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TWO CLASSICAL LATTICE POINT PROBLEMS

BY K. S. GANGADHARAN†

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1·1. Let $r(n)$ be the number of representations of n as a sum of two squares, $d(n)$ the number of divisors of n, and

$$R(x) = \sum_{0 \leqslant n \leqslant x} r(n) = \pi x + \mathrm{P}(x), \tag{1·1·1}$$

$$D(x) = \sum_{1 \leqslant n \leqslant x} d(n) = x \log x + (2\gamma - 1)\, x + \Delta(x), \tag{1·1·2}$$

where γ is Euler's constant. Then $\mathrm{P}(x)$ is the error term in the problem of the lattice points of the circle, and $\Delta(x)$ the error term in Dirichlet's divisor problem, or the problem of the lattice points of the rectangular hyperbola.

1·2. It has been proved that,‡ when $x \to \infty$,

$$\mathrm{P}(x) = \Omega_-(x^{\frac{1}{4}} \log^{\frac{1}{2}} x), \tag{1·2·1}$$

and that

$$\Delta(x) = \Omega_+(x^{\frac{1}{4}} \log^{\frac{1}{2}} x \log\log x). \tag{1·2·2}$$

But, on the Ω_+ side for $\mathrm{P}(x)$, and, on the Ω_- side for $\Delta(x)$, it has only been proved (6) that

$$\overline{\lim_{x \to \infty}} \frac{\mathrm{P}(x)}{x^{\frac{1}{4}}} = +\infty, \tag{1·2·3}$$

and that

$$\underline{\lim_{x \to \infty}} \frac{\Delta(x)}{x^{\frac{1}{4}}} = -\infty. \tag{1·2·4}$$

1·3. The object of this paper is to replace (1·2·3) and (1·2·4) by results of the form

$$\mathrm{P}(x) = \Omega_+(x^{\frac{1}{4}} \rho(x)) \quad \text{as} \quad x \to \infty,$$

and

$$\Delta(x) = \Omega_-(x^{\frac{1}{4}} \delta(x)) \quad \text{as} \quad x \to \infty,$$

† Present address: Department of Mathematics, University College of South Wales, Cathays Park, Cardiff.

‡ See Hardy (4) for a proof of (1·2·2) and Landau (7), pp. 240–49 for a proof of (1·2·1). The method of proof is similar in the two cases, and depends essentially on the application of Dirichlet's theorem on Diophantine approximation and on the use of modified forms of the Phragmén–Lindelöf theorem. The method, however, does not lead to Ω-results of both sign, but Hardy, erroneously believing that it does, states in (4) that $\mathrm{P}(x) = \Omega_\pm(x^{\frac{1}{4}} \log^{\frac{1}{2}} x)$ as $x \to \infty$, and that $\Delta(x) = \Omega_\pm(x^{\frac{1}{4}} \log^{\frac{1}{2}} x \log\log x)$ as $x \to \infty$. In this connexion see note on page 340 of *The Collected Papers of Srinivasa Ramanujan* (Cambridge, 1927).

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respectively, with explicit $\rho(x)$, $\delta(x)$ which tend to infinity with x. We shall prove in fact that

$$\mathrm{P}(x) = \Omega_+(x^{\frac{1}{4}}(\log\log x)^{\frac{1}{4}}(\log\log\log x)^{\frac{1}{4}}) \quad \text{as} \quad x \to \infty, \tag{1·3·1}$$

and that

$$\Delta(x) = \Omega_-(x^{\frac{1}{4}}(\log\log x)^{\frac{1}{4}}(\log\log\log x)^{\frac{1}{4}}) \quad \text{as} \quad x \to \infty. \tag{1·3·2}$$

1·4. Substantially, however, the paper consists of a proof of the theorem stated in § 2·5. The theorem is of a general character and the Ω-results (2·5·1) and (2·5·2) furnished by it are not explicit, but involve an arbitrary function $Q(X)$ belonging to a certain class C_q which is defined in terms of the discussion in §§ 2·1–2·4. Nevertheless, the theorem serves primarily to achieve the object of the paper stated in § 1·3. In fact it is shown in § 2·6 that the results (1·3·1) and (1·3·2) follow from (2·5·1) and (2·5·2), respectively, when Q takes a certain explicit form Q_0 to be determined. But it is not at all certain that (1·3·1) and (1·3·2) are the best possible explicit implications of the theorem. In fact, it is indicated in § 2·7 that there is some justification for believing in the possibility of determining an explicit form Q_1, of Q, which, when substituted in the theorem, will take us beyond (1·3·1) and (1·3·2).

1·5. The method of proof of the theorem is based on an idea due to Ingham. This consists in the use of an argument involving the Fejér kernel as in one of the proofs of Kronecker's Theorem (2).

The preliminary analysis required in the proof of the theorem is contained in Parts 2, 3 and 4 of the paper; the main steps in the proof are given in Part 5.

I wish to express here my gratitude to Mr Ingham for the help and encouragement he gave me in the writing of this paper.

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2·1. Let X be a real variable. Throughout this paper $N = N(X)$ will denote, for each $X \geqslant 1$, the number of square-free integers† q such that $1 \leqslant q \leqslant X$; and the sequence $q_1, q_2, \ldots, q_N$ will represent these integers arranged in ascending order of magnitude.

2·2. For all $X \geqslant 2$, we define S_X as the set of all real numbers η expressible in the form‡

$$\eta = \left|\sqrt{n} + \sum_{\lambda=1}^{N} r_\lambda \sqrt{q_\lambda}\right|,$$

where $n, r_1, r_2, \ldots, r_N$ are any integers subject to the conditions

$$n \geqslant 0; \quad |r_\lambda| \leqslant 1 \ (\lambda = 1, 2, \ldots, N); \quad \sum_{\lambda=1}^{N} r_\lambda^2 \geqslant 2.$$

We then observe the following properties of the set S_X:

(i) $\eta \in S_X \Rightarrow \eta > 0$. (Since at least two of the r_λ must be non-zero, (i) follows easily from the following result§ (**B**): If $\{q\}$ is any finite set of distinct, positive, square-free

† By 'integer' we shall always mean 'rational integer'.

‡ When we use the symbol '$\sqrt{}$' with a non-negative real number we shall always indicate only the non-negative square root of that number.

§ See Besicovitch (1) for a more general result of which this is a special case.

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integers q, then the numbers $\sqrt{q}$, $q \in \{q\}$, are linearly independent with respect to the field of rational numbers.)

(ii) There is a number $\eta \in S_X$ such that $\eta < 1$. (For example, if $\sqrt{n_0} = [\Sigma \sqrt{q_\lambda}]$ then $\eta_0 = |\sqrt{n_0} - \Sigma \sqrt{q_\lambda}| < 1$; also $\eta_0 \in S_X$.)

(iii) There are only a finite number of distinct numbers $\eta \in S_X$ such that $0 < \eta < 1$. (For, if $n > (X^{\frac{3}{2}} + 1)^2$, then $\eta > 1$.)

We deduce from (i), (ii) and (iii) that there is a unique number $\tilde{\eta}(X) \in S_X$ such that

$$0 < \tilde{\eta}(X) < 1 \tag{2·2·1}$$

and

$$\eta \in S_X \Rightarrow \eta \geqslant \tilde{\eta}(X). \tag{2·2·2}$$

2·3. We now write

$$q(X) = -\log \tilde{\eta}(X) \quad (X \geqslant 2),$$

and proceed to establish the following statements:

There is a numerical constant $a > 0$ such that

$$q(X) > aX \quad (X \geqslant 2). \tag{2·3·1}$$

There is a numerical constant $b > 2$ such that

$$q(X) < b^{X/\log X} \quad (X \geqslant 2). \tag{2·3·2}$$

Proof of (2·3·1). Corresponding to each $X \geqslant 2$, there are integers $m', r'_1, r'_2, \ldots, r'_N$, depending on X, and not all zero, such that†

$$\left| m'[\sqrt{X} + 2] + \sum_{\lambda=1}^{N} r'_\lambda \sqrt{q_\lambda} \right| \leqslant 2^{-N}(\sqrt{X} + 2), \tag{2·3·3}$$

and

$$|r'_\lambda| < 2 \quad (\lambda = 1, 2, \ldots, N). \tag{2·3·4}$$

We observe that, for each λ,

$$|m'[\sqrt{X} + 2] + r'_\lambda \sqrt{q_\lambda}| \geqslant 1 \quad \text{unless} \quad m' = r'_\lambda = 0. \tag{2·3·5}$$

Also, since $N \sim 6X/\pi^2$ as $X \to \infty$,

$$2^{-N}(\sqrt{X} + 2) < 1 \quad (X \geqslant X_0), \tag{2·3·6}$$

for some fixed $X_0 \geqslant 2$.

From (2·3·3), (2·3·5) and (2·3·6) we infer that, if $X \geqslant X_0$, then at least two of the r'_λ are non-zero; that is

$$\sum_{\lambda=1}^{N} r'^2_\lambda \geqslant 2 \quad (X \geqslant X_0). \tag{2·3·7}$$

The inequalities (2·3·4) and (2·3·7) imply that, when $X \geqslant X_0$, the number

$$\eta' = \left| m'[\sqrt{X} + 2] + \sum_{\lambda=1}^{N} r'_\lambda \sqrt{q_\lambda} \right| \in S_X,$$

and hence, in view of (2·2·2), that $\eta' \geqslant \tilde{\eta}(X)$. Therefore

$$q(X) = -\log \tilde{\eta}(X) \geqslant N \log 2 - \log(\sqrt{X} + 2) \quad (X \geqslant X_0),$$

† This assertion may be verified as a special case of Minkowski's well-known theorem on linear forms. See, for example, Cassels (3), Appendix B (Theorem III).

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and (2·3·1) follows from this, if we observe that $q(X) > 0$ and increases with X for $X \geqslant 2$, and that $N \sim 6X/\pi^2$, $\log(\sqrt{X}+2) = o(X)$ as $X \to \infty$.

Proof of (2·3·2). We can write

$$\tilde{\eta}(X) = \left|\sqrt{\tilde{n}} + \sum_{\lambda=1}^{N} \tilde{r}_\lambda \sqrt{q_\lambda}\right|, \tag{2·3·8}$$

where $\tilde{n}, \tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_N$ are integers satisfying the conditions

$$\tilde{n} \geqslant 0; \quad |\tilde{r}_\lambda| \leqslant 1 \ (\lambda = 1, 2, \ldots, N); \quad \sum_{\lambda=1}^{N} \tilde{r}_\lambda^2 \geqslant 2.$$

Let $p_1, p_2, \ldots, p_M$, all distinct, be the primes p such that $2 \leqslant p \leqslant X$. Then, each q_λ, (except $q_1 = 1$), can be expressed uniquely as a product of distinct primes of the set $p_1, p_2, \ldots, p_M$. Inserting these expressions for the q_λ in (2·3·8), we can re-write (2·3·8) in the form

$$\tilde{\eta}(X) = |L(\sqrt{\tilde{n}}, \sqrt{q_1}, \sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_M})|, \tag{2·3·9}$$

where $L(x, y, x_1, x_2, \ldots, x_M)$ is a polynomial whose degree in each of the variables $x, y, x_1, x_2, \ldots, x_M$ is not greater than one, and whose coefficients are all integers depending only on $\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_N$.

Let us consider the expression $|L(\pm\sqrt{\tilde{n}}, \pm\sqrt{q_1}, \pm\sqrt{p_1}, \pm\sqrt{p_2}, \ldots, \pm\sqrt{p_M})|$. For every choice of the signs $\pm$ in this expression, we can plainly write

$$|L(\pm\sqrt{\tilde{n}}, \pm\sqrt{q_1}, \pm\sqrt{p_1}, \pm\sqrt{p_2}, \ldots, \pm\sqrt{p_M})| = \left|\sqrt{\tilde{n}} + \sum_{\lambda=1}^{N} (\pm\tilde{r}_\lambda)\sqrt{q_\lambda}\right|, \tag{2·3·10}$$

with an appropriate choice of the signs $\pm$ on the right-hand side. It follows from this remark that $|L(\pm\sqrt{\tilde{n}}, \pm\sqrt{q_1}, \pm\sqrt{p_1}, \pm\sqrt{p_2}, \ldots, \pm\sqrt{p_M})| \in S_X$, and therefore, by (2·2·2) and (2·2·1), that

$$|L(\pm\sqrt{\tilde{n}}, \pm\sqrt{q_1}, \pm\sqrt{p_1}, \pm\sqrt{p_2}, \ldots, \pm\sqrt{p_M})| \geqslant \tilde{\eta}(X) > 0. \tag{2·3·11}$$

On the other hand, in view of (2·3·8) and (2·2·1), (2·3·10) implies also that

$$|L(\pm\sqrt{\tilde{n}}, \pm\sqrt{q_1}, \pm\sqrt{p_1}, \pm\sqrt{p_2}, \ldots, \pm\sqrt{p_M})| \leqslant \tilde{\eta}(X) + 2X^{\frac{3}{2}} < 1 + 2X^{\frac{3}{2}}. \tag{2·3·12}$$

Let us now consider the polynomial

$$\mathscr{L}(x, y, x_1, x_2, \ldots, x_M) = \Pi L(\pm x, \pm y, \pm x_1, \pm x_2, \ldots, \pm x_M),$$

where the product Π is taken over all the 2^{M+2} choices of the signs $\pm$. $\mathscr{L}$ is clearly a polynomial of even degree† in each of the variables $x, y, x_1, x_2, \ldots, x_M$, and its coefficients are all integers. Therefore $\mathscr{L}(\sqrt{\tilde{n}}, \sqrt{q_1}, \sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_M})$ is an integer. Since this integer cannot be zero by virtue of (2·3·11), we may assert that

$$|\mathscr{L}(\sqrt{\tilde{n}}, \sqrt{q_1}, \sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_M})| \geqslant 1. \tag{2·3·13}$$

From (2·3·9), (2·3·12) and (2·3·13) we now obtain

$$\frac{1}{\tilde{\eta}(X)} \leqslant \frac{|\mathscr{L}(\sqrt{\tilde{n}}, \sqrt{q_1}, \sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_M})|}{|L(\sqrt{\tilde{n}}, \sqrt{q_1}, \sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_M})|} < (1 + 2X^{\frac{3}{2}})^{2^{M+2}-1}.$$

Thus $q(X) = -\log\tilde{\eta}(X) < (2^{M+2}-1)\log(1+2X^{\frac{3}{2}})$, and this proves (2·3·2), since $M \sim X/\log X$ as $X \to \infty$.

† $\mathscr{L}$ cannot vanish identically since, by (2·3·11), $\mathscr{L}(\sqrt{\tilde{n}}, \sqrt{q_1}, \sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_M}) \neq 0$.

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2·4. We define C_q to be the class of all functions $Q(X)$ such that, for all $X \geqslant X_Q$ ($\geqslant 2$),

$$Q(X) \text{ is continuous,} \tag{2·4·1}$$

$$X^{-1}Q(X) \text{ increases with } X, \tag{2·4·2}$$

$$Q(X) \geqslant q(X). \tag{2·4·3}$$

We observe first that the class C_q is not empty. In fact:

There is a numerical constant $b > 2$ such that

$$b^{X/\log X} \in C_q. \tag{2·4·4}$$

This result follows immediately from (2·3·2) since $X^{-1}b^{X/\log X}$ is a continuous, increasing function of X for all sufficiently large X.

Next we see that if $Q(X) \in C_q$, then, by (2·4·2), $Q(X)$ is a strictly increasing function of X for $X \geqslant X_Q$. Also, by (2·4·3) and (2·3·1), $Q(X) \to \infty$ as $X \to \infty$. Hence, by (2·4·1), $Q(X)$ has a continuous, strictly increasing inverse $Q^{-1}(X)$ defined for all $X \geqslant X'_Q = Q(X_Q)$ and tending to infinity with X.

2·5. We are now in a position to state the main results of this paper. These are contained in the following

THEOREM. *If $Q(X) \in C_q$, then*

$$\mathrm{P}(x) = \Omega_+(x^{\frac{1}{4}}\{Q^{-1}(\log x)\}^{\frac{1}{4}}) \quad as \quad x \to \infty, \tag{2·5·1}$$

and

$$\Delta(x) = \Omega_-(x^{\frac{1}{4}}\{Q^{-1}(\log x)\}^{\frac{1}{4}} \log\{Q^{-1}(\log x)\}) \quad as \quad x \to \infty, \tag{2·5·2}$$

where Q^{-1} denotes the functional inverse of Q.

The remainder of the paper, with the exception of the next two sections, is concerned with the proof of this theorem.

2·6. The result (2·4·4) informs us that the function $Q_0(X) = b^{X/\log X} \in C_q$, for some numerical constant $b > 2$. We can now readily verify that

$$Q_0^{-1}(X) > \log X \log\log X/\log b \quad (X \geqslant X(b)),$$

and that, if $\beta > 1$ is any numerical constant,

$$Q_0^{-1}(X) < \beta \log X \log\log X/\log b \quad (X \geqslant X(b, \beta)).$$

From these observations it is clear that the results (1·3·1) and (1·3·2) follow from (2·5·1) and (2·5·2) respectively, when we put $Q = Q_0$.

2·7. We now discuss briefly the possibility of using the theorem of § 2·5 to go beyond (1·3·1) and (1·3·2). This possibility obviously depends on that of improving the result (2·3·2) which provides an explicit upper bound for the function $q(X)$.

Let us consider the argument used to prove (2·3·2). Essentially, this argument consists in the use of the identity $q(X) = \log(I/J)$, where

$$I = \frac{|\mathscr{L}(\sqrt{\tilde{n}}, \sqrt{q_1}, \sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_M})|}{|L(\sqrt{\tilde{n}}, \sqrt{q_1}, \sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_M})|}, \quad J = |\mathscr{L}(\sqrt{\tilde{n}}, \sqrt{q_1}, \sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_M})|,$$

and in the determination of an upper bound for I and a lower bound for J. We obtained an upper bound for I using merely the crude fact that every factor

$$|L(\pm\sqrt{\tilde{n}}, \pm\sqrt{q_1}, \pm\sqrt{p_1}, \pm\sqrt{p_2}, \ldots, \pm\sqrt{p_M})|$$

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of I is less than $1+2X^{\frac{3}{2}}$ (see 2·3·12); also, we obtained a lower bound for J from the trivial observation that J is a non-zero integer (see 2·3·13). So it is quite likely that the bounds for I and J which lead to the result (2·3·2) are extremely rough. It is therefore not unreasonable to suppose that it may be possible to improve the result (2·3·2); that is, to replace the index $X/\log X$ in (2·3·2) by an explicit function of X of lower order as $X \to \infty$.

3

3·1. *The function* $g(s)$. The Dirichlet series $\Sigma r(n)\, e^{-s\sqrt{n}}$ is clearly convergent for all $s = \sigma + it$ in the half-plane $\sigma > 0$. We write

$$f(s) = \sum_{n=0}^{\infty} r(n)\, e^{-s\sqrt{n}}, \quad g(s) = \frac{f(s)}{s} - \frac{2\pi}{s^3} \quad (\sigma > 0),$$

and make the following assertions concerning $g(s)$:

LEMMA R_1.
$$\sigma^{\frac{3}{2}} g(\sigma \pm 2\pi i\sqrt{m}) = \frac{1}{4\sqrt{\pi}} \frac{r(m)}{m^{\frac{3}{4}}} e^{\mp\frac{3}{4}\pi i} + O(\sigma Y^{\frac{3}{2}})\dagger$$

$(0 < \sigma < 1;\ Y \geqslant 1;\ 1 \leqslant m \leqslant Y,\ m \text{ an integer})$.

LEMMA R_2. *Let* $R(k, \omega)$ *denote the set of all points* $s = \sigma + it$ *such that*

$$\sigma > 0, \quad |s| \leqslant k, \quad |s \pm 2\pi i\sqrt{n}| \geqslant \omega \quad (n \geqslant 1).$$

Then
$$|g(s)| = O(\omega^{-\frac{3}{2}} k^2) \quad (k \geqslant 2\pi,\ 0 < \omega < 1,\ s \in R(k, \omega)).$$

The proofs of these lemmas depend on the formula (5)

$$\sum_{n=0}^{\infty} r(n)\, e^{-s\sqrt{n}} = 2\pi s \sum_{n=0}^{\infty} \frac{r(n)}{(s^2 + 4\pi^2 n)^{\frac{3}{2}}} \quad (\sigma > 0),$$

where the terms of the second series are taken positive when $s > 0$.

Proof of Lemma R_1. For $0 < \sigma < 1$, $m \geqslant 1$, we have

$$\begin{aligned} 2\pi\sigma^{\frac{3}{2}} \frac{r(m)}{\{(\sigma \pm 2\pi i\sqrt{m})^2 + 4\pi^2 m\}^{\frac{3}{2}}} &= 2\pi \frac{r(m)}{(\pm 4\pi i\sqrt{m})^{\frac{3}{2}}} \left(1 \pm \frac{\sigma}{4\pi i\sqrt{m}}\right)^{-\frac{3}{2}} \\ &= \frac{1}{4\sqrt{\pi}} \frac{r(m)}{m^{\frac{3}{4}}} e^{\mp\frac{3}{4}\pi i} \left(1 \pm \frac{\sigma}{4\pi i\sqrt{m}}\right)^{-\frac{3}{2}} \\ &= \frac{1}{4\sqrt{\pi}} \frac{r(m)}{m^{\frac{3}{4}}} e^{\mp\frac{3}{4}\pi i} \{1 + O(\sigma)\} \\ &= \frac{1}{4\sqrt{\pi}} \frac{r(m)}{m^{\frac{3}{4}}} e^{\mp\frac{3}{4}\pi i} + O(\sigma), \end{aligned} \tag{3·1·1}$$

since
$$\frac{r(m)}{m^{\frac{3}{4}}} = O(1) \quad (m \geqslant 1).$$

† Throughout the paper the symbol O implies an absolute constant and an inequality valid for all specified values of the variables.

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Next, for $0 < \sigma < 1$, $1 \leqslant m \leqslant Y$, we have

$$\begin{aligned}\left|2\pi\sigma^{\frac{3}{2}} \sum_{\substack{n=1\\ n\neq m}}^{\infty} \frac{r(n)}{\{(\sigma \pm 2\pi i\sqrt{m})^2 + 4\pi^2 n\}^{\frac{3}{2}}}\right| &< 2\pi\sigma^{\frac{3}{2}}\left\{\sum_{n=1}^{m-1} r(n) + \sum_{n=m+1}^{\infty} (m+1)^{\frac{3}{2}} \frac{r(n)}{(4\pi^2 n)^{\frac{3}{2}}}\right\}\\ &< 2\pi\sigma^{\frac{3}{2}}\left\{\sum_{n=1}^{m-1} r(n) + (m+1)^{\frac{3}{2}} \sum_{n=1}^{\infty} r(n)\, n^{-\frac{3}{2}}\right\}\\ &< 2\pi\sigma^{\frac{3}{2}}\left\{\sum_{1\leqslant n\leqslant Y} r(n) + (Y+1)^{\frac{3}{2}} \sum_{n=1}^{\infty} r(n)\, n^{-\frac{3}{2}}\right\}\\ &= O(\sigma^{\frac{3}{2}} Y^{\frac{3}{2}}), \qquad (3\cdot1\cdot2)\end{aligned}$$

since $$\sum_{1\leqslant n\leqslant Y} r(n) = O(Y) = O(Y^{\frac{3}{2}}) \quad (Y \geqslant 1).$$

Lemma R_1 now follows at once from (3·1·1) and (3·1·2).

Proof of Lemma R_2. Suppose that $k \geqslant 2\pi$, $0 < \omega < 1$, $s \in R(k, \omega)$. Let $K = k^2/2\pi^2$. Then, since $$|s^2 + 4\pi^2 n| = |s + 2\pi i\sqrt{n}|\, |s - 2\pi i\sqrt{n}| \geqslant 2\pi\omega \quad (n \geqslant 1),$$

and $$|s^2 + 4\pi^2 n| \geqslant 4\pi^2 n - k^2 > 2\pi^2 n \quad (n > K),$$

we have

$$\begin{aligned}|g(s)| &= 2\pi\left|\sum_{n=1}^{\infty} \frac{r(n)}{(s^2 + 4\pi^2 n)^{\frac{3}{2}}}\right|\\ &\leqslant 2\pi\left\{\sum_{1\leqslant n\leqslant K} \frac{r(n)}{|s^2 + 4\pi^2 n|^{\frac{3}{2}}} + \sum_{n>K} \frac{r(n)}{|s^2 + 4\pi^2 n|^{\frac{3}{2}}}\right\}\\ &< 2\pi\left\{(2\pi\omega)^{-\frac{3}{2}} \sum_{1\leqslant n\leqslant K} r(n) + \sum_{n>K} \frac{r(n)}{(2\pi^2 n)^{\frac{3}{2}}}\right\}\\ &< 2\pi\left\{\omega^{-\frac{3}{2}} \sum_{1\leqslant n\leqslant K} r(n) + \sum_{n=1}^{\infty} r(n)\, n^{-\frac{3}{2}}\right\}\\ &= O(\omega^{-\frac{3}{2}} k^2),\end{aligned}$$

and this proves Lemma R_2.

3·2. *The function* $G(s)$. The Dirichlet series $\Sigma d(n)\, e^{-s\sqrt{n}}$ is clearly convergent for all $s = \sigma + it$ in the half-plane $\sigma > 0$. We write

$$F(s) = \sum_{n=1}^{\infty} d(n)\, e^{-s\sqrt{n}}, \quad G(s) = \frac{F(s)}{s} + \frac{4}{s^3}(\log s - 1) - \frac{1}{4s} \quad (\sigma > 0),$$

and make the following assertions concerning $G(s)$:

LEMMA D_1. $$\sigma^{\frac{3}{2}} G(\sigma \pm 4\pi i\sqrt{m}) = \frac{1}{4\sqrt{2}\sqrt{\pi}} \frac{d(m)}{m^{\frac{3}{4}}} e^{\mp\frac{1}{4}\pi i} + O(\sigma^{\frac{1}{2}} Y^2 \log^3 Y)$$

$(0 < \sigma < 1;\ Y \geqslant 2;\ 1 \leqslant m \leqslant Y,\ m$ *an integer*$)$.

LEMMA D_2. *Let* $D(k, \omega)$ *denote the set of all points* $s = \sigma + it$ *such that*

$$0 < \sigma < 1, \quad |s| \leqslant k, \quad |s \pm 4\pi i\sqrt{n}| \geqslant \omega \quad (n \geqslant 1).$$

Then $$|G(s)| = O(\sigma^{-1}\omega^{-\frac{3}{2}} k^{\frac{5}{2}} \log^3 k) \quad (k \geqslant 2\pi,\ 0 < \omega < 1,\ s \in D(k, \omega)).$$

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The proofs of these lemmas depend on the formula†

$$\sum_{n=1}^{\infty} d(n)\, e^{-s\sqrt{n}}$$
$$= -\frac{4}{s^2}(\log s - 1) + \frac{1}{4} + \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{d(n)}{n}\left\{l\left(-\frac{s}{2\pi\sqrt{n}}\right) - \tfrac{1}{2}l\left(-\frac{is}{2\pi\sqrt{n}}\right) - \tfrac{1}{2}l\left(\frac{is}{2\pi\sqrt{n}}\right)\right\} \quad (\sigma > 0),$$

where
$$l(z) = \int_0^{\frac{1}{2}\pi} \frac{\sin\phi\, d\phi}{(1-\frac{1}{2}z\sin\phi)^2}.$$

We need some preliminary results concerning the function $l(z)$. We note first that $l(z)$ is single-valued and regular in the region Z_0 of the z-plane, obtained by making a slit from $z = 2$ to $z = \infty$ along the real axis. We may therefore express $l(z)$ as an absolutely convergent power series

$$\sum_{n=0}^{\infty} l_n z^n$$

when $|z| < 2$. So, if $|z| \leqslant 1$, we have

$$|l(z) - \tfrac{1}{2}l(iz) - \tfrac{1}{2}l(-iz)| = \left|l_1 z + \sum_{n=2}^{\infty} l_n\{z^n - \tfrac{1}{2}(iz)^n - \tfrac{1}{2}(-iz)^n\}\right|$$
$$\leqslant |z|\left\{|l_1| + 2\sum_{n=2}^{\infty}|l_n|\right\}.$$

Thus
$$|l(z) - \tfrac{1}{2}l(iz) - \tfrac{1}{2}l(-iz)| = O(|z|) \quad (|z| \leqslant 1). \tag{3·2·1}$$

Next, using the original integral formula for $l(z)$, we see that, if $\Re z \leqslant 0$,

$$|l(z)| \leqslant \int_0^{\frac{1}{2}\pi} \sin\phi\, d\phi.$$

Thus
$$|l(z)| \leqslant 1 \quad (\Re z \leqslant 0). \tag{3·2·2}$$

Again, evaluating the integral formula for $l(z)$, we have‡

$$l(z) = \frac{8z}{(4-z^2)^{\frac{3}{2}}}\tan^{-1}\sqrt{\left(\frac{2+z}{2-z}\right)} + \frac{4}{4-z^2} \quad (z \in Z_0), \tag{3·2·3}$$

† See Hardy (4). Hardy states the formula in a postscript to his paper. In the paper itself Hardy considers the function defined in the half-plane $\sigma > 0$ by the series $\Sigma d(n) n^{-\frac{1}{4}} e^{-s\sqrt{n}}$ and his analysis implies that

$$\sum_{n=1}^{\infty}\frac{d(n)}{\sqrt{n}}\, e^{-s\sqrt{n}} = -\frac{4\log s}{s} + \{\zeta(\tfrac{1}{2})\}^2 - \tfrac{1}{4}s$$
$$+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{d(n)}{\sqrt{n}}\left\{L\left(-\frac{s}{2\pi\sqrt{n}}\right) + \frac{i}{2}L\left(-\frac{is}{2\pi\sqrt{n}}\right) - \frac{i}{2}L\left(\frac{is}{2\pi\sqrt{n}}\right)\right\} \quad (\sigma > 0),$$

where
$$L(z) = -\pi - z - 2\int_0^{\frac{1}{2}\pi}\frac{d\phi}{1-\frac{1}{2}z\sin\phi}.$$

From this formula the corresponding one for $\Sigma d(n)\, e^{-s\sqrt{n}}$ may be obtained by term-by-term differentiation which is valid since all the series concerned can be shown to converge uniformly in every bounded, closed, region in the half-plane $\sigma > 0$.

‡ In (3·2·3) the relevant branch of the many-valued function on the right-hand side is the principal branch—that is, the branch which takes, at say $z = 1$, the value $\frac{8}{9\sqrt{3}}\pi + \frac{4}{3}$.

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and we shall now use this expression for $l(z)$ to show that

$$\left.\begin{aligned} |l(z)| &= O(\epsilon^{-\frac{3}{2}}\Lambda\log\Lambda) \\ (0<\epsilon<1;\ \Lambda\geqslant 2;\ z\in Z_0;\ |z|\leqslant\Lambda;\ |z-2|\geqslant\epsilon).& \end{aligned}\right\} \tag{3·2·4}$$

and that

$$y^{\frac{3}{2}}l(2\pm iy) = \pi e^{\pm\frac{3}{4}\pi i} + O(\sqrt{y}) \quad (0<y\leqslant 1). \tag{3·2·5}$$

Proof of (3·2·4). Suppose that ϵ, Λ and z satisfy the conditions prescribed in (3·2·4). We may assume also that $\Re z > 0$, in view of (3·2·2).

Observing that $|4-z^2| > 2|z-2|$, by (3·2·3) we get

$$|l(z)| \leqslant 2\sqrt{2}\,\Lambda\epsilon^{-\frac{3}{2}}\left|\tan^{-1}\sqrt{\left(\frac{2+z}{2-z}\right)}\right| + 2\epsilon^{-1}. \tag{3·2·6}$$

Now let
$$\sqrt{\left(\frac{2+z}{2-z}\right)} = \zeta e^{i\xi} \quad (\zeta>0,\ \xi \text{ real}).$$

Then
$$z = 2\frac{\zeta^2 e^{2i\xi}-1}{\zeta^2 e^{2i\xi}+1}.$$

Hence
$$\Lambda^2 \geqslant |z|^2 = 4\frac{\zeta^4-2\zeta^2\cos 2\xi+1}{\zeta^4+2\zeta^2\cos 2\xi+1} = 4\frac{(\zeta^2-1)^2+4\zeta^2\sin^2\xi}{(\zeta^2+1)^2-4\zeta^2\sin^2\xi},$$

so that
$$2\zeta|\sin\xi| \leqslant (\zeta^2+1)\,\Lambda(\Lambda^2+4)^{-\frac{1}{2}}.$$

Thus
$$\begin{aligned}\left|\tan^{-1}\sqrt{\left(\frac{2+z}{2-z}\right)}\right| &= \left|\frac{1}{2i}\log\frac{1+i\zeta e^{i\xi}}{1-i\zeta e^{i\xi}}\right| \\ &\leqslant \frac{1}{2}\left|\log\left|\frac{1+i\zeta e^{i\xi}}{1-i\zeta e^{i\xi}}\right|\right| + \frac{\pi}{2} \\ &\leqslant \frac{1}{4}\left|\log\frac{\zeta^2-2\zeta\sin\xi+1}{\zeta^2+2\zeta\sin\xi+1}\right| + \frac{\pi}{2} \\ &\leqslant \tfrac{1}{4}\log\frac{\zeta^2+2\zeta|\sin\xi|+1}{\zeta^2-2\zeta|\sin\xi|+1} + \frac{\pi}{2} \\ &\leqslant \tfrac{1}{4}\log\frac{1+\Lambda(\Lambda^2+4)^{-\frac{1}{2}}}{1-\Lambda(\Lambda^2+4)^{-\frac{1}{2}}} + \frac{\pi}{2} \\ &< \tfrac{1}{4}\log\frac{2}{1-\Lambda(\Lambda^2+4)^{-\frac{1}{2}}} + \frac{\pi}{2} \\ &< \tfrac{1}{4}\log(2\Lambda^2) + \tfrac{1}{2}\pi, \end{aligned} \tag{3·2·7}$$

since $\Lambda \geqslant 2$. (3·2·4) now follows at once from (3·2·6) and (3·2·7).

Proof of (3·2·5). Putting $z = 2\pm iy$ $(0<y\leqslant 1)$ in (3·2·3), and multiplying out by $y^{\frac{3}{2}}$, we have

$$\begin{aligned} y^{\frac{3}{2}}l(2\pm iy) &= y^{\frac{3}{2}}\frac{8(2\pm iy)}{\{4-(2\pm iy)^2\}^{\frac{3}{2}}}\tan^{-1}\sqrt{\left(\frac{4\pm iy}{\mp iy}\right)} + \frac{4y^{\frac{3}{2}}}{4-(2\pm iy)^2} \\ &= \frac{16}{(\mp 4i)^{\frac{3}{2}}}\frac{1\pm\frac{1}{2}iy}{(1\pm\frac{1}{4}iy)^{\frac{3}{2}}}\left\{\frac{\pi}{2}-\tan^{-1}\sqrt{\left(\frac{\mp iy}{4\pm iy}\right)}\right\} \pm \frac{i\sqrt{y}}{1\pm\frac{1}{4}iy}, \end{aligned}$$

and from this the truth of (3·2·5) is evident.

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Proof of Lemma D_1. We write

$$\Sigma_1 = \frac{\sigma^{\frac{3}{2}}}{2\pi^2 s'} \sum_{1 \leqslant n \leqslant 9Y} \frac{d(n)}{n} l\left(-\frac{s'}{2\pi\sqrt{n}}\right),$$

$$\Sigma_2 = \frac{\sigma^{\frac{3}{2}}}{2\pi^2 s'} \sum_{1 \leqslant n \leqslant 9Y} \frac{d(n)}{n} \left\{-\tfrac{1}{2} l\left(\frac{is'}{2\pi\sqrt{n}}\right)\right\},$$

$$\Sigma_3 = \frac{\sigma^{\frac{3}{2}}}{2\pi^2 s'} \sum_{n > 9Y} \frac{d(n)}{n} \left\{l\left(-\frac{s'}{2\pi\sqrt{n}}\right) - \tfrac{1}{2} l\left(-\frac{is'}{2\pi\sqrt{n}}\right) - \tfrac{1}{2} l\left(\frac{is'}{2\pi\sqrt{n}}\right)\right\},$$

$$\Sigma_4 = \frac{\sigma^{\frac{3}{2}}}{2\pi^2 s'} \sum_{\substack{1 \leqslant n \leqslant 9Y \\ n \neq m}} \frac{d(n)}{n} \left\{-\tfrac{1}{2} l\left(-\frac{is'}{2\pi\sqrt{n}}\right)\right\},$$

$$D = -\frac{\sigma^{\frac{3}{2}}}{4\pi^2 s'} \frac{d(m)}{m} l\left(-\frac{is'}{2\pi\sqrt{m}}\right),$$

where $s' = \sigma + 4\pi i\sqrt{m}$ $(0 < \sigma < 1;\ Y \geqslant 2;\ 1 \leqslant m \leqslant Y,\ m$ an integer),† so that

$$\sigma^{\frac{3}{2}} G(\sigma + 4\pi i\sqrt{m}) = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + D,$$

and we proceed to estimate the expressions Σ_1, Σ_2, Σ_3, Σ_4, and D separately.

(Σ_1). Since

$$\Re \frac{-s'}{2\pi\sqrt{n}} = -\frac{\sigma}{2\pi\sqrt{n}} < 0,$$

by (3·2·2) we have

$$\left| l\left(-\frac{s'}{2\pi\sqrt{n}}\right)\right| \leqslant 1 \quad (n \geqslant 1).$$

Also $|s'| \geqslant 4\pi\sqrt{m} \geqslant 4\pi$. Hence

$$|\Sigma_1| \leqslant \frac{\sigma^{\frac{3}{2}}}{8\pi^3} \sum_{1 \leqslant n \leqslant 9Y} \frac{d(n)}{n} = O(\sigma^{\frac{3}{2}} \log^2 Y). \tag{3·2·8}$$

(Σ_2). Similarly, since

$$\Re \frac{is'}{2\pi\sqrt{n}} = -2\frac{\sqrt{m}}{\sqrt{n}} < 0,$$

we have

$$|\Sigma_2| \leqslant \frac{\sigma^{\frac{3}{2}}}{16\pi^3} \sum_{1 \leqslant n \leqslant 9Y} \frac{d(n)}{n} = O(\sigma^{\frac{3}{2}} \log^2 Y). \tag{3·2·9}$$

(Σ_3). If $n > 9Y$, then $|s'/2\pi\sqrt{n}| < 1$ since $|s'| < 5\pi\sqrt{m} \leqslant 5\pi\sqrt{Y}$. Hence, by (3·2·1), we have

$$\left| l\left(-\frac{s'}{2\pi\sqrt{n}}\right) - \tfrac{1}{2} l\left(-\frac{is'}{2\pi\sqrt{n}}\right) - \tfrac{1}{2} l\left(\frac{is'}{2\pi\sqrt{n}}\right)\right| = O\left(\left|\frac{s'}{2\pi\sqrt{n}}\right|\right) \quad (n > 9Y),$$

and consequently

$$|\Sigma_3| = O\left(\sigma^{\frac{3}{2}} \sum_{n > 9Y} \frac{d(n)}{n^{\frac{3}{2}}}\right) = O\left(\sigma^{\frac{3}{2}} \sum_{n=1}^{\infty} \frac{d(n)}{n^{\frac{3}{2}}}\right) = O(\sigma^{\frac{3}{2}}). \tag{3·2·10}$$

(Σ_4). If $n \neq m$,

$$\left| -\frac{is'}{2\pi\sqrt{n}} - 2\right| = \left| -\frac{i\sigma}{2\pi\sqrt{n}} + 2\frac{\sqrt{m}}{\sqrt{n}} - 2\right| > 2\left|\frac{\sqrt{m}}{\sqrt{n}} - 1\right| > \frac{1}{2m} \geqslant \frac{1}{2Y}.$$

† We do not repeat these conditions on σ, Y and m again in the proof, but they are always to be understood whenever s' occurs.

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Also, for $n \geqslant 1$,

$$\left|-\frac{is'}{2\pi\sqrt{n}}\right| < \frac{5\pi\sqrt{m}}{2\pi\sqrt{n}} \leqslant \frac{5\pi\sqrt{Y}}{2\pi\sqrt{n}} < 3\sqrt{Y}.$$

Thus, taking

$$z = -\frac{is'}{2\pi\sqrt{n}} \ (n \neq m), \quad \Lambda = 3\sqrt{Y}, \quad \epsilon = \frac{1}{2Y}$$

in (3·2·4), we see that

$$\left|l\left(-\frac{is'}{2\pi\sqrt{n}}\right)\right| = O(Y^2 \log Y) \quad (n \neq m),$$

and from this we obtain

$$|\Sigma_4| = O\left(\sigma^{\frac{3}{2}} Y^2 \log Y \sum_{1 \leqslant n \leqslant 9Y} \frac{d(n)}{n}\right) = O(\sigma^{\frac{3}{2}} Y^2 \log^3 Y). \tag{3·2·11}$$

(*D*). We have

$$\begin{aligned} D &= -\frac{\sigma^{\frac{3}{2}}}{4\pi^2(\sigma + 4\pi i\sqrt{m})} \frac{d(m)}{m} l\left(2 - \frac{i\sigma}{2\pi\sqrt{m}}\right) \\ &= -\frac{1}{4\pi^2} \frac{d(m)}{m} \frac{1}{4\pi i\sqrt{m}} \left(1 + \frac{\sigma}{4\pi i\sqrt{m}}\right)^{-1} (2\pi\sqrt{m})^{\frac{3}{2}} \left\{\left(\frac{\sigma}{2\pi\sqrt{m}}\right)^{\frac{3}{2}} l\left(2 - \frac{i\sigma}{2\pi\sqrt{m}}\right)\right\} \\ &= \frac{i}{4\sqrt{2}\,\pi^{\frac{3}{2}}} \frac{d(m)}{m^{\frac{3}{4}}} \{1 + O(\sigma)\} \left\{\left(\frac{\sigma}{2\pi\sqrt{m}}\right)^{\frac{3}{2}} l\left(2 - \frac{i\sigma}{2\pi\sqrt{m}}\right)\right\}. \end{aligned}$$

Now, using (3·2·5), we have

$$\begin{aligned} D &= \frac{i}{4\sqrt{2}\,\pi^{\frac{3}{2}}} \frac{d(m)}{m^{\frac{3}{4}}} \{1 + O(\sigma)\} \{\pi e^{-\frac{3}{4}\pi i} + O(\sqrt{\sigma})\} \\ &= \frac{1}{4\sqrt{2}\sqrt{\pi}} \frac{d(m)}{m^{\frac{3}{4}}} e^{-\frac{1}{4}\pi i} + O(\sqrt{\sigma}), \end{aligned} \tag{3·2·12}$$

since $0 < \sigma < 1$ and $\dfrac{d(m)}{m^{\frac{3}{4}}} = O(1), (m \geqslant 1)$.

From (3·2·8)–(3·2·12) we obtain at once the expression for $\sigma^{\frac{3}{2}} G(\sigma + 4\pi i\sqrt{m})$ given by Lemma D_1. The corresponding expression for $\sigma^{\frac{3}{2}} G(\sigma - 4\pi i\sqrt{m})$ then follows automatically, since $G(\bar{s})$ is equal to the complex conjugate of $G(s)$ for all $s = \sigma + it$ with $\sigma > 0$.

Proof of Lemma D_2. Suppose that $k \geqslant 2\pi$, $0 < \omega < 1$, $s'' = \sigma'' + it'' \in D(k, \omega)$. Let $K' = k^2/4\pi^2$. We write

$$\Sigma_1' = \frac{1}{2\pi^2 s''} \sum_{1 \leqslant n \leqslant K'} \frac{d(n)}{n} l\left(-\frac{s''}{2\pi\sqrt{n}}\right),$$

$$\Sigma_2' = \frac{1}{2\pi^2 s''} \sum_{1 \leqslant n \leqslant K'} \frac{d(n)}{n} \left\{-\tfrac{1}{2} l\left(-\frac{is''}{2\pi\sqrt{n}}\right) - \tfrac{1}{2} l\left(\frac{is''}{2\pi\sqrt{n}}\right)\right\},$$

$$\Sigma_3' = \frac{1}{2\pi^2 s''} \sum_{n > K'} \frac{d(n)}{n} \left\{l\left(-\frac{s''}{2\pi\sqrt{n}}\right) - \tfrac{1}{2} l\left(-\frac{is''}{2\pi\sqrt{n}}\right) - \tfrac{1}{2} l\left(\frac{is''}{2\pi\sqrt{n}}\right)\right\},$$

so that $G(s'') = \Sigma_1' + \Sigma_2' + \Sigma_3'$.

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Using inequalities similar to the ones used in the estimations of Σ_1 and Σ_3 in the proof of Lemma D_1, we can easily see that

$$|\Sigma_1'| = O(\sigma''^{-1}\log^2 k), \tag{3·2·13}$$

and that

$$|\Sigma_3'| = O(1). \tag{3·2·14}$$

It remains to estimate Σ_2'. By hypothesis, for $n \geqslant 1$,

$$\left|\pm\frac{is''}{2\pi\sqrt{n}} - 2\right| = \frac{1}{2\pi\sqrt{n}}|s'' \pm 4\pi i\sqrt{n}| \geqslant \frac{\omega}{2\pi\sqrt{n}}.$$

So, if $1 \leqslant n \leqslant K'$,

$$\left|\pm\frac{is''}{2\pi\sqrt{n}} - 2\right| \geqslant \frac{\omega}{k}.$$

Thus, taking $\quad z = \pm\dfrac{is''}{2\pi\sqrt{n}} \quad (1 \leqslant n \leqslant K'), \quad \Lambda = k, \quad \epsilon = \dfrac{\omega}{k}$

in (3·2·4), we see that

$$\left|l\left(\pm\frac{is''}{2\pi\sqrt{n}}\right)\right| = O(\omega^{-\frac{3}{2}}k^{\frac{5}{2}}\log k) \quad (1 \leqslant n \leqslant K').$$

Hence

$$\begin{aligned}|\Sigma_2'| &= O\left(\sigma''^{-1}\omega^{-\frac{3}{2}}k^{\frac{5}{2}}\log k \sum_{1\leqslant n\leqslant K'}\frac{d(n)}{n}\right)\\ &= O(\sigma''^{-1}\omega^{-\frac{3}{2}}k^{\frac{5}{2}}\log^3 k),\end{aligned} \tag{3·2·15}$$

and now Lemma D_2 follows at once from (3·2·13)–(3·2·15).

3·3. LEMMA R_3. *There is a numerical constant $r_0 > 0$ such that*

$$\sum_{\lambda=1}^{N}\frac{r(q_\lambda)}{q_\lambda^{\frac{3}{4}}} > r_0 X^{\frac{1}{4}} \quad (X \geqslant 1).$$

Proof. Since

$$\sum_{0\leqslant n\leqslant x} r(n) = \pi x + \mathrm{P}(x)$$

and $\mathrm{P}(x) = O(x^{\frac{1}{2}})$ $(x \geqslant 1)$, we have

$$\begin{aligned}\sum_{1\leqslant n\leqslant X}\frac{r(n)}{n^{\frac{3}{4}}} &= 4 + \int_1^X x^{-\frac{3}{4}}d\{\pi x + \mathrm{P}(x)\}\\ &= 4 + \pi\int_1^X x^{-\frac{3}{4}}dx + [x^{-\frac{3}{4}}\mathrm{P}(x)]_1^X - \int_1^X \mathrm{P}(x)\,d(x^{-\frac{3}{4}})\\ &= 4\pi X^{\frac{1}{4}} + O(1) + O\left(\int_1^X x^{\frac{1}{2}}x^{-\frac{7}{4}}dx\right) \quad (X \geqslant 1)\\ &= 4\pi X^{\frac{1}{4}} + O(1) \quad (X \geqslant 1).\end{aligned} \tag{3·3·1}$$

We shall show that (3·3·1) implies Lemma R_3. We write

$$\tilde{r}(n) = \tfrac{1}{4}r(n) \quad (n \geqslant 1).$$

Then $\tilde{r}(n)$ is a multiplicative function of n; that is $\tilde{r}(n_1 n_2) = \tilde{r}(n_1)\tilde{r}(n_2)$ for all positive integers n_1, n_2 such that $(n_1, n_2) = 1$. Also

$$\tilde{r}(1) = 1, \quad \tilde{r}(2^m) = 1 \ \ (m = 1, 2, 3, \ldots),$$

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and, if p is an odd prime,

$$\tilde{r}(p^m) = \begin{cases} m+1 \ (m = 1, 2, 3, \ldots) & \text{if } p \equiv 1 \pmod 4, \\ 1 & \text{if } p \equiv 3 \pmod 4 \text{ and } m\ (>0) \text{ is an even integer}, \\ 0 & \text{if } p \equiv 3 \pmod 4 \text{ and } m\ (>0) \text{ is an odd integer}. \end{cases}$$

We may therefore write

$$\sum_{n=1}^{\infty} \frac{\tilde{r}(n)}{n^s} = (1-2^{-s})^{-1} \prod_{p \equiv 1 \pmod 4} (1-p^{-s})^{-2} \prod_{p \equiv 3 \pmod 4} (1-p^{-2s})^{-1} \quad (s = \sigma + it,\ \sigma > 1),$$

and

$$\sum_{q} \frac{\tilde{r}(q)}{q^s} = (1+2^{-s}) \prod_{p \equiv 1 \pmod 4} (1+2p^{-s}) \quad (s = \sigma + it,\ \sigma > 1),$$

where $\sum_q$ denotes summation over all positive square-free integers q. Thus

$$\frac{\sum_{n=1}^{\infty} \frac{\tilde{r}(n)}{n^s}}{\sum_q \frac{\tilde{r}(q)}{q^s}} = (1-2^{-2s})^{-1} \prod_{p \equiv 1 \pmod 4} (1-3p^{-2s}+2p^{-3s})^{-1} \prod_{p \equiv 3 \pmod 4} (1-p^{-2s})^{-1} \quad (\sigma > 1),$$

and the expression on the right-hand side may be equated, for $\sigma > \frac{1}{2}$, to an absolutely convergent series of the form

$$\sum_{m=1}^{\infty} \frac{\tilde{\rho}(m)}{m^s},$$

since $|2^{-2s}| < \frac{1}{2}$ $(\sigma > \frac{1}{2})$; $|3p^{-2s} - 2p^{-3s}| < 5p^{-2\sigma} < 1$ $(p \geqslant 5,\ \sigma > \frac{1}{2})$; $|p^{-2s}| < \frac{1}{3}$ $(p \geqslant 3,\ \sigma > \frac{1}{2})$. Now equating coefficients of n^{-s} in

$$\sum_{n=1}^{\infty} \frac{\tilde{r}(n)}{n^s} = \sum_q \frac{\tilde{r}(q)}{q^s} \sum_{m=1}^{\infty} \frac{\tilde{\rho}(m)}{m^s} \quad (\sigma > \tfrac{1}{2}),$$

we obtain

$$\tilde{r}(n) = \sum_{\substack{q,\,m \\ qm=n}} \tilde{r}(q)\,\tilde{\rho}(m).$$

Hence

$$\begin{aligned} \sum_{1 \leqslant n \leqslant X} \frac{\tilde{r}(n)}{n^{\frac{3}{4}}} &= \sum_{\substack{q,\,m \\ 1 \leqslant qm \leqslant X}} \frac{\tilde{r}(q)\,\tilde{\rho}(m)}{(qm)^{\frac{3}{4}}} \\ &\leqslant \sum_{\substack{q,\,m \\ 1 \leqslant qm \leqslant X}} \frac{\tilde{r}(q)\,|\tilde{\rho}(m)|}{q^{\frac{3}{4}} m^{\frac{3}{4}}} \\ &\leqslant \sum_{1 \leqslant q \leqslant X} \frac{\tilde{r}(q)}{q^{\frac{3}{4}}} \sum_{m=1}^{\infty} \frac{|\tilde{\rho}(m)|}{m^{\frac{3}{4}}}. \end{aligned}$$

That is

$$\sum_{1 \leqslant n \leqslant X} \frac{r(n)}{n^{\frac{3}{4}}} \leqslant \sum_{\lambda=1}^{N} \frac{r(q_\lambda)}{q_\lambda^{\frac{3}{4}}} \sum_{m=1}^{\infty} \frac{|\tilde{\rho}(m)|}{m^{\frac{3}{4}}},$$

and Lemma R_3 follows from this inequality by virtue of (3·3·1).

LEMMA D_3. *There is a numerical constant $d_0 > 0$ such that*

$$\sum_{\lambda=1}^{N} \frac{d(q_\lambda)}{q_\lambda^{\frac{3}{4}}} > d_0 X^{\frac{1}{4}} \log X \quad (X \geqslant 1).$$

Proof. Let

$$\sum_{1 \leqslant n \leqslant x} d(n) = x \log x + \Delta^*(x).$$

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Then $\Delta^*(x) = O(x)$ $(x \geqslant 1)$. Thus, for $X \geqslant 1$, we have

$$\begin{aligned}\sum_{1\leqslant n\leqslant X} \frac{d(n)}{n^{\frac{3}{4}}} &= 1 + \int_1^X x^{-\frac{3}{4}} d\{x\log x + \Delta^*(x)\} \\ &= 1 + [x^{-\frac{3}{4}}\{x\log x + \Delta^*(x)\}]_1^X + \frac{3}{4}\int_1^X x^{-\frac{7}{4}}\{x\log x + \Delta^*(x)\}\,dx \\ &= X^{\frac{1}{4}}\log X + O(X^{\frac{1}{4}}) + 3X^{\frac{1}{4}}\log X - 3\int_1^X x^{-\frac{3}{4}}\,dx + \frac{3}{4}\int_1^X x^{-\frac{7}{4}}\Delta^*(x)\,dx \\ &= 4X^{\frac{1}{4}}\log X + O(X^{\frac{1}{4}}). \qquad (3\cdot3\cdot2)\end{aligned}$$

Next, since every integer n such that $1 \leqslant n \leqslant X$ can be expressed uniquely in the form $n = \nu^2 q$, $(1 \leqslant q \leqslant X, 1 \leqslant \nu \leqslant X^{\frac{1}{2}})$, where q is a square-free integer and ν an integer, we also have

$$\begin{aligned}\sum_{1\leqslant n\leqslant X} \frac{d(n)}{n^{\frac{3}{4}}} &= \sum_{1\leqslant \nu\leqslant X^{\frac{1}{2}}} \sum_{1\leqslant q\leqslant (X/\nu^2)} \frac{d(\nu^2 q)}{\nu^{\frac{3}{2}} q^{\frac{3}{4}}} \\ &\leqslant \sum_{1\leqslant \nu\leqslant X^{\frac{1}{2}}} \sum_{1\leqslant q\leqslant (X/\nu^2)} \frac{d(\nu^2)}{\nu^{\frac{3}{2}}} \frac{d(q)}{q^{\frac{3}{4}}} \\ &\leqslant \sum_{1\leqslant \nu\leqslant X^{\frac{1}{2}}} \frac{\{d(\nu)\}^2}{\nu^{\frac{3}{2}}} \sum_{1\leqslant q\leqslant X} \frac{d(q)}{q^{\frac{3}{4}}} \\ &< \sum_{m=1}^{\infty} \frac{\{d(m)\}^2}{m^{\frac{3}{2}}} \sum_{\lambda=1}^{N} \frac{d(q_\lambda)}{q_\lambda^{\frac{3}{4}}}. \qquad (3\cdot3\cdot3)\end{aligned}$$

Lemma D_3 follows at once from (3·3·2) and (3·3·3).

4

4·1. *The trigonometric polynomials $T_X(u)$ and $\mathscr{T}_X(u)$.* Let

$$K(\xi) = \tfrac{1}{2}e^{-i\xi} + 1 + \tfrac{1}{2}e^{i\xi} = 2\cos^2 \tfrac{1}{2}\xi,$$

(the Fejér kernel of order 2), and let

$$T_X(u) = \prod_{\lambda=1}^{N} K(2\pi\sqrt{q_\lambda}\,u - \tfrac{3}{4}\pi), \quad \mathscr{T}_X(u) = \prod_{\lambda=1}^{N} K(4\pi\sqrt{q_\lambda}\,u + \tfrac{3}{4}\pi).$$

From the second expression for $K(\xi)$, it is clear that

$$T_X(u),\ \mathscr{T}_X(u) \geqslant 0 \qquad (4\cdot1\cdot1)$$

for all real u, and all $X \geqslant 1$. Next, from the first expression for $K(\xi)$, we can see that

(i) $T_X(u)$ can be expressed as a trigonometric polynomial† in u whose exponents are the 3^N distinct‡ numbers

$$2\pi\sum_{\lambda=1}^{N} t_\lambda\sqrt{q_\lambda} \quad (t_\lambda \text{ integers},\ |t_\lambda| \leqslant 1),$$

† We describe any finite sum of the form $T(u) = \sum_\gamma a_\gamma e^{-i\alpha_\gamma u}$ (a_γ complex, α_γ real and distinct) as a trigonometric polynomial in u whose exponents are the numbers α_γ.

‡ Distinct, by virtue of **(B)**. See §2·2.

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and that

(ii) $\mathscr{T}_X(u)$ can be expressed as a trigonometric polynomial in u whose exponents are the 3^N distinct numbers

$$4\pi \sum_{\lambda=1}^{N} t_\lambda \sqrt{q_\lambda} \quad (t_\lambda \text{ integers}, |t_\lambda| \leqslant 1).$$

In fact, for $X \geqslant 2$, we may write

$$T_X(u) = T_X^{(0)}(u) + T_X^{(1)}(u) + \overline{T}_X^{(1)}(u) + T_X^{(2)}(u),$$

and

$$\mathscr{T}_X(u) = \mathscr{T}_X^{(0)}(u) + \mathscr{T}_X^{(1)}(u) + \overline{\mathscr{T}}_X^{(1)}(u) + \mathscr{T}_X^{(2)}(u),$$

where

$$T_X^{(0)}(u) \equiv 1, \qquad \mathscr{T}_X^{(0)}(u) \equiv 1,$$

$$T_X^{(1)}(u) \equiv \sum_{\lambda=1}^{N} \tfrac{1}{2} e^{\frac{3}{4}\pi i} e^{-2\pi i \sqrt{q_\lambda} u}, \quad \mathscr{T}_X^{(1)}(u) \equiv \sum_{\lambda=1}^{N} \tfrac{1}{2} e^{-\frac{3}{4}\pi i} e^{-4\pi i \sqrt{q_\lambda} u},$$

$$\overline{T}_X^{(1)}(u) \equiv \sum_{\lambda=1}^{N} \tfrac{1}{2} e^{-\frac{3}{4}\pi i} e^{2\pi i \sqrt{q_\lambda} u}, \quad \overline{\mathscr{T}}_X^{(1)}(u) \equiv \sum_{\lambda=1}^{N} \tfrac{1}{2} e^{\frac{3}{4}\pi i} e^{4\pi i \sqrt{q_\lambda} u},$$

$$T_X^{(2)}(u) \equiv \sum_{\tau=1}^{T} b_\tau e^{-2\pi i \beta_\tau u}, \qquad \mathscr{T}_X^{(2)}(u) \equiv \sum_{\tau=1}^{T} \bar{b}_\tau e^{-4\pi i \beta_\tau u}$$

are all trigonometric polynomials in u, the numbers β_τ being the

$$T = 3^N - 2N - 1 < 3^X \quad (X \geqslant 2) \tag{4·1·2}$$

distinct numbers of the form

$$\sum_{\lambda=1}^{N} r_\lambda \sqrt{q_\lambda} \quad \left(r_\lambda \text{ integers}, |r_\lambda| \leqslant 1, \sum_{\lambda=1}^{N} r_\lambda^2 \geqslant 2\right),$$

and the numbers b_τ being all of the form

$$2^{-(h+l)} e^{\frac{3}{4}\pi i(h-l)} \quad (h, l \text{ integers}; 0 \leqslant h, l \leqslant N, h+l \geqslant 2).$$

We then note that† $$|\sqrt{n} \pm \beta_\tau| \geqslant e^{-q(X)} \tag{4·1·3}$$

for all integers $n \geqslant 0$, all $X \geqslant 2$, and $1 \leqslant \tau \leqslant T$. Also, it is clear that

$$|b_\tau| = |\bar{b}_\tau| \leqslant \tfrac{1}{4} \quad (1 \leqslant \tau \leqslant T). \tag{4·1·4}$$

4·2. *A notation.* Let $T(u) = \sum_\gamma a_\gamma e^{-i\alpha_\gamma u}$ (a_γ complex, α_γ real and distinct) be any trigonometric polynomial in u, and let $H(s)$ be any function of the complex variable s. We shall use the notation

$$T \wedge H(s)$$

to represent the sum $\sum_\gamma a_\gamma H(s + i\alpha_\gamma)$ for every s for which this sum exists.

4·3. With the notation described above we prove the following results:

If $I_\theta(s) = s^{-\theta}$ ($s = \sigma + it$), then, for $1 \leqslant \theta \leqslant 2$, $X \geqslant 2$, and all $\sigma > 0$,

$$|T_X \wedge I_\theta(\sigma) - \sigma^{-\theta}| = O(e^{2q(X)} 3^X), \tag{4·3·1}$$

and

$$|\mathscr{T}_X \wedge I_\theta(\sigma) - \sigma^{-\theta}| = O(e^{2q(X)} 3^X). \tag{4·3·2}$$

† $|\sqrt{n} \pm \beta_\tau| \in S_X$ (see §2·2).

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Proof of (4·3·1). Suppose that $1 \leqslant \theta \leqslant 2$, $X \geqslant 2$, $\sigma > 0$. First, we have

$$T_X^{(0)} \wedge I_\theta(\sigma) = \sigma^{-\theta}. \tag{4·3·3}$$

Next,
$$|T_X^{(1)} \wedge I_\theta(\sigma)| = \left| \sum_{\lambda=1}^{N} \tfrac{1}{2} e^{\frac{3}{4}\pi i} (\sigma + 2\pi i \sqrt{q_\lambda})^{-\theta} \right|$$
$$\leqslant \frac{1}{2} \sum_{\lambda=1}^{N} |\sigma + 2\pi i \sqrt{q_\lambda}|^{-\theta} < \frac{1}{2} \frac{N}{(2\pi)^\theta} \leqslant \frac{X}{4\pi}.$$

That is,
$$|T_X^{(1)} \wedge I_\theta(\sigma)| = O(X). \tag{4·3·4}$$

Similarly
$$|\overline{T}_X^{(1)} \wedge I_\theta(\sigma)| = O(X). \tag{4·3·5}$$

Finally, using (4·1·2), (4·1·3) and (4·1·4), we have

$$|T_X^{(2)} \wedge I_\theta(\sigma)| = \left| \sum_{\tau=1}^{T} b_\tau (\sigma + 2\pi i \beta_\tau)^{-\theta} \right|$$
$$\leqslant \frac{1}{4} \sum_{\tau=1}^{T} |\sigma + 2\pi i \beta_\tau|^{-\theta}$$
$$< \frac{1}{4} \sum_{\tau=1}^{T} |\beta_\tau|^{-\theta}$$
$$< \tfrac{1}{4} e^{\theta q(X)} 3^X$$
$$\leqslant \tfrac{1}{4} e^{2q(X)} 3^X.$$

That is,
$$|T_X^{(2)} \wedge I_\theta(\sigma)| = O(e^{2q(X)} 3^X), \tag{4·3·6}$$

and (4·3·1) now follows immediately from (4·3·3)–(4·3·6) since

$$T_X \wedge I_\theta(\sigma) = T_X^{(0)} \wedge I_\theta(\sigma) + T_X^{(1)} \wedge I_\theta(\sigma) + \overline{T}_X^{(1)} \wedge I_\theta(\sigma) + T_X^{(2)} \wedge I_\theta(\sigma).$$

The proof of (4·3·2) is similar.

4·4. *The numbers* σ_X *and* θ_X.

(i) We choose a numerical constant A such that

$$A \geqslant 6 \tag{4·4·1}$$

and
$$e^{aA} \geqslant 3^5, \tag{4·4·2}$$

where a is the constant occurring in (2·3·1), and we define

$$\sigma_X = e^{-Aq(X)} \quad (X \geqslant 2). \tag{4·4·3}$$

Then obviously
$$0 < \sigma_X < 1 \quad (X \geqslant 2). \tag{4·4·4}$$

In fact, by (2·3·1) and (4·4·2),

$$0 < \sigma_X < 3^{-5X} \quad (X \geqslant 2). \tag{4·4·5}$$

(ii) We take an arbitrary function $Q(X) \in C_q$ and we put

$$\theta_X = \frac{1}{2} + \frac{1}{Q(X)} \quad (X \geqslant X_Q). \tag{4·4·6}$$

Then, since $Q(X) \to \infty$ as $X \to \infty$, for some $X_Q'' \geqslant X_Q$ we have

$$\tfrac{1}{2} < \theta_X < 1 \quad (X \geqslant X_Q''). \tag{4·4·7}$$

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4·5. We conclude Part 4 (and also our preliminary analysis) by proving that, as $X \to \infty$,

$$\sigma_X^{\frac{3}{2}}\{T_X \wedge g(\sigma_X)\} = \frac{1}{4\sqrt{\pi}} \sum_{\lambda=1}^{N} \frac{r(q_\lambda)}{q_\lambda^{\frac{3}{4}}} + o(1), \tag{R}$$

$$\sigma_X^{\frac{3}{2}}\{\mathscr{T}_X \wedge G(\sigma_X)\} = -\frac{1}{4\sqrt{2}\sqrt{\pi}} \sum_{\lambda=1}^{N} \frac{d(q_\lambda)}{q_\lambda^{\frac{3}{4}}} + o(1), \tag{D}$$

$$\sigma_X^{\frac{3}{2}}\{T_X \wedge I_{\theta_X+1}(\sigma_X)\} = e^{A\{q(X)/Q(X)\}} + o(1), \tag{I}_R$$

$$\sigma_X^{\frac{3}{2}}\{\mathscr{T}_X \wedge I_{\theta_X+1}(\sigma_X)\} = e^{A\{q(X)/Q(X)\}} + o(1), \tag{I}_D$$

$$\sigma_X^{\frac{3}{2}}\{T_X \wedge I_1(\sigma_X)\} = o(1), \tag{I_1}_R$$

$$\sigma_X^{\frac{3}{2}}\{\mathscr{T}_X \wedge I_1(\sigma_X)\} = o(1). \tag{I_1}_D$$

Proof of (R). First, we have

$$\begin{aligned}\sigma_X^{\frac{3}{2}}\{T_X^{(0)} \wedge g(\sigma_X)\} &= \sigma_X^{\frac{3}{2}} g(\sigma_X) \\ &= \sigma_X^{\frac{3}{2}} \sum_{n=1}^{\infty} \frac{r(n)}{(\sigma_X^2 + 4\pi^2 n)^{\frac{3}{2}}} \\ &= o(1) \quad \text{as} \quad X \to \infty,\end{aligned}$$

since $\sigma_X = o(1)$ as $X \to \infty$, by (4·4·5).

Next, in view of (4·4·4), we may put $Y = X\ (\geqslant 2)$, $\sigma = \sigma_X$, $m = q_\lambda\ (\lambda = 1, 2, \ldots, N)$, in Lemma R_1 of § 3·1. We then have

$$\sigma_X^{\frac{3}{2}} g(\sigma_X + 2\pi i\sqrt{q_\lambda}) = \frac{1}{4\sqrt{\pi}} \frac{r(q_\lambda)}{q_\lambda^{\frac{3}{4}}} e^{-\frac{3}{4}\pi i} + O(\sigma_X X^{\frac{3}{2}}) \quad (X \geqslant 2,\ 1 \leqslant \lambda \leqslant N).$$

Hence

$$\begin{aligned}\sigma_X^{\frac{3}{2}}\{T_X^{(1)} \wedge g(\sigma_X)\} &= \sigma_X^{\frac{3}{2}} \sum_{\lambda=1}^{N} \tfrac{1}{2} e^{\frac{3}{4}\pi i} g(\sigma_X + 2\pi i\sqrt{q_\lambda}) \\ &= \frac{1}{8\sqrt{\pi}} \sum_{\lambda=1}^{N} \frac{r(q_\lambda)}{q_\lambda^{\frac{3}{4}}} + O(N\sigma_X X^{\frac{3}{2}}) \quad (X \geqslant 2) \\ &= \frac{1}{8\sqrt{\pi}} \sum_{\lambda=1}^{N} \frac{r(q_\lambda)}{q_\lambda^{\frac{3}{4}}} + O(\sigma_X X^{\frac{5}{2}}) \quad (X \geqslant 2) \\ &= \frac{1}{8\sqrt{\pi}} \sum_{\lambda=1}^{N} \frac{r(q_\lambda)}{q_\lambda^{\frac{3}{4}}} + o(1) \quad \text{as} \quad X \to \infty,\end{aligned}$$

by (4·4·5). Similarly

$$\sigma_X^{\frac{3}{2}}\{\overline{T}_X^{(1)} \wedge g(\sigma_X)\} = \frac{1}{8\sqrt{\pi}} \sum_{\lambda=1}^{N} \frac{r(q_\lambda)}{q_\lambda^{\frac{3}{4}}} + o(1) \quad \text{as} \quad X \to \infty.$$

The result (R) will now follow if we show that

$$\sigma_X^{\frac{3}{2}}\{T_X^{(2)} \wedge g(\sigma_X)\} = o(1) \quad \text{as} \quad X \to \infty.$$

To do this we use Lemma R_2 of § 3·1. We have

$$|\sigma_X + 2\pi i\beta_\tau| \leqslant |\sigma_X| + 2\pi\,|\beta_\tau| < 1 + 2\pi X^{\frac{3}{2}} < 3\pi X^{\frac{3}{2}}$$

for $X \geqslant 2$ and $1 \leqslant \tau \leqslant T$. Also, by (4·1·3),

$$|\sigma_X + 2\pi i\beta_\tau \pm 2\pi i\sqrt{n}| > 2\pi\,|\beta_\tau \pm \sqrt{n}| \geqslant 2\pi e^{-q(X)}$$

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for $X \geqslant 2$, $1 \leqslant \tau \leqslant T$, and all integers $n \geqslant 0$. So, taking $k = 3\pi X^{\frac{3}{2}}$, $\omega = e^{-q(X)}$, $s = \sigma_X + 2\pi i\beta_\tau$ in Lemma R_2, we obtain

$$|g(\sigma_X + 2\pi i\beta_\tau)| = O(e^{\frac{3}{2}q(X)} X^3) \quad (X \geqslant 2,\ 1 \leqslant \tau \leqslant T).$$

Hence, by (4·1·2), (4·1·4), (4·4·1), (4·4·3) and (4·4·5) we have

$$\begin{aligned}
|\sigma_X^{\frac{3}{2}}\{T_X^{(2)} \wedge g(\sigma_X)\}| &= \left|\sigma_X^{\frac{3}{2}} \sum_{\tau=1}^{T} b_\tau g(\sigma_X + 2\pi i\beta_\tau)\right| \\
&\leqslant \tfrac{1}{4}\sigma_X^{\frac{3}{2}} \sum_{\tau=1}^{T} |g(\sigma_X + 2\pi i\beta_\tau)| \\
&= O(\sigma_X^{\frac{3}{2}} e^{\frac{3}{2}q(X)} X^3 3^X) \quad (X \geqslant 2) \\
&= O(e^{-Aq(X)} 3^{-\frac{1}{2}X} e^{\frac{3}{2}q(X)} X^3 3^X) \quad (X \geqslant 2) \\
&= o(1) \quad \text{as} \quad X \to \infty,
\end{aligned}$$

and this completes the proof of (R).

Proof of (D). The proof is similar to that of (R), and so we shall not give all the details. First, we have

$$\sigma_X^{\frac{3}{2}}\{\mathscr{T}_X^{(0)} \wedge G(\sigma_X)\} = \sigma_X^{\frac{3}{2}} G(\sigma_X) = o(1) \quad \text{as} \quad X \to \infty.$$

Next, using Lemma D_1 of § 3·2, we can show that

$$\sigma_X^{\frac{3}{2}}\{\mathscr{T}_X^{(1)} \wedge G(\sigma_X)\} = -\frac{1}{8\sqrt{2}\sqrt{\pi}} \sum_{\lambda=1}^{N} \frac{d(q_\lambda)}{q_\lambda^{\frac{3}{4}}} + o(1) \quad \text{as} \quad X \to \infty,$$

and that

$$\sigma_X^{\frac{3}{2}}\{\overline{\mathscr{T}}_X^{(1)} \wedge G(\sigma_X)\} = -\frac{1}{8\sqrt{2}\sqrt{\pi}} \sum_{\lambda=1}^{N} \frac{d(q_\lambda)}{q_\lambda^{\frac{3}{4}}} + o(1) \quad \text{as} \quad X \to \infty.$$

To complete the proof we shall use Lemma D_2 to show that

$$\sigma_X^{\frac{3}{2}}\{\mathscr{T}_X^{(2)} \wedge G(\sigma_X)\} = o(1) \quad \text{as} \quad X \to \infty.$$

We have

$$|\sigma_X + 4\pi i\beta_\tau| < 5\pi X^{\frac{3}{2}} \quad (X \geqslant 2,\ 1 \leqslant \tau \leqslant T),$$

and

$$|\sigma_X + 4\pi i\beta_\tau \pm 4\pi i\sqrt{n}| > 4\pi e^{-q(X)}$$

for $X \geqslant 2$, $1 \leqslant \tau \leqslant T$, and all integers $n \geqslant 0$. So, taking

$$k = 5\pi X^{\frac{3}{2}}, \quad \omega = e^{-q(X)}, \quad s = \sigma_X + 4\pi i\beta_\tau$$

in Lemma D_2, we obtain

$$|G(\sigma_X + 4\pi i\beta_\tau)| = O(\sigma_X^{-1} e^{\frac{3}{2}q(X)} X^{\frac{1.5}{4}} \log^3 X) \quad (X \geqslant 2,\ 1 \leqslant \tau \leqslant T).$$

Hence, by (4·1·2), (4·1·4), (4·4·1), (4·4·3) and (4·4·5) we have

$$\begin{aligned}
|\sigma_X^{\frac{3}{2}}\{\mathscr{T}_X^{(2)} \wedge G(\sigma_X)\}| &= \left|\sigma_X^{\frac{3}{2}} \sum_{\tau=1}^{T} b_\tau G(\sigma_X + 4\pi i\beta_\tau)\right| \\
&\leqslant \tfrac{1}{4}\sigma_X^{\frac{3}{2}} \sum_{\tau=1}^{T} |G(\sigma_X + 4\pi i\beta_\tau)| \\
&= O(\sigma_X^{\frac{1}{2}} e^{\frac{3}{2}q(X)} X^{\frac{1.5}{4}} (\log^3 X)\, 3^X) \quad (X \geqslant 2) \\
&= O(e^{-\frac{1}{4}Aq(X)} 3^{-\frac{1}{4}X} e^{\frac{3}{2}q(X)} X^{\frac{1.5}{4}} (\log^3 X)\, 3^X) \quad (X \geqslant 2) \\
&= o(1) \quad \text{as} \quad X \to \infty.
\end{aligned}$$

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Proof of $(I)_R$. Taking $X \geqslant X''_Q$, putting† $\sigma = \sigma_X$, $\theta = \theta_X + 1$ in (4·3·1), multiplying the result throughout by $\sigma_X^{\frac{3}{2}}$, and using (4·4·6), we have

$$|\sigma_X^{\frac{3}{2}}\{T_X \wedge I_{\theta_X+1}(\sigma_X)\} - \sigma_X^{-1/Q(X)}| = O(\sigma_X^{\frac{3}{2}} e^{2q(X)} 3^X) \quad (X \geqslant X''_Q).$$

Hence, by (4·4·1), (4·4·3) and (4·4·5), we have

$$\begin{aligned}|\sigma_X^{\frac{3}{2}}\{T_X \wedge I_{\theta_X+1}(\sigma_X)\} - e^{A\{q(X)/Q(X)\}}| &= O(e^{-Aq(X)} 3^{-\frac{5}{2}X} e^{2q(X)} 3^X) \quad (X \geqslant X''_Q)\\ &= o(1) \quad \text{as} \quad X \to \infty,\end{aligned}$$

and this proves $(I)_R$.

Proof of $(I)_D$. Taking $X \geqslant X''_Q$, we put $\sigma = \sigma_X$, $\theta = \theta_X + 1$ in (4·3·2), and then proceed exactly as in the proof of $(I)_R$.

Proof of $(I_1)_R$. Taking $X \geqslant 2$, putting $\sigma = \sigma_X$, $\theta = 1$ in (4·3·1), and multiplying the result throughout by $\sigma_X^{\frac{3}{2}}$, we have

$$|\sigma_X^{\frac{3}{2}}\{T_X \wedge I_1(\sigma_X)\} - \sigma_X^{\frac{1}{2}}| = O(\sigma_X^{\frac{3}{2}} e^{2q(X)} 3^X) \quad (X \geqslant 2).$$

$(I_1)_R$ follows from this since $\sigma_X^{\frac{1}{2}} = o(1)$ as $X \to \infty$, by (4·4·5), and since the O term here is also $o(1)$ as $X \to \infty$, as in the proof of $(I)_R$.

Proof of $(I_1)_D$. The proof is exactly the same as that of $(I_1)_R$ except that we use (4·3·2) instead of (4·3·1).

5

5·1. *Proof of* (2·5·1). We present the proof in three stages.

Stage 1. We begin by considering the function $u^{-\theta}\{\mathrm{P}(u^2) - 1\}$. Since, by (1·1·1), $\mathrm{P}(u^2) - 1 = -\pi u^2$ $(0 \leqslant u < 1)$, it follows that

$$u^{-\theta}\{\mathrm{P}(u^2) - 1\} = O(1) \quad (0 < u < 1,\ \theta \leqslant 2). \tag{5·1·1}$$

Next, for each fixed $\theta > \frac{1}{2}$, we may assume that $u^{-\theta}\{\mathrm{P}(u^2) - 1\}$ is bounded above for all $u \geqslant 1$, since otherwise, on account of (2·3·1) and (2·4·3), much more than (2·5·1) is true. So, if

$$\mathrm{P}_\theta = \sup_{u>0} u^{-\theta}\{\mathrm{P}(u^2) - 1\},$$

then we may certainly assume that P_θ is finite when $\frac{1}{2} < \theta \leqslant 2$.

Now let

$$\rho_\theta(u) = \mathrm{P}_\theta u^\theta - \mathrm{P}(u^2) + 1 \quad (u \geqslant 0,\ \tfrac{1}{2} < \theta \leqslant 2).$$

Then clearly

$$\rho_\theta(u) \geqslant 0 \quad (u \geqslant 0,\ \tfrac{1}{2} < \theta \leqslant 2). \tag{5·1·2}$$

Also, for all $s = \sigma + it$ $(\sigma > 0)$, we have

$$\begin{aligned}\int_0^\infty \rho_\theta(u)\, e^{-su}\, du &= \mathrm{P}_\theta \int_0^\infty u^\theta e^{-su}\, du - \int_0^\infty \{R(u^2) - \pi u^2\} e^{-su}\, du + \int_0^\infty e^{-su}\, du\\ &= \mathrm{P}_\theta \frac{\Gamma(\theta+1)}{s^{\theta+1}} - \frac{f(s)}{s} + \frac{2\pi}{s^3} + \frac{1}{s}\\ &= \mathrm{P}_\theta \frac{\Gamma(\theta+1)}{s^{\theta+1}} - g(s) + \frac{1}{s}.\end{aligned}$$

Hence, if $T(u)$ is any trigonometric polynomial in u, we see that

$$\begin{aligned}\int_0^\infty \rho_\theta(u)\, e^{-\sigma u}\, T(u)\, du &= \mathrm{P}_\theta \Gamma(\theta+1)\{T \wedge I_{\theta+1}(\sigma)\}\\ &\quad - T \wedge g(\sigma) + T \wedge I_1(\sigma) \quad (\sigma > 0).\end{aligned} \tag{5·1·3}$$

† This is a legitimate substitution because of (4·4·4) and (4·4·7).

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Stage 2. Taking $X \geqslant X_Q''$, putting† $\sigma = \sigma_X$, $\theta = \theta_X$, $T(u) \equiv T_X(u)$ in (5·1·3), and multiplying the result throughout by $\sigma_X^{\frac{3}{2}}$, we have

$$\sigma_X^{\frac{3}{2}} \int_0^\infty \rho_{\theta_X}(u)\, e^{-\sigma_X u}\, T_X(u)\, du = \mathrm{P}_{\theta_X}\, \Gamma(\theta_X + 1)\, (\sigma_X^{\frac{3}{2}}\{T_X \wedge I_{\theta_X+1}(\sigma_X)\}) - \sigma_X^{\frac{3}{2}}\{T_X \wedge g(\sigma_X)\} + \sigma_X^{\frac{3}{2}}\{T_X \wedge I_1(\sigma_X)\} \quad (X \geqslant X_Q''). \tag{5·1·4}$$

Now, by (4·1·1) and (5·1·2), it is clear that the left-hand side of (5·1·4) is greater than or equal to zero. But, using the results (R), $(I)_R$ and $(I_1)_R$ of § 4·5, and noting also that

$$\Gamma(\theta_X + 1) = \Gamma\left(\frac{3}{2} + \frac{1}{Q(X)}\right) = \frac{\sqrt{\pi}}{2} + o(1) \quad \text{as} \quad X \to \infty,$$

we find that the right-hand side of (5·1·4) is equal to

$$\mathrm{P}_{\theta_X}\left(\frac{\sqrt{\pi}}{2} + o(1)\right)(e^{A\{q(X)/Q(X)\}} + o(1)) - \frac{1}{4\sqrt{\pi}} \sum_{\lambda=1}^{N} \frac{r(q_\lambda)}{q_\lambda^{\frac{3}{2}}} + o(1) \quad \text{as} \quad X \to \infty.$$

From these observations we deduce first that, when X is sufficiently large, $\mathrm{P}_{\theta_X} > 0$. Noting this fact, we may then assert more strongly that

$$\mathrm{P}_{\theta_X} \sqrt{\pi}\, e^{2A} - \frac{1}{8\sqrt{\pi}} \sum_{\lambda=1}^{N} \frac{r(q_\lambda)}{q_\lambda^{\frac{3}{2}}} > 0 \quad (X \geqslant X_1),$$

for some $X_1 \geqslant X_Q''$.

The application of Lemma R_3 of § 3·3 now leads at once to the result

$$\mathrm{P}_{\theta_X} > \tilde{r}_0 X^{\frac{1}{4}} \quad (X \geqslant X_1), \tag{5·1·5}$$

where $\tilde{r}_0 = \dfrac{1}{8\pi} r_0\, e^{-2A} > 0$ is a numerical constant.

Stage 3. (5·1·5) implies that, for each $X \geqslant X_1$, there is a number $u_X > 0$ such that

$$u_X^{-\theta_X}\{\mathrm{P}(u_X^2) - 1\} > \tilde{r}_0 X^{\frac{1}{4}}. \tag{5·1·6}$$

Then obviously

$$u_X^{-\theta_X}\, \mathrm{P}(u_X^2) > \tilde{r}_0 X^{\frac{1}{4}} \quad (X \geqslant X_1). \tag{5·1·7}$$

Now, from (5·1·1) and (5·1·6), it is clear that $u_X \geqslant 1$ when X is sufficiently large. Noting this, we then see again from (5·1·6) that

$$u_X \to \infty \quad \text{as} \quad X \to \infty. \tag{5·1·8}$$

We take a fixed $X_2 \geqslant X_1$ so that $2 \log u_X \geqslant X_Q'$ $(X \geqslant X_2)$,‡ and we write

$$v_X = u_X^{1/Q(X)} \quad (X \geqslant X_2).$$

Then
$$2 \log u_X = 2Q(X) \log v_X \leqslant \begin{cases} Q(2X \log v_X) & \text{if} \quad 2 \log v_X \geqslant 1, \\ Q(X) & \text{if} \quad 2 \log v_X < 1, \end{cases}$$

and hence
$$Q^{-1}(2 \log u_X) \leqslant X V_X \quad (X \geqslant X_2), \tag{5·1·9}$$

where $V_X = \max(1, 2 \log v_X)$.

† This is a legitimate substitution because of (4·4·4) and (4·4·7).

‡ This is to ensure the existence of $Q^{-1}(2 \log u_X)$ in (5·1·9). See the last sentence of § 2·4.

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From (5·1·7) and (5·1·9) we see that, for $X \geqslant X_2$,

$$\frac{\mathrm{P}(u_X^2)}{u_X^{\frac{1}{2}}\{Q^{-1}(2\log u_X)\}^{\frac{1}{4}}} = \frac{\mathrm{P}(u_X^2)}{u_X^{\theta_X}} \frac{u_X^{1/Q(X)}}{\{Q^{-1}(2\log u_X)\}^{\frac{1}{4}}}$$

$$> \tilde{r}_0 X^{\frac{1}{4}} \frac{v_X}{X^{\frac{1}{4}} V_X^{\frac{1}{4}}} > 2^{-\frac{1}{4}} \tilde{r}_0,$$

since $v_X > 1$ and $V_X < 2v_X$. (2·5·1) now follows at once, in view of (5·1·8).

5·2. *Proof of* (2·5·2). We proceed in three stages as in the proof of (2·5·1).

Stage 1. We begin by extending the range of definition of the functions $D(x)$ and $\Delta(x)$, which are defined by (1·1·2) only for $x \geqslant 1$. We now define†

$$D(x) = 0 \ \ (0 \leqslant x < 1), \quad \Delta(x) = -x\log x - (2\gamma - 1)x \ \ (0 < x < 1), \quad \Delta(0) = 0.$$

Let us now consider the function $u^{-\theta}\Delta(u^2)$. It is clear that

$$u^{-\theta}\Delta(u^2) = O(1) \quad (0 < u < 1, \ \theta < 1). \tag{5·2·1}$$

Also, for each fixed $\theta > \frac{1}{2}$, we may assume that $-u^{-\theta}\Delta(u^2)$ is bounded above for all $u \geqslant 1$, since otherwise, on account of (2·3·1) and (2·4·3), much more than (2·5·2) is true. So, if

$$\Delta_\theta = \sup_{u>0}\{-u^{-\theta}\Delta(u^2)\},$$

then we may certainly assume that Δ_θ is finite when $\frac{1}{2} < \theta < 1$.

Now let

$$\delta_\theta(u) = \Delta_\theta u^\theta + \Delta(u^2) \quad (u \geqslant 0, \ \tfrac{1}{2} < \theta < 1).$$

Then clearly

$$\delta_\theta(u) \geqslant 0 \quad (u \geqslant 0, \ \tfrac{1}{2} < \theta < 1). \tag{5·2·2}$$

Also, for all $s = \sigma + it$ ($\sigma > 0$), we have

$$\begin{aligned}\int_0^\infty \delta_\theta(u)\, e^{-su}\, du &= \Delta_\theta \int_0^\infty u^\theta\, e^{-su}\, du + \int_0^\infty \{D(u^2) - 2u^2\log u - (2\gamma-1)u^2\}\, e^{-su}\, du \\ &= \Delta_\theta \frac{\Gamma(\theta+1)}{s^{\theta+1}} + \frac{F(s)}{s} - \frac{2}{s^3}(3 - 2\gamma - 2\log s) - (2\gamma-1)\frac{2}{s^3} \\ &= \Delta_\theta \frac{\Gamma(\theta+1)}{s^{\theta+1}} + \frac{F(s)}{s} - \frac{4}{s^3} + \frac{4\log s}{s^3} \\ &= \Delta_\theta \frac{\Gamma(\theta+1)}{s^{\theta+1}} + G(s) + \frac{1}{4s}.\end{aligned}$$

Hence, if $T(u)$ is any trigonometric polynomial in u, we see that

$$\int_0^\infty \delta_\theta(u)\, e^{-\sigma u}\, T(u)\, du = \Delta_\theta \Gamma(\theta+1)\{T \wedge I_{\theta+1}(\sigma)\} + T \wedge G(\sigma) + \tfrac{1}{4}\{T \wedge I_1(\sigma)\} \quad (\sigma > 0). \tag{5·2·3}$$

† With these definitions the relation $D(x) = x\log x + (2\gamma - 1)\, x + \Delta(x)$ holds for $x > 0$. Also we can assert (as we do later in the proof) that

$$\int_0^\infty D(u^2)\, e^{-su}\, du = \frac{F(s)}{s} \quad (\sigma > 0).$$

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Stage 2. Taking $X \geqslant X''_Q$, putting† $\sigma = \sigma_X$, $\theta = \theta_X$, $T(u) \equiv \mathscr{T}_X(u)$ in (5·2·3), and multiplying the result throughout by $\sigma_X^{\frac{3}{2}}$, we have

$$\sigma_X^{\frac{3}{2}} \int_0^\infty \delta_{\theta_X}(u)\, e^{-\sigma_X u}\, \mathscr{T}_X(u)\, du = \Delta_{\theta_X}\, \Gamma(\theta_X + 1)\, (\sigma_X^{\frac{3}{2}}\{\mathscr{T}_X \wedge I_{\theta_X+1}(\sigma_X)\}) + \sigma_X^{\frac{3}{2}}\{\mathscr{T}_X \wedge G(\sigma_X)\} + \tfrac{1}{4}\sigma_X^{\frac{3}{2}}\{\mathscr{T}_X \wedge I_1(\sigma_X)\} \quad (X \geqslant X''_Q). \tag{5·2·4}$$

Now, by (4·1·1) and (5·2·2), it is clear that the left-hand side of (5·2·4) is greater than or equal to zero. But, using the results (D), $(I)_D$ and $(I_1)_D$ of § 4·5, and noting also that

$$\Gamma(\theta_X + 1) = \Gamma\left(\frac{3}{2} + \frac{1}{Q(X)}\right) = \frac{\sqrt{\pi}}{2} + o(1) \quad \text{as} \quad X \to \infty,$$

we find that the right-hand side of (5·2·4) is equal to

$$\Delta_{\theta_X}\left(\frac{\sqrt{\pi}}{2} + o(1)\right)(e^{A\{Q(X)/Q(X)\}} + o(1)) - \frac{1}{4\sqrt{2}\sqrt{\pi}} \sum_{\lambda=1}^{N} \frac{d(q_\lambda)}{q_\lambda^{\frac{3}{2}}} + o(1) \quad \text{as} \quad X \to \infty.$$

From these observations we deduce first that, when X is sufficiently large, $\Delta_{\theta_X} > 0$. Noting this fact, we may then assert more strongly that

$$\Delta_{\theta_X} \sqrt{\pi}\, e^{2A} - \frac{1}{8\sqrt{2}\sqrt{\pi}} \sum_{\lambda=1}^{N} \frac{d(q_\lambda)}{q_\lambda^{\frac{3}{2}}} > 0 \quad (X \geqslant X'_1),$$

for some $X'_1 \geqslant X''_Q$.

The application of Lemma D_3 of § 3·3 now leads at once to the result

$$\Delta_{\theta_X} > \tilde{d}_0 X^{\frac{1}{4}} \log X \quad (X \geqslant X'_1), \tag{5·2·5}$$

where

$$\tilde{d}_0 = \frac{1}{8\pi\sqrt{2}}\, d_0\, e^{-2A} > 0$$

is a numerical constant.

Stage 3. (5·2·5) implies that, for each $X \geqslant X'_1$, there is a number $u'_X > 0$ such that

$$u_X'^{-\theta_X} \Delta(u_X'^2) < -\tilde{d}_0 X^{\frac{1}{4}} \log X. \tag{5·2·6}$$

Now, from (5·2·1) and (5·2·6), it is clear that $u'_X \geqslant 1$ when X is sufficiently large. Noting this, we then see again from (5·2·6) that

$$u'_X \to \infty \quad \text{as} \quad X \to \infty. \tag{5·2·7}$$

We take a fixed $X'_2 \geqslant X'_1$ so that $2 \log u'_X \geqslant X'_Q$ $(X \geqslant X'_2)$, and we write

$$w_X = u_X'^{\{1/Q(X)\}} \quad (X \geqslant X'_2).$$

Then

$$2 \log u'_X = 2Q(X) \log w_X \leqslant \begin{cases} Q(2X \log w_X) & \text{if} \quad 2 \log w_X \geqslant 1, \\ Q(X) & \text{if} \quad 2 \log w_X < 1, \end{cases}$$

and hence

$$Q^{-1}(2 \log u'_X) \leqslant X W_X \quad (X \geqslant X'_2), \tag{5·2·8}$$

† This is a legitimate substitution because of (4·4·4) and (4·4·7).

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where $W_X = \max(1, 2\log w_X)$. Also, for $X \geqslant X_2'$, we have

$$\begin{aligned} 0 < \log\{Q^{-1}(2\log u_X')\} &\leqslant \log X + \log W_X \\ &< \log X + W_X^{\frac{1}{2}} \\ &< W_X^{\frac{1}{2}}\log X + 2W_X^{\frac{1}{2}}\log X \\ &= 3W_X^{\frac{1}{2}}\log X. \end{aligned} \tag{5·2·9}$$

From (5·2·6), (5·2·8) and (5·2·9), we see that, for $X \geqslant X_2'$,

$$\begin{aligned} \frac{\Delta(u_X'^2)}{u_X'^{\frac{1}{2}}\{Q^{-1}(2\log u_X')\}^{\frac{1}{4}}\log\{Q^{-1}(2\log u_X')\}} &= \frac{\Delta(u_X'^2)}{u_X'^{\theta_X}} \frac{u_X'^{\{(1/Q(X)\}}}{\{Q^{-1}(2\log u_X')\}^{\frac{1}{4}}\log\{Q^{-1}(2\log u_X')\}} \\ &< -\tilde{d}_0 X^{\frac{1}{4}}\log X \frac{w_X}{(XW_X)^{\frac{1}{4}}(3W_X^{\frac{1}{2}}\log X)} \\ &= -\tilde{d}_0 \frac{w_X}{3W_X^{\frac{1}{4}}} < -\tfrac{1}{3}2^{-\frac{1}{4}}\tilde{d}_0, \end{aligned}$$

since $w_X > 1$ and $W_X < 2w_X$. (2·5·2) now follows at once, in view of (5·2·7).

REFERENCES

(1) Besicovitch, A. S. On the linear independence of fractional powers of integers. *J. London Math. Soc.* 15 (1940), 3–6.

(2) Bohr, H. and Jessen, B. One more proof of Kronecker's theorem. *J. London Math. Soc.* 7 (1932), 274–5.

(3) Cassels, J. W. S. *An introduction to Diophantine approximation* (Cambridge, 1957).

(4) Hardy, G. H. On Dirichlet's divisor problem. *Proc. London Math. Soc.* (2), 15 (1916), 1–25.

(5) Hardy, G. H. On the expression of a number as the sum of two squares. *Quart. J. Math.* 46 (1915), 263–83.

(6) Ingham, A. E. On two classical lattice point problems. *Proc. Cambridge Philos. Soc.* 36 (1940), 131–8.

(7) Landau, E. *Vorlesungen über Zahlentheorie*, Bd. 2 (Leipzig, 1927).

Department of Mathematics
University of Ceylon

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