mathcal{L}$ is aligned with $\mathcal{X}_1$ via φ_1, there exists $s = s(1,1)$ such that $\varphi_1(L_{i_1})$ is contained in $d_{1,s}^{-1}(\{0\})$. Choose T_1 so that $T_1 > h_{1,s}$, and let $z \in L_{i_1} \cap W$. Choose a neighborhood $\hat{U}_1$ of z so that $\overline{\hat{U}_1} \subset W$. Since $\mathcal{L}$ respects R_1 and the latter is closed, there is an open set $U_1 \subset \hat{U}_1$ such that $L_{i_1} \cap U_1 \neq \varnothing$ and $U_1 \cap R_1 = \varnothing$. This choice ensures that (ii) holds; (iii) holds by the choice of s and T_1, and (i) holds since $U_1 \subset \hat{U}_1$ and $U_0 = W$. Item (iv) is vacuous for $\ell = 1$. This completes the base case.

Step 3: Inductive step. Assume that we have U_ℓ, T_ℓ, i_ℓ satisfying the inductive hypotheses. In view of (1.8) and (ii), there exists $i_{\ell+1}$ so that

$$L_{i_{\ell+1}} \cap L_{i_\ell} \cap U_\ell \neq \varnothing \quad \text{and} \quad L_{i_{\ell+1}} \not\subset R_{\ell+1}.$$

Since for any $k \in \mathbb{N}$ the collection $\mathcal{L}$ is aligned with $\mathcal{X}_k$ via φ_k, for each $k = 1, \dots, \ell+1$ there is $s(k, \ell+1) \in \mathcal{I}$ such that $\varphi_k(L_{i_{\ell+1}}) \subset d_{k,s(k,\ell+1)}^{-1}(\{0\})$. Let

$$T_{\ell+1} > \max(T_\ell \cup \{h_{k,s(k,\ell+1)} : k = 1, \dots, \ell+1\}),$$

and let $z \in L_{i_{\ell+1}} \cap L_{i_\ell} \cap U_\ell$. Since for $k = 1, \dots, \ell$ the functions $d_{k,s(k,\ell)} \circ \varphi_k$ are continuous and vanish at $z \in L_{i_\ell}$, there is a neighborhood $\hat{U}_{\ell+1}$ of z such that $\overline{\hat{U}_{\ell+1}} \subset U_\ell$ and

$$d_{k,s(k,\ell)}(\varphi_k(y)) < f_k(T_{\ell+1}) \quad \text{for all } y \in \hat{U}_{\ell+1}.$$

Again, since $R_{\ell+1}$ is closed and is respected by $\mathcal{L}$, it follows that there is an open set $U_{\ell+1} \subset \hat{U}_{\ell+1}$ such that

$$L_{i_{\ell+1}} \cap U_{\ell+1} \neq \varnothing \quad \text{and} \quad U_{\ell+1} \cap R_{\ell+1} = \varnothing.$$

This choice, and the inductive hypothesis, ensure that (ii) holds for $\ell+1$; (iii) holds for $\ell+1$ by the choice of $s(k, \ell+1)$ and $T_{\ell+1}$, and (i) and (iv) hold for $\ell+1$ since $U_{\ell+1} \subset \hat{U}_{\ell+1}$. This finishes the proof of (a).

Step 4: Part (b). The second part of Theorem 1.5 follows easily from part (a). Indeed, arguing by contradiction, assume that the set (1.11) consists of countably many points $\mathcal{R}_0 \stackrel{\text{def}}{=} \{y_j : j \in \mathbb{N}\}$. Then one can replace $\mathcal{R}$ with $\mathcal{R} \cup \mathcal{R}_0$ which will still satisfy (1.8) and be respected by $\mathcal{L}$ (here we use the fact that every $L \in \mathcal{L}$ is either connected or has no isolated points). Thus the set (1.11) with this new choice of $\mathcal{R}$ will be empty, contradicting Theorem 1.5(a). □

3. Simultaneous approximation

In this section we consider the standard Diophantine system $\mathcal{X}_{m,n}$ as in (1.6). We are going to apply Theorem 1.5 with $\varphi_k = \mathrm{Id}$ for each k and with the collection $\mathcal{L}$ parametrized by the direct

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product of $\mathbb{Z}^m$ and $\mathbb{Z}^n \smallsetminus \{0\}$. Namely we will consider the collection

$$\mathcal{L} \stackrel{\text{def}}{=} \{L_{\mathbf{p},\mathbf{q}} : \mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\},\ \mathbf{p} \in \mathbb{Z}^m\} \quad \text{where } L_{\mathbf{p},\mathbf{q}} \stackrel{\text{def}}{=} \{A \in M_{m,n}(\mathbb{R}) : A\mathbf{q} = \mathbf{p}\}. \tag{3.1}$$

Let us prove the following proposition.

PROPOSITION 3.1. *Let $m, n \in \mathbb{N}$ with $n > 1$, and let Y be a non-empty open subset of $M_{m,n}(\mathbb{R})$. Then the collection*

$$\{L_{\boldsymbol{p},\boldsymbol{q}} \cap Y : \boldsymbol{q} \in \mathbb{Z}^n \smallsetminus \{0\}, \boldsymbol{p} \in \mathbb{Z}^m\} \tag{3.2}$$

is totally dense.

Proof. Clearly $M_{m\times n}(\mathbb{Q})$ is dense in $M_{m,n}(\mathbb{R})$, and, furthermore, $L_{\mathbf{p},\mathbf{q}} \cap M_{m\times n}(\mathbb{Q})$ is dense in $L_{\mathbf{p},\mathbf{q}}$ for any $\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}$ and $\mathbf{p} \in \mathbb{Z}^m$. Moreover, any $B \in M_{m\times n}(\mathbb{Q})$ lies in an $L_{\mathbf{p},\mathbf{q}}$ for some $\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}$ and $\mathbf{p} \in \mathbb{Z}^m$. This implies (1.7).

Now fix $(\mathbf{p}_0, \mathbf{q}_0) \in \mathbb{Z}^m \times (\mathbb{Z}^n \smallsetminus \{0\})$ such that $L_{\mathbf{p}_0,\mathbf{q}_0} \cap Y \neq \varnothing$, and choose an open $W \subset Y$ which intersects $L_{\mathbf{p}_0,\mathbf{q}_0}$ non-trivially. Then we pick an arbitrary

$$B \in M_{m\times n}(\mathbb{Q}) \cap L_{\mathbf{p}_0,\mathbf{q}_0} \cap W,$$

we let E_B be the union of all the subspaces $L_{\mathbf{p},\mathbf{q}}$ containing B (this clearly includes $L_{\mathbf{p}_0,\mathbf{q}_0}$), and we write

$$\begin{aligned} E_B &= \bigcup_{(\mathbf{p},\mathbf{q}) \in \mathbb{Z}^m \times (\mathbb{Z}^n \smallsetminus \{0\}) : B\mathbf{q} = \mathbf{p}} \{A \in M_{m,n}(\mathbb{R}) : A\mathbf{q} = \mathbf{p}\} \\ &= \bigcup_{\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\},\ B\mathbf{q} \in \mathbb{Z}^m} \{A \in M_{m,n}(\mathbb{R}) : A\mathbf{q} = B\mathbf{q}\} \\ &= B + \bigcup_{\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\},\ B\mathbf{q} \in \mathbb{Z}^m} \{C \in M_{m,n}(\mathbb{R}) : C\mathbf{q} = 0\}. \end{aligned}$$

Note that the set of $\mathbf{q} \in \mathbb{Z}^n$ such that $B\mathbf{q} \in \mathbb{Z}^m$ contains $N\mathbb{Z}^n$ for some $N \in \mathbb{N}$. Therefore one has

$$E_B \supset B + \{C \in M_{m\times n}(\mathbb{Q}) : \operatorname{rank} C < n\}.$$

If $n > m$ it is clear that E_B contains $M_{m\times n}(\mathbb{Q})$, which readily implies the total density of the collection (3.2) in an even stronger form: namely, that the union in (1.9) is dense in $M_{m,n}(\mathbb{R})$.

In general we see that $\overline{E_B} = B + R_{<n}$, where

$$R_{<n} \stackrel{\text{def}}{=} \{C \in M_{m,n}(\mathbb{R}) : \operatorname{rank} C < n\},$$

which is a proper algebraic subvariety of $M_{m,n}(\mathbb{R})$ if $n \leqslant m$. Now let us consider the union of the sets E_B over all $B \in M_{m\times n}(\mathbb{Q}) \cap L_{\mathbf{p}_0,\mathbf{q}_0} \cap W$, and then take the closure, which is easily seen to have the following form:

$$\overline{\bigcup_{B \in M_{m\times n}(\mathbb{Q}) \cap L_{\mathbf{p}_0,\mathbf{q}_0} \cap W} E_B} = \{B \in \overline{W} : B\mathbf{q}_0 = \mathbf{p}_0\} + R_{<n}.$$

If $n = 1$ then $E_B = \{B\}$, and the above closure coincides with $L_{\mathbf{p}_0,\mathbf{q}_0} \cap \overline{W}$, and hence there is no total density. Now suppose that $1 < n \leqslant m$, and let $D \in \mathrm{GL}_n(\mathbb{R})$ be such that $\mathbf{q}_0 = D\mathbf{e}_n$. Then one can write

$$\begin{aligned} \{B \in \overline{W} : B\mathbf{q}_0 = \mathbf{p}_0\} + R_{<n} &= \{B \in \overline{W} : BD\mathbf{e}_n = \mathbf{p}_0\} + R_{<n} \\ &= (\{A \in \overline{W}D : A\mathbf{e}_n = \mathbf{p}_0\} + R_{<n})D^{-1}, \end{aligned}$$

since $R_{<n}$ is invariant under right-multiplication by invertible matrices. Thus it suffices to prove the claim under the assumption that $\mathbf{q} = \mathbf{e}_n$, that is, to show that for any non-empty open

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$W \subset M_{m,n}(\mathbb{R})$ and any $\mathbf{p}_0 \in \mathbb{Z}^m$, the set $\{A \in \overline{W} : A\mathbf{e}_n = \mathbf{p}_0\} + R_{<n}$ contains some open neighborhood V of $L_{\mathbf{p}_0,\mathbf{e}_n} \cap W$.

For this it will be convenient to write the elements of $M_{m,n}(\mathbb{R})$ in a column-vector notation. Namely, we can write

$$L_{\mathbf{p}_0,\mathbf{e}_n} \cap W = \{[\mathbf{a}_1 \cdots \mathbf{a}_{n-1}\ \mathbf{p}_0] : [\mathbf{a}_1 \cdots \mathbf{a}_{n-1}] \in W'\}$$

for some non-empty open subset W' of $M_{m\times(n-1)}$. Then it becomes clear that we can take

$$V \stackrel{\text{def}}{=} \{[\mathbf{a}_1 \cdots \mathbf{a}_{n-1}\ \mathbf{b}] : [\mathbf{a}_1 \cdots \mathbf{a}_{n-1}] \in W',\ \mathbf{b} \in \mathbb{R}^m\},$$

simply because any matrix $[\mathbf{a}_1 \cdots \mathbf{a}_{n-1}\ \mathbf{b}]$ with $[\mathbf{a}_1 \cdots \mathbf{a}_{n-1}] \in W'$ can be written as

$$[\mathbf{a}_1 \cdots \mathbf{a}_{n-1}\ \mathbf{p}_0] + [\mathbf{0} \cdots \mathbf{0}\ \mathbf{b} - \mathbf{p}_0] \in (L_{\mathbf{p}_0,\mathbf{e}_n} \cap W) + R_{<n}.$$

This finishes the proof. □

We can now use the above proposition to furnish the following proof.

Proof of Theorem 1.2(a). Indeed, it is clear that the collection $\mathcal{L}$ as in (3.1) is aligned with $\mathcal{X}_{m,n}$, and its total density is provided by Proposition 3.1. Let $\mathcal{R}$ be the collection of all proper rational affine subspaces of $M_{m,n}(\mathbb{R})$. Obviously if L and R are two subspaces of a finite-dimensional vector space such that $L \cap R$ has non-empty interior in L, then $\dim(L \cap R) = \dim(L)$, which implies that $L \subset R$. Hence $\mathcal{L}$ respects any subspace of $M_{m,n}(\mathbb{R})$. This verifies all the conditions of Theorem 1.5; since elements of $\mathcal{L}$ have no isolated points, part (ii) implies the uncountability of $\mathrm{UA}^*_{m,n}(f)$ for any positive non-increasing f. □

More generally, one can strengthen Theorem 1.2(a) using the fact that $\mathcal{L}$ respects closed analytic submanifolds of $M_{m,n}(\mathbb{R})$. Recall that $\Psi : U \to \mathbb{R}^k$, where U is an open subset of $\mathbb{R}^d$ with $d \leqslant k$, is called a *real analytic immersion* if it is injective, each of its coordinate functions $\Psi_i : U \to \mathbb{R}$ $(i = 1, \dots, k)$ is infinitely differentiable, the Taylor series of each Ψ_i converges in a neighborhood of every $\mathbf{x} \in U$, and the derivative mapping $d_{\mathbf{x}}\Psi : \mathbb{R}^d \to \mathbb{R}^k$ has rank d. By a *d-dimensional real analytic submanifold of $\mathbb{R}^k$* we mean a subset $Y \subset \mathbb{R}^k$ such that for every $\mathbf{y} \in Y$ there is a neighborhood $V \subset \mathbb{R}^k$ containing $\mathbf{y}$, an open set $U \subset \mathbb{R}^d$, and a real analytic immersion $\Psi : U \to \mathbb{R}^k$ such that $V \cap Y = \Psi(U)$.

The crucial property, which distinguishes real analytic submanifolds from smooth manifolds and follows easily from the definitions, is given in the following lemma.

Lemma 3.2. *Let L and R be real analytic submanifolds of $\mathbb{R}^k$, equipped with the topology they inherit as subsets of $\mathbb{R}^k$. Suppose that L is connected, R is closed, and $L \cap R$ has non-empty interior in L. Then $L \subset R$.*

To make the paper self-contained we provide the proof of this lemma.

Proof. Let d_1, d_2 denote the dimensions of L and R. Since $L \cap R$ has non-empty interior in L, by invariance of dimension under immersions, we have $d_1 \leqslant d_2$. Let W denote the interior (in L) of $L \cap R$. If W is closed (in L), then by connectedness of L we have $L = W$, and hence $L \subset R$ and there is nothing to prove. We will assume that W is not closed in L and derive a contradiction.

Let $p \in \overline{W} \smallsetminus W$. Since the closure is taken in L we have $p \in L$, and since R is closed we have $p \in R$. By the real analyticity of L and R, there are open subsets $U_1 \subset \mathbb{R}^{d_1}$, $U_2 \subset \mathbb{R}^{d_2}$ and real analytic immerions $\Psi_i : U_i \to \mathbb{R}^k$ such that Ψ_1 parametrizes a neighborhood of p in L, and Ψ_2 parametrizes a neighborhood of p in R. For $i = 1, 2$, let $p_i \in U_i$ such that $\Psi_i(p_i) = p$, and let $W_i = \Psi_i^{-1}(W)$. Then p_1 belongs to the closure of W_1 in U_1, and W_1 is open in U_1; thus by the uniqueness of analytic continuation (see [KP92, § 1.2] or [KN63, § VI.6]), Ψ_i is uniquely

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determined in a neighborhood of p by $\Psi_1|_{W_1}$. In particular, L is uniquely determined in a neighborhood of p by W.

By the inverse function theorem for real analytic immersions (see [KP92, § 1.8]), there is a real analytic inverse Ψ_2^{-1} to Ψ_2. In an abuse of notation, denote by $\mathbb{R}^{d_1}$ the coordinate plane in $\mathbb{R}^{d_2}$ defined as

$$\mathbb{R}^{d_1} \stackrel{\text{def}}{=} \{(x_1, \dots, x_{d_2}) \in \mathbb{R}^{d_2} : x_{d_1+1} = \dots = x_{d_2} = 0\}.$$

By making U_2 smaller and by replacing U_2 with its image under a real analytic diffeomorphism, we can assume that W_2 is open in $U_2 \cap \mathbb{R}^{d_1}$; indeed, to see that such a change of variables exists, see [KP92, Proof of Theorem 1.9.5]. Once again, using the uniqueness of analytic continuation of $\Psi_2|_{W_2}$, we see that $\Psi_2|_{U_2 \cap \mathbb{R}^{d_1}}$ is determined by $\Psi_2|_{W_2}$. Since the two analytic continuations agree, we see that p belongs to the interior of $L \cap R$. □

Arguing as in the above proof of Theorem 1.2(a), we arrive at the following.

Theorem 3.3. *Let $m, n \in \mathbb{N}$ with $n > 1$, and let a non-increasing function $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ and an arbitrary countable collection $\{R_\ell\}$ of proper closed analytic submanifolds of $M_{m,n}(\mathbb{R})$ be given. Then $\mathrm{UA}_{m,n}(f) \smallsetminus \cup_\ell R_\ell$ is uncountable and dense in $M_{m,n}(\mathbb{R})$.*

Likewise, for any weight vector $\boldsymbol{\omega} = (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbb{R}^{m+n}_{>0}$ as in § 1.4, in view of (1.13) the conclusion of the above theorem holds verbatim for the set $\mathrm{UA}_{m,n}(f, \boldsymbol{\omega})$. Moreover, one can take a subset $\mathcal{W}$ of $\mathbb{R}^{m+n}_{>0}$ and say that $A \in M_{m,n}(\mathbb{R})$ is $(f, \mathcal{W})$-*uniform*, denoted with some abuse of notation by $A \in \mathrm{UA}_{m,n}(f, \mathcal{W})$, if for every sufficiently large t one can find $\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}$ and $\mathbf{p} \in \mathbb{Z}^m$ such that for any $\boldsymbol{\omega} = (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{W}$ the inequalities (1.12) hold. Clearly this is stronger than being $(f, \boldsymbol{\omega})$-uniform for any $\boldsymbol{\omega} \in \mathcal{W}$. The next statement follows from Theorem 1.5 as easily as the previous one did. It immediately implies both parts of Theorem 1.7.

Theorem 3.4. *Let $m, n \in \mathbb{N}$ with $n > 1$. Suppose that for any $k \in \mathbb{N}$ we are given a non-increasing function $f_k : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ and a subset $\mathcal{W}_k$ of $\mathbb{R}^{m+n}_{>0}$ such that*

$$\sup_{(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{W}_k} \max_i \alpha_i < \infty \quad \text{and} \quad \inf_{(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{W}_k} \min_j \beta_j > 0. \tag{3.3}$$

Then for any countable collection $\{R_\ell\}$ of proper analytic submanifolds of $M_{m,n}(\mathbb{R})$, the set

$$\bigcap_k \mathrm{UA}_{m,n}(f_k, \mathcal{W}_k) \smallsetminus \bigcup_\ell R_\ell$$

is uncountable and dense in $M_{m,n}(\mathbb{R})$. In particular, the set

$$\bigcap_{\boldsymbol{\omega} \in \mathbb{R}^{m+n}_{>0}} \mathrm{UA}_{m,n}(f, \boldsymbol{\omega}) \smallsetminus \bigcup_\ell R_\ell$$

is uncountable and dense in $M_{m,n}(\mathbb{R})$ for any $\{R_\ell\}$ as above and any non-increasing positive function f.

Proof. Again we take $Y = M_{m,n}(\mathbb{R}) = X_k$, $\mathcal{I} = \mathbb{Z}^m \times (\mathbb{Z}^n \smallsetminus \{0\})$, and $\varphi_k = \mathrm{Id}$ for each k, and, to define the Diophantine systems associated with each X_k, we let

$$h_{k,(\mathbf{p},\mathbf{q})} \stackrel{\text{def}}{=} \sup_{\boldsymbol{\omega} \in \mathcal{W}_k} \|\mathbf{q}\|_{\boldsymbol{\beta}} \quad \text{and} \quad d_{k,(\mathbf{p},\mathbf{q})}(A) \stackrel{\text{def}}{=} \inf_{\boldsymbol{\omega} \in \mathcal{W}_k} \|A\mathbf{q} - \mathbf{p}\|_{\boldsymbol{\alpha}}. \tag{3.4}$$

Because of (3.3), for any $(\mathbf{p}, \mathbf{q})$ and any k the value $h_{k,(\mathbf{p},\mathbf{q})}$ is finite and the function $d_{k,(\mathbf{p},\mathbf{q})}$ is continuous. Thus the same argument, in view of Remark 1.6(2), yields the proof of the first part of the theorem. For the 'in particular' part one simply writes $\mathbb{R}^{m+n}_{>0}$ as the union of countably many subsets $\mathcal{W}_k$ satisfying (3.3). □

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4. Uniformly approximable row vectors on analytic submanifolds and some fractals

Our next goal is to apply Theorem 1.5 to study uniform approximation on submanifolds Y of $M_{m,n}(\mathbb{R})$. It is clear that the method we are using in this paper (which is essentially Khintchine's original argument) is not applicable if $\dim Y = 1$; indeed, there are very few results in the literature dealing with singular vectors on curves, and many open questions (see e.g. [PR22] and the references therein). However, the method does work if $m = 1$, Y is a connected analytic submanifold of $\mathbb{R}^n \cong M_{1,n}(\mathbb{R})$, and the dimension of Y is at least 2 (see Theorem 1.8).

In this section we will generalize the aforementioned theorem, but first let us observe that the situation gets more complicated when $\min(m,n) > 1$. The next proposition gives an example of a three-dimensional submanifold (in fact, an affine subspace) of $M_{2,2}(\mathbb{R})$ which does not contain any f-uniform matrices if f decays rapidly enough. Let us recall that a real number α is badly approximable if

$$\inf_{q\in\mathbb{Z}\smallsetminus\{0\}} |q|\,\operatorname{dist}(\alpha q,\mathbb{Z}) > 0. \tag{4.1}$$

Proposition 4.1. *Let $\alpha\in\mathbb{R}$ be a badly approximable number, let $\lambda=(\sqrt{5}+3)/2$, and let Y be the set of 2×2 matrices of the form $A=\begin{pmatrix}\alpha & *\\ * & *\end{pmatrix}$. Then $\hat\omega(A)\leqslant\lambda$ for any totally irrational $A\in Y$; that is, no totally irrational $A\in Y$ is f-uniform as long as f decays faster than $\phi_{\lambda+\varepsilon}$ for some $\varepsilon>0$.*

Proof. We need to recall the notion of an *ordinary Diophantine exponent*, which is an asymptotic version of (1.5); namely,

$$\begin{aligned}\omega(A) &\stackrel{\text{def}}{=} \sup\left\{a>0 \;\middle|\; \begin{array}{c}\text{for an unbounded set of } t>0\\ \exists\,\mathbf{q}\in\mathbb{Z}^n\smallsetminus\{0\},\ \mathbf{p}\in\mathbb{Z}^m\\ \text{such that (1.2) holds for } f=\phi_a\end{array}\right\}\\ &= \inf\Big\{a>0 : \inf_{\mathbf{q}\in\mathbb{Z}^n\smallsetminus\{0\}} \|\mathbf{q}\|^a\operatorname{dist}(A\mathbf{q},\mathbb{Z}^m)>0\Big\},\end{aligned}$$

where the distance is computed using the supremum norm on $\mathbb{R}^m$. It follows from (4.1) that $\omega(\alpha)\leqslant 1$. Take an arbitrary $\beta\in\mathbb{R}$ such that $1,\alpha,\beta$ are linearly independent over $\mathbb{Q}$, and consider $\mathbf{v}=\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$. Clearly we have

$$\inf_{q\in\mathbb{Z}\smallsetminus\{0\}} q\,\operatorname{dist}(\mathbf{v}q,\mathbb{Z}^2)>0;$$

hence $\omega(\mathbf{v})\leqslant 1$ as well.

Now we will use the inequalities due to Jarník relating the ordinary and the uniform exponents of $\mathbf{v}$ and $\mathbf{v}^T$:

$$\omega(\mathbf{v}) \geqslant \frac{\hat\omega(\mathbf{v})^2}{1-\hat\omega(\mathbf{v})} \tag{4.2}$$

and

$$\hat\omega(\mathbf{v}) + \frac{1}{\hat\omega(\mathbf{v}^T)} = 1 \tag{4.3}$$

(see [Jar38] and [Jar54] respectively). It follows from (4.2) that $\hat\omega(\mathbf{v})\leqslant(\sqrt{5}-1)/2$, and from (4.3) we can obtain $\hat\omega(\mathbf{v}^T)\leqslant(\sqrt{5}+3)/2=\lambda$. Thus for any $A\in M_{2,2}(\mathbb{R})$ with (α,β) as its row vector we have $\hat\omega(A)\leqslant\hat\omega(\mathbf{v}^T)\leqslant\lambda$. □

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Our next result shows that such examples are impossible when $m=1$ and the dimension d of the submanifold Y of $\mathbb{R}^n \cong M_{1,n}$ is at least 2. Furthermore, we will exhibit vectors in Y which are f-uniform of order $d-1$, as defined in § 1.5.

Theorem 4.2. *Let $n \geqslant 2$, let Y be a d-dimensional connected analytic submanifold of $\mathbb{R}^n$, where $d \geqslant 2$, and let $1 \leqslant g \leqslant d-1$. Suppose we are given an arbitrary non-increasing function $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ and a countable collection $\{R_\ell\}$ of proper closed analytic submanifolds of Y. Then the set*

$$Y \cap \mathrm{UA}_{1,n}(f;g) \smallsetminus \bigcup_{\ell \in \mathbb{N}} R_\ell \tag{4.4}$$

is uncountable and dense in Y. In particular, if one in addition assumes that Y is not contained in any proper rational affine subspace of $\mathbb{R}^n$, then the set $Y \cap \mathrm{UA}^*_{1,n}(f;g)$ is uncountable and dense in Y.

To prove Theorem 4.2 we first need to describe the Diophantine system responsible for approximation of order g. Unwrapping Definition 1.9, we can easily see that $\mathbf{x} \in \mathbb{R}^n$ is f-uniform of order g if and only if it is $(\mathcal{X}, f)$-uniform, where $\mathcal{X}$ is the Diophantine system given by

$$\begin{gathered} X = \mathbb{R}^n, \ \ \mathcal{I} = \mathbb{Z}^g \times ((\mathbb{Z}^n)^g \smallsetminus \{\text{linearly dependent } g\text{-tuples}\}), \\ h_{p_1,\dots,p_g,\mathbf{q}_1,\dots,\mathbf{q}_g} = \max_{i=1,\dots,g} \|\mathbf{q}_i\|, \\ d_{p_1,\dots,p_g,\mathbf{q}_1,\dots,\mathbf{q}_g}(\mathbf{x}) = \max_{i=1,\dots,g} |\mathbf{q}_i \cdot \mathbf{x} - p_i|. \end{gathered} \tag{4.5}$$

Note that the zero locus of $d_{p_1,\dots,p_g,\mathbf{q}_1,\dots,\mathbf{q}_g}$ is precisely the intersection of g rational affine hyperplanes $L_{p_i,\mathbf{q}_i}$, which is a codimension g rational affine subspace of $\mathbb{R}^n$; moreover, any rational affine subspace of $\mathbb{R}^n$ of codimension g can be (in many different ways) written as such an intersection. Consequently, the collection $\mathcal{L}$ of all rational affine subspaces of $\mathbb{R}^n$ of codimension g is aligned with $\mathcal{X}$ as in (4.5). It is trivial to check that $\mathcal{L}$ is totally dense and respects any closed analytic submanifold of $\mathbb{R}^n$. Hence the case $Y = \mathbb{R}^n$ of Theorem 4.2 easily follows from Theorem 1.5. Let us now prove the general case where Y is an arbitrary d-dimensional analytic submanifold of $\mathbb{R}^n$. For the proof we will need the following lemma.

Lemma 4.3. *Let Y be a d-dimensional C^1 embedded submanifold of $\mathbb{R}^n$, where $d \geqslant 2$, and let $1 \leqslant g \leqslant d-1$. Then for any $\mathbf{y} \in Y$ there exists an open subset W of $\mathbb{R}^n$ containing $\mathbf{y}$ and a totally dense (in $Y \cap W$) collection $\mathcal{L}_W$ consisting of intersections of rational affine subspaces of $\mathbb{R}^n$ of codimension g with $Y \cap W$.*

Proof. For any $\mathbf{y} \in Y$ one can choose a neighborhood $V \subset \mathbb{R}^n$ containing $\mathbf{y}$, an open set $U \subset \mathbb{R}^d$, and a differentiable embedding $\Psi : U \to \mathbb{R}^n$ such that $V \cap Y = \Psi(U)$. Let $\mathbf{x} \in U$ be such that $\Psi(\mathbf{x}) = \mathbf{y}$; then $d_{\mathbf{x}}\Psi$ has rank d, and by permuting coordinates in $\mathbb{R}^n$ without loss of generality we can assume that the leftmost $d \times d$ submatrix of $d_{\mathbf{x}}\Psi$ is non-singular. Then, in view of the implicit function theorem, we can choose an open subset W of B containing $\mathbf{y}$ such that

$$W \cap Y = \{(x_1, \dots, x_d, \psi(x_1, \dots, x_d)) : (x_1, \dots, x_d) \in O\},$$

where O is an open ball in $\mathbb{R}^d$ and $\psi : O \to \mathbb{R}^{n-d}$ is a differentiable function.

Now define

$$\mathcal{L}_W \stackrel{\text{def}}{=} \{L_M \cap W \cap Y : M \text{ is a rational affine subspace of } \mathbb{R}^d \text{ of codimension } g\},$$

where

$$L_M \stackrel{\text{def}}{=} \{(x_1, \dots, x_n) : (x_1, \dots, x_d) \in M\}. \tag{4.6}$$

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Equivalently, $L_M \cap W \cap Y$ can be written as

$$\{(x_1, \dots, x_d, \psi(x_1, \dots, x_d)) : (x_1, \dots, x_d) \in M \cap O\}. \tag{4.7}$$

Since $d \geqslant 2$ and $1 \leqslant g \leqslant d-1$, the collection

$$\{M \cap O : M \text{ is a rational affine subspace of } \mathbb{R}^d \text{ of codimension } g\}$$

is totally dense in O, which immediately implies that $\mathcal{L}_W$ is totally dense in $Y \cap W$. □

Proof of Theorem 4.2. Our goal is to apply Theorem 1.5 locally in the neighborhood of any given point $\mathbf{y} \in Y$. Choose an open subset W of $\mathbb{R}^n$ containing $\mathbf{y}$ and a collection $\mathcal{L}_W$ as in the above lemma. Since L_M as in (4.6) is a codimension g rational affine subspace of $\mathbb{R}^n$, it is clear that $\mathcal{L}$ is aligned with the Diophantine system (4.5). Now let us recall that Y was assumed to be an analytic manifold; hence the map ψ is real analytic, and therefore every $L_M \cap W \cap Y$ is a connected analytic submanifold of $W \cap Y$. If R is a closed analytic submanifold of Y such that $L_M \cap W \cap R$ has non-empty interior in $L_M \cap W$, it follows that $L_M \subset W \cap R$, which, in view of Lemma 3.2, implies that $L_M \subset R$. Hence $\mathcal{L}_W$ respects R, and we can apply Theorem 1.5(b) and conclude that the intersection of the set (4.4) with W is uncountable, finishing the proof of the theorem. □

Let us now show that the same method can produce uniformly approximable vectors on certain fractals. Below is an example from [KMW21].

Theorem 4.4. *Let $n \geqslant 2$ and let $Y_1, \dots, Y_n$ be perfect subsets of $\mathbb{R}$ such that*

$$\mathbb{Q} \cap Y_k \text{ is dense in } Y_k \text{ for each } k \in \{1, 2\}. \tag{4.8}$$

Let $Y = \prod_{j=1}^n Y_j$. Suppose we are given an arbitrary non-increasing function $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ and a countable collection $\{R_\ell\}$ of proper closed analytic submanifolds of $\mathbb{R}^n$. Then

$$Y \cap \mathrm{UA}_{1,n}(f) \smallsetminus \bigcup_{\ell \in \mathbb{N}} R_\ell$$

is uncountable and dense in Y. In particular, $Y \cap \mathrm{UA}^*_{1,n}(f)$ is uncountable and dense in Y.

Proof. Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the standard basis vectors in $\mathbb{R}^n$, and let $\{A_i\}$ be the collection of all rational affine hyperplanes of $\mathbb{R}^n$ which are normal to one of $\mathbf{e}_1$ and $\mathbf{e}_2$ and have non-trivial intersection with Y; that is, each of the A_i is of the form

$$A_i = \{\mathbf{x} \in \mathbb{R}^n : x_{k_i} = r_i\}, \text{ where } r_i \in \mathbb{Q} \text{ and } k_i \in \{1, 2\};$$

note that necessarily we have $r_i \in Y_{k_i}$. We claim that the collection $\{Y \cap A_i\}$, which is obviously aligned with $\mathcal{X}_{1,n}$, is totally dense. Indeed, (1.7) clearly follows from (4.8). To prove (1.9), take an open subset W of $\mathbb{R}^n$ of the form $I_1 \times \cdots \times I_n$, where I_j are open intervals in $\mathbb{R}$, and suppose that

$$A = \{\mathbf{x} \in \mathbb{R}^n : x_1 = r\} \tag{4.9}$$

intersects with Y non-trivially, that is, we have $r \in Y_1$. Then, again in view of (4.8), the union of subspaces A_j such that $A_j \cap A \cap Y \cap W \neq \varnothing$ is dense in $(I_1 \cap Y_1) \times \mathbb{R} \times \cdots \times \mathbb{R}$, and hence the union of corresponding intersections $Y \cap A_j$ is dense in $Y \cap (I_1 \times \mathbb{R} \times \cdots \times \mathbb{R})$.

It remains to prove that the collection $\{Y \cap A_i\}$ respects an arbitrary closed analytic submanifold R of $\mathbb{R}^n$. Suppose that, for A as in (4.9), the intersection $Y \cap A \cap R$ has a non-empty interior in $Y \cap A$. Take a point $\mathbf{x} = (x_1, \dots, x_n)$ in the interior of $Y \cap A \cap R$; then there exists a neighborhood W of $\mathbf{x}$ in $\mathbb{R}^n$ such that $Y \cap A \cap R$ contains $(\{x_1\} \times Y_2 \times \cdots \times Y_n) \cap W$. Since

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each of the sets Y_i is perfect, there are sequences in $Y \cap A \cap R$ approaching $\mathbf{x}$ from each coordinate direction. This makes it possible to determine all partial derivatives of all orders of any analytic function on A at $(x_1, \dots, x_n)$ by its values on $Y \cap A \cap R$. In particular, since R is a closed analytic manifold, it has to contain A, and hence it is respected by $Y \cap A$.

Now we can conclude that Theorem 1.5 applies and implies the density of $Y \cap \mathrm{UA}_{1,n}(f) \smallsetminus \bigcup_{\ell \in \mathbb{N}} R_\ell$ for any countable set of proper closed analytic submanifolds R_ℓ of $\mathbb{R}^n$. The uncountability follows as well, since the sets $\{Y \cap A_i\}$ have no isolated points. □

Remark 4.5. Our arguments can be adapted to many other fractal sets. We sketch one more example involving a 'rational Koch snowflake' which can be treated by our method. To define it, let $\alpha, \beta, \gamma, \delta$ be positive rational numbers, where $\alpha < \beta < \delta < 1$, and let Y be the attractor of the iterated function system $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$, where $\varphi_i : \mathbb{R}^2 \to \mathbb{R}^2$ is the unique orientation-preserving contracting similarity map sending the points $\mathbf{x}_0 = (0,0)$ and $\mathbf{y}_0 = (1,0)$, to the points $\mathbf{x}_i, \mathbf{y}_i$ defined by

$$\begin{aligned} \mathbf{x}_1 &= (0,0), & \mathbf{y}_1 &= (\alpha, 0) \\ \mathbf{x}_2 &= \mathbf{y}_1, & \mathbf{y}_2 &= (\beta, \gamma) \\ \mathbf{x}_3 &= \mathbf{y}_2, & \mathbf{y}_3 &= (\delta, 0) \\ \mathbf{x}_4 &= \mathbf{y}_3, & \mathbf{y}_4 &= (1,0). \end{aligned}$$

Then Y is a topological curve (continuous image of a segment) of Hausdorff dimension greater than one. The usual version of the Koch snowflake has a similar description, with $\alpha = \frac{1}{3}$, $\beta = \frac{1}{2}$, $\gamma = \frac{\sqrt{3}}{6}$, $\delta = \frac{2}{3}$ (see [Fal04, Figure 0.2]).

For our application, it is important to note that if θ is the slope of the line from $\mathbf{x}_2$ to $\mathbf{y}_2$, then Y contains a dense collection of rational points Y_0, and the horizontal lines and lines of slope θ through points of Y_0 intersect Y in an uncountable perfect set. This property makes it possible to repeat the proof of Theorem 5.2, taking the sets $\{A_i\}$ to be the lines having these two slopes and passing through Y_0.

5. Vectors that are k-singular for all k

In this section we prove a strengthening of Theorem 1.11. To state it, let us consider a definition generalizing the notion of k-singular vectors. Namely, for a non-increasing $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ we say that $\mathbf{x} \in \mathbb{R}^n$ is *f-uniform of degree k* if for any sufficiently large t there exists a polynomial $P \in \mathbb{Z}[X_1, \dots, X_n]$ such that $\deg(P) \leqslant k$, $H(P) \leqslant t$, and $|P(\mathbf{x})| \leqslant f(t)$. Clearly $\mathbf{x}$, viewed as a row vector, is k-singular (as defined in § 1.6) if and only if it is $\varepsilon\phi_{N_k}$-uniform of degree k, where N_k is as in (1.14). Also it is clear that k-algebraic vectors (again see § 1.6 for a definition) are f-uniform of degree k for any positive f.

Let us say that a submanifold $Y \subset \mathbb{R}^n$ is *transcendental* if Y is not contained in an algebraic subvariety defined over $\mathbb{Q}$ or, more concretely, if there is no nonzero polynomial in $\mathbb{Z}[X_1, \dots, X_n]$ which vanishes on Y.

Theorem 5.1. *Let $n \geqslant 2$, and let Y be a transcendental analytic submanifold of $\mathbb{R}^n$ of dimension $d \geqslant 2$. Suppose that for each $k \in \mathbb{N}$ we are given a non-increasing $f_k : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$. Then there is a dense uncountable set of $\mathbf{x} \in Y$ which are not algebraic and are f_k-uniform of degree k for all $k \in \mathbb{N}$. In particular, Y contains a dense uncountable set of $\mathbf{x}$ which are not algebraic and are k-singular for all $k \in \mathbb{N}$.*

This result extends [KMW21, Theorem 1.7]; namely, there it was shown that Y as above contains uncountably many singular vectors.

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The proof is based on the following observation. For $\mathbf{x} \in \mathbb{R}^n$, denote by $\varphi_k(\mathbf{x})$ the vector with coordinates consisting of all non-constant monomials of degree at most k in n variables $x_1, \dots, x_n$, viewed as a row vector in $\mathbb{R}^{N(k,n)}$, and for each k consider the standard Diophantine system $\mathcal{X}_{1,N(k,n)}$ on $X_k = \mathbb{R}^{N(k,n)} \cong M_{1,N(k,n)}$ with distance functions from rational affine hyperplanes and standard heights. Then it is clear that $\mathbf{x} \in \mathbb{R}^n$ is f_k-uniform of degree k if and only if $\varphi_k(\mathbf{x}) \in \mathrm{UA}_{1,N(k,n)}(f_k)$. The proof is as follows.

Proof of Theorem 5.1. Take Y as in the statement of Theorem 5.1 and argue as in the proof of Theorem 4.2. Namely, for any $\mathbf{y} \in Y$ define:

- a neighborhood W of $\mathbf{y}$ such that $Y \cap W$ is a graph of an analytic function $\mathbb{R}^d \to \mathbb{R}^{n-d}$; and
- (using Lemma 4.3 with $g = 1$) a totally dense collection $\mathcal{L}_W$ of intersections of some rational affine hyperplanes of $\mathbb{R}^n$ with $Y \cap W$.

It is clear that $\mathcal{L}_W$ as above is aligned with $\mathcal{X}_{1,N(k,n)}$ via φ_k: indeed, if $\pi : \mathbb{R}^{N(k,n)} \to \mathbb{R}^n$ denotes the projection onto the first n coordinates and L is a rational affine hyperplane in $\mathbb{R}^n$, then $\varphi_k(L)$ is contained in $\pi^{-1}(L)$, which is a rational affine hyperplane in $\mathbb{R}^{N(k,n)}$. Now let $\{R_\ell\}$ be the collection of sets of the form $\{\mathbf{y} \in Y : P(\mathbf{y}) = 0\}$, where P ranges over all nonzero polynomials in $\mathbb{Z}[X_1, \dots, X_n]$. Since Y is assumed to be transcendental, it follows that each of the sets R_ℓ is a proper subset of Y. It is easy to show that $\mathcal{L}_W$ respects R_ℓ: indeed, any element L of $\mathcal{L}_W$ is of the form (4.7) with analytic ψ; furthermore, the fact that $L \cap R_\ell$ has non-empty interior in L implies that $P(x_1, \dots, x_d, \psi(x_1, \dots, x_d)) = 0$ on an open subset of an affine hyperplane M of $\mathbb{R}^d$. Since $d \geqslant 2$ and ψ is analytic, it follows that $L \subset R_\ell$. Hence Theorem 1.5 applies, and we conclude that the set

$$\bigcap_k \varphi_k^{-1}(\,\mathrm{UA}_{1,N(k,n)}(f_k)) \smallsetminus \bigcup_\ell R_\ell,$$

which contains the set of vectors that are not algebraic and are f_k-uniform of degree k for all $k \in \mathbb{N}$, is uncountable and dense in Y for any choice of non-increasing functions $f_k : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$. □

Using similar ideas and arguing as in the proof of Theorem 4.4 and Remark 4.5, one can establish a similar result for fractals.

Theorem 5.2. *Let Y be either the product of perfect subsets $Y_1, \dots, Y_n$ of $\mathbb{R}$ such that (4.8) holds ($n \geqslant 2$), or a rational Koch snowflake ($n = 2$). Then the conclusions of Theorem 5.1 hold for Y.*

6. Sumsets of sets of totally irrational uniformly approximable vectors

Our goal in this section is to prove Theorem 1.12, which states that when $n \geqslant 3$, the sum of $\mathrm{UA}^*_{1,n}(f_1)$ and $\mathrm{UA}^*_{1,n}(f_2)$ is $\mathbb{R}^n$ for any pair of approximating functions f_1, f_2. Clearly, by replacing each of the functions f_i with $\min(f_1, f_2)$, it is enough to take a non-increasing $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ and prove that $\mathrm{UA}^*_{1,n}(f) + \mathrm{UA}^*_{1,n}(f) = \mathbb{R}^n$. Equivalently, since $\mathrm{UA}^*_{1,n}(f)$ coincides with $-\mathrm{UA}^*_{1,n}(f)$, we need to show that for any fixed $\mathbf{z} \in \mathbb{R}^n$ the intersection of $\mathrm{UA}^*_{1,n}(f)$ and its translate $\mathrm{UA}^*_{1,n}(f) - \mathbf{z}$ is not empty. Unwrapping the definitions, we see that this amounts to finding $\mathbf{x} \in \mathbb{R}^n$ such that both $\mathbf{x}$ and $\mathbf{x} + \mathbf{z}$ are totally irrational and such that for every sufficiently large t one can find $\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}$ and $p \in \mathbb{Z}$ with

$$|\mathbf{x} \cdot \mathbf{q} - p| \leqslant f(t) \quad \text{and} \quad \|\mathbf{q}\| \leqslant t,$$

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and also $\mathbf{q}' \in \mathbb{Z}^n \smallsetminus \{0\}$ and $p' \in \mathbb{Z}$ with

$$|(\mathbf{x}-\mathbf{z})\cdot\mathbf{q}' - p'| \leqslant f(t) \quad \text{and} \quad \|\mathbf{q}'\| \leqslant t.$$

Rewriting the two systems of inequalities above as

$$\max(|\mathbf{x}\cdot\mathbf{q}-p|, |(\mathbf{x}-\mathbf{z})\cdot\mathbf{q}' - p'|) \leqslant f(t) \quad \text{and} \quad \max(\|\mathbf{q}\|, \|\mathbf{q}'\|) \leqslant t,$$

we see that the problem reduces to a new Diophantine system

$$\mathcal{X}_{\mathbf{z}} := (\mathbb{R}^n, \{d_{p,p',\mathbf{q},\mathbf{q}'}\}, \{h_{p,p',\mathbf{q},\mathbf{q}'}\})$$

where $d_{p,p',\mathbf{q},\mathbf{q}'}(\mathbf{x}) := \max(|\mathbf{x}\cdot\mathbf{q}-p|, |(\mathbf{x}-\mathbf{z})\cdot\mathbf{q}'-p'|)$, $h_{p,p',\mathbf{q},\mathbf{q}'} := \max(\|\mathbf{q}\|, \|\mathbf{q}'\|)$, and $(p, p', \mathbf{q}, \mathbf{q}')$ runs through $\mathbb{Z}\times\mathbb{Z}\times(\mathbb{Z}^n\smallsetminus\{0\})\times(\mathbb{Z}^n\smallsetminus\{0\})$. Note that the zero locus of $d_{p,p',\mathbf{q},\mathbf{q}'}$ is precisely the intersection $L_{p,\mathbf{q}} \cap (L_{p',\mathbf{q}'} + \mathbf{z})$ of two hyperplanes in $\mathbb{R}^n$; in other words, the collection

$$\mathcal{L}_{\mathbf{z}} \stackrel{\text{def}}{=} \left\{ L_{p,\mathbf{q}} \cap (L_{p',\mathbf{q}'} + \mathbf{z}) \;\middle|\; \begin{array}{c} p, p' \in \mathbb{Z},\ \mathbf{q}, \mathbf{q}' \in \mathbb{Z}^n \smallsetminus \{0\}, \\ \mathbf{q} \text{ and } \mathbf{q}' \text{ are not proportional} \end{array} \right\}$$

is aligned with the Diophantine system $\mathcal{X}_{\mathbf{z}}$.

The crucial step of the proof of Theorem 1.12 will be the following lemma.

Lemma 6.1. *Let $n \geqslant 3$ and $\mathbf{z} \in \mathbb{R}^n$. Then $\mathcal{L}_{\mathbf{z}}$ is totally dense relative to the collection of all affine hyperplanes in $\mathbb{R}^n$.*

Note that we are not able to prove the stronger form of total density, that is, the validity of (1.9) for $\mathcal{L} = \mathcal{L}_{\mathbf{z}}$.

Proof of Lemma 6.1. Let us start with the following elementary sublemma.

Sublemma 6.2. *Let $n \geqslant 3$ and let N be an affine hyperplane of $\mathbb{R}^n$. Then the collection*

$$\mathcal{L}(N) \stackrel{\text{def}}{=} \{M \cap N : M \text{ is a rational affine hyperplane of } \mathbb{R}^n, M \neq N\}$$

is totally dense in N; furthermore, the union in (1.9) is dense in N.

Proof. By permuting coordinates, without loss of generality we can express N in the form

$$x_n = a_1x_1 + \cdots + a_{n-1}x_{n-1} + a_n \quad \text{where } a_i \in \mathbb{R};$$

in this way the projection $\pi : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1})$ maps N bijectively onto $\mathbb{R}^{n-1}$. Furthermore, any $L \in \mathcal{L}(N)$ is of the form

$$\{\mathbf{x} \in \mathbb{R}^n : x_n = a_1x_1 + \cdots + a_{n-1}x_{n-1} + a_n,\ q_1x_1 + \cdots + q_nx_n = p\} \tag{6.1}$$

for some $\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}$ and $p \in \mathbb{Z}$. It suffices to prove that the collection $\pi(\mathcal{L}(N))$ of affine hyperplanes of $\mathbb{R}^{n-1}$ is totally dense in $\mathbb{R}^{n-1}$. By considering the case $q_n = 0$ in (6.1), it is easy to see that $\pi(\mathcal{L}(N))$ contains the collection of all rational affine hyperplanes of $\mathbb{R}^{n-1}$. The latter collection, in additional to being totally dense, has the following property: for any affine hyperplane L of $\mathbb{R}^{n-1}$, any open $W \subset \mathbb{R}^{n-1}$ with $W \cap L \neq \varnothing$ and any $\mathbf{y} \in \mathbb{Q}^{n-1} \smallsetminus L$ there exists $\mathbf{y}' \in \mathbb{Q}^{n-1}$ such that the (rational) line passing through $\mathbf{y}$ and $\mathbf{y}'$ intersects $W \cap L$. This clearly implies that the union of all rational affine hyperplanes touching $W \cap L \neq \varnothing$ is dense in $\mathbb{R}^{n-1}$, and hence the total density of $\pi(\mathcal{L}(N))$. □

The above sublemma implies, in particular, that for any $N = L_{p',\mathbf{q}'} + \mathbf{z}$ the union

$$\bigcup_{p\in\mathbb{Z},\, \mathbf{q}\in\mathbb{Z}^n\setminus\mathbb{R}\mathbf{q}} L_{p,\mathbf{q}} \cap N$$

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is dense in N. Since the union of all rational affine hyperplanes translated by $\mathbf{z}$ is dense in $\mathbb{R}^n$, it follows that $\bigcup_{L\in\mathcal{L}_\mathbf{z}} L$ is dense in $\mathbb{R}^n$, i.e. condition (1.7) holds.

Now take $L\in\mathcal{L}_\mathbf{z}$, that is, $L = L_{p,\mathbf{q}}\cap(L_{p',\mathbf{q}'}+\mathbf{z})$ such that $\mathbf{q}$ and $\mathbf{q}'$ are not proportional, and choose an open subset W of $\mathbb{R}^n$ with $L\cap W\neq\varnothing$. For brevity we write $M\stackrel{\text{def}}{=}L_{p,\mathbf{q}}$ and $N\stackrel{\text{def}}{=}L_{p',\mathbf{q}'}+\mathbf{z}$. It follows from Sublemma 6.2 that the union of $P\cap N$ over all rational affine hyperplanes $P\neq N$ that satisfy $P\cap W\cap L\neq\varnothing$ is dense in N. Applying a translation by $-\mathbf{z}$ to the above conclusion one gets that the union of $(P+\mathbf{z})\cap M$ over all rational affine hyperplanes P with $P+\mathbf{z}\neq M$ and $P\cap W\cap L\neq\varnothing$ is dense in M. It follows that the closure of the union of all $L'\in\mathcal{L}_\mathbf{z}$ such that $L'\cap W\cap L\neq\varnothing$ contains $M\cup N$, and thus it cannot be a subset of a single affine hyperplane. □

Now, since $\mathcal{L}_\mathbf{z}$ obviously respects any affine subspace in $\mathbb{R}^n$, we can apply Theorem 1.5 to $\mathcal{X}_\mathbf{z}$ with $\mathcal{R}=\{R_\ell\}$ being an arbitrary countable collection of affine hyperplanes of $\mathbb{R}^n$. This immediately yields the proof of Theorem 1.12 in a slightly more general form, with $\mathrm{UA}^*_{1,n}(f_k)$, $k=1,2$, in (1.15) replaced by $\mathrm{UA}_{1,n}(f_k)\smallsetminus\bigcup_\ell R_\ell$ for $\mathcal{R}$ as above.

Remark 6.3. It is not hard to show, by adapting the above proof, that for any $n\geqslant 3$, any positive non-increasing $f_1,\dots,f_{n-1}$, and any $\mathbf{z}_1,\dots,\mathbf{z}_{n-1}$, we have

$$\bigcap_{k=1}^{n-1}(\mathrm{UA}^*_{1,n}(f_k)-\mathbf{z}_k)\neq\varnothing.$$

Note that our method gives no information on the intersection of translates of $\mathrm{UA}^*_{1,2}(f)$. In particular, it is an open problem to determine whether for any non-increasing $f:\mathbb{R}_{>0}\to\mathbb{R}_{>0}$ the sumset of $\mathrm{UA}^*_{1,2}(f)$ with itself coincides with $\mathbb{R}^2$.

7. Transference and improved rates for vectors in analytic submanifolds of $\mathbb{R}^n$

Given $\mathbf{x}=(x_1,\dots,x_n)\in\mathbb{R}^n$ and real numbers τ,ε,t,η, define the following parallelepipeds:

$$\Pi^{\tau,\varepsilon}\stackrel{\text{def}}{=}\Big\{(z_0,z_1,\dots,z_n)\in\mathbb{R}^{n+1}:\ \max_{1\leqslant j\leqslant n}|z_j|\leqslant\tau,\ |z_0+z_1x_1+\dots+z_nx_n|\leqslant\varepsilon\Big\}$$

and

$$\Pi_{t,\eta}\stackrel{\text{def}}{=}\Big\{(z_0,z_1,\dots,z_n)\in\mathbb{R}^{n+1}:\ |z_0|\leqslant t,\ \max_{1\leqslant j\leqslant n}|z_0x_j-z_j|\leqslant\eta\Big\}.$$

Note that $\Pi^{\tau,\varepsilon}$ encodes information about approximations to $\mathbf{x}$ as a linear form, while $\Pi_{t,\eta}$ encodes approximations to $\mathbf{x}$ as a vector.

With these definitions, we have the following standard transference result:

Lemma 7.1. *Let $1\leqslant g\leqslant n$, suppose that τ,ε,t,η satisfy*

$$\frac{\eta}{t}=\frac{\varepsilon}{\tau}\tag{7.1}$$

and

$$\eta^{n-g}t=(4n)^{2n}\tau^g,\tag{7.2}$$

and assume that the parallelepiped $\Pi^{\tau,\varepsilon}$ contains g linearly independent integer points. Then $\Pi_{t,\eta}$ contains a nonzero integer point.

Proof. Let $\mathbf{z}_1,\dots,\mathbf{z}_g$ be linearly independent integer points in $\Pi^{\tau,\varepsilon}$, let

$$L\stackrel{\text{def}}{=}\operatorname{span}(\mathbf{z}_1,\dots,\mathbf{z}_g)$$

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be the space generated by these points, and denote the orthogonal complement of L by $L^\perp$. Clearly $\Lambda \stackrel{\text{def}}{=} L \cap \mathbb{Z}^{n+1}$ is a lattice in L, and since $L^\perp$ is rational, $\Lambda^\perp = L^\perp \cap \mathbb{Z}^{n+1}$ is a lattice in $L^\perp$. It is well known that the covolumes $\operatorname{covol}(L/\Lambda)$ and $\operatorname{covol}(L^\perp/\Lambda^\perp)$ are equal to each other. Now define $\Omega \stackrel{\text{def}}{=} L \cap \Pi^{\tau,\varepsilon}$ and $\Omega^\perp \stackrel{\text{def}}{=} L^\perp \cap \Pi_{t,\eta}$, and denote by $\operatorname{vol}_g \Omega$ and $\operatorname{vol}_{n+1-g}\Omega^\perp$, respectively, the g-dimensional and $(n-g+1)$-dimensional volumes of Ω and $\Omega^\perp$.

Denote the vector $(1, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$ by $\mathbf{x}_0$, and let

$$M = \{(z_0, z_1, \dots, z_n) \in \mathbb{R}^{n+1} : \, z_0 + z_1 x_1 + \dots + z_n x_n = 0\}$$

be the hyperplane orthogonal to $\mathbf{x}_0$. Let us define

$$\theta_{\max} \stackrel{\text{def}}{=} \max_{\mathbf{u}\in L,\, \mathbf{v}\in M} (\text{angle between } \mathbf{u} \text{ and } \mathbf{v})$$

to be the maximal angle between vectors in L and M, and

$$\theta_{\min} = \min_{\mathbf{u}\in L^\perp} (\text{angle between } \mathbf{u} \text{ and } \mathbf{x}_0).$$

It is easy to see that $\theta_{\min} \leqslant \theta_{\max}$. Indeed, let $\mathbf{u}_0$ be the minimizer in the definition of $\theta_{\min}$, let V be the two-dimensional subspace generated by $\mathbf{x}_0$ and $\mathbf{u}_0$, and let $\mathbf{u}_0^\perp$ and $\mathbf{x}_0^\perp$ be unit vectors in V perpendicular to $\mathbf{u}_0$ and $\mathbf{x}_0$ respectively. Then

$$\theta_{\min} = \text{ angle } (\mathbf{u}_0, \mathbf{x}_0) = \text{ angle } (\mathbf{u}_0^\perp, \mathbf{x}_0^\perp) \leqslant \theta_{\max}.$$

Let us now consider two cases. Suppose that $\sin\theta_{\min} \leqslant \frac{\eta}{t}$. Then we have a lower bound

$$\operatorname{vol}_{n-g+1}\Omega^\perp \geqslant t\eta^{n-g}, \tag{7.3}$$

and an upper bound

$$\operatorname{vol}_g \Omega \leqslant 2n^{g/2}\tau^g. \tag{7.4}$$

Note that Ω contains the convex hull of $\{\pm\mathbf{z}_1, \dots, \pm\mathbf{z}_g\}$, and the parallelepiped

$$\left\{\mathbf{z} = \lambda_1\mathbf{z}_1 + \dots + \lambda_g\mathbf{z}_g, \ \max_{1\leqslant i\leqslant g} |\lambda_i| \leqslant \frac{1}{2}\right\} \subset \frac{g}{2}\cdot\Omega$$

contains a fundamental domain for Λ. Therefore we have

$$\begin{aligned}\operatorname{covol}(L/\Lambda) = \operatorname{covol}(L^\perp/\Lambda^\perp) &\leqslant \frac{g^g \cdot \operatorname{vol}_g\Omega}{2^g} \stackrel{(7.4)}{\leqslant} \frac{g^g n^{g/2}\tau^g}{2^{g-1}} \\ &\stackrel{(7.2)}{=} \frac{g^g n^{g/2-2n}}{2^{4n+g-1}} \cdot t\eta^{n-g} \leqslant \frac{1}{2^{n-g+1}} \cdot t\eta^{n-g} \stackrel{(7.3)}{\leqslant} \frac{\operatorname{vol}_{n-g+1}\Omega^\perp}{2^{n-g+1}}.\end{aligned}$$

By the Minkowski Convex Body Theorem there exists a nonzero integer point in $\Omega^\perp \subset \Pi_{t,\eta}$. Hence we are done in this case.

Now suppose that $\sin\theta_{\min} > \eta/t$. Then

$$\operatorname{vol}_{n-g+1}\Omega^\perp \geqslant \frac{\eta^{n-g+1}}{\sin\theta_{\min}} \tag{7.5}$$

and

$$\operatorname{vol}_g \Omega \leqslant n^{g/2}\tau^{g-1} \cdot \frac{\varepsilon}{\sin\theta_{\max}}. \tag{7.6}$$

Arguing as in the previous case but using (7.5) and (7.6) instead of (7.3) and (7.4), we obtain

$$\begin{aligned}\operatorname{covol}(L/\Lambda) = \operatorname{covol}(L^\perp/\Lambda^\perp) &\leqslant \frac{g!}{2^g} \cdot \operatorname{vol}_g\Omega \leqslant \frac{g! n^{g/2}}{2^g} \cdot \frac{\tau^{g-1}\varepsilon}{\sin\theta_{\max}} \\ &\leqslant \frac{\eta}{t} \cdot \frac{\eta^{n-g}t}{2^{n-g+1}\sin\theta_{\min}} \leqslant \frac{\operatorname{vol}_{n-g+1}\Omega^\perp}{2^{n-g+1}},\end{aligned}$$

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and, again by the Minkowski Convex Body Theorem, we get a nonzero integer point in $\Omega^\perp \subset \Pi_{t,\eta}$. □

Now let us state and prove a generalization of Theorem 1.13.

Theorem 7.2. *Let $n \geqslant 2$, let Y be a d-dimensional connected analytic submanifold of $\mathbb{R}^n \cong M_{n,1}(\mathbb{R})$, where $d \geqslant 2$, and let $\{R_\ell\}$ be a countable collection of proper closed analytic submanifolds of Y. Suppose that a continuous non-decreasing function f satisfies (1.17). Then the set*

$$Y \cap \mathrm{UA}_{n,1}(f) \smallsetminus \bigcup_{\ell \in \mathbb{N}} R_\ell$$

is uncountable and dense in Y. In particular, if one in addition assumes that Y is not contained in any proper rational affine subspace of $\mathbb{R}^n$, then the set $Y \cap \mathrm{UA}^*_{n,1}(f)$ is uncountable and dense in Y.

Proof. For $t > 0$ and $g \stackrel{\text{def}}{=} d-1$, let us set $\eta \stackrel{\text{def}}{=} f(t)$ and $\tau \stackrel{\text{def}}{=} ((\eta^{n-g}t)/(4n)^{2n})^{1/g}$, so that (7.2) is satisfied, and the function $t \mapsto \tau$ is continuous. By (1.17), we have that $\tau \to \infty$ monotonically when $t \to \infty$. The map $t \mapsto \tau$ is thus bijective, and hence one can consider the inverse map $\tau \mapsto t(\tau)$ and define a positive function

$$\varepsilon = h(\tau) \stackrel{\text{def}}{=} \frac{\tau f(t(\tau))}{t(\tau)},$$

so that (7.1) holds. Note that h is non-increasing, since so is the function

$$h(\tau(t)) = \frac{(f(t)^{n-g}t/(4n)^{2n})^{1/g} f(t)}{t} = \frac{f(t)^{n/g}}{(4n)^{2n/g} t^{1-1/g}}.$$

We will prove the implication

$$\mathbf{x}^T \in \mathrm{UA}_{1,n}(h; g) \implies \mathbf{x} \in \mathrm{UA}_{n,1}(f). \tag{7.7}$$

This, in view of Theorem 4.2, will immediately imply the conclusion of Theorem 7.2.

To prove (7.7), note that $\mathbf{x}^T \in \mathrm{UA}_{1,n}(h; g)$ amounts to saying that for any large enough τ the parallelepiped $\Pi^{\tau,\varepsilon}$ contains g linearly independent integer points. In view of Lemma 7.1 we get that for all t large enough, the parallelepiped $\Pi_{t,\eta}$ contains a nonzero integer point $\mathbf{z} = (q, \mathbf{p})$. For t large enough, the first coordinate q will be nonzero; thus we can find $q \in \mathbb{Z} \smallsetminus \{0\}$ and $\mathbf{p} \in \mathbb{Z}^n$ such that $q \leqslant t$ and $\|q\mathbf{x} - \mathbf{p}\| \leqslant f(t)$. □

8. Further applications

Most of the results described in this section are not proved in this paper; the proofs will appear elsewhere.

8.1 Irrationality measure functions and the Kan–Moshchevitin phenomenon

Another convenient way to describe various results in the theory of Diophantine approximation is through irrationality measure functions. Namely, given $A \in M_{m,n}(\mathbb{R})$, one defines its *irrationality measure function* by

$$\psi_A : \mathbb{R}_{>0} \to \mathbb{R}_{>0}, \quad \psi_A(t) \stackrel{\text{def}}{=} \inf\{\|A\mathbf{q} - \mathbf{p}\| : \mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\},\ \|\mathbf{q}\| \leqslant t,\ \mathbf{p} \in \mathbb{Z}^m\}.$$

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Then it is easy to see that A is f-uniform[4] if and only if $\psi_A(t) \leqslant f(t)$ for all large enough t. Similarly, for an arbitrary Diophantine system $\mathcal{X} = (X, \mathcal{D}, \mathcal{H})$ one can introduce the irrationality measure function associated to $x \in X$:

$$\psi_x(t) \stackrel{\text{def}}{=} \inf\{d_s(x) : h_s \leqslant t\}.$$

Then $x \in \mathrm{UA}_{\mathcal{X}}(f)$ if and only if $\psi_x(t) \leqslant f(t)$ for all large enough t.

In 2009 the following result was proved by Kan and Moshchevitin [KM10]: in the standard Diophantine system corresponding to approximations to one real number ($m = n = 1$), for any two different real numbers x, y with

$$\psi_x(t) > 0, \quad \psi_y(t) > 0 \quad \forall\, t$$

(a condition equivalent to $x, y \notin \mathbb{Q}$), the difference $\psi_x(t) - \psi_y(t)$ changes its sign infinitely often as $t \to \infty$. This phenomenon is not present when $\max(m, n) > 1$. More generally, as long as the assumptions of our main theorem (Theorem 1.5) are satisfied, there exist pairs of points whose irrationality measure functions do not exhibit the pattern of infinitely many changes of sign. More precisely, suppose a Diophantine system

$$\mathcal{X} = (X, \mathcal{D} = \{d_s : s \in \mathcal{I}\}, \mathcal{H} = \{h_s : s \in \mathcal{I}\})$$

and a countable collection $\mathcal{L}$ of closed subsets of X satisfy the assumptions of Theorem 1.5 with $Y = X$, $\varphi_k \equiv \mathrm{Id}$ and $\mathcal{R} = \{{d_s}^{-1}(0) : s \in \mathcal{I}\}$. Then for any $x \in X$ such that

$$d_s(x) \neq 0 \quad \Longleftrightarrow \quad \psi_x(t) > 0 \ \ \forall\, t$$

one can apply the theorem with $f = \psi_x$ and conclude that there exists a dense set of $y \in X$ such that $\psi_y(t) > 0$ for all t and $\psi_y(t) < \psi_x(t)$ for all large enough t.

8.2 Inhomogeneous approximation and approximation with restrictions

It is natural to consider the problems of uniform approximation for systems of linear forms in the inhomogeneous set-up; that is, fix $\mathbf{b} \in \mathbb{R}^m$ and, instead of (1.2), look for non-trivial integer solutions of the system

$$\|A\mathbf{q} + \mathbf{b} - \mathbf{p}\| \leqslant f(t) \quad \text{and} \quad \|\mathbf{q}\| \leqslant t. \tag{8.1}$$

This calls for the consideration of the collection

$$\{L_{\mathbf{p},\mathbf{q},\mathbf{b}} : \mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\},\ \mathbf{p} \in \mathbb{Z}^m\} \quad \text{where } L_{\mathbf{p},\mathbf{q},\mathbf{b}} \stackrel{\text{def}}{=} \{A \in M_{m,n}(\mathbb{R}) : A\mathbf{q} + \mathbf{b} = \mathbf{p}\}, \tag{8.2}$$

and the proof of its total density for arbitrary $\mathbf{b} \in \mathbb{R}^n$. And indeed it turns out to be possible to achieve this when $n > 1$, and thereby construct f-uniform systems of affine forms with an arbitrary fixed translation part. Moreover, in a forthcoming joint work with Hong and Neckrasov we take subsets P of $\mathbb{R}^m$ and Q of $\mathbb{Z}^n$, and say that $A \in M_{m,n}(\mathbb{R})$ is *f-uniform with respect to* (P, Q) if for all large enough $t > 0$ there exist $\mathbf{q} \in Q$ and $\mathbf{p} \in P$ satisfying (1.2). The classical theory of homogeneous approximation corresponds to $P = \mathbb{Z}^m$ and $Q = \mathbb{Z}^n \smallsetminus \{0\}$. The following theorem can be proved (work in progress).

Theorem 8.1. *The collection $\{L_{\mathbf{p},\mathbf{q}} : \mathbf{p} \in P,\ \mathbf{q} \in Q\}$ of subspaces of $M_{m,n}(\mathbb{R})$ is totally dense if:*

(a) *$Q = \mathbb{Z}^n \smallsetminus \{0\}$ and P is a subgroup of $\mathbb{Z}^m$ of rank $> m - n + 1$ (restricted numerators);*
(b) *$Q = Q_1 \times \cdots \times Q_n \subset \mathbb{Z} \times \cdots \times \mathbb{Z}$, where at least two of the sets Q_i are infinite (restricted denominators), and P is of bounded Hausdorff distance from $\mathbb{R}^m$.*

[4] Likewise one can also use the function ψ_A to study *asymptotic approximation*: requiring that $\psi_A(t) \leqslant f(t)$ for an unbounded set of $t > 0$, one gets a definition of *f-approximable* systems of linear forms A.

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Consequently (modulo Theorem 1.5) for any non-increasing $f:\mathbb{R}_{>0}\to\mathbb{R}_{>0}$ the set of $m\times n$ matrices that are f-uniform with respect to (P,Q) and not contained in a given countable family of proper analytic submanifolds of $M_{m,n}(\mathbb{R})$ is uncountable and dense.

Note that both (a) and (b) implicitly assume that $n>1$, and the set-up of (b) includes inhomogeneous approximation by letting $P=\mathbb{Z}^m+\mathbf{b}$ for a fixed $\mathbf{b}\in\mathbb{R}^m$. When $n=1$, it is clear that for any fixed $\mathbf{b}\in\mathbb{R}^m$ the collection (8.2) is not totally dense. On the other hand one can study a doubly metric version of the problem, that is, the set of pairs $(A,\mathbf{b})$ such that (8.1) has a non-trivial integer solution for large enough t. It was recently documented by the second named author [Mos24] that for any $m\in\mathbb{N}$ and any non-increasing $f:\mathbb{R}_{>0}\to\mathbb{R}_{>0}$ there exists a dense and uncountable set of pairs $(\mathbf{x},\mathbf{y})\in\mathbb{R}^m\times\mathbb{R}^m\cong M_{m,2}$ such that the system of inequalities

$$\|q\mathbf{x}+\mathbf{y}-\mathbf{p}\|\leqslant f(t)\quad\text{and}\quad|q|\leqslant t$$

has a nonzero integer solution $(\mathbf{p},q)$ for all large enough t. In fact, this statement follows from a old result by Khintchine [Khi47] which is not very well known. See also [MN25, § 3.3] for a discussion.

8.3 Rational approximations to linear subspaces

In this subsection we fix $d\in\mathbb{N}_{\geqslant 2}$ and $a=1,\dots,d-1$, and consider a certain Diophantine system on $X=\mathrm{Gr}_{d,a}$, the Grassmannian of a-dimensional subspaces of $\mathbb{R}^d$. Following the set-up of [Sch67], one can study the problems of approximation of an a-dimensional subspace A of $\mathbb{R}^d$ by b-dimensional rational subspaces B of $\mathbb{R}^d$, where $1\leqslant b<d$, in terms of the so-called *first angle* between the subspaces. The latter (or rather, formally speaking, the sine of the angle) is defined as follows:

$$\measuredangle_1(A,B)\stackrel{\text{def}}{=}\min_{\mathbf{x}\in A\smallsetminus\{0\},\,\mathbf{y}\in B\smallsetminus\{0\}}\frac{\|\mathbf{x}\wedge\mathbf{y}\|}{\|\mathbf{x}\|\cdot\|\mathbf{y}\|},$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$ and on $\bigwedge^2(\mathbb{R}^d)$.

Note that $\measuredangle_1(A,B)=0$ if and only if $\dim(A\cap B)>0$. Using the terminology introduced in [Nec22], let us say that $A\in\mathrm{Gr}_{d,a}$ is *completely irrational* if for any $(d-a)$-dimensional rational subspace R we have $A\cap R=\{0\}$. We also define the *height* $H(B)$ of a rational subspace $B\subset\mathbb{R}^d$ of dimension b in the natural way as the covolume of the b-dimensional lattice $B\cap\mathbb{Z}^d$. Using the methods of this paper, namely Theorem 1.5, it is possible to prove the following theorem.

Theorem 8.2. *Let $1\leqslant a,b<d$ with $\max(a,b)>1$. Then for any non-increasing $f:\mathbb{R}_{>0}\to\mathbb{R}_{>0}$ the set of completely irrational $A\in\mathrm{Gr}_{d,a}$ such that the system of inequalities*

$$H(B)\leqslant t,\quad\measuredangle_1(A,B)\leqslant f(t)$$

has a solution in b-dimensional rational subspaces B for all large enough t is uncountable and dense.

The case $d=4$ and $a=b=2$ is a recent result of Chebotarenko [Che24]. The above theorem, as well as several generalizations, will be proved in a forthcoming work.

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Conflicts of interest

None.

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Uniform approximation: weights, intersections, rates

Dmitry Kleinbock kleinboc@brandeis.edu
Department of Mathematics, Brandeis University, Waltham, MA 02454-9110, USA

Nikolay Moshchevitin nikolai.moshchevitin@tuwien.ac.at
Institut für diskrete Mathematik und Geometrie, Technische Universität Wien, Freihaus, Wiedner Hauptstraße 8, A-1040, Wien

Jacqueline M. Warren j4warren@ucsd.edu
Department of Mathematics, University of California San Diego, CA 92093-0112, USA

Barak Weiss barakw@tauex.tau.ac.il
Department of Mathematics, Tel Aviv University, Tel Aviv 69978, Israel

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