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102.38 A very elementary short proof of Conway's little theorem

In this short Note, we give a very elementary, brief and self-contained proof of the following interesting theorem.

Theorem (Conway's Little Theorem): A triangle is equilateral if, and only if, the ratio between the length of any two sides is rational and the ratio between any two angles is also rational.

This theorem first appeared in [1, last comment], and recently with some added witty remarks (e.g., why should only Fermat have a little theorem? .....) has re-appeared in [2], and Conway this time calls it his little theorem.

We should admit that the re-publication of the theorem in [2], with those added witty remarks, made us read the theorem and its proof in [2] more carefully. Since the theorem is a nice characterisation of equilateral triangles (perhaps, an unusual characterisation in the classical sense of Euclidean geometry), one is tempted to present a geometric proof which applies tools as elementary as possible. Before proceeding with our promised proof, let us recall that as a consequence of two rather sophisticated theorems in [3, Theorems 3.9 and 3.11], it is shown in [3, Corollary 3.12] that the only rational values of the trigonometric functions of θ are given by: $\sin\theta$, $\cos\theta = 0, \pm 1, \pm\frac{1}{2}$, $\sec\theta$, $\operatorname{cosec}\theta = \pm 1, \pm 2$, $\tan\theta$, $\cot\theta = 0, \pm 1$, where $\theta = r\pi$, with r rational.

In what follows, we first give a quick self-contained proof of this interesting fact; see also [4, p. 358]. It is clear that we may, without loss, prove this only for the cosine function since, for example,

$$\sin\theta = \cos\left(\tfrac{\pi}{2} - \theta\right) \qquad \text{and} \qquad \tan\theta = \frac{\cos\left(\frac{\pi}{2} - \theta\right)}{\cos\theta}.$$

Let us now suppose that $\cos\theta$ is rational and $r = \frac{a}{b}$, where a, b are positive coprime integers. Note that we may, without loss, assume that r is positive. For any positive integer m we have $m = bq + t$, where q, t are non-negative integers with $0 \leqslant t < b$. Hence $\cos m\theta = \pm\cos t\theta$, where $0 \leqslant t < b$. This implies that for all positive integers m, $\cos m\theta$ belongs to a fixed finite set of rational numbers. Note that, for $0 \leqslant t < b$, $\cos t\theta$ takes only a finite number of rational values.

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Now consider the sequence $\{\cos 2^k\theta : k \geqslant 1\}$. By what we have already observed, this sequence consists of only finitely many distinct rational numbers. Therefore, among the non-zero rational numbers which appear in this sequence, we may select the rational $\frac{p}{q}$ (in its lowest terms) with greatest denominator. Now let $\frac{p}{q} = \cos\alpha$, where $\alpha = 2^n\theta$ for some n.

Clearly $\cos 2\alpha = 2\cos^2\alpha - 1 = \dfrac{2p^2 - q^2}{q^2}$ is also a member of this sequence. But the denominator of the latter fraction is q^2 when q is odd and $\frac{1}{2}q^2$ when q is even. Since in any case the denominator of this fraction should be less than or equal to q, we infer that $q = 1$ or $q = 2$. Consequently, $\cos\theta = 0, \pm 1, \pm\frac{1}{2}$ and we are done. Now let us return to the proof of the theorem.

Proof: In view of our assumption, the triangle is similar to a triangle with rational sides. Hence we may, without loss, assume that the triangle itself has rational sides. Since the sum of the values of the angles of any triangle is π, we infer that each angle of the triangle is a rational multiple of π. Now by invoking the law of cosines, we immediately infer the cosine of each angle of the triangle is a rational number. But, by what we have observed above, if the cosines of all the angles in the triangle are to be rational, they must each be equal to $\frac{1}{2}$. Hence we are done.

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