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## 100.33 A surprising relationship between Pell-Lucas, Pell, square and centred square numbers

For a positive integer $k \geqslant 3$, the $n$th term of the $k$th polygonal number, $F(n; k)$, and the $n$th term of the $k$th centred polygonal number, $C(n; k)$, are given by

$$F(n; k) = \frac{(k-2)n^2 + (4-k)n}{2},$$

$$C(n; k) = \frac{kn^2 - kn + 2}{2},$$

respectively. It was shown in [1] that the numbers that are both $k$-gonal and centred $k$-gonal are given by

$$u(n;k) = \frac{k}{16(k-2)}\left\{\frac{-2k^2+18k-32}{k} + \left[k-1+\sqrt{k(k-2)}\right]^{2n+1} + \left[k-1-\sqrt{k(k-2)}\right]^{2n+1}\right\}$$

for $n \geqslant 0$ and $k \geqslant 3$. The Pell numbers are defined recursively by $P_0 = 0$, $P_1 = 1$ and $P_{n+2} = 2P_{n+1} + P_n$ for all $n \geqslant 0$. The Pell-Lucas numbers are defined recursively by $Q_0 = 2$, $Q_1 = 2$ and $Q_{n+2} = 2Q_{n+1} + Q_n$ for all $n \geqslant 0$. The purpose of this Note is to show an unexpected relationship between Pell-Lucas numbers, Pell numbers and $u(n; 4)$ – the square numbers that are centred square numbers.

The Binet formula for $Q_n$ is

$$Q_n = r^n + s^n$$

for $n \geqslant 0$ where $r = 1 + \sqrt{2}$ and $s = 1 - \sqrt{2}$. Since $r^2 = 3 + 2\sqrt{2}$, $s^2 = 3 - 2\sqrt{2}$ and $rs = -1$,

$$\begin{aligned} u(n; 4) &= \frac{1}{8}\left[2 + \left(3 + 2\sqrt{2}\right)^{2n+1} + \left(3 - 2\sqrt{2}\right)^{2n+1}\right] \\ &= \frac{1}{8}\left[2 + \left(r^2\right)^{2n+1} + \left(s^2\right)^{2n+1}\right] \qquad (1) \\ &= \frac{1}{8}\left(2 + Q_{4n+2}\right). \end{aligned}$$

Since $Q_m^2 - Q_{2m} = 2(-1)^m$ and $Q_m^2 - 8P_m^2 = 4(-1)^m$, (1) can be written as

$$u(n; 4) = \tfrac{1}{8}\left(4 + Q_{2n+1}^2\right) = P_{2n+1}^2.$$

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## 100.34 Tilings from a square template with asymmetric edges

*Question*: How many distinct shapes that tile the plane can be formed on a square template with identical asymmetric edges?

Shapes formed on a square template with identical asymmetric edges can tessellate only if they have two edges that are 'ins' and two edges that are 'outs'. Further, through rotations or flips, any shape with two 'in' edges and two 'out' edges can be arranged so that an 'out' will be in a chosen fixed position. With one edge fixed, there are three ways to position the other 'out' edge and for each position it can have two orientations. Once the second 'out' edge is fixed, there are two slots left for 'in' edges. Each of them can take two orientations. Thus there are $3 \times 2 \times 2 \times 2 = 24$ shapes that need to be considered. These 24 shapes are shown as puzzle pieces in Figure 1, with sides also distinguished by letters. To answer the question we need to determine whether each shape tessellates and to eliminate duplicates.

One way to determine which of the 24 shapes tessellate is, by trial and error, to fit copies of each shape together, which was the author's initial approach. A more elegant alternative is to analyse each shape to see if it satisfies conditions of a type known to tessellate. All tiles with shapeable edges that tile isohedrally, that is, edges fit together in only one way throughout a periodic tiling, belong to one of 28 types, named after mathematician Heinrich Heesch (1906-1996). Eleven of them are quadrilateral types but only five play a role in the following discussion, those that are limited to translations, glide reflections, and corner rotations of 90°. (Corner rotations of 60° and 120° do not form squares and centre-point rotations have symmetry so are not used as edges for this question.) A TTTT type translates or moves the top edge to the bottom and the left edge to the right edge. A TGTG type translates one pair of opposite edges while the other pair are flipped versions of each other. In a G1G2G1G2 type both opposite pairs contain edges that are flipped versions of one another, while a G1G1G2G2 type flips one edge and rotates it to form an adjacent edge and does the same for the other adjacent pair. Finally, in a C4C4C4C4 type one edge is rotated 90° to form an adjacent edge and the other two edges are formed in the same way. If a shape can be shown to fit one of these types, it must tessellate.

To determine if a shape fits one of these Heesch types, there are six isometries that relate edges.

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