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A look back at Ramanujan's Notebooks

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1. Ramanujan's notebooks were the theme of a lecture (20) given by G. N. Watson to the London Mathematical Society in 1931. At that time, he and B. M. Wilson were collaborating on a critical edition; Watson was anticipating that the task might take a further 5 years. Unhappily, Wilson died, and the task of editing the notebooks has never been completed.

According to Watson (loc. cit.) it appears that one of Ramanujan's notebooks was left with Hardy when Ramanujan returned to India in 1919. After Ramanujan's death, this notebook was sent to Madras; in return, a manuscript copy by Mr T. A. Satagopan of the second notebook and various loose papers was sent to Hardy. I understand that the three notebooks have remained in the University of Madras ever since; a facsimile edition was published in 1957 by the Tata Institute for Fundamental Research. The first and second notebooks seem to contain substantially similar material, the second notebook being perhaps Ramanujan's edited and improved version of the first; the third notebook is short and comparatively scrappy.

In October–November 1928 (according to the title page; see also (20) and (22)) G. N. Watson transcribed the Satagopan copy; the resulting volume, written in Indian ink on one side of 288 foolscap pages, is now in the library of Oxford's Mathematical Institute. In contrast to the facsimile, Watson's writing is quite easily legible. The contents of this copy are as follows:

pp. 2–176 Ramanujan's second notebook.
pp. 231–263 Various manuscripts.
pp. 264–288 Ramanujan's three scholarship holder's reports of 1913–14.

In a rather superficial examination, I have not spotted any differences, except in the pagination and the ordering of material, between Watson's copy and the facsimile edition. (In any critical edition of the notebooks, the ordering of the material will present problems; as is common enough in a semi-private notebook, Ramanujan writes at all angles, and seems not to have always used the pages in the order in which they are bound.)

In this note, I am concerned with the 'various manuscripts', which seem, on the face of it, not to be available anywhere else. They support the view that Ramanujan's insight into the arithmetic of modular forms was even greater than has been realized.

2. There appeared to be eight separate fragments, which I will refer to as [I] to [VIII]; the most remarkable are [V] [VI] and [VII]. I will quote [V] and [VI] in sections 3 and 4 below, they are simply statements of unproved results. The other

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fragments are parts of published and unpublished papers; in the rest of this section I will describe, rather briefly, what they contain.

Fragments [I] and [III], pp. 231–4 and 237–41, are about reciprocal functions. The formulae are general, and unrigorous. Hardy describes Ramanujan's work in this field in Chapter XI of (3); fragment [III] starts with a set of equations similar to (11.22.2) to (11.22.4) of (3).

Fragment [II], pp. 235–6, was published as (13).

Fragment [IV], pp. 242–6, is entitled 'Approximate sums of series involving prime numbers'. It is somewhat in the style of (11); asymptotic formulae as $s \to 0$ are derived for sums like $\Sigma q_r^{-s} \log(p_r)$, where p_r is the rth prime and q_r is the product of the first r primes.

Fragment [VII], pp. 252–7, may be part of a longer manuscript from which (14) was extracted; it does not seem to be Hardy's source for the posthumous paper (16), as the proofs are different. It begins by proving congruences for $p(n)$ modulo 5 and 25, *precisely* as in section 4 of (14); it then goes on to sketch the proofs of congruences modulo higher powers of 5, and finally proves the congruences modulo 7 and 49 as in (14) but with more details; nothing is said about higher powers of 7. The sketches of proof for congruences modulo higher powers of 5 (though exiguous) appear correct, and indeed the arguments are very similar to those of Watson (23); Watson himself implies in (22) (but not in (23)) that Ramanujan had proofs for such congruences.

Recently, more beautiful congruence properties of the coefficients of modular forms have been discovered and explained – see in particular (1), (19) and (18). In this context, it is notable that Ramanujan's assertions are actually significantly stronger than those reported by Watson; he states that, if λ is a positive odd integer, then

$$\sum_0^\infty p\left(\frac{19.5^\lambda+1}{24}+5^\lambda n\right)x^n \equiv c_\lambda 5^\lambda\{(1-x)(1-x^2)(1-x^3)\ldots\}^{19} \quad \text{modulo } 5^{\lambda+1},$$

and if λ is a strictly positive even integer then

$$\sum_0^\infty p\left(\frac{23.5^\lambda+1}{24}+5^\lambda n\right)x^n \equiv c_\lambda 5^\lambda\{(1-x)(1-x^2)(1-x^3)\ldots\}^{23} \quad \text{modulo } 5^{\lambda+1}$$

where c_λ in both cases is a constant. With deference to Atkin (1) section 7, this seems to be an Atkin-type property rather than a Ramanujan-type property!

Fragment [VIII], pp. 258–62, is clearly the end of (11), which (as Hardy explains in the Collected Works) was not published complete as the London Mathematical Society was in financial difficulties at the time. The published version of (11) ends with section 52 – this fragment contains sections 63–68.

3. Fragment [V], pp. 247–9, is a list of assertions that (the transforms of) various modular forms have Euler products – very much in the spirit of sections 18–22 of (12) (one of the most beautiful papers published by this Society). Ramanujan realized that, even when a modular form did not itself have an Euler product, one could regain the Euler product property by taking an appropriate linear combination of forms of the same level. Nowadays of course this is well known – see for instance Chapter 4 of (8) – but I do not know a reference earlier than Hecke's classic papers (4), (5).

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Though it is nowadays a matter of (non-trivial) routine to obtain such formulae, I quote the fragment complete. I have changed the notation to that of (12), but I have not altered (or checked) any of the numbers; recollect that

$$f(q) = q^{1/24}(1-q)(1-q^2)(1-q^3)\dots,$$

$$P = 1 - 24\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \dots\right),$$

$$Q = 1 + 240\left(\frac{x}{1-x} + \frac{2^3x^2}{1-x^2} + \frac{3^3x^3}{1-x^3} + \dots\right),$$

$$R = 1 - 504\left(\frac{x}{1-x} + \frac{2^5x^2}{1-x^2} + \frac{3^5x^3}{1-x^3} + \dots\right),$$

so that Q, R, Q^2, QR, Q^2R are the Eisenstein series E_4, E_6, E_8, E_{10}, E_{14}, and

$$1728f^{24} = 1728\Delta = Q^3 - R^2.$$

"I. Suppose that A and B are any two integers such that $A^2 + 3B^2 = p$ and

$$A \equiv 1 \pmod 3,$$

p being a prime of the form $6k+1$. Let

$$\Sigma a_0(n)\, q^{n/6} = f^4, \quad \Sigma a_1(n)\, q^{n/6} = f^4P,$$
$$\Sigma a_2(n)\, q^{n/6} = f^4Q, \quad \Sigma a_3(n)\, q^{n/6} = f^4R,$$
$$\Sigma a_4(n)\, q^{n/6} = f^4Q^2 + 288\sqrt{70}f^{20}, \quad \Sigma a_5(n)\, q^{n/6} = f^4QR,$$
$$\Sigma a_7(n)\, q^{n/6} = f^4Q^2R + 10080\sqrt{286}f^{20}R.$$

In all these cases

$$\Sigma a_K(n)\, n^{-s} = \Pi(1 - a_K(p)\, p^{-s} + p^{2K+1-2s})^{-1},$$

where p assumes all prime values greater than 3.

If $K = 0, 2, 3, 5$ then $a_K(p) = 0$ for $p \equiv 5$ (6),

$$a_K(p) = (A + \sqrt{-3}B)^{2K+1} + (A - \sqrt{-3}B)^{2K+1}$$

for $p \equiv 1$ (6). For all values of n, $a_1(n) = na_0(n)$. But $a_4(p)$ and $a_7(p)$ do not seem to have such simple laws.

II. Suppose again that A and B are defined as in I and let

$$\Sigma a_0(n)\, q^{n/3} = f^8, \quad \Sigma a_1(n)\, q^{n/3} = f^8P,$$
$$\Sigma a_2(n)\, q^{n/3} = f^8Q + 6\sqrt{10}f^{16}, \quad \Sigma a_3(n)\, q^{n/3} = f^8R,$$
$$\Sigma a_4(n)\, q^{n/3} = f^8Q^2 + 6\sqrt{70}f^{16}Q,$$
$$\Sigma a_5(n)\, q^{n/3} = f^8QR + 12\sqrt{55}f^{16}R,$$
$$\Sigma a_7(n)\, q^{n/3} = f^8Q^2R + 12\sqrt{910}f^{16}QR.$$

In all these cases

$$\Sigma a_K(n)\, n^{-s} = \Pi(1 - a_K(p)\, p^{-s} + p^{2K+3-2s})^{-1},$$

where p assumes all prime values except 3.

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If $K = 0$ or 3, then $a_K(p) = 0$ for $p \equiv 2\,(3)$, $a_K(p) = (A+\sqrt{-3}B)^{2K+3} + (A-\sqrt{-3}B)^{2K+3}$ for $p \equiv 1\,(3)$. $a_1(n) = na_0(n)$.

III. Suppose that C and D are integers such that $C^2+4D^2 = p$, where p is of the form $4k+1$.

$$\Sigma a_0(n)\,q^{n/4} = f^6, \quad \Sigma a_1(n)\,q^{n/4} = f^6P,$$
$$\Sigma a_2(n)\,q^{n/4} = f^6Q, \quad \Sigma a_3(n)\,q^{n/4} = f^6R + 24i\sqrt{35}\,f^{18},$$
$$\Sigma a_4(n)\,q^{n/4} = f^6Q^2, \quad \Sigma a_5(n)\,q^{n/4} = f^6QR + 24i\sqrt{1155}\,f^{18}Q,$$
$$\Sigma a_7(n)\,q^{n/4} = f^6Q^2R + 120i\sqrt{3003}\,f^{18}Q^2, \quad \text{where} \quad i^2 = -1.$$

In all these cases

$$\Sigma a_K(n)\,n^{-s} = \Pi_1\Pi_2,$$

where

$$\Pi_1 = \Pi(1 - a_K(p)\,p^{-s} - p^{2K+2-2s})^{-1},$$

p assuming primes of the form $4k-1$,

and

$$\Pi_2 = \Pi(1 - a_K(p)\,p^{-s} + p^{2K+2-2s})^{-1}$$

p assuming all primes of the form $4k+1$.

If $K = 0, 2$ or 4 then $a_K(p) = 0$ for $p \equiv 3\,(4)$, $a_K(p) = (C+2iD)^{2K+2} + (C-2iD)^{2K+2}$ for $p \equiv 1\,(4)$. $a_1(n) = na_0(n)$.

IV. Let

$$\Sigma a_0(n)\,q^{n/12} = f^2, \quad \Sigma a_1(n)\,q^{n/12} = f^2P, \quad \Sigma a_2(n)\,q^{n/12} = f^2Q + 48f^{10},$$
$$\Sigma a_3(n)\,q^{n/12} = f^2R + 360i\sqrt{3}\,f^{14}, \quad \Sigma a_4(n)\,q^{n/12} = f^2Q^2 + 672f^{10}Q,$$
$$\Sigma a_5(n)\,q^{n/12} = f^2QR + 96\sqrt{1045}\,f^{10}R + 216i\sqrt{7315}\,f^{14}Q + 103680i\sqrt{7}\,f^{22},$$
$$\Sigma a_7(n)\,q^{n/12} = f^2Q^2R + 48\sqrt{(910.2911)}\,f^{10}QR + 216i\sqrt{(5005.2911)}\,f^{14}Q^2 - 471744i\sqrt{22}\,f^{22}Q.$$

. . . . . ."

4. Fragment VI, pp. 250–1, is a list of 40 identities involving the 'Rogers–Ramanujan' functions

$$G(x) = \Pi \frac{1}{(1-x^{5n+1})(1-x^{5n+4})} = 1 + \frac{x}{1-x} + \frac{x^4}{(1-x)(1-x^2)} + \dots,$$

$$H(x) = \Pi \frac{1}{(1-x^{5n+2})(1-x^{5n+3})} = 1 + \frac{x^2}{1-x} + \frac{x^6}{(1-x)(1-x^2)} + \dots.$$

This is without doubt the list of identities referred to by Watson in (21). In his paper, Watson proves about a third of them; it would be nice to have a beautiful proof of the rest. A dull proof would have little value – in fact, all the functions involved in the identities are essentially theta functions, so modular forms of known level with poles of bounded order at known places, so the identities may presumably be verified by just checking that the first hundred or so powers of x are correct. In (15), Ramanujan seems to imply that he proved at least some of them, and indeed some appear too complicated to guess, even for one with his incredible instinct for formulae.

I have corrected one blatant misprint (where $=$ had turned into $-$); and I have used a synthetic notation, for ease of printing.

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Write

$$g(n) = G(x^n), \quad h(n) = H(x^n),$$

$$p(n) = (1-x^n)(1-x^{3n})(1-x^{5n})(1-x^{7n})\ldots,$$

$$q(n) = (1-x^n)(1-x^{2n})(1-x^{3n})(1-x^{4n})\ldots,$$

$$r(n) = 1 + 2x^n + 2x^{4n} + 2x^{9n} + 2x^{16n} + \ldots,$$

and write $g^*(n) = G(-x^n)$ and h^*, p^*, q^*, r^* similarly for the result of changing x^n in h, p, q, r to $-x^n$; so $p(n)\,p^*(n) = p(2n)$, $q(n)\,g(n)\,h(n) = q(5n)$, and so forth.

Ramanujan's 40 identities are as follows:

$$[g(1)]^{11}\,h(1) - x^2 g(1)\,[h(1)]^{11} = 1 + 11x[g(1)\,h(1)]^6, \tag{1}$$

$$g(1)\,g(4) + xh(1)\,h(4) = [p^*(1)]^2 = r(1)/q(2), \tag{2}$$

$$g(1)\,g(4) - xh(1)\,h(4) = r(5)/q(2), \tag{3}$$

$$g(11)\,h(1) - x^2 g(1)\,h(11) = 1, \tag{4}$$

$$g(16)\,h(1) - x^3 g(1)\,h(16) = p^*(2), \tag{5}$$

$$g(1)\,g(9) + x^2 h(1)\,h(9) = q(3)^2/q(1)\,q(9), \tag{6}$$

$$g(2)\,g(3) + xh(2)\,h(3) = p(3)/p(1), \tag{7}$$

$$g(6)\,h(1) - xg(1)\,h(6) = p(1)/p(3), \tag{8}$$

$$g(7)\,h(2) - xg(2)\,h(7) = p(1)/p(7), \tag{9}$$

$$g(1)\,g(14) + x^3 h(1)\,h(14) = p(7)/p(1), \tag{10}$$

$$g(8)\,h(3) - xg(3)\,h(8) = p(1)\,p(4)/p(3)\,p(12), \tag{11}$$

$$g(1)\,g(24) + x^5 h(1)\,h(24) = p(3)\,p(12)/p(1)\,p(4), \tag{12}$$

$$g(9)\,h(4) - xg(4)\,h(9) = p(1)\,p(6)/p(3)\,p(18), \tag{13}$$

$$g(36)\,h(1) - x^7 g(1)\,h(36) = p(6)\,p(9)/p(2)\,p(3), \tag{14}$$

$$g(6)\,g(14) + x^4 h(6)\,h(14) = g(42)\,h(2) - x^8 g(2)\,h(42), \tag{15}$$

$$= [p^*(1)\,p(3)\,p^*(7)\,p(21) - p(1)\,p^*(3)\,p(7)\,p^*(21)]/2x, \tag{16}$$

$$g(2)\,g(13) + x^3 h(2)\,h(13) = g(26)\,h(1) - x^5 g(1)\,h(26), \tag{17}$$

$$= [p(13)^2 - xp(1)^2]^{\frac{1}{2}}/[p(13)\,p(1)]^{\frac{1}{2}}, \tag{18}$$

$$g(2)\,g(38) + x^8 h(2)\,h(38) = p^*(1)^2 p^*(19)^2/4x - p(1)^2 p(19)^2/4x - x^4/p(2)^2 p(38)^2, \tag{19}$$

$$g(31)\,h(1) - x^6 g(1)\,h(31) = p^*(1)\,p^*(31)/2x - p(1)\,p(31)/2x + x^3/p(2)\,p(62), \tag{20}$$

$$[g(1)\,g(39) + x^8 h(1)\,h(39)]\,q(1)\,q(39) = [g(13)\,h(3) - x^2 g(3)\,h(13)]\,q(3)\,q(13), \tag{21}$$

$$= [r^*(3)\,r^*(13) - r^*(1)\,r^*(39)]/2x, \tag{22}$$

$$g(1)\,h^*(1) + g^*(1)\,h(1) = 2/p(2)^2 = 2(1 + x^2 + x^6 + x^{12} + \ldots)/q(2), \tag{23}$$

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$$g(1)\,h^*(1)-g^*(1)\,h(1)=2x(1+x^{10}+x^{30}+x^{60}+\ldots)/q(2), \tag{24}$$

$$g^*(1)\,g^*(4)+xh^*(1)\,h^*(4)=p^*(2), \tag{25}$$

$$g^*(2)\,g^*(3)+xh^*(2)\,h^*(3)=p^*(1)\,p^*(6)/p^*(2)\,p^*(3), \tag{26}$$

$$g^*(6)\,h^*(1)-xh^*(6)\,g^*(1)=p^*(2)\,p^*(3)/p^*(1)\,p^*(6), \tag{27}$$

$$g^*(1)\,g(9)-x^2h^*(1)\,h(9)=p(1)\,p^*(9)/p(6), \tag{28}$$

$$g(11)\,h^*(1)+x^2g^*(1)\,h(11)=[p^*(2)\,p^*(22)\,p(4)\,p(44)-2x^3]/p(2)\,p(4)\,p(22)\,p(44) \tag{29}$$

$$g(1)\,g(44)+x^9h(1)\,h(44)=U,\quad g(4)\,g(11)+x^3h(4)\,h(11)=V, \tag{30), (31}$$

where

$$U^2+xV^2=p^*(1)^3\,p^*(11)^3,\quad UV+x=p^*(1)^2\,p^*(11)^2,$$

$$g(17)\,h(2)-x^3g(2)\,h(17)=U,\quad g(1)\,g(34)+x^7h(1)\,h(34)=V, \tag{32), (33}$$

where

$$U/V=p(1)/p(17),\quad U^4V^4-xU^2V^2=[p(17)^3+x^2p(1)^3]^2/p(1)^3\,p(17)^3,$$

$$[g(2)\,g(23)+x^5h(2)\,h(23)]\,[g(46)\,h(1)-x^9g(1)\,h(46)]=p(1)\,p(23)+x+[2x^2/p(1)\,p(23)], \tag{34}$$

$$[g(19)\,h(4)-x^3g(4)\,h(19)]/[g(76)\,h^*(1)+x^{15}g^*(1)\,h(76)]=p(2)/p(38), \tag{35}$$

$$[g(2)\,g(33)+x^7h(2)\,h(33)]/[g(66)\,h(1)-x^{13}g(1)\,h(66)]=p(3)/p(11), \tag{36}$$

$$[g(3)\,g(22)+x^5h(3)\,h(22)]/[g(11)\,h(6)-xg(6)\,h(11)]=p(33)/p(1), \tag{37}$$

$$[g(1)\,g(54)+x^{11}h(1)\,h(54)]/[g(27)\,h(2)-x^5g(2)\,h(27)]=p(3)\,p(27)/p(1)\,p(9), \tag{38}$$

$$[g(1)\,g^*(19)-x^4h(1)\,h^*(19)]\,[g^*(1)\,g(19)-x^4h^*(1)\,h^*(19)]=[g(2)\,g(38)+x^8h(2)\,h(38)], \tag{39}$$

$$[g(1)\,g(94)+x^{19}h(1)\,h(94)]\,[g(47)\,h(2)-x^9g(2)\,h(47)]$$
$$=p(1)\,p(47)+2x^2+[2x^4/p(1)\,p(47)]+[4x^2p(1)\,p(47)+9x^4+8x^6/p(1)\,p(47)]^{\frac{1}{2}}. \tag{40}$$

Of the above identities, Watson (21) gives proofs for (2), (3), (4), (23), (24), (29), (30) and (31). The first identity, and one or two others of the same sort (stated by Ramanujan in the Notebook, particularly in section 10 of Chapters 19 and 20) have been proved by Rogers (17), Darling (2) and Mordell (7). Identity (39) has been garbled at some stage of the copying process.

REFERENCES

(1) ATKIN, A. O. L. Congruences for modular forms, in *Computers in mathematical research*, (Churchhouse, R. F. and Herz, J.-C. editors) pp. 8–19. (North Holland, Amsterdam, 1968.)

(2) DARLING, H. B. C. Proofs of certain identities and congruences enunciated by S. Ramanujan, *Proc. London Math. Soc.* (2) **19** (1921), 350–372.

(3) HARDY, G. H. *Ramanujan* (Cambridge, 1940).

(4) HECKE, E. Neuere Fortschritte in der Theorie der elliptischen Modulfunktionen. *Comptes rendus du Congrès international des Mathematiciens, Oslo* 1936, pp. 140–156.

(5) HECKE, E. Über Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung. I. *Math. Ann.* **114** (1937), 1–28.

(6) MORDELL, L. J. On Mr Ramanujan's empirical expansions of modular functions. *Proc. Cambridge Philos. Soc.* **19** (1917), 117–124.

https://doi.org/10.1017/S0305004100051501 Published online by Cambridge University Press

(7) MORDELL, L. J. Note on certain modular relations considered by Messrs Ramanujan, Darling and Rogers. *Proc. London Math. Soc.* (2), **20** (1922), 408–416.
(8) OGG, A. Modular forms and Dirichlet series (Benjamin, New York, 1969.)
(9) RAMANUJAN, S. Collected Papers (Cambridge, 1927).
(10) RAMANUJAN S. *Notebooks of Srinivasa Ramanujan* (Tata Institute for Fundamental Research, Bombay, 1957).
(11) RAMANUJAN, S. Highly composite numbers. *Proc. London Math. Soc.* (2), **14** (1915), 347–409.
(12) RAMANUJAN, S. On certain arithmetical functions. *Trans. Cambridge Philos. Soc.* **22**, No. 9 (1916), 159–184.
(13) RAMANUJAN, S. A proof of Bertrand's postulate. *J. Indian Math. Soc.* **11** (1919), 181–182.
(14) RAMANUJAN, S. Some properties of $p(n)$, the number of partitions of n. *Proc. Cambridge Philos. Soc.* **19** (1919), 207–210.
(15) RAMANUJAN, S. Algebraic relations between certain infinite products. *Proc. London Math. Soc.* (2), **18** (1920), xx.
(16) RAMANUJAN, S. Congruence properties of partitions. *Math. Z.* **9** (1921), 147–153.
(17) ROGERS, L. J. On a type of modular relation. *Proc. London Math. Soc.* (2), **19** (1921), 387–397.
(18) SERRE, J.-P. Formes modulaires et fonctions zêta p-adiques, in *Modular Functions of One Variable III*, 191–268. *Lecture Notes in Mathematics* 350 (Springer-Verlag Heidelberg, 1973).
(19) SWINNERTON-DYER, H. P. F. On l-adic representations and congruences for coefficients of modular forms, in *Modular functions of one variable III*, 1–55. *Lecture Notes in Mathematics* 350 (Springer-Verlag Heidelberg, 1973).
(20) WATSON, G. N. Ramanujan's note books. *J. London Math. Soc.* **6** (1931), 137–153.
(21) WATSON, G. N. Proof of certain identities in combinatory analysis. *J. Indian Math. Soc.* **20** (1933), 57–69.
(22) WATSON, G. N. Über Ramanujansche Kongruenzeigenschaften der Zerfällungsanzahlen. *Math. Z.* **39** (1935), 712–731.
(23) WATSON, G. N. Ramanujans Vermutung über Zerfällungsanzahlen. *J. Zeine Angew. Math.* **179** (1938), 97–128.

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