Ubiquitous cousins and parental slides

TONY CRILLY and COLIN R. FLETCHER

1. *Introduction*

In a previous article [1] we examined the concept of blood relations in a family tree and we used mathematics to highlight and simplify the descriptions involved, such as child, parent, great-uncle, second cousin, third cousin once removed.

A family tree involving blood relationship (*consanguinity*) is one characteristic we might wish to highlight, but now we examine family trees in which non-blood relationships occur. As before, the concept of a mathematical graph and mathematical tree will be useful. For example, a family tree will be a connected graph (between any two vertices there is always a path of edges joining them). The edges of a family tree will link a child with a parent, and so the edges have a natural direction.

The following example of a family tree (Figure 1) is one which we shall meet later. Note that it is not a *mathematical* tree since it contains a cycle (a closed path of edges).

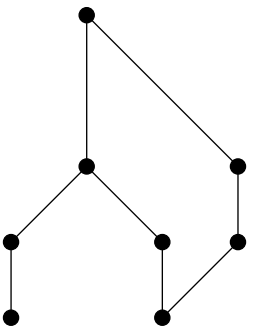


FIGURE 1: Family tree as a mathematical graph

How do we treat non-blood relations, when two individuals in a family tree do not have a common ancestor? In this case there is a break in the bloodline as occurs when *A* and *B* are parents but are not blood related. For this we consider a V-tree, a graph having the shape of a V (Figure 2(a)), where *A* and *B* are the parents of *C*.

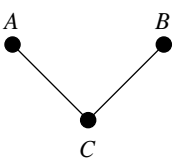


FIGURE 2(a)

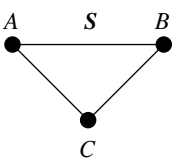


FIGURE 2(b)

https://doi.org/10.1017/mag.2021.5 Published online by Cambridge University Press

If A and B are non-blood related then we introduce the notion of a 'parental slide' $\boldsymbol{S}$, an edge linking the parents A and B. (See Figure 2(b), and also Section 5 for the reasoning behind this.)

In [1] we showed that family trees exist in which a pair of first cousins can also be second cousins. In what follows we classify such family trees (up to isomorphism), and then we go on to deal with non-blood relations.

2. *The Darwin-Wedgwood family tree*

To illustrate our investigation we use the family entanglements of the Darwin and Wedgwood families, two prominent interrelated English families (a very detailed Darwin-Wedgwood-Galton family tree is given in [2]).

In Table 1 we use the abbreviation D = Darwin, W = Wedgwood (using surnames at birth):

Darwin and Wedgwood Abbreviations		
William E D	1839-1914	WD
Margaret W	1843-1937	MW
Charles D	1809-1882	CD
Emma W	1808-1896	EW
Josiah W III	1795-1880	JW3
Caroline S D	1800-1888	CSD
Robert W D	1766-1848	RD
Susannah W	1765-1817	SW
Josiah W II	1769-1843	JW2
Erasmus D	1731-1802	ED
Josiah W I	1730-1795	JW1

TABLE 1: The Darwin-Wedgwood family

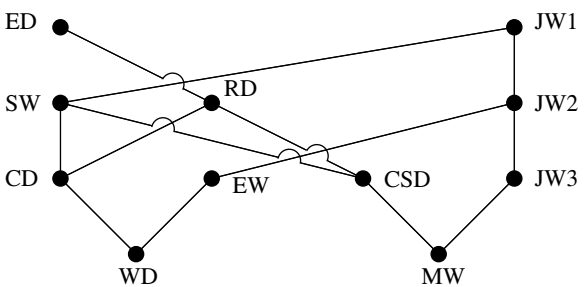


FIGURE 3: The Darwin and Wedgwood family tree

https://doi.org/10.1017/mag.2021.5 Published online by Cambridge University Press

In this connected graph, Figure 3, the edges (downwards) can be interpreted as parent-child, or (upwards) as child-parent. The family tree is sufficiently entangled to make the mathematics interesting: there are three examples of a Darwin marrying a Wedgwood. Moreover WD and MW at the bottom of the tree are blood relatives (since they have an ancestor in common). A curiosity is that they are both first cousins (ancestor RD in common) and second cousins (ancestor JW1 in common). We note that the family tree is not a mathematical tree, since there exist cycles (for example, CD, RD, CSD, SW, CD).

Not every pair in this family tree is blood related though. For instance, at the top of the tree, ED is not blood related to JW1 (as far as we know). How can this kind of relationship be expressed?

3. *Basic definitions*

For A and B to be mth cousins they must have a common ancestor Z who is $m + 1$ edges away from both. But all ancestors of Z are common ancestors of A and B, so in the definition of mth cousins we must include a qualifying *minimality clause*. This states that a deletion of any edge would cause a break in one of the paths from A to Z or from B to Z.

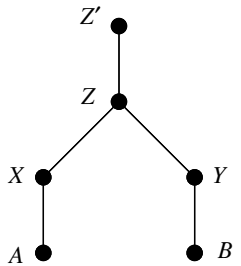


FIGURE 4: The minimality clause

In Figure 4, a deletion of the edge ZZ' does not cause a break in the path from A to Z or from B to Z, but a deletion of an edge from the path from A to Z (say) does.

Introducing minimality prevents mth cousins automatically being $(m + 1)$ th cousins, and $(m + 2)$ th cousins and so on. However this is not a numerical minimality, in the sense that first cousins are prevented from being second cousins or any higher order of cousins, as we have seen from the Darwin-Wegewood family graph.

These considerations lead to

Definition 1: A and B are mth cousins if they have a common ancestor Z where the paths from A to Z and B to Z both contain $m + 1$ edges and these paths are *minimal*.

We have not yet discussed what possible values m can take. The case $m = 1$ is the traditional first cousin relationship and higher values of m provide the definition of second cousin, third cousin and so on.

https://doi.org/10.1017/mag.2021.5 Published online by Cambridge University Press

We might though consider other values of m. If $m = 0$ for instance, A and B are siblings and we may describe them as 0th cousins. Next we try negative values of m. For the value $m = -1$ there would be no edges between A and Z and between B and Z, so this value of m can be identified as the condition that A and B are identical. We can interpret $m = -2$ as there being one edge *downwards* from A to Z and B to Z. This shows that A and B have a common descendant one generation down (they have a child).

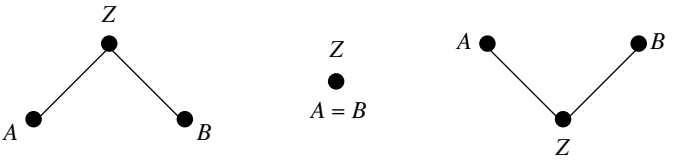


FIGURE 5: $m = 0, m = -1, m = -2$

We can have similar meanings when m is negative and $|m| > 2$, but we will not pursue these cases.

We next consider the possibility that paths from A to Z and B to Z contain a different number of edges. This is where the 'removed' terminology comes into play.

Definition 2: We say that A is the mth cousin of B n times ($n \geqslant 0$) removed if the path from Z to B contains $m + 1$ edges and the path from Z to A contains $m + n + 1$ edges, (or if the path from Z to A contains $m + 1$ edges and the path from Z to B contains $m + n + 1$ edges) and the paths are minimal (see Figure 6).

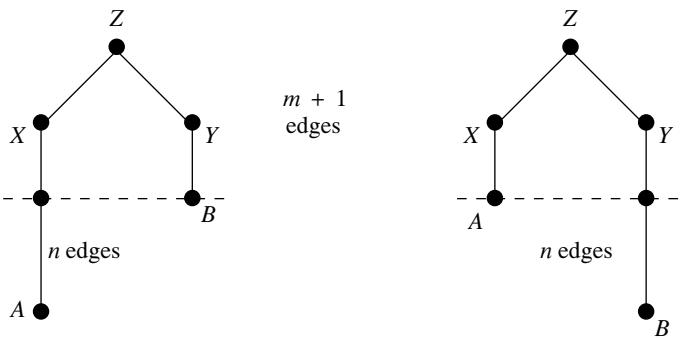


FIGURE 6: The relationship of mth cousin n times removed

This leads us to a mathematical observation: If A and B are blood related and consequently have an ancestor in common, every blood relation can be expressed in the form of being an mth cousin n times removed. The broad concept of 'cousin' pervades the whole field of blood relationships. No matter how simple or how complicated the relationship of two blood related family members seems, it can always be expressed in terms of cousins. One must be an mth cousin n times removed of the other.

https://doi.org/10.1017/mag.2021.5 Published online by Cambridge University Press

This mathematical observation is capable of producing some strange nomenclature in the case of close relatives. If A is a niece of B then we would have the tree (Figure 7):

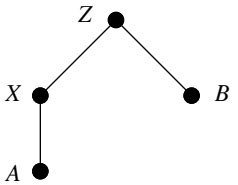


FIGURE 7: The niece-uncle relationship

Since siblings X and B are 0 th cousins, A is the 0 th cousin of B once removed. Even stranger is the description which occurs in the case of A being the daughter of B. This leads us back to negative m for now B is the common ancestor Z and so B and Z are (-1) th cousins (identity) so that 'daughter' is equivalent to A being a '(-1) th cousin of B once removed' (see Figure 8):

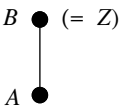


FIGURE 8

4. *First and second cousins*

From Figure 3 we have observed that it is possible for two members to be both first and second cousins. We can turn the problem around and discuss the question of finding all basic family trees containing two members A and B having the property of being both first and second cousins.

We suppose A and B are first cousins with ancestor Z_1 and that Z_2 is the ancestor that makes them second cousins. Thus we have the basic template (Figure 9):

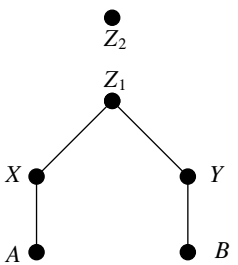


FIGURE 9: Template for first and second cousins

Possible solutions will be split between two categories:

(i) There is at least one edge connecting Z_1 with Z_2,

(ii) No edge connects Z_1 with Z_2.

https://doi.org/10.1017/mag.2021.5 Published online by Cambridge University Press

Case (i). This means that either Z_1 is a child of Z_2 (as in the positioning of Z_2 in Figure 9) or Z_1 is a parent of Z_2. We deal with the latter case first. For this we would have the graph (Figure 10) where W, U are ancestors of A and W', U' are ancestors of B.

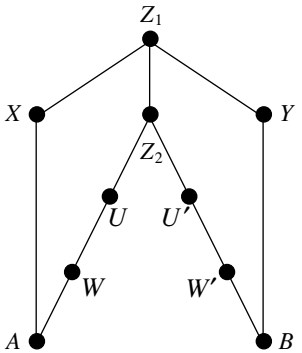


FIGURE 10

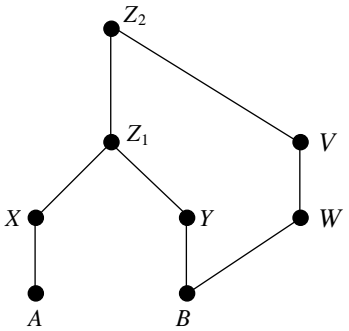
If X is at W then the parents of W are Z_1 and U. But Z_1 is a grandparent of U (let us say a grandfather), then U is female and is the daughter of Z_1's child. This falls foul of the man's son's daughter ban or the man's daughter's daughter ban of the *Marriage Act* (1949, England and Wales) [3]. In other words, marriages are prohibited between a man and his granddaughter, and between a woman and her grandson. So we reject this possibility.

Similarly Y cannot be at W', so the graph of Figure 10 remains, and is one solution to the question under discussion. We shall call this graph H_1.

We turn now to the alternative scenario of Z_1 being a child of Z_2. First we note that A and B cannot both be connected to Z_2 via Z_1 by minimality (a condition which prevents an mth cousin *automatically* being an $(m + 1)$th cousin, see Section 3).

Suppose the path connecting Z_2 and A is Z_2Z_1XA. Since A and B are second cousins there must be a path connecting B with Z_2 not passing through Z_1. Let Z_2V be the first edge of this path. An edge VY is not possible since V and Z_1 are siblings, and the existence of an edge VY would mean Y_1 is their child. All that remains is to insert a further vertex W between V and B. We will label this graph by H_2 so this (or graphs isomorphic to it) will be the second graph representing case (i) (see Figure 11).

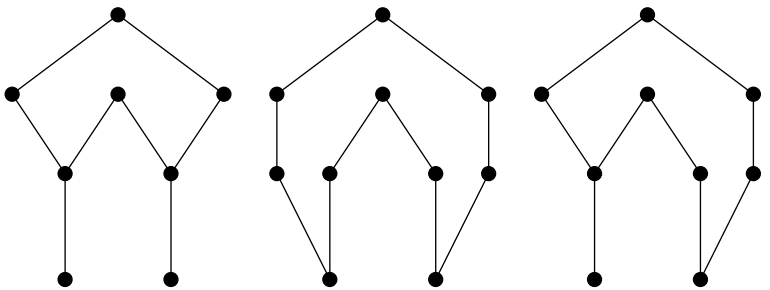
https://doi.org/10.1017/mag.2021.5 Published online by Cambridge University Press



H_2

FIGURE 11

We now cover the case (ii). Similar arguments as above shows that there are three non-isomorphic graphs in which there is no edge between Z_1 and Z_2. We shall denote these graphs L_1, L_2, L_3 (see Figure 12).



L_1 L_2 L_3

FIGURE 12

This means that a graph which contains two individuals who are both first and second cousins must contain one of five possible graphs as a subgraph, as displayed in Figures 10, 11, and 12.

The real life case of the Darwin-Wedgwood family in which WD and MW are first and second cousins has the graphs L_3 and H_2 (see Figure 13).

https://doi.org/10.1017/mag.2021.5 Published online by Cambridge University Press

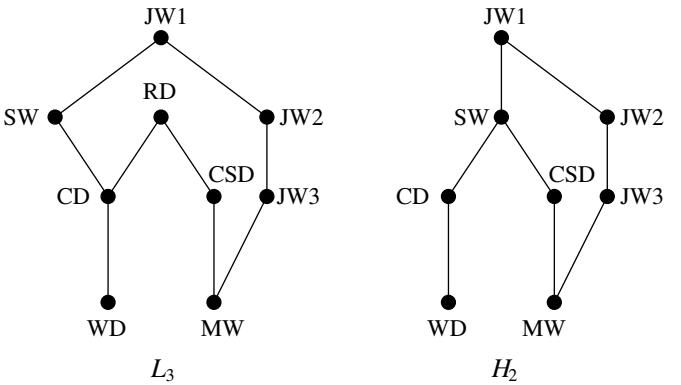


FIGURE 13: Darwin-Wedgwood family

5. *Non-blood relations*

We first consider the case of three vertices and two edges. There are three possible tree graphs (see Figure 14).

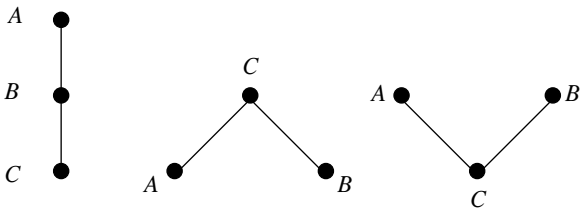


FIGURE 14

The first is child, parent, grandparent, and all are blood related. The second is parent and two children, also blood relations.

The third case is of two parents and a child. Here A and C are blood related and so are B and C, but A and B are not necessarily blood related. So the only place in a family tree where the blood line could break down is where we have a V-tree.

We have two cases to consider:

(i) A and B blood related

(ii) A and B not *blood* related

In the first case A and B have a common ancestor and we can simply write down the relationship, that A and B are mth cousins, n times removed for some integer values of m, n, and give the appropriate path.

In the second case if A and B are not blood related then we introduce the parental slide $\boldsymbol{S}$, an edge from one parent to the other (see Figure 2(b)). What will happen is that a path in a family tree is going to be split up into blood related subsets joined by parental slides $\boldsymbol{S}$. See for example, Figure 15, where C is a parent of B and a grandparent of A, D is a child of C, and E is a

https://doi.org/10.1017/mag.2021.5 Published online by Cambridge University Press

child of D and F. The $\boldsymbol{S}$ edge indicates that the parents D and F are not blood related, and G is a parent of F and H.

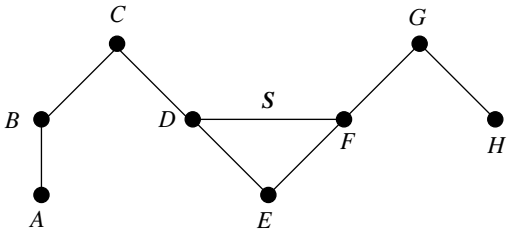


FIGURE 15

The path from A to H would be written:

$$(A, B, C, D)\boldsymbol{S}(F, G, H).$$

The individuals within the brackets are blood related. The path A, B, C, D, E, F does not form a blood related set. The individuals A, B, C, D, E are blood related but F is not blood related to A, B, C, D (as F and A have no common ancestor). To get from D to F we need to use the parental slide $\boldsymbol{S}$ otherwise the blood relationship within the set is broken.

The relationship of A and H is thus in two parts. A is the 0th cousin of D once removed, and F is the 0th cousin of H. We can simplify the notation by writing $[m, n]$ for the relationship of mth cousin n times removed. So the relationship of A and H can be written [0, 1][0, 0].

We add, for the sake of completeness, that a non-blood related couple without children must also have a slide edge $\boldsymbol{S}$ between them otherwise the family tree would be disconnected.

6. *Parental slides in the Darwin-Wedgwood family tree*

From Figure 3, we seek to express the relationship between RD and MW (Robert W Darwin and Margaret Wedgwood).

There are four instances of the problematic V-tree here, parents SW and RD (two children CD and CSD), CD and EW (with child WD), and CSD and JW3 (with child MW). Each of the latter two sets of parents are blood related, for both are first cousins with common ancestor JW1. As for parents SW and RD the family tree (Figure 3) shows no evidence that they are blood related and without such evidence we have to suppose that they are not. So we can draw a parental slide $\boldsymbol{S}$ between them.

We consider the relationship of RD and MW by finding paths between them. There are the paths

(i) RD, CSD, MW
This is a blood related subset and so the path can be written as (RD,CSD,MW) and RD and MW are (-1)th cousins twice removed, expressed simply by $[-1, 2]$.

https://doi.org/10.1017/mag.2021.5 Published online by Cambridge University Press

(ii) RD, SW, JW1, JW2, JW3, MW

As already noted, RD and SW are not blood related parents and are connected by an edge, the slide ***S***. All the other individuals along the path from SW to MW are blood related, and so the path is written

(RD) ***S*** (SW, JW1, JW2, JW3, MW).

The relationship of RD and MW is in two parts. (RD) is the identity (-1) th cousin, and SW and MW are 0 th cousins twice removed. So the relationship can be written $[-1,\ 0]\,[0,\ 2]$.

Then RD and MW are connected by (i) $[-1, 2]$ and (ii) by $[-1, 0]\,[0, 2]$. This relationship is an example of a blood related pair who also have a relationship containing a parental slide. So the existence of a parental slide in the statement of a relationship between two individuals does not mean that they are not blood related. The converse does hold; if two individuals are not blood related then there must be a parental slide in the statement of their relationship. It should be stressed at this point we are talking only of individuals within one family tree. (For more on this see Section 8).

The paths (i), (ii) from RD to MW are not the only ones, for we also have the path RD, CD, WD, EW, JW2, JW3, MW. At first glance this looks like a blood related set, but the blood relationship between CD and EW flatters to deceive. Even though CD and EW are blood related (grandparent JW1), RD and EW are not. Moreover we cannot put a parental slide between CD and EW (because they are blood relatives). The only way around this is to send the path from CD to EW via their common ancestor JW1. The path now becomes

RD, CD, SW, JW1, JW2, EW, JW2, JW3, MW.

The edge from JW2 to EW and back is redundant and can therefore be omitted. The path is reduced to

RD, CD, SW, JW1, JW2, JW3, MW.

Now RD and SW are not blood related, and we have a parental slide ***S*** to separate the path into two distinct blood related parts

(RD) ***S*** (SW, JW1, JW2, JW3, MW),

which is the same path (ii) we had initially.

In Section 2, we posed the question of how can the relationship of ED and JW1 be expressed even though they are not blood related. To see how this is done and the use of the parental slide we first take a path from ED to JW1, namely

ED, RD, CSD, MW, JW3, JW2, JW1.

There is a break in the blood relationship between MW and JW3, (even though CSD and JW3 are first cousins).

So the path from CSD to JW3 has to go via their common ancestor JW1

ED, RD, CSD, SW, JW1, JW2, JW3, JW2, JW1.

https://doi.org/10.1017/mag.2021.5 Published online by Cambridge University Press

This simplifies to

$$\text{ED, RD, CSD, SW, JW1.}$$

There is another break between CSD and SW, so we use the slide $\boldsymbol{S}$

$$\text{(ED, RD) } \boldsymbol{S} \text{ (SW, JW1)}$$

so the relationship of ED and JW1 is

$$[-1,\ 1]\,[-1,\ 1]\,.$$

7. *The relation of transitivity*

We have touched upon the symmetric property of being a cousin but the transitive relation is more interesting.

We consider the simple case of siblings (0 th cousins), and we ask the question: If A and B are 0 th cousins and B and C are 0 th cousins what possible relations can exist between A and C? The tree takes the form shown in Figure 16.

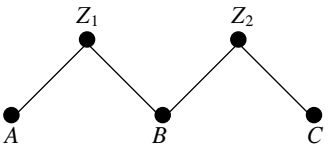


FIGURE 16

We note immediately that the problematic V-tree appears so the only possible break in the bloodline occurs between Z_1 and Z_2. Formally we have to consider the three cases

(i) $Z_1 = Z_2$

(ii) Z_1 and Z_2 being blood related

(iii) Z_1 and Z_2 *not* being blood related.

In case (i), A and C are 0 th cousins and so transitivity follows. In case (ii), Z_1 and Z_2 are m th cousins n times removed for some m and n. By adding the edges AZ_1 and Z_2C we see that A and C are $(m + 1)$ th cousins n times removed. These are not 0 th cousins (unless $m = -1$, $n = 0$, which is the previous case $Z_1 = Z_2$).

Finally we consider case (iii), and suppose Z_1 and Z_2 not being blood related. Then we have the tree in Figure 17.

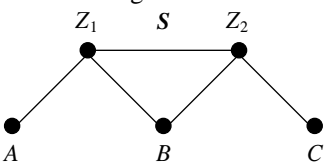


FIGURE 17

The path from A to C is $(AZ_1)\boldsymbol{S}(Z_2C)$ where the separate parts AZ_1 and Z_2C are blood related, and the relationship can be expressed as $[-1,\ 1]\,[-1,\ 1]$.

https://doi.org/10.1017/mag.2021.5 Published online by Cambridge University Press

8. *The existence of relationships*

We have seen examples of paths in a family tree expressed as a succession of blood related sets interspersed with parental slides. These paths can be translated into a succession of relationships of the form 'm th cousins, n times removed', that is,

$$[m_1, n_1]\,[m_2, n_2]\,[m_3, n_3]\,\ldots\,[m_t, n_t]\,.$$

What we have not done is to show that such relationships always exist between two individuals in a family tree.

One way to define a family tree is in terms of the underlying connected graph, connected because there needs to be a path of edges from one individual to another. This provides the existence of a path but, as we saw in Section 5, a given path may need to be amended to give us a succession of the form $[m, n]$.

To prove the existence of such amended paths we first note a relevant fact about the graph of a family tree, that there has to be a limit to a succession of edges going up the tree from child to parent because of a lack of information when we delve deeply into the past.

Given two individuals and the path P between them, the only place we lose the blood relationship is in a V-tree. If two parents are not blood related we insert a slide $\boldsymbol{S}$. This leads us from one blood related set to another. The only complication arises when the two parents in a V-tree are blood related. In this case we can connect them via a common ancestor.

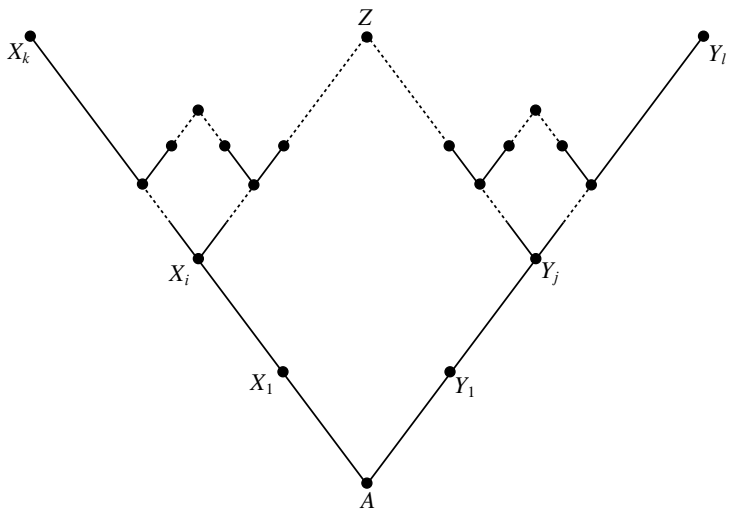


FIGURE 18

Suppose there is a path from X_k via a succession of edges to X_1 and A, and from A and Y_1 via a succession of edges to Y_l (see Figure 18). This gives us the V-tree X_1AY_1, and if X_1 and Y_1 are blood related, then they have a

https://doi.org/10.1017/mag.2021.5 Published online by Cambridge University Press

common ancestor Z, and there are edges linking X_1 and Z, and Z and Y_1. There are special cases to consider (such as Z lying on the path X_kX_1 or Y_1Y_l, but we shall only consider the general case where the path X_1Z leaves the path X_1X_k at X_i, and ZY_1 joins Y_lY_1 at Y_j as indicated in Figure 18. As we saw in the Example in Section 5 we can now change the original path

$$X_k \ldots X_1AY_1 \ldots Y_l \quad \text{to} \quad X_k \ldots X_i \ldots Z \ldots Y_j \ldots Y_l.$$

On the face of it the situation now seems worse because to get rid of one V-tree we appear to have produced two more at X_i and Y_j. Of course these may give us a slide S separating two blood related sets, but if not then we use again the same procedure introducing common ancestors in each V-tree which would lead this time to four V-trees. This is where the aforesaid limit comes in. Look at the child A of the first V-tree we examined, and the children X_i and Y_j in the second and third V-trees. Each use of the procedure moves the children up along edges to an ancestor. This process must come to an end.

Finally, we have an existence result.

Proposition: Given two individuals in a family tree (a connected graph) the relationship between them can be expressed in the form:

$$[m_1, n_1]\,[m_2, n_2]\,[m_3, n_3] \ldots [m_t, n_t]\,.$$

References

1. Tony Crilly and Colin R. Fletcher, How to choose your relations, *Math. Gaz*. **94** (July 2010) pp. 193-202.
2. https://en.wikipedia.org/wiki/Darwin-Wedgwood_family
3. https://en.wikipedia.org/wiki/Prohibited_degree_of_kinship

10.1017/mag.2021.5

© The Mathematical Association 2021

TONY CRILLY
10 Lemsford Road, St Albans AL1 3PB
e-mail: *tonycrilly42@gmail.com*
COLIN R. FLETCHER
Atalaya, Lon Glanfred, Llandre, Aberystwyth SY24 5BY
e-mail: *Atalaya1@btinternet.com*

https://doi.org/10.1017/mag.2021.5 Published online by Cambridge University Press