Ergod. Th. & Dynam. Sys. (2018), **38**, 2571–2617 © *Cambridge University Press, 2017*
doi:10.1017/etds.2016.129

The local rotation set is an interval

JONATHAN CONEJEROS

Institut de Mathématiques de Jussieu – Paris Rive Gauche, UPMC, 4 place Jussieu, Case 247, 75252 Paris Cedex 5, France
(e-mail: jaconeje@uc.cl)

(*Received* 30 *July* 2015 *and accepted in revised form* 3 *November* 2016)

Abstract. Let $\mathrm{Homeo}_0(\mathbb{R}^2; 0)$ be the set of all homeomorphisms of the plane that are isotopic to the identity and which fix zero. Recently, in Le Roux [L'ensemble de rotation autour d'un point fixe. *Astérisque* (350) (2013), 1–109], Le Roux gave the definition of the *local rotation set of an isotopy* I in $\mathrm{Homeo}_0(\mathbb{R}^2; 0)$ from the identity to a homeomorphism f and he asked if this set is always an interval. In this article, we give a positive answer to this question and to the analogous question in the case of the open annulus.

1. *Introduction*

The concept of *rotation number* was introduced by Poincaré to study the dynamics of orientation-preserving homeomorphisms of the circle $\mathbb{T}^1$ (in the context of torus flows, see [**Poi85**]). More precisely, for every orientation-preserving homeomorphism h of the circle $\mathbb{T}^1$, H. Poincaré defined a number $\rho(h)$, measuring the 'asymptotic speed of rotation of the orbits of h around $\mathbb{T}^1$'. He proved that this number provides information about the dynamics of h. The construction of Poincaré can be generalized for homeomorphisms of surfaces. In the case of the two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, this notion of a rotation set was introduced by Misiurewicz and Ziemian (see [**MZ89**]). For a torus homeomorphism f which is isotopic to the identity, the rotation set associated to some lift $\widetilde{f}$ of f, denoted by $\rho(\widetilde{f})$, is a subset of $\mathbb{R}^2$. In this, and in many other articles, the relation between the rotation set and the dynamics of f is studied. For example, if $(p_1/q, p_2/q)$ is a rational point in the interior of $\rho(\widetilde{f})$, then there exists a q-periodic point for f (see [**Fra88**]). In [**LM91**], it is proved that if $\rho(\widetilde{f})$ has non-empty interior, then f has positive entropy. More recently, using transverse foliations to the dynamics, one can give more precise descriptions of the dynamics of some homeomorphisms of the torus (see, for example, [**Dav13**, **LT15**]).

1.1. *Main results.*

In this article, we are interested in the local case (that is, around a fixed point) and in the case of the open annulus $\mathbb{A} = \mathbb{T}^1 \times \mathbb{R}$. In the local case, firstly,

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

Naĭshul' proved that for diffeomorphisms f which fix zero and whose differential at zero is a rotation, the angle of the rotation is a topological invariant (see [**Naĭ82**]). In [**GP95**] Gambaudo and Pécou gave a simple proof of Naĭshul's result. Next, Le Calvez (see [**LeC03**]) and then Le Calvez and Yoccoz (see [**LY97**]) defined the *local rotation number* of some homeomorphisms. These works provide settings where the local rotation set is a single number. More recently, Le Roux defined *the local rotation set* for a general homeomorphism isotopic to the identity of the plane which fixes zero (see [**LeR13**]). An easy but important example is the family of *fibered rotations*. For a continuous function $\alpha : (0, +\infty) \to \mathbb{R}$, we define the fibered rotation as

$$h_\alpha : (r, \theta) \mapsto (r, \theta + \alpha(r)).$$

In this case, the local rotation set of h_α (which will be defined shortly) coincides with the limit points of the function α at zero. Hence, for every compact interval in $[-\infty, +\infty]$, there exists a fibered rotation whose local rotation set is this interval.

A natural question is whether the local rotation set is always an interval. The following theorem gives a positive answer to the above question. We denote by $\mathrm{Homeo}_0(\mathbb{R}^2; 0)$ the set of all homeomorphisms of the plane which are isotopic to the identity and fix zero.

THEOREM A. *Let I be an isotopy in* $\mathrm{Homeo}_0(\mathbb{R}^2; 0)$ *starting from the identity. Then the local rotation set of I, $\rho_{\mathrm{loc}}(I)$, is an interval. More precisely, every rational number belonging to the interior of the convex hull of $\rho_{\mathrm{loc}}(I)$ is included in the rotation set of a compact invariant set arbitrarily close to zero.*

Let us turn to the case of the open annulus $\mathbb{A} = \mathbb{T}^1 \times \mathbb{R}$. The notion of rotation set was introduced by Franks (see [**Fra96**]; see also [**LeR13**]). On the other hand, Le Calvez introduced the *rotation set of recurrent points* (see [**LeC01**]). Moreover, these sets coincide if the latter is non-empty and the homeomorphism satisfies the intersection property (see [**Wan14**]). Our second result is the following. We denote by $\mathrm{Homeo}_0(\mathbb{A})$ the set of all homeomorphisms of the open annulus which are isotopic to the identity.

THEOREM B. *Let I be an isotopy in* $\mathrm{Homeo}_0(\mathbb{A})$ *starting from the identity. Then the rotation set of I in the open annulus, $\rho_{\mathrm{ann}}(I)$, is an interval.*

1.2. *Analogous results for some compact surfaces.* Let us recall briefly analogous results of Theorems A and B for some compact surfaces. In the case of the closed annulus $\overline{\mathbb{A}} = \mathbb{T}^1 \times [0, 1]$, one can easily give a positive answer to the above question. Let f be a homeomorphism of the closed annulus which is isotopic to the identity and let I be an isotopy from the identity to f. There is a lift $\widetilde{f} : \mathbb{R} \times [0, 1] \to \mathbb{R} \times [0, 1]$ of f associated to I. The *rotation set of I* is a compact interval of $\mathbb{R}$, which measures the asymptotic speeds of rotation of the orbits of f around the annulus. More precisely, let $n \geq 1$ be an integer. We define

$$\rho_n := \left\{ \frac{p_1'(\widetilde{f}^n(\widetilde{z})) - p_1'(\widetilde{z})}{n} : \widetilde{z} \in \mathbb{R} \times [0, 1] \right\},$$

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

where $p_1' : \mathbb{R} \times [0, 1] \to \mathbb{R}$ is the projection on the first coordinate. *The rotation set of* I is

$$\rho(I) := \bigcap_{m \geq 1} \mathrm{Cl}\Big(\bigcup_{n \geq m} \rho_n \Big).$$

More precisely, ρ belongs to $\rho(I)$ if

$$\rho = \lim_{k \to +\infty} \frac{p_1'(\widetilde{f}^{n_k}(\widetilde{z}_k)) - p_1'(\widetilde{z}_k)}{n_k},$$

where $(\widetilde{z}_k)_{k \in \mathbb{N}}$ is a sequence in $\mathbb{R} \times [0, 1]$ and $(n_k)_{k \in \mathbb{N}}$ is a sequence of integers which goes to $+\infty$. Let us prove that $\rho(I)$ is an interval. Since $\overline{\mathbb{A}}$ is a connected set, the set ρ_n is a connected subset of $\mathbb{R}$. Using that ρ_{kn} is contained in ρ_n for every integer $k \geq 1$, we deduce that $\bigcup_{n \geq m} \rho_n$ is an interval for all integer $m \geq 1$, and thus as $\rho(I)$ is a decreasing sequence of intervals, it is an interval.

In the case of the two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, Misiurewicz and Ziemian proved in [**MZ89**] that the rotation set is a compact convex subset of $\mathbb{R}^2$. However, it is not known which compact and convex subsets of the plane can be realized as rotation sets of homeomorphisms of $\mathbb{T}^2$. It was conjectured by Franks and Misiurewicz in [**FM90**] that a line segment L could not be realized as a rotation set of torus homeomorphisms in the following cases: (i) if L has irrational slope and a rational point in its interior; (ii) if L has rational slope but no rational points; and (iii) if L has irrational slope and no rational points. Recently, Le Calvez and Tal proved case (i) (see [**LT15**]), and Ávila has announced a counter-example for case (iii).

At first sight, the local case and the open annulus case may seem very close to the closed annulus case. However, if we try the straightforward adaptation of the notion of rotation set of the closed annulus, the invariance by topological conjugacy fails. For example the two contractions mapping $z \mapsto \frac{1}{2}z$ and $z \mapsto \frac{i}{2}z$ are conjugate but they will have $\{0\}$ and $\{\frac{1}{4}\}$ by local rotation set, respectively (see Remark 2). A solution for this is 'to select good orbits', but the resulting definition is a little more complex (see the definition of $\rho_K(I)$ below) and the easy argument for the case of the closed annulus fails to prove connectedness of the local rotation set.

1.3. *Strategy of the proofs.* Now we will give the precise definitions and the strategy of the proof of Theorem A (these can be adapted to prove Theorem B). Let f be in $\mathrm{Homeo}_0(\mathbb{R}^2; 0)$, that is, a homeomorphism of the plane $\mathbb{R}^2$ which is isotopic to the identity and which fixes zero. Let $I = (f_t)_{t \in [0,1]}$ be an isotopy from the identity to f in $\mathrm{Homeo}_0(\mathbb{R}^2; 0)$, that is, a continuous path $t \mapsto f_t$ from $[0, 1]$ to $\mathrm{Homeo}_0(\mathbb{R}^2; 0)$ (endowed with the compact-open topology), which connects the identity to $f_1 = f$. Let $\pi : \mathbb{R} \times (0, +\infty) \to \mathbb{R}^2 \backslash \{0\}$ be the universal covering of $\mathbb{R}^2 \backslash \{0\}$ and let $\widetilde{I} = (\widetilde{f}_t)_{t \in [0,1]}$ be the lift of the isotopy I to the universal covering of $\mathbb{R}^2 \backslash \{0\}$ such that $\widetilde{f}_0 = \mathrm{Id}$. Define $\widetilde{f} := \widetilde{f}_1$. Given $z \in \mathbb{R}^2 \backslash \{0\}$ and an integer $n \geq 1$, we consider $\rho_n(z)$ the *average change of angular coordinate along the trajectory of* z *for the isotopy* I: that is,

$$\rho_n(z) := \frac{1}{n}(p_1(\widetilde{f}^n(\widetilde{z})) - p_1(\widetilde{z})),$$

where $p_1 : \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ is the projection on the first coordinate and $\widetilde{z}$ is a point in $\pi^{-1}(z)$. If we try the straightforward adaptation of the notion of rotation set of the closed

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

annulus to the local case or open annulus case, the invariance by topological conjugacy fails. A solution for this is 'to select good orbits'. In the case of the open annulus, we consider segments of orbits whose endpoints belong to a compact set of $\mathbb{A}$ (see Figure 1). In the local case (that is, around a fixed point zero), we consider segments of orbits which are close to zero, but whose endpoints are not close enough to zero (Figure 1; see the definitions of $\rho_K(I)$ and $\rho_{V,W}(I)$ below).

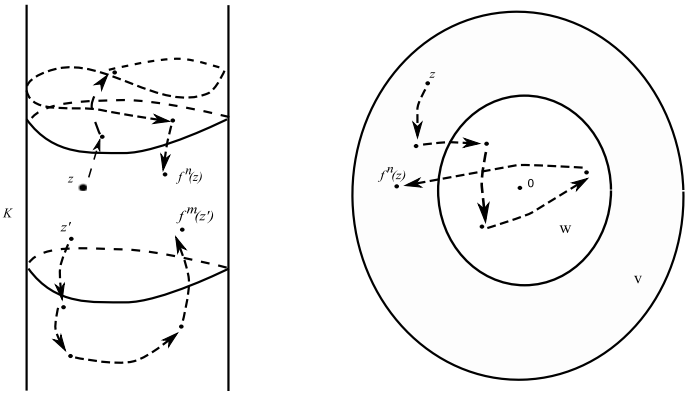


FIGURE 1. Left: good orbits in the open annulus. Right: good orbits around a fixed point.

For a compact set K in $\mathbb{R}^2\backslash\{0\}$ we define

$$\rho_K(I) := \bigcap_{m\geq 1} \mathrm{Cl}\Big(\bigcup_{n\geq m}\{\rho_n(z) : z \in K,\ f^n(z) \in K\}\Big),$$

where the closure is taken in $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$. More precisely, ρ belongs to $\rho_K(I)$ if

$$\rho = \lim_{k\to+\infty} \rho_{n_k}(z_k),$$

where $(z_k)_{k\in\mathbb{N}}$ is a sequence in K and $(n_k)_{k\in\mathbb{N}}$ is a sequence of integers which goes to $+\infty$ and such that z_k and $f^{n_k}(z_k)$ belong to K. We will give the definition of the *local rotation set of* I due to Le Roux (see [**LeR13**]). Given two neighborhoods V and W of zero with $W \subset V$, we define

$$\rho_{V,W}(I) := \bigcap_{m\geq 1} \mathrm{Cl}\Big(\bigcup_{n\geq m}\{\rho_n(z) : z \notin W,\ f^n(z) \notin W,\ \text{and } z, \ldots, f^n(z) \in V\}\Big).$$

The *local rotation set of the isotopy* I is defined as

$$\rho_{\mathrm{loc}}(I) := \bigcap_{V} \mathrm{Cl}\Big(\bigcup_{W\subset V} \rho_{V,W}(I)\Big),$$

where $W \subset V$ are neighborhoods of zero. Note that $\rho_{V,W}(I)$ is analogous to the above definition of $\rho_K(I)$ with $K = V\backslash W$. Using the usual properties of rotation sets, we can reduce the proof of Theorem A to the following theorem.

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

THEOREM A*. *Let I be an isotopy in* $\mathrm{Homeo}_0(\mathbb{R}^2; 0)$ *from the identity to a homeomorphism f. Suppose that the local rotation set $\rho_{\mathrm{loc}}(I)$ contains both positive and negative real numbers. Then zero belongs to $\rho_{\mathrm{loc}}(I)$. More precisely, for every V neighborhood of zero, there exists an f-invariant and compact set K in $\mathbb{R}^2\backslash\{0\}$ contained in V, such that zero belongs to $\rho_K(I)$.*

This theorem is an improvement of a result due to Le Roux. Under the hypotheses of Theorem A* and assuming, furthermore, that f satisfies the *local intersection property*, that is, every Jordan curve γ non-contractible in $\mathbb{R}^2\backslash\{0\}$ meets its image under f (that is, $f(\gamma) \cap \gamma \neq \emptyset$), he proved that V contains a *contractible fixed point z under I*, that is, a fixed point of f such that its trajectory under the isotopy I, $z \mapsto f_t(z)$ is a loop contractible in $\mathbb{R}^2\backslash\{0\}$ (see [**LeR13**] and, for the case of the open annulus, see [**LeC05**]). Thus we see that, in this case, we take $K = \{z\}$.

On the other hand, let us examine the opposite case when f does not have any contractible fixed point close enough to zero. We can apply a generalization of Brouwer's translation theorem due to Le Calvez (see [**LeC05**]), and we obtain an oriented foliation $\mathcal{F}$ defined in $\mathbb{R}^2\backslash\{0\}$ which is positively transverse to the isotopy I. Let γ be a closed leaf of $\mathcal{F}$. Then γ is a Jordan curve, it is *essential* (that is, non-contractible in $\mathbb{R}^2\backslash\{0\}$) and it is *$f$-free* (that is, $f(\gamma) \cap \gamma = \emptyset$). Using the fact that $\rho_{\mathrm{loc}}(I)$ contains both positive and negative real numbers, we obtain the following situation (see Proposition 3.3). In every neighborhood of zero, there exist three closed leaves γ_0, γ_1 and γ_2 of $\mathcal{F}$ such that:

(i) the curve γ_1 separates γ_0 and γ_2;
(ii) the set of the points x such that all iterates of x under f remain between γ_0 and γ_1 is non-empty and its rotation set is contained in $(0, +\infty)$; and
(iii) the set of the points x such that all iterates of x under f remain between γ_1 and γ_2 is non-empty and its rotation set is contained in $(-\infty, 0)$.

(For the case of the open annulus, see Figure 2.) Thus Theorem A* will be a consequence of the following theorem. This is the main and new result of this article. For E a subset of $\mathbb{R}^2\backslash\{0\}$, we write $\Theta(E)$ for the *maximal invariant set of E*, that is, the set of the points x of E such that all iterates of x under f remain in E. We remark that $\mathbb{R}^2\backslash\{0\}$ is homeomorphic to the open annulus $\mathbb{A}$, and so all above definitions are valid in $\mathbb{A}$.

THEOREM C. *Let I be an isotopy in* $\mathrm{Homeo}_0(\mathbb{A})$ *from the identity to a homeomorphism f. Assume that there exist γ_0, γ_1 and γ_2 which are three essential, pairwise disjoint and f-free Jordan curves in the annulus $\mathbb{A}$ such that γ_1 separates γ_0 of γ_2. For $i \in \{0, 1\}$, let $\Theta(A_i)$ be the maximal invariant set of the annulus A_i delimited by γ_i and γ_{i+1}. Suppose that:*

(i) *the sets $\Theta(A_0)$ and $\Theta(A_1)$ are non-empty; and*
(ii) *the set $\rho_{\Theta(A_0)}(I)$ is contained in either $(0, +\infty)$ or $(-\infty, 0)$ and $\rho_{\Theta(A_1)}(I)$ is contained in the other one.*

Let $\Theta(A)$ be the maximal invariant set of $A = A_0 \cup A_1$. Then

$$0 \in \rho_{\Theta(A)}(I).$$

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

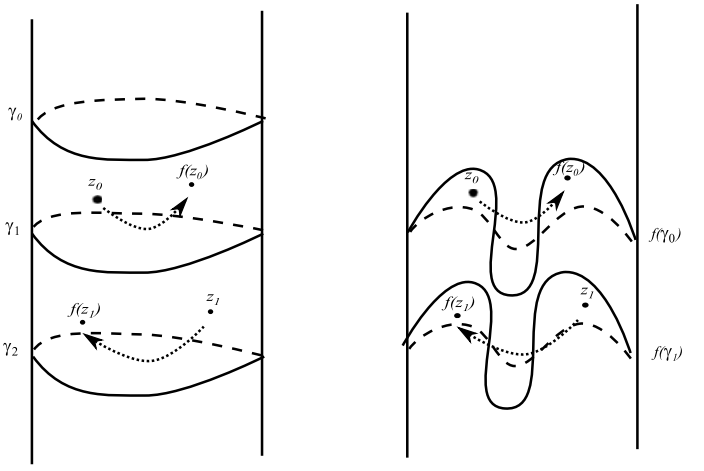


FIGURE 2. Theorem C.

Another consequence of Theorem C is the following theorem.

THEOREM D. *Let I be an isotopy in* $\mathrm{Homeo}_0(\mathbb{A})$ *from the identity to a homeomorphism f. Assume that there exist γ^+ and γ^- which are two essential, disjoint and f-free Jordan curves in the annulus $\mathbb{A}$. Let $\Theta(A)$ be the maximal invariant set of the annulus A delimited by γ^+ and γ^- and suppose that $\Theta(A)$ is non-empty. Then $\rho_{\Theta(A)}(I)$ is an interval.*

Here is an outline of this article. In §2, we will recall some definitions and results that we will use in the proof of Theorems A and B, in particular, Le Calvez's generalization of Brouwer's translation theorem. In §3, we will prove Theorem A assuming Theorem C. In §4, we will prove Theorem B assuming Theorem C. In §5, we will prove Theorem D. Finally, in §6, we will prove Theorem C.

2. *Preliminary results*

In this section, we will recall some definitions and results that we will use in the rest of the article, in particular, the generalization of Brouwer's translation theorem due to Le Calvez.

2.1. *Foliations.* Let M be an oriented surface. By an *oriented singular foliation* $\mathcal{F}$ on M we mean a closed set $\mathrm{Sing}(\mathcal{F})$, called *the set of singularities of* $\mathcal{F}$, together with an oriented foliation $\mathcal{F}'$ on $M \setminus \mathrm{Sing}(\mathcal{F})$: that is, $\mathcal{F}'$ is a partition of $M \setminus \mathrm{Sing}(\mathcal{F})$ into connected oriented 1-manifolds (circles or lines) called *leaves of* $\mathcal{F}$, such that, for every z in $M \setminus \mathrm{Sing}(\mathcal{F})$, there exists a neighborhood U of z, called a *trivializing neighborhood*, and an orientation-preserving homeomorphism ψ, called a *trivialization chart at* z, mapping U to an open set U' of $\mathbb{R}^2$ such that ψ maps the foliation induced by $\mathcal{F}$ in U to the oriented foliation by vertical lines oriented upwards. By a theorem of Whitney (see [**Whi33**, **Whi41**]), all singular foliations $\mathcal{F}$ can be embedded in flows: that is, $\mathcal{F}$ is the set of (oriented) orbits of some *topological flow* $\phi : M \times \mathbb{R} \to M$ (where the set of singularities of $\mathcal{F}$ coincides with the set of fixed points of ϕ). Therefore, we can

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

define the α-limit and ω-limit sets of each leaf of $\mathcal{F}$ as follows. If l is a leaf of $\mathcal{F}$ and z is a point of l, then

$$\omega(l) := \bigcap_{t\in[0,+\infty)} \mathrm{Cl}\{\phi(z,t') : t' \geq t\} \quad \text{and} \quad \alpha(l) := \bigcap_{t\in(-\infty,0]} \mathrm{Cl}\{\phi(z,t') : t' \leq t\}.$$

Let $\mathcal{F}$ be an oriented singular foliation on M. We will say that an arc γ in $M \setminus \mathrm{Sing}(\mathcal{F})$ is *positively transverse* to $\mathcal{F}$ if, for every $t_0 \in [0, 1]$ and a trivialization chart ψ at $\gamma(t_0)$, the first coordinate of the application $t \mapsto \psi(\gamma(t))$ is increasing in a neighborhood of t_0.

2.2. *Isotopies.* An *isotopy* $I = (f_t)_{t\in[0,1]}$ on M is a family $(f_t)_{t\in[0,1]}$ of homeomorphisms of M such that the map $(z, t) \mapsto f_t(z)$ is continuous on $M \times [0, 1]$. This implies that the map $(z, t) \mapsto f_t^{-1}(z)$ is continuous, and then we can define the inverse isotopy $I^{-1} = (f_t^{-1})_{t\in[0,1]}$. Two isotopies $I = (f_t)_{t\in[0,1]}$ and $I' = (f'_t)_{t\in[0,1]}$ are said to be homotopic if there exists a family $(f_{t,s})_{(t,s)\in[0,1]^2}$ of homeomorphisms of M satisfying:

- $f_{t,0} = f_t$ for every $t \in [0, 1]$;
- $f_{t,1} = f'_t$ for every $t \in [0, 1]$; and
- the map $(z, t, s) \mapsto f_{t,s}(z)$ is continuous on $M \times [0, 1]^2$.

Let $I = (f_t)_{t\in[0,1]}$ and $I' = (f'_t)_{t\in[0,1]}$ be two isotopies on M. Then $I * I' = (g_t)_{t\in[0,1]}$ denotes the isotopy defined as

$$g_t = \begin{cases} f_{2t} & \text{if } 0 \leq t \leq \frac{1}{2}, \\ f'_{2t-1} \circ f_1 & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

In particular, if q is an integer, we define I^q as the isotopy $I * \cdots * I$ (q times) if $q \geq 1$ and as $I^{-1} * \cdots * I^{-1}$ ($-q$ times) if $q \leq -1$. Let M be an oriented surface, and let $I = (f_t)_{t\in[0,1]}$ be an isotopy from $f_0 = \mathrm{Id}_M$ to a homeomorphism $f_1 = f$. Let $\pi : \widetilde{M} \to M$ be the universal covering of M and let $\widetilde{I} = (\widetilde{f}_t)_{t\in[0,1]}$ be the lift of the isotopy I such that $\widetilde{f}_0 = \mathrm{Id}_{\widetilde{M}}$. The homeomorphism $\widetilde{f} = \widetilde{f}_1$ is called the *lift of f associated to I*. Let $z \in M$. We call the *trajectory of z along I* the arc $t \mapsto f_t(z)$. A fixed point of $f = f_1$ is said to be *contractible for I* if its trajectory under I is a loop homotopic to a constant loop in M. This definition depends only on the lift $\widetilde{f}$ associated to I. In fact, it is easy to see that the set of contractible fixed points of I coincides with the image of the set of fixed points of $\widetilde{f}$ under π.

2.3. *The existence of a transverse foliation.* Let $I = (f_t)_{t\in[0,1]}$ be an isotopy from the identity to a homeomorphism $f_1 = f$. We say that an oriented foliation $\mathcal{F}$ on M is *transverse to I* if, for every $z \in M$, there exists an arc which is homotopic with fixed endpoints to the trajectory of z along I and is positively transverse to $\mathcal{F}$. We know that '$\mathcal{F}$ is transverse to I' persists on small perturbations of $\mathcal{F}$ (see [**LeR13**, Proposition 3.6.3]). More precisely, let $\mathcal{F}$ be an oriented foliation on M. Let $\psi : [0, 1]^2 \to M$ be a 'compact chart', that is, an injective map, which maps each vertical line oriented upwards in $[0, 1]^2$ to a segment of one leaf of $\mathcal{F}$. For every real number $\epsilon > 0$, we write $\mathcal{O}(\psi, \epsilon)$ for the set of foliations $\mathcal{F}'$ which admit a compact chart ψ' satisfying

$$\sup_{(x,y)\in[0,1]^2} d_M(\psi(x, y), \psi'(x, y)) < \epsilon,$$

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

where d_M is a metric on M which induces the topology of M. A family of compact charts $(\psi_\alpha)_\alpha$ is said to be locally finite in M if every point $z \in M$ belongs to only a finite number of $\psi_\alpha([0, 1]^2)$ (more details can be found in [**LeR13**]).

PROPOSITION 2.1. (Le Roux, [**LeR13**, Proposition 3.6.3]) *Let M be the complement of a finite set in an oriented compact surface. Let I be an isotopy on M. Let $\mathcal{F}$ be an oriented foliation on M transverse to I. Suppose that there exists a family $(\psi_\alpha)_\alpha$ of compact charts of $\mathcal{F}$ that is locally finite in M and a positive real sequence $(\epsilon_\alpha)_\alpha$. Then every oriented foliation $\mathcal{F}'$ on M belonging to the Whitney's open set*

$$\mathcal{O}((\psi_\alpha), (\epsilon_\alpha)) = \bigcap_\alpha \mathcal{O}(\psi_\alpha, \epsilon_\alpha)$$

is positively transverse to I too.

The following theorem, due to Le Calvez, is a generalization of Brouwer's translation theorem.

THEOREM 2.2. (Le Calvez, [**LeC05**, Theorem 1.3]) *Let I be an isotopy on M from the identity to a homeomorphism f. If I does not have any contractible fixed point, then there exists an oriented foliation $\mathcal{F}$ on M which is transverse to I.*

Remark 1. By a remark due to Le Calvez (see [**LeC05**]), the oriented foliation $\mathcal{F}$ is transverse to the isotopy I if and only if, in the universal covering, each leaf $\widetilde{l}$ of the lifted foliation $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ is a *Brouwer's line* for $\widetilde{f}$, which is the lift of f associated to I. More precisely, by Schoenflies' theorem, there exists an orientation-preserving homeomorphism h of $\mathbb{R}^2$ such that $h \circ \widetilde{l}(t) = (0, t)$, for every $t \in \mathbb{R}$. Then we write $R(\widetilde{l}) = h^{-1}((0, +\infty) \times \mathbb{R})$ and $L(\widetilde{l}) = h^{-1}((-\infty, 0) \times \mathbb{R})$. We say that $\widetilde{l}$ is a Brouwer's line for $\widetilde{f}$ if $\widetilde{f}^{-1}(\widetilde{l}) \subset L(\widetilde{l})$ and $\widetilde{f}(\widetilde{l}) \subset R(\widetilde{l})$.

Since the set of contractible fixed points is usually non-empty, one needs some additional modifications before using the previous theorem. In the local case, we can use a result of Le Roux.

THEOREM 2.3. (Le Roux, [**LeR13**, Appendix A]) *Let I be an isotopy in* $\mathrm{Homeo}_0(\mathbb{R}^2; 0)$ *from the identity to a homeomorphism f. Suppose that I does not have any contractible fixed point $z \neq 0$ on a neighborhood U of zero. Then there exists a homeomorphism $\bar{f}$ of the plane which coincides with f in a neighborhood $U' \subset U$ of zero and which does not have any contractible fixed point in $\mathbb{R}^2 \setminus \{0\}$.*

2.4. *Dynamics of the transverse foliation with two singularities.* In this subsection, we consider the open annulus $\mathbb{A} = \mathbb{T}^1 \times \mathbb{R}$. We denote by N (respectively, S) the upper (respectively, lower) end of $\mathbb{A}$ and by $\hat{\mathbb{A}}$ the end compactification of $\mathbb{A}$, which is a topological space homeomorphic to the two-dimensional sphere. Suppose that the isotopy $I' = (f'_t)_{t\in[0,1]}$ from the identity to f has no contractible fixed points. Thus, we can apply Theorem 2.2 and obtain an oriented foliation $\mathcal{F}'$ on $\mathbb{A}$ which is transverse to I'. Then one can extend I' to an isotopy $I = (f_t)_{t\in[0,1]}$ on $\hat{\mathbb{A}}$ that fixes the ends of $\mathbb{A}$, that is, $f_t(N) = N$ and $f_t(S) = S$ for every $t \in [0, 1]$. Similarly, the foliation $\mathcal{F}'$ can be extended to an

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

oriented singular foliation $\mathcal{F}$ on $\hat{\mathbb{A}}$ where the singularities are the ends of $\mathbb{A}$. In this case, Le Calvez proved that $\mathcal{F}$ is also *locally transverse to* I at each end of $\mathbb{A}$ (see [**LeC08**]): that is, for every neighborhood $\hat{U}$ of an end $\hat{z}_0$ of $\mathbb{A}$, there exists a neighborhood $\hat{V}$ contained in $\hat{U}$ such that, for every $z \in \hat{V}\backslash\{\hat{z}_0\}$, there is an arc contained in $\hat{U}$ which is positively transverse to $\mathcal{F}$ and homotopic with fixed endpoints in $\hat{U}\backslash\{\hat{z}_0\}$ to the trajectory of z along I.

PROPOSITION 2.4. (Le Calvez, [**LeC08**, Proposition 3.4]) *The oriented singular foliation $\mathcal{F}$ is locally transverse to I at each end of $\mathbb{A}$.*

On the other hand, we can apply the theorem of Poincaré–Bendixson to $\mathcal{F}$ seen as a flow (see [**dMP82**]).

THEOREM 2.5. (Poincaré–Bendixson' theorem) *Let $\mathcal{F}$ be an oriented singular foliation on $\hat{\mathbb{A}}$ whose singularities are the ends of $\mathbb{A}$. For every leaf l of $\mathcal{F}$, one of the following possibilities holds.*

(i) *The ω-limit set of l is a closed leaf of $\mathcal{F}$.*
(ii) *The ω-limit set of l is reduced to an end of $\mathbb{A}$. or*
(iii) *The ω-limit set of l is constituted of ends of $\mathbb{A}$ and leaves l' joining two ends (eventually the same) of $\mathbb{A}$.*

One has the same possibilities for the α-limit set of l. Moreover, for every non-closed leaf, either:

(iv) *the sets $\omega(l)$ and $\alpha(l)$ are reduced to the same end of $\mathbb{A}$; or*
(v) *the sets $\omega(l)$ and $\alpha(l)$ are disjoint.*

We recall the following fact (see for example [**Hae62**, §3.2]).

(1) The union of the closed leaves of $\mathcal{F}$ is closed.

The Poincaré–Bendixson theorem and the transversality of $\mathcal{F}$ and I give the following properties.

(2) Every closed leaf γ separates N and S: that is, N and S belong to distinct connected components of $\hat{\mathbb{A}}\backslash\gamma$.
(3) Every closed leaf γ is f-free: that is, $f(\gamma) \cap \gamma = \emptyset$.

The following result is due to Le Roux (see [**LeR00**, **LeR04**]).

(4) Either the foliation $\mathcal{F}$ has recurrence, that is, there is a leaf l of $\mathcal{F}$ such that $\alpha(l) \cup \omega(l)$ is not contained in $\{N, S\}$, or there is a leaf l of $\mathcal{F}$ which joins N and S: that is, either $\alpha(l) = \{N\}$ and $\omega(l) = \{S\}$ or $\alpha(l) = \{S\}$ and $\omega(l) = \{N\}$.

We deduce the following properties in a neighborhood of an end $\hat{z}_0$ of $\mathbb{A}$. We say that $\hat{z}_0$ *is accumulated by closed leaves of* $\mathcal{F}$ if every neighborhood of $\hat{z}_0$ contains a closed leaf of $\mathcal{F}$.

LEMMA 2.6. (Non-accumulated case) *Let $\mathcal{F}$ be an oriented singular foliation on $\hat{\mathbb{A}}$ whose singularities are the ends of $\mathbb{A}$. Suppose that the end $\hat{z}_0$ is not accumulated by closed leaves of $\mathcal{F}$. Then there exists a leaf l of $\mathcal{F}$ such that $\alpha(l) = \{\hat{z}_0\}$ or $\omega(l) = \{\hat{z}_0\}$.*

Proof. By Property (4), we can suppose that $\mathcal{F}$ has recurrence. Firstly, suppose furthermore that $\mathcal{F}$ has closed leaves. Since these do not accumulate at $\hat{z}_0$, we can consider the 'last' closed leaf γ: that is, γ is a closed leaf of $\mathcal{F}$ and there are no closed leaves of $\mathcal{F}$ in $U_\gamma^{\hat{z}_0}$, the connected component of $\hat{\mathbb{A}}\backslash\gamma$ containing $\hat{z}_0$. Hence, if l is a leaf of $\mathcal{F}$ contained

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

in $U_\gamma^{\hat{z}_0}$, then Theorem 2.5 (items (iv) and (v)) implies that the α-limit set or the ω-limit set of l contains $\hat{z}_0$. Suppose that the ω-limit set of l contains $\hat{z}_0$. If the ω-limit set of l is reduced to $\hat{z}_0$, then we have finished. Otherwise, item (iii) from Theorem 2.5 provides a leaf l' from $\hat{z}_0$ to $\hat{z}_0$ because l is contained in $U_\gamma^{\hat{z}_0}$. This completes the proof in the case when $\mathcal{F}$ has closed leaves. Finally, suppose that the foliation $\mathcal{F}$ has no closed leaves. Since $\mathcal{F}$ has recurrence, there exists a leaf l of $\mathcal{F}$ such that $\alpha(l) \cup \omega(l)$ is not contained in $\{N, S\}$. Hence, by Theorem 2.5 (items (iv) and (v)), either $\alpha(l)$ or $\omega(l)$ contains $\hat{z}_0$. One concludes as in the first part. This completes the proof. □

LEMMA 2.7. (Accumulated case) *Let $\mathcal{F}$ be an oriented singular foliation on $\hat{\mathbb{A}}$ whose singularities are the ends of $\mathbb{A}$. Suppose that the end $\hat{z}_0$ is accumulated by closed leaves of $\mathcal{F}$. Let us fix a closed leaf γ of $\mathcal{F}$ and put $\hat{U}$ the closure of the connected component of the complement of γ containing $\hat{z}_0$. Let $\mathcal{C}$ be the union of all closed leaves of $\mathcal{F}$ contained in $\hat{U}$. Let A be a connected component of the complement (in $\hat{U}$) of $\mathcal{C} \cup \{0\}$. Then the set A is delimited by two closed leaves $\partial^+ A$ and $\partial^- A$ and, for every leaf l of $\mathcal{F}$ included in A, either $\alpha(l) = \partial^+ A$ and $\omega(l) = \partial^- A$ or $\alpha(l) = \partial^- A$ and $\omega(l) = \partial^+ A$. Moreover, all these leaves have the same α-limit and ω-limit sets.*

Proof. Using a chart centered at $\hat{z}_0$, we can suppose that $\mathcal{F}$ is a foliation on a neighborhood of $0 \in \mathbb{R}^2$. Let $\mathcal{C}$ be the union of all closed leaves of $\mathcal{F}$ contained in $\hat{U}$. This is a closed set of $\hat{U}\backslash\{0\}$ (Property (1)). The closed leaves are totally ordered by the relation of inclusion of the connected components of their complements containing zero. By Schoenflies' theorem, we can construct a homeomorphism of the plane that maps each closed leaf of $\mathcal{F}$ to a Euclidean circle centered at zero. Let A be a connected component of the complement (in $\hat{U}$) of $\mathcal{C} \cup \{0\}$. The set A is delimited by two closed leaves $\partial^+ A$ and $\partial^- A$, which are non-contractible in $\mathbb{R}^2\backslash\{0\}$ (Property (2)). Hence, by Theorem 2.5 (items (iv) and (v)), for every leaf l of $\mathcal{F}$ included in A, either $\alpha(l) = \partial^+ A$ and $\omega(l) = \partial^- A$ or $\alpha(l) = \partial^- A$ and $\omega(l) = \partial^+ A$. Moreover, all these leaves have the same α-limit and ω-limit sets. Hence, the foliation on the closure of A is homeomorphic to a Reeb component or a spiral on $\mathbb{T}^1 \times [0, 1]$. □

3. *Local case*

In this section, we will introduce the definition of the local rotation set and we will show Theorem A assuming Theorem C.

3.1. *Definitions.*

3.1.1. *The punctured plane.* We denote by $\mathbb{R}^2$ the plane endowed with its usual topology and orientation. We denote by zero the point $(0, 0)$ of the plane $\mathbb{R}^2$. Let γ be a non-contractible Jordan curve in $\mathbb{R}^2\backslash\{0\}$. We call the *interior of* γ (respectively, the *exterior of* γ) the connected component of $\mathbb{R}^2\backslash\gamma$ which contains (respectively, does not contain) zero.

3.1.2. *The set* $\mathrm{Homeo}_0(\mathbb{R}^2; 0)$. We say that a homeomorphism f of the plane *fixes* zero, if $f(0) = 0$. We denote by $\mathrm{Homeo}_0(\mathbb{R}^2; 0)$ the set of all homeomorphisms of the

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

plane that are isotopic to the identity and which fix zero. We recall that the fundamental group of $\mathrm{Homeo}_0(\mathbb{R}^2; 0)$ is isomorphic to $\mathbb{Z}$ (see [**Ham74**]). More precisely, consider $J = (R_t)_{t\in[0,1]}$ the isotopy defined by

$$R_t(x, y) := (x\cos(2\pi t) + y\sin(2\pi t), -x\sin(2\pi t) + y\cos(2\pi t)).$$

If $I = (f_t)_{t\in[0,1]}$ and $I' = (f'_t)_{t\in[0,1]}$ are two isotopies in $\mathrm{Homeo}_0(\mathbb{R}^2; 0)$ that are both from the identity to the same homeomorphism $f_1 = f'_1$, then there exists a unique integer $q \in \mathbb{Z}$ such that I' is homotopic (with fixed endpoints) to $J^q * I$.

3.1.3. *The local rotation set.* In this paragraph, we consider a homeomorphism f in $\mathrm{Homeo}_0(\mathbb{R}^2; 0)$. Let $I = (f_t)_{t\in[0,1]}$ be an isotopy in $\mathrm{Homeo}_0(\mathbb{R}^2; 0)$ from the identity to f. We will give the definition of the *local rotation set of* I due to Le Roux (see [**LeR13**]). Let $\pi : \mathbb{R}\times(0, +\infty) \to \mathbb{R}^2\backslash\{0\}$ be the universal covering of $\mathbb{R}^2\backslash\{0\}$ and let $\widetilde{I} = (\widetilde{f}_t)_{t\in[0,1]}$ be the lift of the isotopy I to the universal covering of $\mathbb{R}^2\backslash\{0\}$ such that $\widetilde{f}_0 = \mathrm{Id}$. Define $\widetilde{f} := \widetilde{f}_1$. Given $z \in \mathbb{R}^2\backslash\{0\}$ and an integer $n \geq 1$, we consider $\rho_n(z)$ the *average change of angular coordinate along the trajectory of* z *for the isotopy* I, that is,

$$\rho_n(z) := \frac{1}{n}(p_1(\widetilde{f}^n(\widetilde{z})) - p_1(\widetilde{z})),$$

where $p_1 : \mathbb{R}\times(0, +\infty) \to \mathbb{R}$ is the projection on the first coordinate and $\widetilde{z}$ is a point in $\pi^{-1}(z)$.

Remark 2. If we try to generalize the notion of rotation set of the closed annulus to the local case, we will define

$$\bigcap_V \bigcap_{m\geq 1} \mathrm{Cl}\Big(\bigcup_{n\geq m}\{\rho_n(z) : f^i(z) \in V \text{ for all } i \in \{0, \ldots, n\}\}\Big).$$

However, in this setting, the invariance by conjugation fails. For example, the two contractions mapping $z \mapsto (1/2)z$ and $z \mapsto (i/2)z$ are conjugate but they will have $\{0\}$ and $\{\frac{1}{4}\}$ by local rotation set, respectively.

A solution for this is 'to select good orbits'. We consider the segments of orbits which are close to zero but whose endpoints are not close enough to zero. Given two neighborhoods V and W of zero with $W \subset V$, we define the *local rotation set of* I *relative to* V *and* W by

$$\rho_{V,W}(I) := \bigcap_{m\geq 1} \mathrm{Cl}\Big(\bigcup_{n\geq m}\{\rho_n(z) : z \notin W,\ f^n(z) \notin W,\ \text{and } z, \ldots, f^n(z) \in V\}\Big),$$

where the closure is taken in $\overline{\mathbb{R}} := \mathbb{R}\cup\{+\infty\}\cup\{-\infty\}$. Next we define the *local rotation set of* I *relative to* V by

$$\rho_V(I) := \mathrm{Cl}\Big(\bigcup_W \rho_{V,W}(I)\Big),$$

where W is a neighborhood of zero included in V. Finally, we define the local rotation set of the isotopy I by

$$\rho_{\mathrm{loc}}(I) := \bigcap_V \rho_V(I),$$

where V is a neighborhood of zero.

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

Remark 3. The local rotation set $\rho_{\mathrm{loc}}(I)$ only depends on the choice of the homotopy class of I: that is, if I_1 and I_2 are two isotopies from the identity to the same homeomorphism which are homotopic, then $\rho_{\mathrm{loc}}(I_1) = \rho_{\mathrm{loc}}(I_2)$.

The local rotation set verifies the following properties.

PROPOSITION 3.1. *Let I be an isotopy in* $\mathrm{Homeo}_0(\mathbb{R}^2; 0)$ *from the identity to f. For every integer p, let $J^p = (R_{pt})_{t\in[0,1]}$. Then:*

(1) *the local rotation set of I is invariant by oriented topological conjugacy; and*
(2) *for every $p, q \in \mathbb{Z}$, $\rho_{\mathrm{loc}}(J^p * I^q) = q\ \rho_{\mathrm{loc}}(I) + p$.*

Remark 4. Any of the following properties imply that zero belongs to $\rho_{\mathrm{loc}}(I)$.

(1) Every neighborhood of zero contains a contractible fixed point $z \neq 0$ of I.
(2) Every neighborhood of zero contains a compact and f-invariant set K not containing zero such that zero belongs to $\rho_K(I)$.
(3) Every neighborhood V of zero contains a compact (non-invariant) set K not containing zero such that $\rho_K(I) \subset \rho_V(I)$ and zero belongs to $\rho_K(I)$.

Note that (1) $\Rightarrow$ (2) $\Rightarrow$ (3), and (3) implies that zero belongs to $\rho_{\mathrm{loc}}(I)$.

3.2. *Main result: proof of Theorem A.* The purpose of this section is to deduce Theorem A from Theorem C.

Let I be an isotopy in $\mathrm{Homeo}_0(\mathbb{R}^2; 0)$ from the identity to a homeomorphism f. Since the local rotation set $\rho_{\mathrm{loc}}(I)$ is closed and $\mathbb{Q}$ is dense in $\mathbb{R}$, in order to prove Theorem A, it is sufficient to show that every irreducible rational number p/q which belongs to the interior of the convex hull of $\rho_{\mathrm{loc}}(I)$ belongs to the local rotation set. Moreover, by considering $J^p * I^q$ instead of I, we may assume that $p = 0$ and $q = 1$. This follows from the formula relating the local rotation sets of I and $J^p * I^q$ (Proposition 3.1). Thus, we reduced the proof of Theorem A to proving the following theorem.

THEOREM A*. *Let I be an isotopy in* $\mathrm{Homeo}_0(\mathbb{R}^2; 0)$ *from the identity to a homeomorphism f. Suppose that the local rotation set $\rho_{\mathrm{loc}}(I)$ contains both positive and negative real numbers. Then zero belongs to $\rho_{\mathrm{loc}}(I)$. More precisely, for every V neighborhood of zero, there exists an f-invariant and compact set K in $\mathbb{R}^2\backslash\{0\}$ contained in V such that zero belongs to $\rho_K(I)$.*

Remark 5. Theorem A* implies that Property (2) of Remark 4 holds.

Example. There is a homeomorphism f of $\mathbb{R}^2$ which satisfies Property (2) but not Property (1) of Remark 4. To construct f, we start with a 'twice' Reeb's homeomorphism f_1' defined on the compact annulus $D_0 = \{z \in \mathbb{R}^2 : 1 \leq \|z\| \leq 3\}$ in $\mathbb{R}^2$ satisfying:

(i) f_1' leaves invariant the circles c_i, where $i \in \{1, 2, 3\}$ and $c_i = \{z \in \mathbb{R}^2 : \|z\| = i\} \subset D_0$;
(ii) $f_1'|_{c_1}$ and $f_1'|_{c_3}$ act as a rotation R_β with angle $\beta > 0$ and $f_1'|_{c_2}$ acts as a rotation R_α with angle $\alpha < 0$; and
(iii) f_1' does not have fixed points in D_0.

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

Let $I' = (f'_t)_{t\in[0,1]}$ be the natural isotopy on D_0 from the identity of D_0 to f'_1. We define $f_t : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f_t(z) = \begin{cases} \dfrac{f'_t(3^n z)}{3^n} & \text{if } \dfrac{1}{3^n} \le \|z\| \le \dfrac{1}{3^{n-1}},\ n \in \mathbb{Z}, \\ f_t(0) = 0. \end{cases}$$

Let $I = (f_t)_{t\in[0,1]}$ be the isotopy from the identity to $f = f_1$. For every integer $n \in \mathbb{N}$, let us put $D_n = (1/3^n)D_0$. Then zero belongs to $\rho_{D_n}(I)$ but f does not have any fixed point $z \neq 0$.

For the proof of Theorem A*, one introduces the following property.

Definition 3.2. We say that a homeomorphism f in $\mathrm{Homeo}_0(\mathbb{R}^2; 0)$ satisfies the *local intersection property* if there exists a neighborhood V of zero, such that every Jordan curve γ contained in V and non-contractible in $\mathbb{R}^2\backslash\{0\}$ meets its image under f: that is, $f(\gamma) \cap \gamma \neq \emptyset$.

Frédéric Le Roux proved Theorem A* (see [**LeR13**]) in the case when the homeomorphism f satisfies the local intersection property. He proved something better: in this case the local rotation set 'detects' some fixed points: that is, Property (1) of Remark 4 holds. This follows from the following proposition which will be proved in the next subsection.

PROPOSITION 3.3. *Let I be an isotopy in* $\mathrm{Homeo}_0(\mathbb{R}^2; 0)$ *from the identity to a homeomorphism f. Suppose that the local rotation set, $\rho_{\mathrm{loc}}(I)$, contains both positive and negative real numbers. Then one of the following cases holds.*

Case (a). Every neighborhood of zero contains a contractible fixed point $z \neq 0$.

Case (b). The homeomorphism f does not satisfy the local intersection property, and every neighborhood of zero contains three pairwise disjoint, f-free Jordan curves γ_0, γ_1 and γ_2 which are non-contractible in $\mathbb{R}^2\backslash\{0\}$ with γ_{i+1} included in the interior of γ_i for $i \in \{0, 1\}$ and such that the closed annulus A_i delimited by γ_i and γ_{i+1}, $i \in \{0, 1\}$, satisfies:

(i) *the maximal f-invariant set $\Theta(A_i)$ of A_i is non-empty; and*

(ii) *the set $\rho_{\Theta(A_0)}(I)$ is contained in one of the sets $(0, +\infty)$ or $(-\infty, 0)$ and the set $\rho_{\Theta(A_1)}(I)$ is contained in the other one.*

Note that, in the previous example, the free curves are essential loops transverse to the Reeb component. We recall that this proposition is an adaptation of the following result for homeomorphisms of the closed annulus $\mathbb{T}^1 \times [0, 1]$. This result was proved by Le Calvez for diffeomorphisms in [**LeC91**] but, by Theorem 2.2, it is also valid for homeomorphisms.

THEOREM 3.4. (Le Calvez [**LeC91**, **LeC05**]) *Let $f : \mathbb{T}^1 \times [0, 1] \to \mathbb{T}^1 \times [0, 1]$ be a homeomorphism isotopic to the identity with a lift $\widetilde{f} : \mathbb{R} \times [0, 1] \to \mathbb{R} \times [0, 1]$ that has no fixed points and whose rotation set contains both negative and positive real numbers. Then there exists a finite non-empty family $\{\gamma_i\}$ of essential, pairwise disjoint and f-free Jordan curves such that the maximal invariant sets of the closed annulus delimited by two consecutive curves has rotation set contained either in $(0, +\infty)$ or in $(-\infty, 0)$.*

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

Now we show that Proposition 3.3 and Theorem C imply Theorem A*.

End of the proof of Theorem A (assuming Theorem C).* Let I be an isotopy in $\mathrm{Homeo}_0(\mathbb{R}^2; 0)$ from the identity to a homeomorphism f. Suppose that the local rotation set $\rho_{\mathrm{loc}}(I)$ contains both positive and negative real numbers. One considers the different cases of Proposition 3.3. In case (a), we have finished. On the other hand, if case (b) of Proposition 3.3 holds, we can apply Theorem C, which says that zero belongs to $\rho_{\Theta(A)}(I)$, where $\Theta(A)$ is the maximal invariant set of $A = A_0 \cup A_1$. This completes the proof of Theorem A*. □

3.3. *Proof of Proposition 3.3.* Let I and f satisfy the hypotheses of the proposition. We prove Proposition 3.3 assuming that case (a) does not hold, that is, there is a neighborhood of zero that does not contain any contractible fixed point. By Theorem 2.3, changing f in the complement of a Jordan domain containing zero, we may suppose that f does not have any contractible fixed point in $\mathbb{R}^2\backslash\{0\}$. Thus, we can apply Theorem 2.2 and obtain an oriented foliation $\mathcal{F}$ defined in $\mathbb{R}^2\backslash\{0\}$. The foliation $\mathcal{F}$ can be extended to an oriented singular foliation on the end compactification of $\mathbb{R}^2\backslash\{0\}$ where the singularities are the ends of $\mathbb{R}^2\backslash\{0\}$. In this case, by Proposition 2.4, $\mathcal{F}$ is also locally transverse to I at zero.

LEMMA 3.5. *In this case, the closed leaves of $\mathcal{F}$ accumulate at zero.*

Proof. Otherwise, by Lemma 2.6, there exists a leaf l of $\mathcal{F}$ whose ω-limit or α-limit set is reduced to $\{0\}$. The following lemma implies that either $\rho_{\mathrm{loc}}(I) \subset [0, +\infty]$ or $\rho_{\mathrm{loc}}(I) \subset [-\infty, 0]$. This contradicts our hypotheses. □

LEMMA 3.6. *Let I be an isotopy in* $\mathrm{Homeo}_0(\mathbb{R}^2; 0)$ *and let $\mathcal{F}$ be a foliation positively transverse to I. Suppose that $\mathcal{F}$ admits a leaf l such that either $\omega(l) = \{0\}$ or $\alpha(l) = \{0\}$. Then either $\rho_{\mathrm{loc}}(I) \subset [0, +\infty]$ or $\rho_{\mathrm{loc}}(I) \subset [-\infty, 0]$.*

Proof. Without loss of generality, we may suppose that $\alpha(l) = \{0\}$. In addition, conjugating f by Φ (given by Schoenflies' theorem), we may suppose that the negative half-leaf l^- containing one point of l is contained in the positive x-axis. Let U be a Euclidean circle centered at zero whose boundary meets l^-. By Proposition 2.4, $\mathcal{F}$ is locally transverse to I at zero. Let V be a neighborhood of zero contained in U given by the local transversality of $\mathcal{F}$ to I at zero: the trajectory of each $z \in V$ along I is homotopic, with fixed endpoints, to an arc α_z which is positively transverse to $\mathcal{F}$ and included in U. In particular, the arcs α_z can cross l^- only downwards. More precisely, let $\widetilde{f}$ be the lift of f associated to I and let $\widetilde{\mathcal{F}}$ be the lift of $\mathcal{F}$. Let $\widetilde{U}$ and $\widetilde{V}$ be lifts of the sets $U\backslash\{0\}$ and $V\backslash\{0\}$, respectively. Let $\widetilde{l^-}$ be the lift of l^- contained in the line $\{0\} \times (0, +\infty)$. Let $z \in V\backslash\{0\}$ and let $n \in \mathbb{N}$ be such that $\{z, \ldots, f^{n-1}(z)\} \subset V$. Let $\widetilde{z}$ be the lift of z such that $0 < p_1(\widetilde{z}) \leq 1$ and let $\alpha_{\widetilde{z}}$ be the lift of the arc α_z from $\widetilde{z}$. Since the arc $\alpha_{\widetilde{f}^{n-1}(\widetilde{z})} * \cdots * \alpha_{\widetilde{z}}$ is positively transverse to $\widetilde{\mathcal{F}}$ and does not meet the boundary of $\widetilde{U}$, we obtain that $p_1(\widetilde{f}^n(\widetilde{z})) > 0$ and thus

$$\rho_n(z) = \frac{1}{n}(p_1(\widetilde{f}^n(\widetilde{z})) - p_1(\widetilde{z})) \geq -\frac{1}{n}.$$

This implies the inclusion $\rho_V(I) \subset [0, +\infty]$, and so $\rho_{\mathrm{loc}}(I) \subset [0, +\infty]$. This completes the proof. □

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

In what follows, we suppose that the closed leaves of $\mathcal{F}$ accumulate at zero and that all closed leaves of $\mathcal{F}$ are Euclidean circles centered at zero (see Lemma 2.7). We remark that $\mathcal{F}$ cannot coincide with the oriented foliation in Euclidean circles centered at zero on a neighborhood of zero. Otherwise, by transversality, for every $z \in \mathbb{R}^2\backslash\{0\}$ close to zero, either $\|f(z)\| < \|z\|$ or $\|z\| < \|f(z)\|$. Thus f is conjugated to a homothety. In this case, $\rho_{\mathrm{loc}}(I)$ is empty (see [**LeR13**]), which contradicts our hypothesis.

We will consider the set $\mathcal{A}$ of all closed annuli A whose boundary components are closed leaves of $\mathcal{F}$ and which do not contain any closed leaf of $\mathcal{F}$ in their interiors. By the Poincaré–Bendixson theorem (Theorem 2.5), the foliation on A is a Reeb component or a spiral (see Lemma 2.7).

For every annulus $A \in \mathcal{A}$, we write $\partial^+ A$ and $\partial^- A$ for its two boundary components with $\partial^- A$ included in the interior of $\partial^+ A$.

Definition 3.7. We say that the foliation $\mathcal{F}$ goes from $\partial^+ A$ to $\partial^- A$ (respectively, from $\partial^- A$ to $\partial^+ A$) on the annulus $A \in \mathcal{A}$ if all leaves of $\mathcal{F}$ in the interior of A have $\partial^+(A)$ (respectively, $\partial^-(A)$) as their α-limit set and $\partial^-(A)$ (respectively, $\partial^+ A$) as their ω-limit set.

Consider the following lemma.

LEMMA 3.8. (Le Calvez, [**LeC91**]) *Suppose that the foliation $\mathcal{F}$ goes from $\partial^- A$ to $\partial^+ A$ on the annulus $A \in \mathcal{A}$. Let $\Theta(A)$ the maximal invariant set of A. Then the set $\rho_{\Theta(A)}(I)$ (eventually empty) is contained in* $(0, +\infty)$.

Proof. Since each boundary component of the annulus A is f-free, we deduce that $\Theta(A)$ is included in the interior of A. For $z \in \Theta(A)$, by transversality of $\mathcal{F}$ and I, there exists an arc α_z joining z and $f(z)$, which is homotopic (with fixed endpoints) to the trajectory of z along the isotopy I and is transverse to the foliation $\mathcal{F}$. Moreover, from the fact that z and $f(z)$ are both in the interior of A, we deduce that α_z is also included in the interior of A. We can find an annulus A' whose boundary components are Jordan curves that are not contractible in $\mathbb{R}^2\backslash\{0\}$ such that the set $\Theta(A) = f(\Theta(A))$ is included in A'. Considering a change of coordinates, we can assume that the boundary components of A' are Euclidean circles centered at zero and that the foliation induced by $\mathcal{F}$ on A' coincides with the radial foliation with the leaves backward at zero. By transversality, for every $z \in \Theta(A)$ and every integer $n \geq 1$, we have that $\rho_n(z)$ is positive. Since $\Theta(A)$ is a compact set, we deduce that $\rho_{\Theta(A)}(I)$ is included in $(0, +\infty)$. This completes the proof. □

In the following, $\mathcal{A}'$ denotes the subset of $\mathcal{A}$ consisting of the annuli $A \in \mathcal{A}$ whose maximal invariant set is non-empty. From the above proposition, we can consider the following definition.

Definition 3.9. We say that $A \in \mathcal{A}'$ is a positive (respectively, negative) annulus if $\rho_{\Theta(A)}(I) \subset (0, +\infty)$ (respectively, $\rho_{\Theta(A)}(I) \subset (-\infty, 0)$).

LEMMA 3.10. *Under the hypotheses of Proposition 3.3, assume, furthermore, that Case (a) does not hold. Then both positive and negative annuli accumulate at zero.*

Proof. Suppose, by contradiction, that there exists a neighborhood of zero which does not contain any positive annulus (the other case is proved in an analogous way).

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

By Lemma 3.8, we know that all annuli A on which the foliation $\mathcal{F}$ goes from $\partial^- A$ to $\partial^+ A$ on A has maximal invariant set empty. The next proposition provides a foliation $\mathcal{F}'$ which is also transverse to I and such that all leaves of $\mathcal{F}'$ included in a neighborhood of zero are spirals whose ω-limit set is $\{0\}$. Hence, by Lemma 3.6, the local rotation set $\rho_{\text{loc}}(I)$ is contained in $[-\infty, 0]$, which contradicts our hypotheses. This completes the proof of the lemma. □

PROPOSITION 3.11. *Let $\mathcal{F}$ be an oriented singular foliation whose singularities are the ends of $\mathbb{R}^2\backslash\{0\}$ transverse to an isotopy I. Suppose that the closed leaves of $\mathcal{F}$ accumulate at zero. Let γ be a closed leaf of $\mathcal{F}$. Suppose that the interior of γ does not contain any positive annulus. Then there exists an oriented foliation $\mathcal{F}'$ which is transverse to I and such that all leaves of $\mathcal{F}'$ contained in the interior of γ are spirals whose ω-limit set is* $\{0\}$.

We will need the following lemma.

LEMMA 3.12. *The set $\mathcal{A}'$ is locally finite in $\mathbb{R}^2\backslash\{0\}$: that is, every compact set in $\mathbb{R}^2\backslash\{0\}$ can meet only a finite number of elements of $\mathcal{A}'$.*

Proof. Let $\mathcal{A}''$ be the subset of $\mathcal{A}$ constituted of the annuli $A \in \mathcal{A}$ that meet its image under f: that is, $f(A) \cap A \neq \emptyset$. Remark that $\mathcal{A}' \subset \mathcal{A}''$, and so it suffices to prove that the set $\mathcal{A}''$ is locally finite in $\mathbb{R}^2\backslash\{0\}$. This will be a consequence of the following lemma.

LEMMA 3.13. *Let γ^+ and γ^- be two closed leaves of $\mathcal{F}$ with γ^- contained in the interior of γ^+. Let A_{γ^+,γ^-} be the closed annulus delimited by γ^+ and γ^-. Then there are a finite number of elements of $\mathcal{A}''$ meeting A_{γ^+,γ^-}.*

Proof. We know that the space of closed leaves of $\mathcal{F}$ included in A_{γ^+,γ^-} is compact and that every closed leaf of $\mathcal{F}$ does not meet its image under f (Properties (1) and (4) of §2.4). Hence, we can consider the real number $\epsilon > 0$ defined as

$$\epsilon := \min_{\gamma} \operatorname{dist}(\gamma,\ f(\gamma)),$$

where γ is a closed leaf of $\mathcal{F}$ contained in A_{γ^+,γ^-} and, for C, D which are two subsets of $\mathbb{R}^2$, $\operatorname{dist}(C, D) := \inf_{c\in C, d\in D} \|c - d\|$ and $\|\cdot\|$ is the Euclidean distance in $\mathbb{R}^2$. Let us prove that each $A \in \mathcal{A}''$ satisfies $\operatorname{dist}(\partial^+ A, \partial^- A) \geq \epsilon$. Indeed, if $f(A) \cap A \neq \emptyset$, then $f(\partial^+ A) \cap A \neq \emptyset$ or $f(\partial^- A) \cap A \neq \emptyset$. Suppose that the first case holds (the proof is analogous in the second case). Let $z \in \partial^+ A$ such that $f(z) \in A$. Hence, there exist real numbers $t^- \leq 1 \leq t^+$ such that $z^- = t^- f(z) \in \partial^- A$ and $z^+ = t^+ f(z) \in \partial^+ A$. Using the fact that $\partial^- A$ and $\partial^+ A$ are Euclidean circles centered at zero, we deduce that

$$\epsilon \leq \operatorname{dist}(f(\partial^+ A), \partial^+ A) \leq \|f(z) - z^+\| \leq \|z^- - z^+\| = \operatorname{dist}(\partial^- A, \partial^+ A).$$

Since the annulus A_{γ^+,γ^-} can be covered by a finite number of annuli A satisfying $\operatorname{dist}(\partial^+ A, \partial^- A) \geq \epsilon$, the proof follows. □

This completes the proof of Lemma 3.12. □

Proof of Proposition 3.11. The proof will be divided in two cases.

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

Case 1. The interior of γ does not contain any annulus A on which the foliation $\mathcal{F}$ goes from $\partial^- A$ to $\partial^+ A$.

We can find a small perturbation $\mathcal{F}'$ of $\mathcal{F}$ which 'breaks' all closed leaves of $\mathcal{F}$ in the interior of γ as follows. Let $\mathcal{C}$ be the union of all closed leaves of $\mathcal{F}$ included in the closure of the interior of γ. Let C be a connected component of $\mathcal{C}$. We write $A^+(C)$ and $A^-(C)$ for the two closed annuli in $\mathcal{A}$ whose boundary meets C and $A^-(C)$ included in the interior of $\partial^+ A^+(C)$. Recall that we are assuming that the foliation $\mathcal{F}$ goes from $\partial^+ A$ to $\partial^- A$ on all annuli $A \in \mathcal{A}'$ in the interior of γ. We will use the following lemma. Let us begin by recalling Whitney's topology and Proposition 2.1. Let $\psi : [0, 1]^2 \to \mathbb{R}^2 \backslash \{0\}$ be a compact chart, that is, an injective map, which maps each vertical line oriented upwards in $[0, 1]^2$ to a segment of a leaf of $\mathcal{F}$. By Proposition 2.1, there exists a real number $\epsilon > 0$, such that every oriented foliation $\mathcal{F}'$ belonging to the Whitney's open set $\mathcal{O}(\psi, \epsilon)$, that is, the set of foliations $\mathcal{F}'$ which admit a compact chart ψ' satisfying

$$\sup_{(x,y)\in[0,1]^2} \|\psi(x, y) - \psi'(x, y)\| < \epsilon,$$

is positively transverse to I.

LEMMA 3.14. *In this situation, there exists an oriented foliation $\mathcal{F}_C$ arbitrarily close to $\mathcal{F}$ (in Whitney's topology) which is positively transverse to I and 'breaks' all closed leaves of $\mathcal{F}$ in C. More precisely, $\mathcal{F}_C$ is positively transverse to I and coincides with $\mathcal{F}$ on the complement of the interior of the annulus $A_C := A^+(C) \cup C \cup A^-(C)$. Moreover, A_C is a closed annulus whose boundary components are $\partial^+ A_C = \partial^+ A^+(C)$ and $\partial^- A_C = \partial^- A^-(C)$ (two closed leaves of $\mathcal{F}_C$) and which does not contain any closed leaf of $\mathcal{F}_C$ in its interior, and $\mathcal{F}_C$ goes from $\partial^+ A_C$ to $\partial^- A_C$ on A_C.*

Proof. We will assume that the closed leaves in C are oriented clockwise. Let $\psi : [0, 1]^2 \to \mathbb{R}^2 \backslash \{0\}$ be a compact chart meeting C, that is, $\psi(\{0\} \times [0, 1])$ is contained in the interior of $A^+(C)$ and $\psi(\{1\} \times [0, 1])$ is contained in the interior of $A^-(C)$. Let $((x_n^+, y_n^+))_{n\in\mathbb{N}}$ be a sequence in $[0, 1]^2$ which is increasing in the first coordinate such that, for every integer $n \geq 1$, the point $z_n^+ = \psi(x_n^+, y_n^+)$ is in a leaf l^+ of $\mathcal{F}$ contained in the interior of $A^+(C)$ and z_{n+1}^+ belongs to the positive half-leaf containing z_n^+. Moreover, the sequence $((x_n^+, y_n^+))_{n\in\mathbb{N}}$ converges to (x_∞^+, y_∞^+) with $z^+ = \psi(x_\infty^+, y_\infty^+)$ in $\partial^- A^+(C)$, which is contained in the boundary of C. Let $((x_n^-, y_n^-))_{n\in\mathbb{N}}$ be a sequence in $[0, 1]^2$ which is decreasing in the first coordinate such that, for every integer $n \geq 1$, the point $z_n^- = \psi(x_n^-, y_n^-)$ is in a leaf l^- of $\mathcal{F}$ contained in the interior of $A^-(C)$ and z_{n+1}^- belongs to the negative half-leaf containing z_n^-. Moreover, the sequence $((x_n^-, y_n^-))_{n\in\mathbb{N}}$ converges to (x_∞^-, y_∞^-) with $z^- = \psi(x_\infty^-, y_\infty^-)$ (eventually $z^- = z^+$ if C is a closed leaf of $\mathcal{F}$) in $\partial^+ A^-(C)$, which is contained in the boundary of C. For every integer n, let $a_n^+ = \psi(x_n^+, 0)$, $a_n^- = \psi(x_n^-, 0)$, $b_n^+ = \psi(x_n^+, 1)$ and $b_n^- = \psi(x_n^-, 1)$. Let $a^- = \psi(x_\infty^-, 0)$ and $b^+ = \psi(x_\infty^+, 1)$. Now we perturb $\mathcal{F}$ as in Figure 3 to obtain the expected foliation. More precisely, for every integer $n \geq 1$, let $\varphi_n : [0, 1]^2 \to [0, 1]^2$ be an orientation-preserving homeomorphism of $[0, 1]^2$ satisfying:

- φ_n is the identity on $[0, 1] \times \{0\} \cup [0, x_{n-1}^+] \times [0, 1] \cup [x_{n-1}^-, 1] \times [0, 1]$;
- $\varphi_n(x_n^+, 1) = (x_\infty^+, 1)$;

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

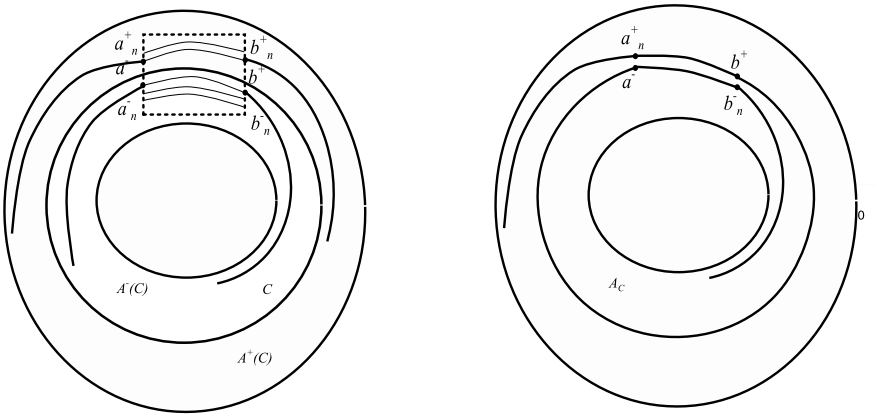


FIGURE 3. Left: foliation $\mathcal{F}$. Right: foliation $\mathcal{F}_C$.

- $\varphi_n(x_\infty^-, 1) = (x_n^-, 1)$;
- for every $x \in [x_\infty^+, x_\infty^-]$, if $\varphi_n(x, 1) = (x', 1)$, then $x < x'$; and
- $\sup_{(x,y)\in[0,1]^2} \|\varphi_n(x, y) - (x, y)\| \leq \min\{x_\infty^+ - x_n^+, x_n^- - x_\infty^-\}$.

Note that for every real number $\epsilon > 0$, there exists an integer n such that the foliation $\mathcal{F}_n$ which coincides with $\mathcal{F}$ on the complement of $\psi([0, 1]^2)$ and admits the compact chart $\psi' = \psi \circ \varphi_n$ belongs to $\mathcal{O}(\psi, \epsilon)$. By Proposition 2.1, for every large enough integer n, the foliation $\mathcal{F}_n$ is positively transverse to I and, by construction, it goes from $\partial^+ A_C$ to $\partial^- A_C$ on the annulus A_C. This completes the proof of the lemma. □

Remark 6. Let C be a connected component of $\mathcal{C}$. Suppose that γ^+ and γ^- are in C with γ^- contained in the interior of γ^+. As in the proof of the previous lemma, we can break all closed leaves of $\mathcal{F}$ in the interior of γ^+ that intersect C but not γ^+ by considering a compact chart $\psi : [0, 1]^2 \to \mathbb{R}^2\backslash\{0\}$ satisfying that $\psi(\{0\} \times [0, 1])$ is contained in γ^+ and $\psi(\{1\} \times [0, 1])$ is contained in the interior of $A^-(C)$. We can break all closed leaves of $\mathcal{F}$ in the interior of the open annulus delimited by γ^+ and γ^- but not γ^+ and γ^- by considering a compact chart $\psi : [0, 1]^2 \to \mathbb{R}^2\backslash\{0\}$ satisfying that $\psi(\{0\} \times [0, 1])$ is contained in γ^+ and $\psi(\{1\} \times [0, 1])$ is contained in γ^-. In these cases, A_C is delimited by γ^+ and $\partial^- A^-(C)$ and γ^+ and γ^-, respectively.

Construction of the foliation $\mathcal{F}'$. The foliation $\mathcal{F}'$ is constructed as the limit of a family of foliations $(\mathcal{F}'_\alpha)_\alpha$, which is stationary on every compact set of $\mathbb{R}^2\backslash\{0\}$. Firstly, suppose that the foliation induced by $\mathcal{F}$ in a neighborhood U of zero is the foliation by closed leaves. Let $(\gamma_n)_{n\in\mathbb{N}}$ be an ordered sequence in U of closed leaves that accumulate at zero (eventually $\gamma = \gamma_0$ if U is the interior of γ). For every integer $n \geq 1$, we can break all closed leaves of $\mathcal{F}$ in the interior of the open annulus delimited by γ_{2n} and γ_{2n+1} but not γ_{2n} and γ_{2n+1}. Next, let C be the connected component of $\mathcal{C}$ containing γ. We can break all closed leaves of $\mathcal{F}$ in the interior of γ that intersect C but not γ. Hence the connected components of $\mathcal{C}$ accumulate at zero. Let $(C_\alpha)_\alpha$ denote the ordered family of all such components, that is, $C_\alpha \prec C_{\alpha'}$ if C_α is contained in the unbounded connected

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

component of $\mathbb{R}^2\backslash C_{\alpha'}$. By the previous lemma, for every $\alpha \in A$, we obtain a foliation $\mathcal{F}_\alpha$, which coincides with the foliation $\mathcal{F}$ on the union of C_α, the bounded connected component of $\mathbb{R}^2\backslash C_\alpha$ and the complement of the interior of γ, and which goes from γ to $\partial^- A^+(C_\alpha)$. Iterating this process, we can construct the family $(\mathcal{F}_\alpha)_\alpha$. Let $\mathcal{F}'$ be the limit of the sequence $(\mathcal{F}_\alpha)_\alpha$. By construction, $\mathcal{F}'$ belongs to the Whitney open set $\mathcal{O}((\psi_\alpha), (\epsilon_\alpha))$, where ψ_α is the compact chart and ϵ_α is the positive real number provided by the previous lemma. From Proposition 2.1, the foliation $\mathcal{F}'$ is also transverse to I and, by construction, all leaves of $\mathcal{F}'$ contained is the interior of γ are spirals whose ω-limit set is $\{0\}$. This completes the proof in Case 1.

Case 2. In the other case, every annulus A in the interior of γ on which $\mathcal{F}$ goes from $\partial^- A$ to $\partial^+ A$ have maximal invariant set empty (we are assuming this). We will construct an intermediary foliation $\mathcal{F}_\infty$ transverse to the isotopy I, as in Case 1 above. The construction has three steps.

1st step. We can replace $\mathcal{F}$ on each annulus $A \in \mathcal{A}\backslash\mathcal{A}''$ (that is, an annulus that does not meet its image under f), by the foliation in Euclidean circles centered at zero. We remark that this foliation is also transverse to I.

2nd step. Recall that, by hypothesis, every annulus A (included in the interior of γ) on which the foliation $\mathcal{F}$ goes from $\partial^- A$ to $\partial^+ A$ has maximal invariant set empty. In this case, we will use the following lemma.

LEMMA 3.15. (Le Roux, [**LeR13**]) *Suppose that the maximal invariant set of the annulus $A \in \mathcal{A}$ is empty. Then there exists an annulus $\hat{A}$ containing A whose boundary is the union of $\partial^+ A$ and $f^n(\partial^- A)$ for some integer n and a foliation $\hat{\mathcal{F}}$ whose leaves $\hat{\gamma}$ in $\hat{A}$ are closed and such that $f(\hat{\gamma})$ is situated on the right of the oriented leaf $\hat{\gamma}$.*

Sketch of the proof. For two disjoint Jordan curves α and β, we write $\alpha < \beta$ if β is contained in the interior of α. Since the maximal invariant set of A is empty, we can suppose that there exists an integer n such that $\partial^- A < f^n(\partial^+ A)$. Since the closed leaves $\partial^+ A$ and $\partial^- A$ are f-free, they project to two simple curves on the quotient torus

$$\left(\bigcup_{n\in\mathbb{Z}} f^n(A_{\partial^+ A, f(\partial^+ A)})\right) \Big/ f,$$

where $A_{\partial^+ A, f(\partial^+ A)}$ is the annulus delimited by the curves $\partial^+ A$ and $f(\partial^+ A)$. The curves $\partial^+ A$ and $\partial^- A$ are homotopic, so we can find a family of curves $(\alpha_t)_{t\in[0,1]}$ with $\alpha_0 = \partial^+ A$ and $\alpha_1 = \partial^- A$ lifting this isotopy. For every $t \in [0, 1]$, choose a curve β_t such that

$$\alpha_t < \beta_t < f(\alpha_t).$$

In particular, for every $t \in [0, 1]$ the annulus delimited by the curves α_t and β_t is f-free. By compactness, we can find real numbers $0 = t_0 < t_1 < \cdots < t_n = 1$ such that, for every $i \in \{0, \ldots, n-1\}$,

$$\alpha_{t_{i+1}} < \beta_{t_i} < f(\alpha_{t_{i+1}}),$$

and, in particular, the annulus delimited by the curves β_{t_i} and $f(\alpha_{t_{i+1}})$ is f-free. For $i \in \{0, \ldots, n\}$, we write $\gamma_i = f^i(\alpha_{t_i})$ and $\delta_i = f^i(\beta_{t_i})$.

$$\partial^+ A = \gamma_0 < \delta_0 < \cdots < \delta_{n-1} < \gamma_n = f^n(\partial^- A).$$

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

Moreover, every two consecutive curves of this sequence delimit a f-free annulus since, in the first step, we can replace the foliation $\mathcal{F}$ on each of these annuli by the foliation in closed leaves. These foliations can be pasted to obtain a foliation $\hat{\mathcal{F}}$ whose leaves in the annulus $\hat{A}$ are closed. This completes the proof. □

By the previous lemma, we obtain a foliation in closed leaves on the annulus $\hat{A}$ delimited by ∂^+A and $f^n(\partial^-A)$. This foliation can be pasted with the foliation $\mathcal{F}$ on the exterior of ∂^+A and with the foliation $(f^n)_*\mathcal{F}$ in the interior of $f^n(\partial^-A)$ to obtain a new foliation $\mathcal{F}_{A,n}$ that is transverse to the isotopy I.

3rd step. Construction of the foliation $\mathcal{F}_\infty$. The intermediary foliation $\mathcal{F}_\infty$ is constructed as the limit of a sequence of foliations $(\mathcal{F}_n)_{n\in\mathbb{N}}$ that is stationary on every compact set of $\mathbb{R}^2\backslash\{0\}$. By the first step and the fact that $\mathcal{A}'$ is locally finite (Lemma 3.12), the set of annuli A on which $\mathcal{F}$ goes from ∂^-A to ∂^+A accumulate only at zero and is a countable set. Let $(A_n)_{n\in\mathbb{N}}$ denote the ordered family of all such annuli: that is, for every integer $n \geq 1$, the annulus A_{n-1} is contained in the unbounded connected component of $\mathbb{R}^2\backslash A_n$. By hypothesis the maximal invariant set of A_0 is empty so, by the second step, we obtain a foliation $\mathcal{F}_0 = \mathcal{F}_{A_0,m_0}$ which coincides with the foliation in closed leaves on $\hat{A}_0$ (annulus delimited by ∂^+A_0 and $f^{m_0}(\partial^-A_0)$ for some integer m_0), with $\mathcal{F}$ on the exterior of ∂^+A_0, and with the foliation $(f^{m_0})_*\mathcal{F}$ in the interior of $f^{m_0}(\partial^-A)$. In a similar way, we obtain a foliation $\mathcal{F}_1 := (\mathcal{F}_0)_{f^{m_0}(A_1),m_1}$. Iterating this process, we can construct the sequence $(\mathcal{F}_n)_{n\in\mathbb{N}}$. Let $\mathcal{F}_\infty$ be the limit of the sequence $(\mathcal{F}_n)_{n\in\mathbb{N}}$. Since, by construction, each leaf l_∞ of $\mathcal{F}_\infty$ is an iterate of one leaf l of either $\mathcal{F}$ or the foliation $\hat{\mathcal{F}}$ provided by the previous lemma, (that is, $l_\infty = f^n(l)$), the remark following Theorem 2.2 implies that the limit foliation obtained, $\mathcal{F}_\infty$, is also transverse to I. Note that the foliation $\mathcal{F}_\infty$ is as in Case 1. This completes the proof. □

End of the proof of Proposition 3.3. Suppose that Case (a) does not hold. Let $\mathcal{F}$ be an oriented foliation transverse to the isotopy I. By Lemma 3.5, the closed leaves of $\mathcal{F}$ accumulate at zero so f does not satisfy the local intersection property. Let V be a neighborhood of zero. By Lemma 3.10, both negative and positive annuli accumulate at zero. Hence there is a negative annulus A^- and a positive annulus A^+ included in V. The next lemma implies that Case (b) of the proposition holds. This completes the proof of Proposition 3.3. □

LEMMA 3.16. *Let $\mathcal{F}$ be an oriented foliation transverse to the isotopy I. Suppose that there exists both negative and positive annuli. Then there exist three Jordan curves as in Case (b) of Proposition 3.3. In other words, the hypothesis of Theorem C is valid.*

Proof. Let A^+ and A^- be a positive and a negative annulus, respectively. Let γ^+ and γ^- be two closed leaves of $\mathcal{F}$ such that A^+ and A^- are contained in the closed annulus A_{γ^+,γ^-} delimited by γ^+ and γ^-. We know by Lemma 3.13 that there exists a finite number of elements of $\mathcal{A}'$ meeting A_{γ^+,γ^-}. Hence there is a positive annulus A_0^+ and a negative annulus A_1^- such that the annulus A (eventually an empty set) 'between' A_0^+ and A_1^- has maximal invariant set empty. Thus the curves $\partial^+A_0^+$, $\partial^+A_1^-$ and $\partial^-A_1^-$ satisfy Case (b) of Proposition 3.3 because $\rho_{A\cup A_1^-}(I) = \rho_{A_1^-}(I)$. This completes the proof of the lemma. □

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

4. *Case of the open annulus*

In this section, we will introduce the definition of the rotation set in the open annulus and we will show Theorem B assuming Theorem C.

4.1. *Definitions.*

4.1.1. *The open annulus.* We will denote by $\mathbb{T}^1$ the unit circle of the plane and by $\mathbb{A} := \mathbb{T}^1 \times \mathbb{R}$ the open annulus. We endow the annulus $\mathbb{A}$ with its usual topology and orientation. We denote by N (respectively, S) the upper (respectively, lower) end of $\mathbb{A}$ and by $\hat{\mathbb{A}}$ the end compactification of $\mathbb{A}$, which is a topological space homeomorphic to the two-dimensional sphere. A Jordan curve γ in $\mathbb{A}$ is called *essential* if its complement has two unbounded connected components. We denote by U_γ^N the upper one and by U_γ^S the lower one. We denote $\hat{U}_\gamma^N = U_\gamma^N \cup \{N\}$ and $\hat{U}_\gamma^S = U_\gamma^S \cup \{S\}$.

4.1.2. *The set* $\mathrm{Homeo}_0(\mathbb{A})$. We denote by $\mathrm{Homeo}_0(\mathbb{A})$ the set of all homeomorphisms of the open annulus $\mathbb{A}$ which are isotopic to the identity. We recall that every homeomorphism f in $\mathrm{Homeo}_0(\mathbb{A})$ can be extended to a homeomorphism $\hat{f}$ of the end compactification of $\mathbb{A}$ and this homeomorphism fixes both ends of $\mathbb{A}$.

4.1.3. *The rotation set of the annulus.* In this paragraph, we consider a homeomorphism f in $\mathrm{Homeo}_0(\mathbb{A})$. Let I be an isotopy in $\mathrm{Homeo}_0(\mathbb{A})$ from the identity to f. We will give the definition of the *rotation set of* I due to Franks (see [**Fra96**]).

If $z \in \mathbb{A}$, we consider $\rho_n(z)$ the *average change of angular coordinate along the trajectory of* z *for the isotopy* I. Next, for a compact set K in $\mathbb{A}$, we define

$$\rho_K(I) := \bigcap_{m\geq 1} \mathrm{Cl}\left(\bigcup_{n\geq m} \{\rho_n(z) : z \in K,\ f^n(z) \in K\} \right),$$

where the closure is taken in $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$.

Finally, we define the rotation set of the isotopy I by

$$\rho_{\mathrm{ann}}(I) := \mathrm{Cl}\left(\bigcup_K \rho_K(I) \right),$$

where K is a compact set in $\mathbb{A}$.

Remark 7. The rotation set $\rho_{\mathrm{ann}}(I)$ only depends on the choice of the homotopy class of I.

As above, the rotation set verifies the following properties.

PROPOSITION 4.1. *Let I be an isotopy in* $\mathrm{Homeo}_0(\mathbb{A})$ *from the identity to f. For every integer p, let $J^p = (R_{pt})_{t\in[0,1]}$. Then:*

(1) *the rotation set of I is invariant by oriented topological conjugacy; and*
(2) *for every $p, q \in \mathbb{Z}$, $\rho_{\mathrm{ann}}(J^p * I^q) = q\ \rho_{\mathrm{ann}}(I) + p$.*

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

4.1.4. *Other rotation sets in the open annulus.* In this paragraph, we introduce other rotation sets in $\mathbb{A}$ such as the *measured rotation set* of an f-invariant compact subset of $\mathbb{A}$ and also *the local rotation set relative to a neighborhood* at each end of $\mathbb{A}$.

Given a compact and f-invariant set K in $\mathbb{A}$, we will denote by $\mathcal{M}_f(K)$ the set of all f-invariant Borel probability measures whose support is contained in K. This is a compact and convex set in the weak topology. If $\mu \in \mathcal{M}_f(K)$, we define its *rotation number* as

$$\rho(\mu) := \int_K \rho_1 \, d\mu,$$

where ρ_1 is the displacement function, that is, $\rho_1(z) := p_1(\widetilde{f}(\widetilde{z})) - p_1(\widetilde{z})$, where $\widetilde{f}$ is a lift of f and $\widetilde{z}$ is a point in $\pi^{-1}(z)$. Then we define the *measured rotation set of* K as

$$\rho_{\mathrm{mes}}(K, I) := \{\rho(\mu) : \mu \in \mathcal{M}_f(K)\}.$$

This set is always a compact interval. More precisely, we have the following result.

LEMMA 4.2. *Let K be a compact and f-invariant set in $\mathbb{A}$. Then:*

(1) *the set $\rho_K(I)$ is a compact set in $\mathbb{R}$; if K is a connected set, then $\rho_K(I)$ is a compact interval; and*
(2) *the set $\rho_{\mathrm{mes}}(K, I)$ coincides with the convex hull of $\rho_K(I)$.*

On the other hand, let $\hat{z}_0$ be an end of $\mathbb{A}$ and let $\hat{V}$ be a neighborhood of $\hat{z}_0$. We consider an orientation-preserving homeomorphism ψ mapping $\hat{V}$ to a neighborhood of $0 \in \mathbb{R}^2$ such that $\psi(\hat{z}_0) = 0$. We write

$$\rho_{\hat{V}}(I) := (-1)^{i(\hat{z}_0)} \rho_{\psi(\hat{V})}(\psi I \psi^{-1}),$$

where $i(N) = -1$, $i(S) = +1$ and $\rho_{\psi(\hat{V})}(\psi I \psi^{-1})$ is the local rotation set relative to $\psi(\hat{V})$ defined in §3.1.3.

Remark 8. By Proposition 3.1, this definition does not depend on the choice of the homeomorphism ψ.

4.2. *Main result: proof of Theorem B.* The purpose of this subsection is to prove Theorem B, assuming Theorem C.

As we have seen in the first paragraph of §3.2, by density of $\mathbb{Q}$ in $\mathbb{R}$ and replacing the isotopy I by $I^p * J^q$, in order to proof Theorem B, it suffices to prove the following theorem.

THEOREM B*. *Let I be an isotopy in* $\mathrm{Homeo}_0(\mathbb{A})$ *from the identity to a homeomorphism f. Suppose that the rotation set $\rho_{\mathrm{ann}}(I)$ contains both positive and negative real numbers. Then there exists a compact (a priori non-invariant) set K in $\mathbb{A}$ such that zero belongs to $\rho_K(I)$.*

The following example satisfies Property (3) but not Property (2) from the analogous Remark 4 for the open annulus. This proves that Theorem B* is sharp: that is, the compact set K such that zero belongs to $\rho_K(I)$ can be non-invariant.

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

Example. There exist a homeomorphism f of $\mathbb{A}$ and an isotopy I from the identity to f satisfying the following conditions.

(i) There exists no f-invariant compact set K in $\mathbb{A}$ such that $0 \in \rho_K(I)$.

(ii) $\{-1, 1\} \subset \rho_{\text{ann}}(I)$.

Let us fix two real numbers $0 < R^- < R^+$. To construct f, we start with a homeomorphism f_1' defined on $\mathbb{R}^2 \cup \{\infty\}$, the one-point compactification of $\mathbb{R}^2$, satisfying:

(i) f_1' acts as a rotation R_{+1} (respectively, R_{-1}) on the disk $\{z \in \mathbb{R}^2 : \|z\| \le R^-\}$ (respectively, $\{z \in \mathbb{R}^2 : \|z\| \ge R^+\}$); and

(ii) f_1' is a Reeb's homeomorphism on $\{z \in \mathbb{R}^2 : R^- \le \|z\| \le R^+\}$.

Let us define the following equivalence relation on $\mathbb{R}^2 \cup \{\infty\}$.

$$z = (x, y) \sim \begin{cases} 0 & \text{if } \|z\| \le R^- \text{ and } y \le 0, \\ (0, y) & \text{if } \|z\| \le R^- \text{ and } y \ge 0. \end{cases}$$

$$z = (x, y) \sim \begin{cases} \infty & \text{if } \|z\| \ge R^+ \text{ and } y \le 0, \\ (0, y) & \text{if } \|z\| \ge R^+ \text{ and } y \ge 0. \end{cases}$$

Let $\hat{\mathbb{A}}' = \mathbb{R}^2 \cup \{\infty\}/\sim$ be the quotient space of $\mathbb{R}^2 \cup \{\infty\}$. For $z \in \mathbb{R}^2 \cup \{\infty\}$, we write $[z]$ for the equivalence class of z. Now, it is easy to check that $\mathbb{A}' = \hat{\mathbb{A}}' \setminus \{[0], [\infty]\}$ is homeomorphic to the open annulus $\mathbb{A}$ and that f_1' induces a homeomorphism f on $\mathbb{A}'$. Moreover, f is isotopic to the identity on $\mathbb{A}'$ and there exists an isotopy I on the annulus $\mathbb{A}'$ from the identity to f such that $\{-1, 1\} \subset \rho_{\text{ann}}(I)$.

4.2.1. *Classification.* The proof of Theorem B* considers several cases described in the following proposition.

PROPOSITION 4.3. *Let I be an isotopy in* $\text{Homeo}_0(\mathbb{A})$ *from the identity to a homeomorphism f. Suppose that the rotation set $\rho_{\text{ann}}(I)$ contains both positive and negative real numbers. Then one of the following holds.*

Case (a). There exists a compact invariant set K in $\mathbb{A}$ such that zero belongs to $\rho_K(I)$.

Case (b). The homeomorphism f satisfies the local intersection property at least at one of the ends of $\mathbb{A}$. There exists an essential f-free Jordan curve γ in $\mathbb{A}$, such that the local rotation set of I at N relative to $\hat{U}_\gamma^N$, $\rho_{\hat{U}_\gamma^N}(I)$, is contained in one of the sets $[0, +\infty]$ *or* $[-\infty, 0]$ *and the local rotation set at S relative to $\hat{U}_\gamma^S$, $\rho_{\hat{U}_\gamma^S}(I)$, is contained in the other one.*

Case (c). The homeomorphism f satisfies the local intersection property at both ends of $\mathbb{A}$. There exist two f-free essential Jordan curves γ_N and γ_S with $\gamma_N \subset U_{\gamma_S}^N$ such that the closed annulus A delimited by γ_N and γ_S satisfies:

(i) *the maximal f-invariant set of A is not empty; and*

(ii) *the sets $\rho_{\hat{U}_{\gamma_N}^N}(I)$ and $\rho_{\hat{U}_{\gamma_S}^S}(I)$ are contained in one of the sets* $[0, +\infty]$ *or* $[-\infty, 0]$ *and the set $\rho_A(I)$ is contained in the interior of the other one.*

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

We prove this proposition assuming that Case (a) does not hold. Then we may assume that I has no contractible fixed points. We apply Theorem 2.2 to obtain an oriented foliation on the annulus $\mathbb{A}$ which is transverse to I. We can see $\mathcal{F}$ as an oriented singular foliation on $\hat{\mathbb{A}}$, the end compactification of $\mathbb{A}$, whose singularities are the ends of $\mathbb{A}$.

LEMMA 4.4. *In this situation, there exists an oriented foliation $\mathcal{F}'$ arbitrarily close to $\mathcal{F}$ (in Whitney's topology) which has a closed leaf.*

Proof. We claim that $\mathcal{F}$ would have recurrence (see Property (4) of §2.4). Otherwise, by Property (4) of §2.4, there exists a leaf of $\mathcal{F}$ joining the two ends of $\mathbb{A}$, which implies that either $\rho_{\text{ann}}(I) \subset [-\infty, 0]$ or $\rho_{\text{ann}}(I) \subset [0, +\infty]$. This contradicts our hypotheses. Hence, changing the orientation of $\mathcal{F}$ by its inverse, if necessary, we may suppose that there is a leaf l of $\mathcal{F}$ whose ω-limit set contains a point $z \in \mathbb{A}$. Let l_z be the leaf of $\mathcal{F}$ containing z. Note that the ω-limit and α-limit sets of l_z are reduced to S or N. Without loss of generality, we may assume that this end is S. Let $\psi : [0, 1]^2 \to \mathbb{A}$ be a compact chart at z and let $(x_\infty, y_\infty) \in [0, 1]^2$ be such that $\psi(x_\infty, y_\infty) = z$. Let $((x_n, y_n))_{n\in\mathbb{N}}$ be a sequence in $[0, 1]^2$ such that, for every integer $n \geq 1$, the point $z_n = \psi(x_n, y_n)$ is in $l \cap \psi([0, 1]^2)$, the point z_{n+1} belongs to the positive half-leaf containing z_n, the connected components of $l \cap \psi([0, 1]^2)$ containing z_n and z_{n+1} are consecutive and the sequence $(z_n)_{n\in\mathbb{N}}$ converges to z. For every integer $n \geq 1$, let $a_n = \psi(x_n, 0)$ and $b_n = \psi(x_n, 1)$. Now we perturb $\mathcal{F}$ as in Figure 4. More precisely, for every integer $n \geq 1$, let $\varphi_n : [0, 1]^2 \to [0, 1]^2$ be an orientation-preserving homeomorphism of $[0, 1]^2$ satisfying:

- $\varphi_n(x_{n+1}, 0) = (x_{n+1}, 0)$;
- $\varphi_n(x_{n+1}, 1) = (x_n, 1)$; and
- $\sup_{(x,y)\in[0,1]^2} \|\varphi_n(x, y) - (x, y)\| \leq |x_n - x_{n+1}|$.

Note that for every real number $\epsilon > 0$, there exists an integer n such that the foliation $\mathcal{F}_n$ that coincides with $\mathcal{F}$ on the complement of $\psi([0, 1]^2)$ and admits the compact chart $\psi' = \psi \circ \varphi_n$ belongs to $\mathcal{O}(\psi, \epsilon)$, that is, the set of all foliations $\mathcal{F}'$ that admits a compact chart ψ' satisfying

$$\sup_{(x,y)\in[0,1]^2} \|\psi(x, y) - \psi'(x, y)\| < \epsilon.$$

By Proposition 2.1, for every large enough integer n, the foliation $\mathcal{F}_n$ is positively transverse to I and, by construction, it admits a closed leaf. This completes the proof of the lemma. □

From Proposition 2.1, the foliation $\mathcal{F}'$ is also transverse to I. Let us assume from now on that $\mathcal{F}$ has closed leaves. Moreover, $\mathcal{F}$ cannot have only closed leaves. Otherwise f is conjugated to a homothety.

LEMMA 4.5. *Under the hypotheses of Proposition 4.3, assume, furthermore, that Case (a) does not hold. Then the closed leaves of $\mathcal{F}$ do not accumulate at least at one of the two ends of $\mathbb{A}$.*

Since the closed leaves of $\mathcal{F}$ are f-free essential curves (Properties (2) and (3) from §2.4), this lemma will follow from Theorem C. As in §3.3, we consider the set $\mathcal{A}$ of all closed annuli A whose boundary components are closed leaves of $\mathcal{F}$ and which do not

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

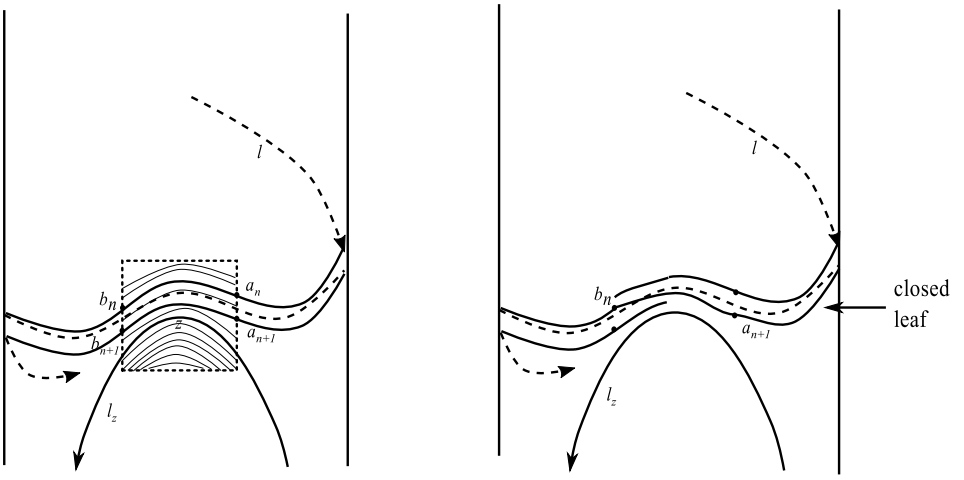


FIGURE 4. Left: foliation $\mathcal{F}$. Right: foliation $\mathcal{F}'$.

contain any closed leaf of $\mathcal{F}$ in their interior. For every annulus $A \in \mathcal{A}$, we write $\partial^+ A$ and $\partial^- A$ for its two boundary components with $\partial^- A$ included in $U^S_{\partial^+ A}$. Let $\mathcal{A}'$ be the subset of $\mathcal{A}$ consisting of the annuli A whose maximal invariant set is not empty. We say that $A \in \mathcal{A}'$ is a positive (respectively, negative) annulus if $\rho_{\Theta(A)}(I) \subset (0, +\infty)$ (respectively, $\rho_{\Theta(A)}(I) \subset (-\infty, 0)$) (see Lemma 3.8).

Proof of Lemma 4.5. By contradiction, we suppose that the closed leaves of $\mathcal{F}$ accumulate at both ends of $\mathbb{A}$. We claim that there are both positive and negative annuli. By contradiction, suppose that all annuli of $\mathcal{A}'$ are negative (or positive). Then the next proposition provides an oriented foliation $\mathcal{F}'$ which is also transverse to I and such that each leaf joins the two ends of $\mathbb{A}$: that is, its α-limit set is reduced to N and the ω-limit set is reduced to S. This contradicts the fact that the rotation set $\rho_{\text{ann}}(I)$ contains both positive and negative real numbers. If there are both positive and negative annuli in $\mathbb{A}$, then Lemma 3.16 provides two annuli A_0 and A_1 which satisfy the hypothesis of Theorem C. We conclude (assuming Theorem C) that zero belongs to $\rho_{\Theta(A)}(I)$, where $\Theta(A)$ is the maximal invariant set of the closed annulus $A = A_0 \cup A_1$. This contradicts the fact that Case (a) of Proposition 4.3 does not hold. This completes the proof. □

Let us state the global version of Proposition 3.11.

PROPOSITION 4.6. *Let $\mathcal{F}$ be an oriented singular foliation whose singularities are the ends of $\mathbb{A}$ transverse to an isotopy I. Suppose that the closed leaves of $\mathcal{F}$ accumulate at both ends of $\mathbb{A}$. Suppose that all annuli $A' \in \mathcal{A}'$ are negative. Then there exists an oriented foliation $\mathcal{F}'$ transverse to I such that each leaf of $\mathcal{F}'$ joins the two ends of $\mathbb{A}$: that is, its α-limit set is $\{N\}$ and its ω-limit set is $\{S\}$.*

Proof. The proof follows as in Proposition 3.11. We will divide the proof into parts. It will be useful for other purposes. Let us fix two closed leaves, γ_S and γ_N, with γ_N contained in $U^N_{\gamma_S}$. By Proposition 3.11 (applied at both ends of $\mathbb{A}$), we can find an oriented foliation

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

$\mathcal{F}''$ transverse to I and such that all leaves of $\mathcal{F}''$ that are contained in $U^S_{\gamma_S}$ (respectively, $U^N_{\gamma_N}$) are spirals whose ω-limit set is $\{S\}$ (respectively, α-limit set is $\{N\}$). In particular, the foliation $\mathcal{F}''$ satisfies the following properties.

(i) All annuli $A \in \mathcal{A}'$ contained in A_{γ_N,γ_S}, the closed annulus delimited by γ_N and γ_S, are negative.
(ii) There is a leaf l_S contained in $U^S_{\gamma_S}$ whose α-limit set is γ_S and ω-limit set is $\{S\}$.
(iii) There is a leaf l_N contained in $U^N_{\gamma_N}$ whose α-limit set is $\{N\}$ and ω-limit set is γ_N.

Now, to obtain the foliation $\mathcal{F}'$, we will use the following lemma.

LEMMA 4.7. *Let $\mathcal{F}''$ be an oriented foliation transverse to an isotopy I. Suppose that $\mathcal{F}''$ satisfies either $\mathcal{A}'$ is empty or the above Property (i).*

(a) *If $\mathcal{F}''$ satisfies furthermore Property (ii), then there exists a foliation $\mathcal{F}'$ transverse to I which admits a leaf whose α-limit set is γ_N and whose ω-limit set is $\{S\}$.*
(b) *If $\mathcal{F}''$ satisfies, furthermore, Properties (ii) and (iii), then there exists a foliation $\mathcal{F}'$ transverse to I which admits a leaf joining the two ends of $\mathbb{A}$.*

Sketch of the proof. The proof follows as in the proof of Proposition 3.11.

Let us prove (a). The proof will be divided in two cases.

Case 1. The annulus A_{γ_N,γ_S} does not contain any annulus A on which the foliation $\mathcal{F}''$ goes from $\partial^- A$ to $\partial^+ A$.

In this case, using Lemma 3.14 and the remark following Lemma 3.14, we can find the foliation $\mathcal{F}'$ 'breaking' all closed leaves of $\mathcal{F}''$ in A_{γ_N,γ_S} but not γ_N.

Case 2. In the other case, every annulus $A \in \mathcal{A}$ has maximal invariant set empty or every annulus $A \in \mathcal{A}'$ on which the foliation $\mathcal{F}''$ goes from $\partial^- A$ to $\partial^+ A$ has maximal invariant set empty (we are assuming this). In this case, we will construct an intermediary foliation $\mathcal{F}''_\infty$ transverse to the isotopy I, as in Case 1 above. The construction has three steps.

1st step. We can replace $\mathcal{F}''$ on each annulus $A \in \mathcal{A}\backslash\mathcal{A}''$ (that is, an annulus that does not meet its image under f) by the foliation in closed leaves. We remark that this foliation is also transverse to I.

2nd step. Recall that, by hypothesis, every annulus $A \in \mathcal{A}$ has maximal invariant set empty or every annulus $A \in \mathcal{A}'$ on which the foliation $\mathcal{F}''$ goes from $\partial^- A$ to $\partial^+ A$ has maximal invariant set empty. Hence, by Lemma 3.15 (applied to an annulus A in $\mathcal{A}$ whose maximal invariant set is empty), there exist an annulus $\hat{A}$ containing A whose boundary components are the union of $\partial^+ A$ and $f^n(\partial^- A)$ for some integer n and a foliation $\hat{\mathcal{F}}$ whose leaves in $\hat{A}$ are closed leaves. This foliation can be pasted with $\mathcal{F}''$ on $U^N_{\partial^+ A}$ and with $(f^n)_*\mathcal{F}''$ on $U^S_{f^n(\partial^- A)}$ to obtain a new foliation $\mathcal{F}_{A,n}$ that is transverse to the isotopy I.

3rd step. Construction of the foliation $\mathcal{F}''_\infty$. The intermediary foliation $\mathcal{F}''_\infty$ is constructed as the limit of a finite sequence of foliations $(\mathcal{F}''_i)_{i=0}^m$, which is stationary on every compact set of $\mathbb{A}$. By the first step and the fact that there are a number finite of elements of $\mathcal{A}'$ meeting the annulus A_{γ_N,γ_S} (Lemma 3.13), the set of annuli A in A_{γ_N,γ_S} on which $\mathcal{F}$ goes from $\partial^- A$ to $\partial^+ A$ is a finite set. Let $(A_i)_{i=0}^m$ denote the ordered family of all such annuli: that is, for every integer $i \in \{1, \ldots, m\}$, the annulus A_{i-1} is contained in the upper connected

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

component of $\mathbb{A}\backslash A_i$. By hypothesis, the maximal invariant set of A_0 is empty so, by the second step, we obtain a foliation $\mathcal{F}_0 = \mathcal{F}_{A_0,n_0}$ which coincides with the foliation in closed leaves on $\hat{A}_0$ (annulus delimited by $\partial^+ A_0$ and $f^{n_0}(\partial^- A_0)$ for some integer n_0), with $\mathcal{F}''$ on $U^N_{\partial^+ A_0}$ and with the foliation $(f^{n_0})_*\mathcal{F}''$ on $U^S_{f^{n_0}(\partial^- A_0)}$. In a similar way, we obtain a foliation $\mathcal{F}_1 := (\mathcal{F}_0)_{f^{n_0}(A_1),n_1}$. Iterating this process, we can construct the finite sequence $(\mathcal{F}''_i)_{i=0}^m$. Since, by construction, each leaf l_m of $\mathcal{F}''_m$ is an iterate of one leaf l of either $\mathcal{F}''$ or the foliation $\hat{\mathcal{F}}''$ provided by the Lemma 3.15 (that is, $l_m = f^n(l)$), the remark following Theorem 2.2 implies that the foliation obtained, $\mathcal{F}''_m$, is also transverse to I. Note that the foliation $\mathcal{F}''_m$ is as in Case 1. This completes the proof of Assertion (a).

Let us prove (b). From Assertion (a), we obtain an oriented foliation that admits a closed leaf γ_N and two leaves l_N and l_S contained in $U^N_{\gamma_N}$ and $U^S_{\gamma_N}$, respectively, whose α-limit set is $\{N\}$ and ω-limit set is γ_N and whose α-limit set is γ_N and ω-limit set is $\{S\}$, respectively. Now, we can find the foliation $\mathcal{F}'$ 'breaking' the closed leaf γ_N. □

This completes the proof of Proposition 4.6. □

Let us assume from now on, without loss of generality, that the closed leaves of $\mathcal{F}$ do not accumulate at S. Let γ_S be 'the last' closed leaf of $\mathcal{F}$: that is, γ_S is a closed leaf of $\mathcal{F}$ such that $U^S_{\gamma_S}$ does not contain any closed leaf of $\mathcal{F}$. We have the following result.

LEMMA 4.8. *There exists an oriented foliation $\mathcal{F}'$ arbitrarily close to $\mathcal{F}$ (in Whitney's topology) that coincides with $\mathcal{F}$ in $\mathrm{Cl}(\hat{U}^N_{\gamma_S})$ and admits a leaf either whose ω-limit set is reduced to S and α-limit set is γ_S or whose α-limit set is reduced to S and ω-limit set is γ_S.*

Proof. Let z' be an element in γ_S and let Σ be an arc positively transverse containing z' in its interior. Since γ_S is a closed leaf of $\mathcal{F}$ and there is no closed leaf in $U^S_{\gamma_S}$, there is a leaf l of $\mathcal{F}$ which meets $\Sigma \cap U^S_{\gamma_S}$ at least two times. From Theorem 2.5, we deduce that either $\alpha(l)$ or $\omega(l)$ is γ_S. Without loss of generality, suppose that we are in the first case. Since there are not closed leaves of $\mathcal{F}$ in $U^S_{\gamma_S}$, we have two possibilities (items (ii) or (iii) of Theorem 2.5): the set $\omega(l)$ is reduced to S, or there is a leaf l_S of $\mathcal{F}$ from S to S contained in $\omega(l)$. In the first case, the foliation $\mathcal{F}$ satisfies the conclusion of the lemma. In the second case, let z be an element in l_S. Let $\psi : [0, 1]^2 \to \mathbb{A}$ be a compact chart at z and let $(x_\infty, y_\infty) \in [0, 1]^2$ be such that $\psi(x_\infty, y_\infty) = z$. Let $((x_n, y_n))_{n\in\mathbb{N}}$ be a sequence in $[0, 1]^2$ such that, for every integer $n \geq 1$, the point $z_n = \psi(x_n, y_n)$ is in $l \cap \psi([0, 1]^2)$, the point z_{n+1} belongs to the positive half-leaf containing z_n and the sequence $(z_n)_{n\in\mathbb{N}}$ converges to z. For every integer $n \geq 1$, let $a_n = \psi(x_n, 0)$ and $b_n = \psi(x_n, 1)$ and let $b = \psi(x_\infty, 1)$. Now we perturb $\mathcal{F}$ as in Figure 5 to obtain the expected foliation. More precisely, for every integer $n \geq 1$, let $\varphi_n : [0, 1]^2 \to [0, 1]^2$ be an orientation-preserving homeomorphism of $[0, 1]^2$ satisfying:

- $\varphi_n(x_n, 0) = (x_n, 0)$;
- $\varphi_n(x_n, 1) = (x_\infty, 1)$; and
- $\sup_{(x,y)\in[0,1]^2} \|\varphi_n(x, y) - (x, y)\| \leq |x_n - x_\infty|$.

Note that, for every real number $\epsilon > 0$, there exists an integer n such that the foliation $\mathcal{F}_n$ that coincides with $\mathcal{F}$ on the complement of $\psi([0, 1]^2)$ and admits the compact chart $\psi' = \psi \circ \varphi_n$ belongs to $\mathcal{O}(\psi, \epsilon)$. By Proposition 2.1, for every large enough integer n

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

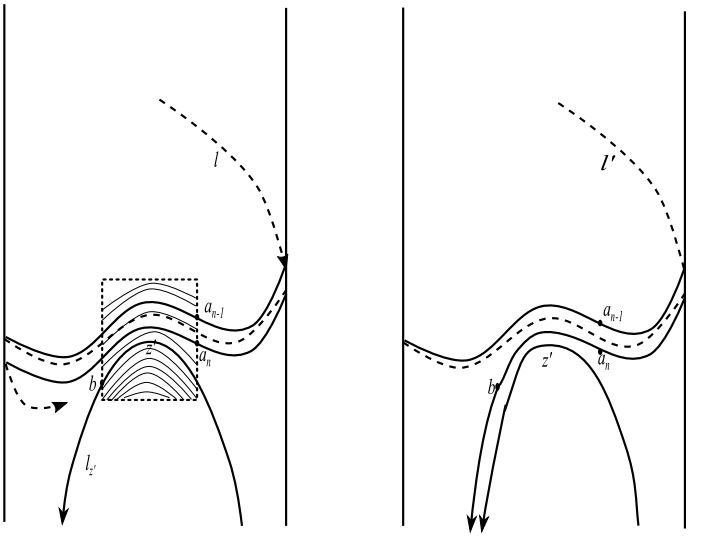


FIGURE 5. Left: foliation $\mathcal{F}$. Right: foliation $\mathcal{F}'$.

the foliation $\mathcal{F}_n$ is positively transverse to I and, by construction, it admits a leaf whose ω-limit set is reduced to S and whose α-limit set is γ_S. This completes the proof of the lemma. □

Using reasoning analogous to that of Lemma 3.6, we obtain the following result.

LEMMA 4.9. *The set* $\rho_{\hat{U}^S_{\gamma_S}}(I)$ *is contained either in* $[0, +\infty]$ *or in* $[-\infty, 0]$.

End of the proof of Proposition 4.3. Under the hypothesis of Proposition 4.3, assume, furthermore, that Case (a) does not hold. Let $\mathcal{F}$ be an oriented foliation transverse to I. By Lemma 4.4, we can assume that the foliation $\mathcal{F}$ has closed leaves but, by Lemma 4.5, these do not accumulate at least at one end of $\mathbb{A}$. Let us assume that the closed leaves of $\mathcal{F}$ do not accumulate at S. Let γ_S be 'the last' closed leaf of $\mathcal{F}$: that is, γ_S is a closed leaf of $\mathcal{F}$ such that $U^S_{\gamma_S}$ does not contain any closed leaf of $\mathcal{F}$. By Lemma 4.8, perturbing $\mathcal{F}$ and changing the orientation of $\mathcal{F}$ we can assume that:

(i) the foliation $\mathcal{F}$ admits a leaf l_S contained in $U^S_{\gamma_S}$ whose α-limit set is γ_S and ω-limit set is $\{S\}$.

Moreover, by the previous lemma:

(ii) the set $\rho_{\hat{U}^S_{\gamma_S}}(I)$ is contained in $[-\infty, 0]$.

We have two cases.

Case 1. The closed leaves of $\mathcal{F}$ *do not accumulate at* N *either.*

As above, we consider 'the first' closed leaf of $\mathcal{F}$, denoted by γ_N, that is, $\gamma_N \subset U^N_{\gamma_S}$ and there are no closed leaves of $\mathcal{F}$ outside the closed annulus delimited by γ_N and γ_S. From Lemma 4.8 (applied to N), we can assume that the foliation $\mathcal{F}$ admits a leaf l_N whose ω-limit set is reduced to N and α-limit set is γ_N or whose α-limit set is reduced to N and ω-limit set is γ_N. We have two subcases.

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

Subcase 1.1. (iii) The leaf l_N satisfies $\alpha(l_N) = \{N\}$ and $\omega(l_N) = \gamma_N$.

Let A_{γ_N,γ_S} be the closed annulus delimited by γ_N and γ_S. We will prove the following assertions.

(1) There is a positive annulus contained in A_{γ_N,γ_S}.

(2) Every annulus $A \in \mathcal{A}'$ is positive.

(3) f satisfies the local intersection property at least at one end of $\mathbb{A}$.

Let us prove (1). Suppose, by contradiction, that every annulus $A \in \mathcal{A}$ either has maximal invariant set empty or is negative. Lemma 4.7 provides an oriented foliation $\mathcal{F}'$ that is also transverse to I and admits a leaf whose α-limits set is $\{N\}$ and ω-limit set is $\{S\}$. This implies that $\rho_{\text{ann}}(I) \subset [-\infty, 0]$. This contradicts our hypotheses and completes the proof of Assertion (1).

Let us prove (2). Suppose, by contradiction, that there are both positive and negative annuli. By Lemma 3.16, we can find two annuli A_0 and A_1 contained in A_{γ_N,γ_S} satisfying the hypothesis of Theorem C. Hence Theorem C implies that zero belongs to $\rho_{\Theta(A)}(I)$, where $\Theta(A)$ is the maximal invariant set of $A = A_0 \cup A_1$. This is a contradiction, because we are assuming that Case (a) does not hold. This completes the proof of Assertion (2).

Let us prove (3). By our assumptions on the leaves l_N and l_S (items (i)–(iii)) and Lemma 4.9:

(iv) $\rho_{\hat{U}^S_{\gamma_S}}(I) \cup \rho_{\hat{U}^N_{\gamma_N}}(I) \subset [-\infty, 0]$.

Moreover, by Assertion (2):

(v) $\rho_{\Theta(A_{\gamma_N,\gamma_S})}(I) \subset (0, +\infty)$.

Now Assertion (3) will be a consequence of the following claim.

CLAIM. *Under our hypotheses. Suppose that f does not satisfy the local intersection property at S. Then f restricted to $\hat{U}^S_{\gamma_S}$ is a homothety. In particular, $\rho_{\hat{U}^S_{\gamma_N}}(I)$ is included in $[0, +\infty]$.*

Proof of the claim. Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of essential f-free curves such that:

- for every integer $n \in \mathbb{N}$, $\gamma_{n+1} \subset U^S_{\gamma_n} \subset U^S_{\gamma_S}$; and
- $\bigcap_{n\in\mathbb{N}} U^S_{\gamma_n} = \emptyset$.

For each integer n, we write $\Theta(A_n)$ for the maximal invariant set of A_n, the closed annulus delimited by γ_S and γ_n. Firstly, suppose that there exists a positive integer n_0 such that $\Theta(A_{n_0})$ is not empty. Then, by item (iv), $\rho_{\Theta(A_{n_0})}(I) \subset (-\infty, 0)$ and we know that $\rho_{\Theta(A_{\gamma_N,\gamma_S})}(I) \subset (0, +\infty)$ (item (v)). Hence Theorem C implies that zero belongs to the rotation set relative to the maximal invariant set of the closed annulus delimited by γ_N and γ_{n_0}. This again contradicts the fact that Case (a) does not hold. Now suppose that, for every integer n, $\Theta(A_n)$ is empty. Considering f instead of f^{-1}, we can assume that γ_S satisfies $f(\gamma_S) \subset U^S_{\gamma_S}$. Hence for each integer n, there is an integer $k(n)$ such that $f^{k(n)}(\gamma_S) \subset U^S_{\gamma_n}$. Therefore

$$\bigcap_{n\in\mathbb{N}} f^n(U^S_{\gamma_S}) \subset \bigcap_{n\in\mathbb{N}} U^S_{\gamma_n} = \emptyset.$$

This proves that f restricted to $\hat{U}^S_{\gamma_S}$ is a homothety and completes the proof of the claim. □

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

Let us continue proving Assertion (3). Suppose, by contradiction, that f does not satisfy the local intersection property at any of the ends of $\mathbb{A}$. Then the previous claim (applied to both ends of $\mathbb{A}$) implies that $\rho_{\text{ann}}(I)$ is contained in $[0, +\infty]$. This contradicts our hypotheses and completes the proof of Assertion (3).

To complete the proof of Proposition 4.3 in Subcase 1.1, note that if f satisfies the local intersection property at the other end of $\mathbb{A}$, then Case (c) of Proposition 4.3 holds. Otherwise, by the previous claim, Case (b) of Proposition 4.3 holds. This completes the proof of Proposition 4.3 in Subcase 1.1.

Subcase 1.2. (vi) The leaf l_N satisfies $\alpha(l_N) = \gamma_N$ and $\omega(l_N) = \{N\}$.

Let A_{γ_N,γ_S} be the closed annulus delimited by γ_N and γ_S.

CLAIM. *In this situation, there exists an oriented foliation $\mathcal{F}'$ transverse to I that has a unique closed leaf γ and two leaves l'_N and l'_S whose α-limit set is γ and ω-limit set is $\{N\}$ and whose α-limit set is γ and ω-limit set is $\{S\}$, respectively.*

Proof. By our assumptions on the leaves l_N and l_S (items (i) and (vi)), if the set $\mathcal{A}'$ is empty, then Lemma 4.7 (applied to S) provides the oriented foliation $\mathcal{F}'$. Suppose now that the set $\mathcal{A}'$ is not empty. We will claim that either all annuli are positive or all annuli are negative. Suppose, by contradiction, that there are both positive and negative annuli in A_{γ_N,γ_S}. By Lemma 3.16, we can find two annuli A_0 and A_1 contained in A_{γ_N,γ_S} satisfying the hypothesis of Theorem C. Hence Theorem C implies that zero belongs to $\rho_{\Theta(A)}(I)$, where $\Theta(A)$ is the maximal invariant set of $A = A_0 \cup A_1$. This is a contradiction, because we are assuming that Case (a) of Proposition 4.3 does not hold. If either all annuli are positive or all annuli are negative, then Lemma 4.7 (applied at an end of $\mathbb{A}$) provides the oriented foliation $\mathcal{F}'$. This completes the proof of the claim. □

By our assumptions on the leaves l_N and l_S (items (i) and (vi)), by Lemma 4.9 we know that $\rho_{\hat{U}^N_\gamma}(I)$ is contained in $[0, +\infty]$ and that $\rho_{\hat{U}^S_\gamma}(I)$ is contained in $[-\infty, 0]$. Let us prove, by contradiction, that f satisfies the local intersection property at least at one of the ends of $\mathbb{A}$. Otherwise, since $\rho_{\text{ann}}(I)$ contains both positive and negative real numbers, we can find two f-free essential Jordan curves γ^+ and γ^- close enough to the ends N and S, respectively, such that the maximal invariant set $\Theta(A_{\gamma^+,\gamma})$ (respectively, $\Theta(A_{\gamma,\gamma^-})$) of the annulus delimited by γ^+ and γ (respectively, γ and γ^-) is not empty. Moreover, $\rho_{\Theta(A_{\gamma_+,\gamma})}(I)$ is contained in $[0, +\infty]$ and $\rho_{\Theta(A_{\gamma,\gamma_-})}(I)$ is contained in $[-\infty, 0]$. Therefore Theorem C implies that zero belongs to $\rho_{\Theta(A_{\gamma^+,\gamma^-})}(I)$, where $\Theta(A_{\gamma^+,\gamma^-})$ is the maximal invariant set of the compact annulus delimited by γ^+ and γ^-. This contradicts that Case (a) of Proposition 4.3 does not hold. This proves that Case (b) of Proposition 4.3 holds and completes the proof of Proposition 4.3 in Subcase 1.2.

This completes the proof of Proposition 4.3 in Case 1.

Case 2. The closed leaves of $\mathcal{F}$ accumulate at N.

As above, we consider 'the last' closed leaf of $\mathcal{F}$ near to S, denoted by γ_S. As above, from Lemma 4.8, we can assume that there is a leaf l_S satisfying that $\omega(l_S) = \{S\}$ and $\alpha(l_S) = \gamma_S$. Since $\rho_{\text{ann}}(I)$ contains both negative and positive real numbers, the following assertions hold.

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

(1) There is a positive annulus contained in $U^N_{\gamma_S}$.
(2) Every annulus $A \in \mathcal{A}'$ is positive.
(3) f satisfies the local intersection property at S.

Let us prove (1). Suppose, by contradiction, that every annulus $A \in \mathcal{A}$ contained in $U^N_{\gamma_S}$ either has maximal invariant set empty or is negative. Proposition 3.11 provides an oriented foliation $\mathcal{F}'$ which is also transverse to I and admits a unique closed leaf γ_S and a leaf whose α-limit set is $\{N\}$ and ω-limit set is γ_S. Now we can find a foliation transverse to I 'breaking' the closed leaf γ_S (see Lemma 4.7). This foliation admits a leaf l which joins both ends of $\mathbb{A}$. This implies that $\rho_{\text{ann}}(I) \subset [-\infty, 0]$. This contradicts our hypotheses and completes the proof of Assertion (1).

Let us prove (2). Suppose, by contradiction, that there are both positive and negative annuli. By Lemma 3.16, we can find two annuli A_0 and A_1 contained in $U^N_{\gamma_S}$ satisfying the hypothesis of Theorem C. Hence Theorem C implies that zero belongs to $\rho_{\Theta(A)}(I)$, where $\Theta(A)$ is the maximal invariant set of $A = A_0 \cup A_1$. This is a contradiction, because we are assuming that Case (a) does not hold. This completes the proof of Assertion (2).

Let us prove (3). By our assumptions on the leaf l_S (items (i) and (ii)):

- $\rho_{\hat{U}^S_{\gamma_S}}(I) \subset [-\infty, 0]$.

Let γ_N be a closed leaf of $\mathcal{F}$ with γ_N included in $U^N_{\gamma_S}$. From above Assertion (2):

- $\rho_{\Theta(A_{\gamma_N,\gamma_S})}(I) \subset (0, +\infty)$.

Assertion (3) is now a consequence of the claim in Subcase 1.1. This completes the proof of Assertion (3).

To complete the proof of Proposition 4.3 in Case 2, note that, by Proposition 3.11, we can find an oriented foliation $\mathcal{F}'$ transverse to I that has a unique closed leaf γ and such that all leaves of $\mathcal{F}'$ contained in $U^N_{\gamma_S}$ are spirals whose ω-limit set is $\{N\}$. From Lemma 4.9, we know that $\rho_{\hat{U}^N_\gamma}(I)$ is contained in $[0, +\infty]$ and that $\rho_{\hat{U}^S_\gamma}(I)$ is contained in $[-\infty, 0]$ (item (ii)). This proves that Case (b) of Proposition 4.3 holds. This completes the proof of Proposition 4.3 in Case 2. □

Proposition 4.3 is now proved.

4.2.2. *An intermediate result.* The purpose of this paragraph is to prove the following proposition and its corollary.

PROPOSITION 4.10. *Let I and f be as in Theorem B*. Assume, furthermore, that:*
(A_1) *the homeomorphism f satisfies the local intersection property at S;*
(A_2) *there exists a Jordan domain $\hat{V}$ containing S (in $\hat{\mathbb{A}}$) such that $\hat{f}(\text{Cl}(\hat{V})) \subset \hat{V}$; and and*
(A_3) *the set $\rho_{\hat{V}}(I)$ is included in $[0, +\infty]$.*

Then either:
(1) *there exists a compact set K in $\mathbb{A}$ such that zero belongs to $\rho_K(I)$; or*
(2) *there exists a compact set K in $\mathbb{A}$ contained in $\mathbb{A}\setminus\hat{V}$ such that the set $\rho_K(I)$ intersects $[0, +\infty]$.*

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

Remark 9. We can replace (A_2) by the hypothesis $\mathrm{Cl}(\hat{V}) \subset \hat{f}(\hat{V})$ by changing the isotopy I by I^{-1}. We can replace (A_3) by the hypothesis $\rho_{\hat{V}}(I) \subset [-\infty, 0]$ by conjugating I by the orientation-reversing homeomorphism $(u, r) \mapsto (-u, r)$ of the annulus $\mathbb{A}$.

Let us put $\Theta^+(\hat{V})$ (respectively, $\Theta^-(\hat{V}^c)$) the forward (respectively, backward) $\hat{f}$-invariant set of $\hat{V}$ (respectively, $\hat{V}^c$): that is,

$$\Theta^+(\hat{V}) := \bigcap_{n\in\mathbb{N}} \hat{f}^n(\hat{V}) \quad \text{and} \quad \Theta^-(\hat{V}^c) := \bigcap_{n\in\mathbb{N}} \hat{f}^{-n}(\hat{V}^c).$$

Using Hypothesis (A_2), one proves that the set $\Theta^+(\hat{V})$ (respectively, $\Theta^-(\hat{V}^c)$) is a continuum, its complement is a connected set of $\hat{\mathbb{A}}$ and it contains S (respectively, N). Moreover it satisfies the following properties.

LEMMA 4.11. *Under the hypotheses of Proposition 4.10:*
(a) *the set $\Theta^+(\hat{V})$ is not reduced to S; and*
(b) *the set $\Theta^-(\hat{V}^c)$ is not reduced to N.*

Proof. Let us prove (a). Since f satisfies the local intersection property at S, there is an open neighborhood $\hat{U}$ of S which does not contain any f-free essential Jordan curve. Hence, for every integer $n \geq 0$, the curve $\hat{f}^n(\partial\hat{V})$ is not contained in $\hat{U}$ because it is a free essential Jordan curve by Hypothesis (A_2). Hence, by Hypothesis (A_2), $(\hat{f}^n(\mathrm{Cl}(\hat{V})\backslash\hat{U}))_{n\in\mathbb{N}}$ is a decreasing sequence of compact sets in $\mathbb{A}$, and so

$$\emptyset \neq \bigcap_{n\in\mathbb{N}} \hat{f}^n(\mathrm{Cl}(\hat{V})\backslash\hat{U}) \subset \Theta^+(\hat{V}).$$

This proves item (a).

Now let us prove (b). Suppose, by contradiction, that $\Theta^-(\hat{V}^c) = \{N\}$. So, by Hypothesis (A_2), for every compact set K in $\mathbb{A}$, there exists an integer $n_K = n \geq 0$ such that K is included in $\hat{f}^{-n}(\hat{V})$. Hence, by Hypotheses (A_2) and (A_3),

$$\rho_K(I) \subset \rho_{\hat{f}^{-n}(\hat{V})}(I) = \rho_{\hat{V}}(I) \subset [0, +\infty].$$

This contradicts the fact that $\rho_{\mathrm{ann}}(I)$ contains some negative real number. This proves item (b). □

Recall that $\pi : \widetilde{\mathbb{A}} \to \mathbb{A}$ is the universal covering of $\mathbb{A}$ and that $p_1 : \widetilde{\mathbb{A}} \to \mathbb{R}$ is the projection onto the first coordinate.

Proof of Proposition 4.10. Let us consider $V := \hat{V} \cap \mathbb{A}$. Let Γ, $\widetilde{V}$, $\Theta^+(\widetilde{V})$ and $\Theta^-(\widetilde{V}^c)$ be lifts of $\mathrm{Fr}(V)$, V, $\Theta^+(\hat{V})\backslash\{S\}$ and $\Theta^-(\hat{V}^c)\backslash\{N\}$, respectively.

Let X be the connected component of $\widetilde{\mathbb{A}}\backslash(\Theta^+(\widetilde{V}) \cup \Theta^-(\widetilde{V}^c))$ containing Γ. The set X is open, connected and $\widetilde{f}$-invariant: that is, $\widetilde{f}(X) = X$, where $\widetilde{f}$ is the lift of f associated to I. Let $\alpha : [0, 1] \to \widetilde{\mathbb{A}}$ be a simple arc satisfying:
(i) the endpoints $\alpha(0)$ and $\alpha(1)$ lie in $\Theta^+(\widetilde{V})$ and $\Theta^-(\widetilde{V}^c)$, respectively;
(ii) if $\dot{\alpha}$ denotes the interior of α, that is, $\dot{\alpha}$ is the image of $(0, 1)$ under α, then the set $\dot{\alpha}$ is included in X; and
(iii) the intersection $\alpha \cap \Gamma$ is a singleton.

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

The arc α separates X because $\pi(X)$ is an open annulus and α joins its two ends (see the paragraph that follows the definitions of $\Theta^+(\hat{V})$ and $\Theta^-(\hat{V}^c)$). Let us denote $R_X(\alpha)$ (respectively, $L_X(\alpha)$) as the connected component of $X\backslash\alpha$ on the right (respectively, left) of α (considering the standard orientation). Since $\dot{\alpha}$ is contained in X and X is $\widetilde{f}$-invariant, one deduces that, for every integer $n \geq 0$, the set $\widetilde{f}^n(\dot{\alpha})$ is included in X, which gives three possibilities:

- $\widetilde{f}^n(\dot{\alpha}) \cap \alpha \neq \emptyset$;
- $\widetilde{f}^n(\dot{\alpha}) \subset L_X(\alpha)$; or
- $\widetilde{f}^n(\dot{\alpha}) \subset R_X(\alpha)$.

To conclude the proof, it suffices to consider the following three cases.

Case 1. Suppose that there exists a sequence of integers $(n_k)_{k\in\mathbb{N}}$ which converges to $+\infty$ such that, for all $k \in \mathbb{N}$, $\widetilde{f}^{n_k}(\dot{\alpha}) \cap \alpha \neq \emptyset$. Then zero belongs to $\rho_{\pi(\alpha)}(I)$. This proves that Assertion (1) of Proposition 4.10 holds.

Case 2. Suppose that there exists a sequence of integers $(n_k)_{k\in\mathbb{N}}$ which converges to $+\infty$ such that, for all $k \in \mathbb{N}$, $\widetilde{f}^{n_k}(\dot{\alpha}) \subset L_X(\alpha)$. Let Γ^r be the connected component of $\Gamma\backslash\alpha$ contained in $R_X(\alpha)$. From the fact that $\widetilde{f}^{n_k}(\alpha \cap \Gamma) \in L_X(\alpha)$ and that $\widetilde{f}^{n_k}(\Gamma^r)$ is a connected set in X which is unbounded to the right and contained in $\widetilde{V}$ (by Hypothesis (A_2)), we deduce that

$$\widetilde{f}^{n_k}(\Gamma^r) \cap \alpha \cap \widetilde{V} \neq \emptyset.$$

Therefore, there exist a real constant $M > 0$ independent of k and a sequence of points $(\widetilde{z}_k)_{k\in\mathbb{N}}$, where $\widetilde{z}_k \in \Gamma^r$ for all $k \in \mathbb{N}$, such that

$$p_1(\widetilde{f}^{n_k}(\widetilde{z}_k)) < p_1(\widetilde{z}_k) + M.$$

Hence, if ρ is a limit point of the sequence $(\rho_{n_k}(\widetilde{z}_k))_{k\in\mathbb{N}}$, where

$$\rho_{n_k}(\widetilde{z}_k) := \frac{1}{n_k}(p_1(\widetilde{f}^{n_k}(\widetilde{z}_k)) - p_1(\widetilde{z}_k)),$$

then $\rho \leq 0$. This implies that $\rho = 0$ because $\rho_{\hat{V}}(I) \subset [0, +\infty]$. Thus zero belongs to $\rho_K(I)$, where $K = \mathrm{Fr}(V) \cup (\pi(\alpha) \cap V)$. This proves that Assertion (1) of Proposition 4.10 holds.

Case 3. Suppose that there exists a sequence of integers $(n_k)_{k\in\mathbb{N}}$ which converges to $+\infty$ such that, for all $k \in \mathbb{N}$, $\widetilde{f}^{n_k}(\dot{\alpha}) \subset R_X(\alpha)$. Let Γ^r be the connected component of $\Gamma\backslash\alpha$ contained in $R_X(\alpha)$. Since, for all $k \in \mathbb{N}$, the arc $\widetilde{f}^{n_k}(\alpha)$ meets $\Theta^+(\widetilde{V})$ and $\Theta^-(\widetilde{V}^c)$, we deduce that

$$\widetilde{f}^{n_k}(\dot{\alpha} \cap \widetilde{V}^c) \cap \Gamma^r \neq \emptyset.$$

Therefore, there exist a real constant $M > 0$ independent of k and a sequence of points $(\widetilde{z}_k)_{k\in\mathbb{N}}$, where $\widetilde{z}_k \in \alpha \cap \widetilde{V}^c$, such that

$$p_1(\widetilde{z}_k) < p_1(\widetilde{f}^{n_k}(\widetilde{z}_k)) + M.$$

If ρ is a limit point of the sequence $(\rho_{n_k}(\widetilde{z}_k))_{k\in\mathbb{N}}$, where

$$\rho_{n_k}(\widetilde{z}_k) := \frac{1}{n_k}(p_1(\widetilde{f}^{n_k}(\widetilde{z}_k)) - p_1(\widetilde{z}_k)),$$

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

then $\rho \geq 0$. Hence, if K is the compact set $\mathrm{Fr}(V) \cup (\pi(\alpha) \cap V^c)$ of $\mathbb{A}$, then $\rho_K(I)$ contains positive real numbers. This proves that Assertion (2) of Proposition 4.10 holds. This completes the proof of Proposition 4.10. □

As a consequence, we have the following corollary.

COROLLARY 4.12. *Let $I = (f_t)_{t\in[0,1]}$ and f be as in Theorem B*. Assume, furthermore, that:*

(A'_1) *the homeomorphism f satisfies the local intersection property at S;*

(A'_2) *there exist two f-free essential Jordan curves γ_N and γ_S with $\gamma_N \subset U^N_{\gamma_S}$ such that the closed annulus A delimited by γ_N and γ_S has maximal invariant set $\Theta(A)$ non-empty; and*

(A'_3) *the set $\rho_{\hat{U}^S_{\gamma_S}}(I)$ is included in $[0, +\infty]$ and $\rho_A(I)$ is contained in $[-\infty, 0]$.*

Then there exists a compact set K in $\mathbb{A}$ such that zero belongs to $\rho_K(I)$.

Proof. Changing I by I^{-1} and conjugating I by the orientation-reversing homeomorphism $(u, r) \mapsto (-u, r)$ of the annulus $\mathbb{A}$, we may suppose that γ_N is attracting: that is, $f(\gamma_N) \subset U^S_{\gamma_N}$ and that hypothesis (A'_3) holds. Let us put $A_0 = \mathrm{Cl}(U^S_{\gamma_N})$ and, for every integer $n \geq 1$, we define $A_n = f^{-n}(U^S_{\gamma_N})$. Note that $\mathbb{A}' = \bigcup_{n\in\mathbb{N}} A_n$ is an open annulus and $f(\mathbb{A}') = \mathbb{A}'$. Let us put $f' = f|_{\mathbb{A}'}$. We note that f' is isotopic to the identity of $\mathbb{A}'$ and, since $f' = f|_{\mathbb{A}'}$, there is an isotopy I' on the annulus $\mathbb{A}'$ from the identity of $\mathbb{A}'$ to the homeomorphism f' such that, for every compact set K contained in $\mathbb{A}'$, $\rho_K(I) = \rho_K(I')$. By Proposition 4.10 applied to I' and f' on $\mathbb{A}'$, we know that:

(1) there exists a compact set K' in $\mathbb{A}'$ such that zero belongs to $\rho_K(I')$; or

(2) there exists a compact set K' in $\mathbb{A}'$ contained in $\mathbb{A}'\backslash U^S_{\gamma_S}$ such that the set $\rho_{K'}(I')$ intersects $[0, +\infty]$.

If Assertion (1) holds, then we have finished. Suppose now that Assertion (2) holds. Since K' is a compact set contained in $\mathbb{A}'\backslash U^S_{\gamma_S}$, there exists a positive integer n such that K' is included in $A \cup \cdots \cup f'^{-n}(A)$. Hence $\rho_{K'}(I')$ is included in the union of $\rho_{f'^{-i}(A)}(I')$. Since, for every i, $\rho_{f'^{-i}(A)}(I') = \rho_A(I')$ and $\rho_A(I) = \rho_A(I')$, we deduce that $\rho_A(I)$ intersects $[0, +\infty]$. Moreover, by hypothesis, we know that $\rho_A(I)$ is included in $[-\infty, 0]$. Therefore zero belongs to $\rho_A(I)$. This completes the proof. □

4.3. *Proof of Theorem B*.* In this subsection, we finish the proof of Theorem B*, using Theorem C. Let I be an isotopy in $\mathrm{Homeo}_0(\mathbb{A})$ from the identity to a homeomorphism f. Suppose that $\rho_{\mathrm{ann}}(I)$ contains both positive and negative real numbers. One considers the different cases of Proposition 4.3. Case (a) is complete.

Suppose now that Case (b) of Proposition 4.3 holds. Let g be the orientation-reversing homeomorphism of $\mathbb{A}$ defined by $(u, r) \mapsto (-u, r)$. Considering I^{-1} instead of I, or conjugating I by g or considering $gI^{-1}g^{-1}$ instead of I, we can assume that Hypotheses (A_1)–(A_3) of Proposition 4.10 hold. Thus, either:

(1) there exists a compact set K of $\mathbb{A}$ such that zero belongs to $\rho_K(I)$; or

(2) there exists a compact set K contained in $\mathrm{Cl}(U^N_\gamma)$ of $\mathbb{A}$ such that the rotation set $\rho_K(I)$ intersects $[0, +\infty]$.

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

If Assertion (1) holds, then we have finished. Suppose now that Assertion (2) holds. Since $K \subset \mathrm{Cl}(U_\gamma^N)$ and $\hat{f}(\mathrm{Cl}(U_\gamma^S)) \subset U_\gamma^S$, we deduce that $\rho_K(I) \subset \rho_{\hat{U}_\gamma^N}(I) \subset [-\infty, 0]$. This last inclusion holds because Case (b) of Proposition 4.3 and Hypothesis (A_3) of Proposition 4.10 hold. Therefore zero belongs to $\rho_K(I)$. This completes the proof when the Case (b) of Proposition 4.3 holds.

Lastly, suppose that Case (c) of Proposition 4.3 holds. Now, by Corollary 4.12, there exists a compact set K of $\mathbb{A}$ such that zero belongs to $\rho_K(I)$. This completes the proof of Theorem B*.

5. *Proof of Theorem D*

The purpose of this section is to prove Theorem D assuming Theorem C. Let I, f, γ^+ and γ^- be as in the hypotheses of Theorem D. Without loss of generality, we suppose that γ^- is contained in $U_{\gamma^+}^S$. Let $\Theta(A)$ be the maximal invariant set of the annulus A delimited by γ^+ and γ^-. As we have seen in the first paragraph of §3.2, we can suppose that $\rho_{\Theta(A)}(I)$ contains both positive and negative real numbers. We want to prove that zero belongs to $\rho_{\Theta(A)}(I)$.

A f-free essential Jordan curve γ in $\mathbb{A}$ is said to be attracting if $f(\gamma) \subset U_\gamma^S$ and repelling if $f^{-1}(\gamma) \subset U_\gamma^S$. The proof of the next lemma is easy.

LEMMA 5.1. *Let γ and γ' be two disjoint f-free essential curves in $\mathbb{A}$. Suppose that γ is attracting and γ' is repelling. Then the maximal invariant set of the closed annulus delimited by γ and γ' is a connected set.*

We remark that Lemma 4.2 implies Theorem D when $\Theta(A)$ is a connected set. By the previous lemma and considering I^{-1} instead of I (if necessary), we can assume that both γ^+ and γ^- are attracting Jordan curves. Hence the sets

$$\Theta^+(\hat{U}_{\gamma^-}^S) := \bigcap_{n\in\mathbb{N}} \hat{f}^n(\hat{U}_{\gamma^-}^S) \quad \text{and} \quad \Theta^-(\hat{U}_{\gamma^+}^N) := \bigcap_{n\in\mathbb{N}} \hat{f}^{-n}(\hat{U}_{\gamma^+}^N)$$

are continua and their complements are connected sets of $\hat{\mathbb{A}}$ and contain S and N, respectively. Let us write $\mathbb{A}'$ for the set $\hat{\mathbb{A}}\backslash(\Theta^+(\hat{U}_{\gamma^-}^S) \cup \Theta^-(\hat{U}_{\gamma^+}^N))$. Then $\mathbb{A}'$ is homeomorphic to an open annulus and $f(\mathbb{A}') = \mathbb{A}'$. Let us write $f' = f|_{\mathbb{A}'}$. We note that f' is isotopic to the identity of $\mathbb{A}'$, so there is an isotopy I' on the annulus $\mathbb{A}'$ from the identity of $\mathbb{A}'$ to the homeomorphism f' such that, for every compact set K contained in $\mathbb{A}'$, $\rho_K(I) = \rho_K(I')$. Since $f' = f|_{\mathbb{A}'}$ does not satisfy the local intersection property at either N' or S', where N' and S' are the ends of $\mathbb{A}'$, by Proposition 4.3 there exists an f'-invariant and compact set K in $\mathbb{A}'$ such that zero belongs to $\rho_K(I')$. Since K is an invariant set in $\mathbb{A}'$ under f', we deduce that K is included in $\Theta(A)$. Therefore

$$0 \in \rho_K(I') \subset \rho_{\Theta(A)}(I') = \rho_{\Theta(A)}(I).$$

This proves Theorem D.

6. *Dynamics in the closed annulus: Theorem C*

The purpose of this section is to prove Theorem C. Let I, f, γ_0, γ_1 and γ_2 be as in the hypotheses of the theorem. Without loss of generality, for $i \in \{0, 1\}$, we suppose that γ_{i+1}

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

is contained in $U^S_{\gamma_i}$. Let $\Theta(A_i)$ be the maximal invariant set of the closed annulus A_i delimited by γ_i and γ_{i+1}. Let $\Theta(A)$ be the maximal invariant set of the annulus $A = A_0 \cup A_1$. We want to prove that zero belongs to $\rho_{\Theta(A)}(I)$.

6.1. *Reduction of Theorem C.* We recall that Lemma 4.2 implies Theorem C when $\Theta(A)$ is a connected set. Thus, from now on, we assume that $\Theta(A)$ is a disconnected set. This implies that the f-free essential Jordan curves γ_0, γ_1 and γ_2 are either all attracting or all repelling (see §5). Indeed, changing I into I^{-1}, we can suppose that γ_0 is attracting. If γ_2 is repelling, then, by Lemma 5.1, $\Theta(A)$ is a connected set. If γ_1 is repelling and γ_2 attracting, then, by Lemma 5.1, both $\Theta(A_0)$ and $\Theta(A_1)$ are connected sets which separate the annulus. In this case, $\Theta(A)$ is the union of $\Theta(A_0)$, of $\Theta(A_1)$ and of the bounded connected component of the complement of $\Theta(A_0) \cup \Theta(A_1)$ and so $\Theta(A)$ is a connected set. Therefore, changing I into I^{-1} and conjugating it by a change of coordinates (given by Schoenflies' theorem), we can assume that γ_0, γ_1 and γ_2 are attracting Euclidean circles.

We will work in $\widetilde{\mathbb{A}} := \mathbb{R} \times \mathbb{R}$, the universal covering of $\mathbb{A} = \mathbb{T}^1 \times \mathbb{R}$, where $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. We recall that $\pi : \widetilde{\mathbb{A}} \to \mathbb{A}$ is the canonical projection of $\widetilde{\mathbb{A}}$ onto $\mathbb{A}$ and that $p_1 : \widetilde{\mathbb{A}} \to \mathbb{R}$ is the projection onto the first coordinate. Let $\widetilde{f}$ be the lift of f associated to I: that is, if $\widetilde{I} = (\widetilde{f}_t)_{t\in[0,1]}$ is the unique lift of I such that $\widetilde{f}_0$ is the identity of $\widetilde{\mathbb{A}}$, then $\widetilde{f} := \widetilde{f}_1$. Recall that $\widetilde{f} : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ is a homeomorphism which is isotopic to the identity and commutes with the translation $T : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ defined by $T(x, y) = (x + 1, y)$. For $j \in \{0, 1, 2\}$, we define $\Gamma_j := \pi^{-1}(\gamma_j)$ and let $\Theta(\widetilde{A}) := \pi^{-1}(\Theta(A))$. If γ is an essential Jordan curve in $\mathbb{A}$, and $\Gamma = \pi^{-1}(\gamma)$, we will write U^+_Γ (respectively, U^-_Γ) for the upper (respectively, lower) unbounded connected component of the complement of Γ. From now on, we will write $\rho(\widetilde{f})$ instead of $\rho_{\text{ann}}(I)$.

Since $\rho(\widetilde{f}^q) = q\rho(\widetilde{f})$ for every $q \in \mathbb{Z}$, if $\Theta(\widetilde{A})$ is a disconnected and non-empty set, then considering an iterate of $\widetilde{f}$ instead of $\widetilde{f}$, we can assume the following hypotheses.

(H_1) The sets Γ_0, Γ_1 and Γ_2 are horizontal straight lines such that, for $i \in \{0, 1\}$, $\Gamma_i \subset U^+_{\Gamma_{i+1}}$ and thus they are attracting: that is, $\widetilde{f}(\Gamma_j) \subset U^-_{\Gamma_j}$ for $j \in \{0, 1, 2\}$.

We are assuming that $\Theta(\widetilde{A})$ is a disconnected and non-empty set. This implies, by Hypothesis (H_1), that there exists an integer n_0 such that, for every integer $n \geq n_0$, $\widetilde{f}^n(\Gamma_0) \cap \Gamma_2 \neq \emptyset$. Then, by considering $\widetilde{f}^{n_0}$ instead of $\widetilde{f}$, we can make the following assumption.

(H_2) For every integer $n \geq 1$, $\widetilde{f}^n(\Gamma_0) \cap \Gamma_2 \neq \emptyset$.

For $i \in \{0, 1\}$, we define by $\Theta(\widetilde{A}_i)$ the maximal invariant set of $\widetilde{A}_i$ which is the topological closed band delimited by Γ_i and Γ_{i+1}.

(H_3) The sets $\Theta(\widetilde{A}_0)$ and $\Theta(\widetilde{A}_1)$ are non-empty and

$$\rho_{\Theta(\widetilde{A}_0)}(\widetilde{f}) \subset (0, +\infty) \quad \text{and} \quad \rho_{\Theta(\widetilde{A}_1)}(\widetilde{f}) \subset (-\infty, 0).$$

Thus we have reduced Theorem C to the following theorem.

THEOREM C*. *Let $\tilde{f} : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ be a homeomorphism, isotopic to the identity, that commutes with the translation T and satisfies Hypotheses (H_1) to (H_3). Let $\Theta(\widetilde{A})$ be the invariant maximal set of $\widetilde{A} := \widetilde{A}_0 \cup \widetilde{A}_1$. Then*

$$0 \in \rho_{\Theta(\widetilde{A})}(\widetilde{f}).$$

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

Outline of the proof of Theorem C.* In what follows, using Hypotheses (H_1) and (H_2), we begin by constructing unstable sets and stable sets of the band $\widetilde{A}_i$ delimited by Γ_i and Γ_{i+1}. Then we consider their connected components, called unstable branches and stable branches: these are connected and compact sets, contained in $\widetilde{A}_i$ and meeting Γ_{i+1} but not Γ_i and Γ_i but not Γ_{i+1}, respectively. Let $\Lambda_0^-(x)$ denote the unstable branch of $x \in \Theta(\widetilde{A}_0)$ for the band $\widetilde{A}_0$ and let $\Lambda_1^+(y)$ denote the stable branch of $y \in \Theta(\widetilde{A}_1)$ for the band $\widetilde{A}_1$.

Next, in §6.3, we will study the sequence of iterates of $\Lambda_0^-(x)$. Using Hypothesis (H_3) (and, in particular, that zero does not belong to $\rho_{\Theta(\widetilde{A}_0)}(\widetilde{f})$), we will prove the following properties.

(1) For every compact set $\widetilde{K}$ included in $\widetilde{A}_0$, $\widetilde{f}^n(\widetilde{K})$ is disjoint from $\widetilde{K}$ for every large enough integer n.
(2) The sequence $(\widetilde{f}^{-n}(\Lambda_0^-(x))_{n\in\mathbb{N}}$ goes uniformly to the left.

Finally, in §6.5, we will finish the proof of Theorem C*. From the previous properties, Theorem C* will be a consequence of the following proposition.

PROPOSITION 6.1. *For every $x \in \Theta(\widetilde{A}_0)$, there exists $y \in \Theta(\widetilde{A}_1)$ such that, for every sufficiently large integer n,*

$$\widetilde{f}^n(\Lambda_0^-(x)) \cap \Lambda_1^+(y) \neq \emptyset.$$

We will explain the end of the proof of Theorem C*. If $\widetilde{z} \in \widetilde{f}^n(\Lambda_0^-(x)) \cap \Lambda_1^+(y)$, then, by definitions of $\Lambda_0^-(x)$ and $\Lambda_1^+(y)$ (see Remark 10 following Proposition 6.3) and the fact that Γ_0 and Γ_2 are $\widetilde{f}$-free (Hypothesis (H_1)), we deduce that $\widetilde{z} \in \Theta(\widetilde{A})$. Moreover, by Property (2) above (Lemma 6.10) and its symmetric property for stable branches (Remark 11 following Lemma 6.10), the positive and negative iterates of $\widetilde{z}$ go to the left. Fix a vertical band K that separates $\widetilde{A}$, where the diameter of its projection on the first coordinate (independent of $\widetilde{z}$) is large enough such that the orbit of $\widetilde{z}$ cannot go from 'the left of K' to 'the right of K' without intersecting K and that is sufficiently on the left of $\widetilde{z}$. Since the positive and negative iterates of $\widetilde{z}$ go to the left, there are integers $n^- \geq 0$ and $n^+ \geq 0$ such that $\widetilde{f}^{-n^-}(\widetilde{z})$ and $\widetilde{f}^{n^+}(\widetilde{z})$ belong to K. Since K can be chosen arbitrarily 'sufficiently on the left to $\widetilde{z}$', this implies that zero belongs to $\rho_{\Theta(\widetilde{A})}(\widetilde{f})$.

6.2. *Stable branches and unstable branches.* In this subsection, we consider a homeomorphism $\widetilde{f} : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$, isotopic to the identity, that commutes with the translation T and satisfies Hypotheses (H_1) and (H_2). For $i \in \{0, 1\}$, we define by $\widetilde{A}_i$ the topological closed band delimited by Γ_i and Γ_{i+1}. Following an idea of Birkhoff, in this subsection we will construct *unstable branches* and *stable branches* of the band $\widetilde{A}_i$.

6.2.1. *Stable sets and unstable sets.* Let us begin with the definition of an *unstable set* (respectively, *stable set*) of the band $\widetilde{A}_i$ for $\widetilde{f}$.

Definition 6.2. We say that Λ_i^- is an *unstable set* of the band $\widetilde{A}_i$ for $\widetilde{f}$ if it satisfies the following properties.

(1) The set Λ_i^- is a closed subset of $\widetilde{A}_i$.
(2) $\Lambda_i^- \cap \Gamma_{i+1} \neq \emptyset$.
(3) The set Λ_i^- is negatively invariant under $\widetilde{f}$: that is, $\widetilde{f}^{-1}(\Lambda_i^-) \subset \Lambda_i^-$.

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

Similarly, we say that Λ_i^+ is a *stable set* of the band $\widetilde{A}_i$ for $\widetilde{f}$ if it satisfies the following properties.

(1) The set Λ_i^+ is a closed subset of $\widetilde{A}_i$.
(2) $\Lambda_i^+ \cap \Gamma_i \neq \emptyset$.
(3) The set Λ_i^+ is positively invariant under $\widetilde{f}$: that is, $\widetilde{f}(\Lambda_i^+) \subset \Lambda_i^+$.

We prove the following proposition.

PROPOSITION 6.3. *Let $\widetilde{f} : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ be a homeomorphism, isotopic to the identity, commuting with the translation T and satisfying Hypotheses (H_1) and (H_2). For $i \in \{0, 1\}$ and for every $n \in \mathbb{Z}$, we define*

$$\Lambda_{n,i}^- := \widetilde{f}^n(\mathrm{Cl}(U_{\Gamma_i}^-)) \cap \mathrm{Cl}(U_{\Gamma_{i+1}}^+) \quad \text{and} \quad \Lambda_{n,i}^+ := \mathrm{Cl}(U_{\Gamma_i}^-) \cap \widetilde{f}^{-n}(\mathrm{Cl}(U_{\Gamma_{i+1}}^+)).$$

The sets (see Figure 6)

$$\Lambda_i^- := \bigcap_{n\in\mathbb{N}} \Lambda_{n,i}^- \quad \text{and} \quad \Lambda_i^+ := \bigcap_{n\in\mathbb{N}} \Lambda_{n,i}^+$$

are, respectively, an unstable set and a stable set of the band $\widetilde{A}_i$ for $\widetilde{f}$.

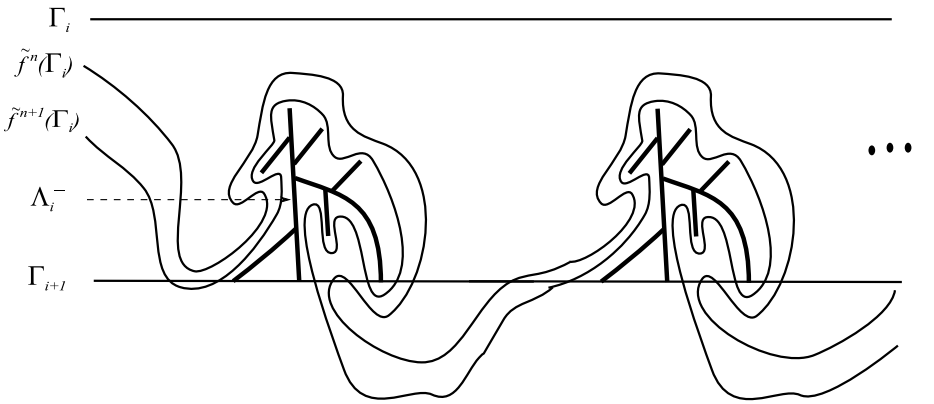


FIGURE 6. Unstable set.

Remark 10. The set Λ_i^- (respectively, Λ_i^+) consists of the points in $\widetilde{A}_i$ whose negative (respectively, positive) iterates under $\widetilde{f}$ remain in the band $\widetilde{A}_i$.

Proof of Proposition 6.3. We will prove that Λ_i^- is an unstable set of the band $\widetilde{A}_i$ for $\tilde{f}$ (one shows, similarly, that Λ_i^+ is a stable set of the band $\widetilde{A}_i$ for $\tilde{f}$). Item 1 of the above definition is a direct consequence of the definition of Λ_i^- and the fact that Γ_i is an attracting line. On the other hand, as the horizontal straight line Γ_{i+1} is attracting, $\widetilde{f}^{-1}(\mathrm{Cl}(U_{\Gamma_{i+1}}^+)) \subset \mathrm{Cl}(U_{\Gamma_{i+1}}^+)$, and so

$$\widetilde{f}^{-1}(\Lambda_i^-) = \bigcap_{n\in\mathbb{N}} \widetilde{f}^{n-1}(\mathrm{Cl}(U_{\Gamma_i}^-)) \cap \widetilde{f}^{-1}(\mathrm{Cl}(U_{\Gamma_{i+1}}^+)) \subset \Lambda_i^-.$$

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

We have proved items 1 and 3 of the above definition. Next, let us prove item 2, that is, we will prove that $\Lambda_i^- \cap \Gamma_{i+1} \neq \emptyset$, by contradiction. Suppose that for every $\tilde{z} \in \Gamma_{i+1}$, $\tilde{z} \notin \Lambda_j^-$. Then we can find the smallest integer $n_0 = n_0(\tilde{z})$ such that $\widetilde{f}^{-n_0(\tilde{z})}(\tilde{z}) \in U_{\Gamma_i}^+$. Since $\widetilde{f}$ is continuous and commutes with the translation T, $n_0(\tilde{z}) \geq n_0(\tilde{z}')$ for every $\tilde{z}'$ close to $\tilde{z}$ and $n_0(\tilde{z}) = n_0(T(\tilde{z}))$. By compactness, we can define the integer

$$n_0 = \sup_{\tilde{z} \in \Gamma_{i+1}} n_0(\tilde{z}) = \sup\{n_0(\tilde{z}) : \tilde{z} \in \Gamma_{i+1} \text{ and } p_1(\tilde{z}) \in [0, 1]\},$$

which satisfies $\widetilde{f}^{-n_0}(\Gamma_{i+1}) \subset U_{\Gamma_i}^+$. This contradicts Hypothesis (H_2), so item 2 is proved. □

6.2.2. *Stable branches and unstable branches.* Now we state and prove the properties verified by unstable branches (we have the 'symmetric properties' for stable branches) that we will use in the next subsections.

Definition 6.4. For $i \in \{0, 1\}$, we call any connected component of the unstable set Λ_i^- an *unstable branch of the band* $\widetilde{A}_i$. For $x \in \Lambda_i^-$, we denote $\Lambda_i^-(x)$, and we say that it is *the unstable branch of* x, the connected component of Λ_i^- containing x.

We have the following properties.

PROPOSITION 6.5. *Let $\widetilde{f} : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ be a homeomorphism, isotopic to the identity, that commutes with the translation T and satisfies Hypotheses (H_1) and (H_2). If $x \in \Lambda_i^-$, then:*

(1) *the set $\Lambda_i^-(x)$ is a compact subset of $\widetilde{A}_i$;*
(2) *$\Lambda_j^-(x) \cap \Gamma_{i+1} \neq \emptyset$;*
(3) *$\widetilde{f}^{-1}(\Lambda_i^-(x)) \subset \Lambda_i^-(\widetilde{f}^{-1}(x))$; and*
(4) *there exists a real number $M_i^- > 0$ (independent of x) such that*

$$\operatorname{diam} p_1(\Lambda_i^-(x)) < M_i^-.$$

For the proof of Proposition 6.5, we will use the following lemma.

LEMMA 6.6. *For $x \in \Lambda_i^-$, we write $\Lambda_{n,i}^-(x)$ for the connected component of $\Lambda_{n,i}^-$ containing x. Then*

$$\Lambda_i^-(x) = \bigcap_{n \in \mathbb{N}} \Lambda_{n,i}^-(x).$$

Proof of Lemma 6.6. First, for every n in $\mathbb{N}$, from Proposition 6.3 we deduce that $\Lambda_i^-(x) \subset \Lambda_{n,i}^-(x)$. This implies that $\Lambda_i^-(x) \subset \bigcap_{n \in \mathbb{N}} \Lambda_{n,i}^-(x)$. To prove the reverse inclusion, we need the following claim.

CLAIM. *$(\Lambda_{n,i}^-(x))_{n \geq 1}$ is a decreasing sequence of compact and connected sets contained in Λ_i^-.*

Proof of the claim. By definition, for every integer $n \geq 1$, the set $\Lambda_{n,i}^-(x)$ is connected and, since Γ_i is an attracting line, we deduce that

$$\Lambda_{n+1,i}^- = \widetilde{f}^{n+1}(\mathrm{Cl}(U_{\Gamma_i}^-)) \cap \mathrm{Cl}(U_{\Gamma_{i+1}}^+) \subset \widetilde{f}^n(\mathrm{Cl}(U_{\Gamma_i}^-)) \cap \mathrm{Cl}(U_{\Gamma_{i+1}}^+) = \Lambda_{n,i}^-.$$

This implies that $\Lambda_{n+1,i}^-(x) \subset \Lambda_{n,i}^-(x)$, that is, the sequence is decreasing. Now, compactness follows from the fact that $\widetilde{f}^n(\Gamma_0) \cap \Gamma_2 \neq \emptyset$ (Hypothesis (H_2)) and that $\widetilde{f}$ commutes with T. This proves the claim. □

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

From the claim, the intersection $\bigcap_{n\in\mathbb{N}} \Lambda^-_{n,i}(x)$ is a compact, connected set contained in Λ^-_i that contains x. We conclude that $\bigcap_{n\in\mathbb{N}} \Lambda^-_{n,i}(x) \subset \Lambda^-_i(x)$. This completes the proof of the lemma. □

Proof of Proposition 6.5. Proof of item (1). We proved (see the above claim) that $\Lambda^-_{n,i}(x)$ is a compact set for every integer $n \geq 1$, so it provides compactness of $\Lambda^-_i(x)$. Moreover, by Proposition 6.3, $\Lambda^-_i(x)$ is contained in $\widetilde{A}_i$.

Proof of item (2). We know that $(\Lambda^-_{n,i}(x))_{n\geq 1}$ is a decreasing sequence of compact and connected sets. Moreover, for every integer $n \geq 1$, $\Lambda^-_{n,i}(x) \cap \Gamma_{i+1} \neq \emptyset$. Therefore $(\Lambda^-_{n,i}(x) \cap \Gamma_{i+1})_{n\geq 1}$ is a decreasing sequence of compact non-empty sets and so, by Proposition 6.3,

$$\Lambda^-_i(x) \cap \Gamma_{i+1} = \bigcap_{n\in\mathbb{N}} \Lambda^-_{n,i}(x) \cap \Gamma_{i+1} \neq \emptyset.$$

This proves item (2).

Proof of item (3). By Proposition 6.3, $\widetilde{f}^{-1}(\Lambda^-_i) \subset \Lambda^-_i$. By connectedness, we deduce that $\widetilde{f}^{-1}(\Lambda^-_i(x)) \subset \Lambda^-_i(\widetilde{f}^{-1}(x))$.

Proof of item (4). Let x be in Λ^-_i. Since, by Lemma 6.6, $\Lambda^-_i(x)$ is contained in $\Lambda^-_{1,i}(x)$, it suffices to check the property for $\Lambda^-_{1,i}$. We will use the following result of Kerékjártó (see [**LY97**]). Let U_1 and U_2 be two Jordan domains in the two-dimensional sphere, that is, connected open sets whose boundary is homeomorphic to the circle. Then each connected component of $U_1 \cap U_2$ is also a Jordan domain.

Kerékjártó's result, applied to $U_1 = \widetilde{f}(U^-_{\Gamma_i})$ and $U_2 = U^+_{\Gamma_{i+1}}$, states that the connected component of $\widetilde{f}(U^-_{\Gamma_i}) \cap U^+_{\Gamma_{i+1}}$ containing x is a Jordan domain. We call this set X. Moreover, by Hypothesis (H_2) and the fact that $\widetilde{f}$ commutes with T, the set X is bounded in $\widetilde{\mathbb{A}}$: that is, its closure is compact. Hence, there exists an arc Γ'_i, contained in Γ_i, such that $\widetilde{f}(\Gamma'_i)$ is a connected component of $\mathrm{Fr}(X) \cap \widetilde{f}(\Gamma_i)$, with $p_1(\widetilde{f}(\Gamma'_i))$ maximal with respect to the inclusion. Let Γ'_{i+1} be the closure of the bounded connected component of $\Gamma_{i+1} \setminus \widetilde{f}(\Gamma'_i)$. Let X' be the domain delimited by the circle $\Gamma'_{i+1} \cup \widetilde{f}(\Gamma'_i)$.

- $\Lambda_{1,i}(x)$ is contained in the closure of X'.
- $\mathrm{diam}\, p_1(X') \leq \mathrm{diam}\, p_1(\widetilde{f}(\Gamma'_i))$.

Hence it remains to prove that there exists M'_i (independent of Γ'_i) such that $\mathrm{diam}\, p_1(\widetilde{f}(\Gamma'_i)) < M'_i$. Let

$$M_0 := \sup_{\widetilde{z}\in\widetilde{A}_i} |p_1(\widetilde{f}(\widetilde{z})) - p_1(\widetilde{z})|.$$

Suppose that Γ'_i is the arc with endpoints $\widetilde{z}_0$ and $\widetilde{z}_1$, with $p_1(\widetilde{z}_0) < p_1(\widetilde{z}_1)$. By Hypothesis (H_2) and the fact that $\widetilde{f}$ commutes with T, $p_1(\widetilde{z}_1) \leq p_1(\widetilde{z}_0) + 1$. Therefore,

$$\mathrm{diam}\, p_1(\widetilde{f}(\Gamma'_i)) \leq 2M_0 + 1.$$

This proves item (4) and completes the proof of the proposition. □

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

6.3. *Consequence of Hypothesis* (H_3). In this subsection, we consider a homeomorphism $\widetilde{f} : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$, isotopic to the identity, that commutes with the translation T and satisfies Hypotheses (H_1)–(H_3). Using Hypothesis (H_3), we will study the dynamics of unstable branches (we have the symmetric properties for stable branches).

Let us begin by recalling that, for $i \in \{0, 1\}$, the inclusion $\Theta(\widetilde{A}_i) \subset \widetilde{A}_i$ implies that $\rho_{\Theta(\widetilde{A}_i)}(\widetilde{f}) \subset \rho_{\widetilde{A}_i}(\widetilde{f})$, but we cannot use Hypothesis (H_3) to conclude that $\rho_{\widetilde{A}_0}(\widetilde{f}) \subset (0, +\infty)$ and $\rho_{\widetilde{A}_1}(\widetilde{f}) \subset (-\infty, 0)$. However, using the fact that the straight lines Γ_i and Γ_{i+1} are attracting, we can prove that the above inclusions are true.

PROPOSITION 6.7. *For $i \in \{0, 1\}$,*

$$\rho_{\widetilde{A}_i}(\widetilde{f}) \subset \rho_{\mathrm{mes}}(\Theta(\widetilde{A}_i), \widetilde{f}) = \mathrm{Conv}\, \rho_{\Theta(\widetilde{A}_i)}(\widetilde{f}).$$

In particular, $\rho_{\widetilde{A}_0}(\widetilde{f}) \subset (0, +\infty)$ and $\rho_{\widetilde{A}_1}(\widetilde{f}) \subset (-\infty, 0)$.

Proof. Recall that we are using the notation $\rho_{\widetilde{A}_i}(\widetilde{f})$ instead of $\rho_{\pi(\widetilde{A}_i)}(I)$. Let ρ be an element in $\rho_{\widetilde{A}_i}(\widetilde{f})$. By definition and the fact that $\gamma_i = \pi(\Gamma_i)$ and $\gamma_{i+1} = \pi(\Gamma_{i+1})$ are f-free, there exist a sequence of points $(z_k)_{k\in\mathbb{N}}$ and a sequence of integers $(n_k)_{k\in\mathbb{N}}$ satisfying:

(i) for every $k \in \mathbb{N}$ and integer $j \in \{0, \ldots, n_k\}$, $f^j(z_k) \in A_i = \pi(\widetilde{A}_i)$;

(ii) the sequence $(n_k)_{k\in\mathbb{N}}$ converges to $+\infty$; and

(iii) the sequence $(\rho_{n_k}(z_k))_{k\in\mathbb{N}}$ converges to ρ.

Let us consider the sequence of Borel probability measures $(\mu_k)_{k\in\mathbb{N}}$, where, for every integer k, μ_k is defined as

$$\mu_k := \frac{1}{n_k} \sum_{i=0}^{n_k-1} \delta_{f^i(z_k)},$$

where δ_z $(z \in \mathbb{A})$ denotes the *Dirac's measure at* z. By item (i), these measures have their supports included in A_i, and so, changing the sequence $(\mu_k)_{k\in\mathbb{N}}$ by one of its subsequences, we may suppose that the sequence $(\mu_k)_{k\in\mathbb{N}}$ converges. Let μ be its limit. We know that the measure μ is invariant (under f), and so its support $\mathrm{Supp}(\mu)$ is contained in the maximal f-invariant set of A_i that is denoted by $\Theta(A_i)$. Moreover,

$$\rho_{n_k}(z_k) = \int \rho_1 \, d\mu_k \to \int \rho_1 \, d\mu = \rho(\mu).$$

From item (iii) and the uniqueness of the limit, we deduce that ρ belongs to $\rho_{\mathrm{mes}}(\Theta(\widetilde{A}_i), \widetilde{f})$. This proves that $\rho_{\widetilde{A}_i}(\widetilde{f}) \subset \rho_{\mathrm{mes}}(\Theta(\widetilde{A}_i), \widetilde{f})$. On the other hand, by Lemma 4.2, we know that $\rho_{\mathrm{mes}}(\Theta(\widetilde{A}_i), \widetilde{f}) = \mathrm{Conv}\, \rho_{\Theta(\widetilde{A}_i)}(\widetilde{f})$. Thus, using Hypothesis (H_3), $\rho_{\widetilde{A}_0}(\widetilde{f}) \subset (0, +\infty)$ and $\rho_{\widetilde{A}_1}(\widetilde{f}) \subset (-\infty, 0)$, because $\rho_{\widetilde{A}_i}(\widetilde{f})$ is a compact set. This completes the proof of the proposition. □

The following lemmas is an immediate consequence of the above proposition (and, in particular, the fact that zero does not belong to $\rho_{\widetilde{A}_0}(\widetilde{f})$).

LEMMA 6.8. *For every real number $M > 0$, there exists $n_0 = n_0(M) \in \mathbb{N}$ such that, for every compact set $\widetilde{K}$ included in $\widetilde{A}_0$ with* $\mathrm{diam}\, p_1(\widetilde{K}) \le M$ *and, for every integer n, $|n| \ge n_0$,*

$$\widetilde{f}^n(\widetilde{K}) \cap \widetilde{K} = \emptyset.$$

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

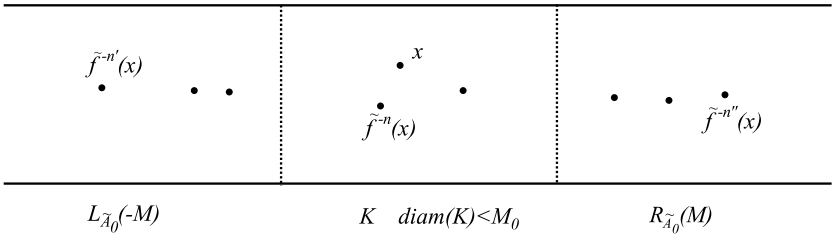


FIGURE 7. The orbit of x.

Proof. Suppose, by contradiction, that there is a real number $M_0 > 0$ and a sequence of integers $(n_k)_{k\in\mathbb{N}}$ which tends to $+\infty$ and there are points $\widetilde{z}_k \in \widetilde{A}_0$ such that $\widetilde{f}^{n_k}(\widetilde{z}_k) \in \widetilde{A}_0$ and

$$|p_1(\widetilde{f}^{n_k}(\widetilde{z}_k)) - p_1(\widetilde{z}_k)| \le M_0.$$

Hence,

$$|\rho_{n_k}(\widetilde{z}_k, \widetilde{f})| := \frac{1}{n_k}|p_1(\widetilde{f}^{n_k}(\widetilde{z}_k)) - p_1(\widetilde{z}_k)| \le \frac{M_0}{n_k}.$$

Letting k tend to infinity, one deduces that zero belongs to $\rho_{\widetilde{A}_0}(\widetilde{f})$. This contradicts Proposition 6.7 and completes the proof. □

In what follows, for every real number M and $i \in \{0, 1\}$, we write

$$L_{\widetilde{A}_i}(M) := \{\widetilde{z} \in \widetilde{A}_i : p_1(\widetilde{z}) < M\} \quad \text{and} \quad R_{\widetilde{A}_i}(M) := \{\widetilde{z} \in \widetilde{A}_i : p_1(\widetilde{z}) > M\}.$$

LEMMA 6.9. *Let $x \in \Theta(\widetilde{A}_0)$. For every real number $M > 0$, there exists $n_0 \in \mathbb{N}$ such that, for every integer $n \ge n_0$,*

$$\widetilde{f}^{-n}(x) \subset L_{\widetilde{A}_0}(-M) \quad \text{and} \quad \widetilde{f}^{n}(x) \subset R_{\widetilde{A}_0}(M).$$

In other words, $\lim_{n\to+\infty} p_1(\widetilde{f}^{-n}(x)) = -\infty$ *and* $\lim_{n\to+\infty} p_1(\widetilde{f}^{n}(x)) = +\infty$.

Proof. In both cases the proofs are analogous, so we only prove the first inclusion. Let

$$M_0 := \sup_{\widetilde{z}\in\widetilde{A}_0} |p_1(\widetilde{f}(\widetilde{z})) - p_1(\widetilde{z})|.$$

Let M be a positive real number such that $2M > M_0$ and $-M < p_1(x) < M$. Let us put $\widetilde{K} = \mathrm{Cl}(R_{\widetilde{A}_0}(-M)) \cap \mathrm{Cl}(L_{\widetilde{A}_0}(M))$. By Lemma 6.8, there is an integer n_0 such that, for every integer $n \ge n_0$, $\widetilde{f}^{-n}(x) \notin \widetilde{K}$: that is, $\widetilde{f}^{-n}(x) \in L_{\widetilde{A}_0}(-M) \cup R_{\widetilde{A}_0}(M)$. Suppose that there are two integers $n'' > n' \ge n_0$ such that $\widetilde{f}^{-n'}(x) \in L_{\widetilde{A}_0}(-M)$ and $\widetilde{f}^{-n''}(x) \in R_{\widetilde{A}_0}(M)$. The following claim contradicts the choice of n_0 (see Figure 7).

CLAIM. *There is an integer $n'' > n > n' \ge n_0$ such that $\widetilde{f}^{-n}(x) \in \widetilde{K}$.*

Proof. Let

$$n - 1 := \max\{m \in \{n', \dots, n''\} : \widetilde{f}^{-m}(x) \in L_{\widetilde{A}_0}(-M)\}.$$

(From the choice of n' and n'', such an integer exists). By the definition of n, $p_1(\widetilde{f}^{-n+1}(x)) < -M \le p_1(\widetilde{f}^{-n}(x))$, and so, in order to prove the claim, it suffices to check that $\widetilde{f}^{-n}(x) < M$. Recall that, for every integer m, the point $\widetilde{f}^{m}(x)$ is in $\widetilde{A}_0$ and so

$$p_1(\widetilde{f}^{-n}(x)) = p_1(\widetilde{f}^{-n}(x)) - p_1(\widetilde{f}^{-n+1}(x)) + p_1(\widetilde{f}^{-n+1}(x)) \le M_0 - M < M.$$

This proves the claim. □

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

We conclude that the sequence $(f^{-n}(x))_{n\geq n_0}$ is contained either in $L_{\widetilde{A}_0}(-M)$ or in $R_{\widetilde{A}_0}(M)$. Since $\rho_{\Theta(\widetilde{A}_0)}(\widetilde{f}) \subset (0, +\infty)$, the second inclusion cannot be possible, which completes the proof of the lemma. □

LEMMA 6.10. *Let $x \in \Theta(\widetilde{A}_0)$. Then the sequence $\{\widetilde{f}^{-n}(\Lambda_0^-(x))\}_{n\in\mathbb{N}}$ goes uniformly to the left: that is, for every real number M, there exists $n_0 \in \mathbb{N}$ such that, for every integer $n \geq n_0$,*

$$\widetilde{f}^{-n}(\Lambda_0^-(x)) \subset L_{\widetilde{A}_0}(M).$$

In other words, for every $\widetilde{z} \in \Lambda_0^-(x)$, $\lim_{n\to+\infty} p_1(\widetilde{f}^{-n}(\widetilde{z})) = -\infty$.

Proof. Let $n \in \mathbb{N}$. Since $\widetilde{f}^{-n}(\Lambda_0^-(x)) \subset \Lambda_0^-(\widetilde{f}^{-n}(x))$, Proposition 6.5 provides a real number M_0^- (independent of n) such that, for every $\widetilde{z} \in \Lambda_0^-(x)$,

$$|p_1(\widetilde{f}^{-n}(\widetilde{z})) - p_1(\widetilde{f}^{-n}(x))| < M_0^-.$$

On the other hand, by Lemma 6.9, $\lim_{n\to+\infty} p_1(\widetilde{f}^{-n}(x)) = -\infty$. This completes the proof of the lemma. □

Remark 11. The symmetric result for stable branches of Lemma 6.10 is the following. Let $y \in \Theta(\widetilde{A}_1)$. For every $\widetilde{z} \in \Lambda_1^+(y)$, $\lim_{n\to+\infty} p_1(\widetilde{f}^n(\widetilde{z})) = -\infty$.

6.4. *Proof of Proposition 6.1.* In this section, we prove Proposition 6.1 which is reformulated as follows.

PROPOSITION 6.11. (Proposition 6.1) *Let $\widetilde{f} : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ be a homeomorphism, isotopic to the identity, that commutes with the translation T and satisfies Hypotheses (H_1) to (H_3). For every $x \in \Theta(\widetilde{A}_0)$, there exist $y \in \Theta(\widetilde{A}_1)$ and a positive integer n_0 such that, for every integer $n \geq n_0$,*

$$\widetilde{f}^n(\Lambda_0^-(x)) \cap \Lambda_1^+(y) \neq \emptyset.$$

We note that if $\widetilde{z} \in \widetilde{f}^n(\Lambda_0^-(x)) \cap \Lambda_1^+(y)$, then, by definition (see Remark 10) and Hypothesis (H_1), the point $\widetilde{z} \in \Theta(\widetilde{A})$. By Lemma 6.10 and its symmetric result for stable branches (see Remark 11), the sequence of positive and negative iterates of $\widetilde{z}$ goes to the left. We will prove, in the next subsection, that this implies Theorem C*.

Proof of Proposition 6.11. Let $x \in \Theta(\widetilde{A}_0)$. Since $\Lambda_0^-(x)$ is a compact set, we can consider a real number M such that $\Lambda_0^-(x) \subset L_{\widetilde{A}_0}(M)$. Let M_1^+ be the positive real constant provided by Proposition 6.5 such that

$$\max_{y'\in\Theta(\widetilde{A}_1)} \operatorname{diam} p_1(\Lambda_1^+(y')) < M_1^+.$$

Let us fix $y \in \Theta(\widetilde{A}_1)$ such that $p_1(y) > M + M_1^+$. We recall that, by the choice of y,

$$\Lambda_0^-(x) \subset L_{\widetilde{A}_0}(M) \quad \text{and} \quad \Lambda_1^+(y) \subset R_{\widetilde{A}_1}(M). \tag{1}$$

LEMMA 6.12. *For every integer $n \geq 0$,*

$$\widetilde{f}^n(\Gamma_1) \cap \Lambda_1^+(y) \neq \emptyset.$$

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

Proof. Suppose, by contradiction, that there exists a positive integer n' such that $\widetilde{f}^{n'}(\Gamma_1) \cap \Lambda_1^+(y) = \emptyset$. This implies that $\Lambda_1^+(y)$ is contained in one of the connected components of $\widetilde{\mathbb{A}} \backslash \widetilde{f}^{n'}(\Gamma_1)$. Since Γ_1 is an attracting line, that is, $\widetilde{f}^{n'}(\Gamma_1) \subset U_{\Gamma_1}^-$, and, since $\Lambda_1^+(y)$ meets Γ_1 (Hypothesis (H_1) and Proposition 6.5), we conclude that $\Lambda_1^+(y) \subset U^+_{\widetilde{f}^{n'}(\Gamma_1)}$, and so $y \in U^+_{\widetilde{f}^{n'}(\Gamma_1)} = \widetilde{f}^{n'}(U_{\Gamma_1}^+)$. This contradicts the choice of y because $y \in \Theta(\widetilde{A}_1) \subset \bigcap_{n\in\mathbb{N}} \widetilde{f}^n(U_{\Gamma_1}^-)$. □

From Proposition 6.5, we know that $\Lambda_1^+(y) \cap \Gamma_1$ is a non-empty set. Let us fix $\widetilde{z} \in \Lambda_1^+(y) \cap \Gamma_1$ and consider the vertical line segment $\alpha : [0, 1] \to \widetilde{\mathbb{A}}$ satisfying:

- $\alpha(0) = \widetilde{z}$;
- $\alpha(1) \in \Gamma_0$; and
- $\alpha((0, 1)) \subset \mathrm{Int}(\widetilde{A}_0)$.

Let $R_{\widetilde{A}_0}(\alpha)$ (respectively, $L_{\widetilde{A}_0}(\alpha)$) be the connected component of $\widetilde{A}_0 \backslash \alpha$ which is unbounded to the right (respectively, to the left). Let $\widetilde{K} = \Lambda_0^-(x) \cup \alpha$. Since $\widetilde{K}$ is a compact subset of $\widetilde{A}_0$, by Lemma 6.8 there is a positive integer n_0 such that, for every integer $n \geq n_0$,

$$\widetilde{f}^n(\Lambda_0^-(x)) \cap \alpha \subset \widetilde{f}^n(\widetilde{K}) \cap \widetilde{K} = \emptyset. \tag{2}$$

Moreover, by Lemma 6.9, we can assume that n_0 is such that, for every integer $n \geq n_0$,

$$\widetilde{f}^n(x) \in R_{\widetilde{A}_0}(\alpha). \tag{3}$$

From now on, we fix such an integer n_0. Let us consider $\Lambda_0^{\pm}(x) := \Lambda_0^-(x) \cup \Lambda_0^+(x)$. This is a connected and compact subset of $\widetilde{A}_0$ which separates the band $\widetilde{A}_0$ (it meets both Γ_0 and Γ_1). We can define its right $R_{\widetilde{A}_0}(\Lambda_0^{\pm}(x))$: that is, the connected component of $\widetilde{A}_0 \backslash \Lambda_0^{\pm}(x)$ which is unbounded to the right.

Let us define

$$\Gamma_1^r := \mathrm{Cl}(R_{\widetilde{A}_0}(\Lambda_0^{\pm}(x))) \cap \Gamma_1 \quad \text{and} \quad \Gamma_1^l := \Gamma_1 \backslash \Gamma_1^r.$$

(See Figure 8). Note that

$$\{\widetilde{z}_0\} = \Gamma_1^r \cap \Lambda_0^-(x) \subset R_{\widetilde{A}_0}(\Lambda_0^{\pm}(x)) \cap \Lambda_0^{\pm}(x). \tag{4}$$

Proposition 6.11 will be a consequence of the following lemma.

LEMMA 6.13. *For every integer $n \geq n_0$,*

$$(\widetilde{f}^n(\Lambda_0^-(x)) \cup \widetilde{f}^n(\Gamma_1^l)) \cap \Lambda_1^+(y) \neq \emptyset.$$

Proof. Fix an integer $n \geq n_0$. We have two cases. If $\widetilde{f}^n(\Gamma_1^r) \cap \Lambda_1^+(y) = \emptyset$, then, by Lemma 6.12 and the fact that $\Gamma_1 = \Gamma_1^l \cup \Gamma_1^r$, $\widetilde{f}^n(\Gamma_1^l) \cap \Lambda_1^+(y) \neq \emptyset$. This completes the proof in this case.

Now suppose that $\widetilde{f}^n(\Gamma_1^r) \cap \Lambda_1^+(y) \neq \emptyset$. Let $\Gamma_1^{r'}$ be a closed line contained in Γ_1^r, unbounded to the right such that if $\widetilde{z}_1$ is its endpoint, then $\widetilde{f}^n(\widetilde{z}_1) \in \Lambda_1^+(y)$, but

$$\widetilde{f}^n(\Gamma_1^{r'} \backslash \{\widetilde{z}_1\}) \cap \Lambda_1^+(y) = \emptyset.$$

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

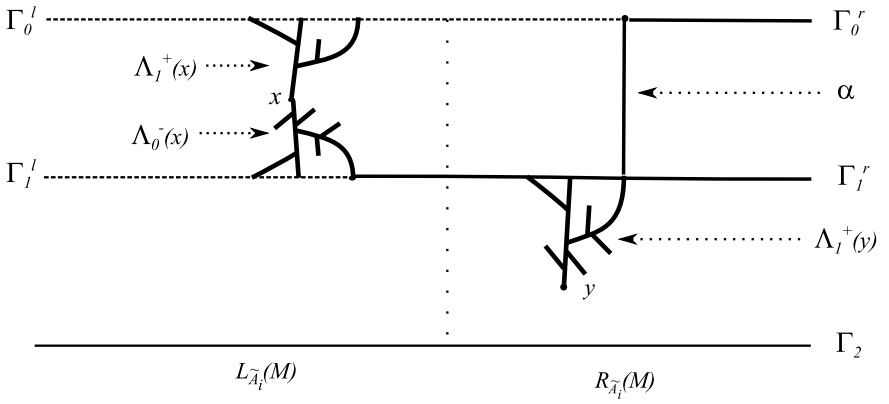


FIGURE 8. Proof of Proposition 6.11.

Let $\Gamma_0^l = \Gamma_0 \cap L_{\widetilde{A}_0}(\alpha)$ and $\Gamma_0^r = \Gamma_0 \backslash \Gamma_0^l$ (see Figure 9). Let

$$C_n := \Lambda_1^+(y) \cup \widetilde{f}^n(\Gamma_1^{r'}) \cup \alpha \cup \Gamma_0^r.$$

Remark that C_n is a connected set. Moreover, we can assume that it does not meet $\widetilde{f}^n(\Lambda_0^-(x))$. Otherwise, $\widetilde{f}^n(\Lambda_0^-(x)) \cap \Lambda_1^+(y) \neq \emptyset$ because $\widetilde{f}^n(\Lambda_0^-(x))$ does not meet α and Γ_0 (relation (2) and Remark 10 (it follows Proposition 6.3), respectively). Moreover, if $\widetilde{f}^n(\Lambda_0^-(x))$ meets $\widetilde{f}^n(\Gamma_1^{r'})$, then $\widetilde{z}_0 = \widetilde{z}_1$. Hence, by definition, $\widetilde{f}^n(\widetilde{z}_1) \in \widetilde{f}^n(\Lambda_0^-(x)) \cap \Lambda_1^+(y)$. This finishes the proof in this case.

Since Γ_1 is an attracting line, that is, $\widetilde{f}^n(\Gamma_1) \subset U_{\Gamma_1}^-$ and $\Lambda_1^+(y)$ is contained in $\widetilde{A}_1$,

$$\mathrm{Int}(R_{\widetilde{A}_0}(\alpha)) \subset \mathrm{Int}(\widetilde{A}_0) \backslash \alpha \subset \widetilde{\mathbb{A}} \backslash C_n.$$

Let X_n be the connected component of $\widetilde{\mathbb{A}} \backslash C_n$ that contains $\mathrm{Int}(R_{\widetilde{A}_0}(\alpha))$. Note that since C_n is a closed subset of $\widetilde{\mathbb{A}}$, X_n is an open set of $\widetilde{\mathbb{A}}$ and $\mathrm{Fr}(X_n)$ is contained in C_n. Hence X_n is bounded to the left because $\Lambda_1^+(y) \cup \alpha$ and $\widetilde{f}^n(\Gamma_1^r)$ are bounded to the left. On the other hand, since, by relation (3), $\widetilde{f}^n(x) \in R_{\widetilde{A}_0}(\alpha)$ and $\widetilde{f}^n(\Lambda_0^-(x))$ does not meet C_n, $\widetilde{f}^n(\Lambda_0^-(x))$ is contained in X_n. Therefore the curve $\widetilde{f}^n(\Gamma_1^l)$ which is unbounded to the left must meet $\mathrm{Fr}(X_n)$. As $\mathrm{Fr}(X_n) \subset C_n$, it follows that

$$\widetilde{f}^n(\Gamma_1^l) \cap C_n \neq \emptyset.$$

Since Γ_1 is an attracting line and since $\Gamma_1^l \cap \Gamma_1^r = \emptyset$ and $\widetilde{f}$ is a homeomorphism, $\widetilde{f}^n(\Gamma_1) \cap (\alpha \cup \Gamma_0) = \emptyset$ and $\widetilde{f}^n(\Gamma_1^l) \cap \widetilde{f}^n(\Gamma_1^r) = \emptyset$, respectively. Therefore, we conclude that $\widetilde{f}^n(\Gamma_1^l) \cap \Lambda_1^+(y) \neq \emptyset$. This completes the proof of the lemma. □

End of the proof of Proposition 6.11. Suppose, by contradiction, that there exists a sequence of integers $(n_k)_{k\in\mathbb{N}}$ which converges to $+\infty$ such that

$$\widetilde{f}^{n_k}(\Lambda_0^-(x)) \cap \Lambda_1^+(y) = \emptyset.$$

From Lemma 6.13, we obtain a sequence of points $(\widetilde{z}_k)_{k\in\mathbb{N}}$ with $\widetilde{z}_k \in \Gamma_1^l$ and $\widetilde{f}^{n_k}(\widetilde{z}_k) \in \Lambda_1^+(y)$. Recall that, by (1), $\Gamma_1^l \subset L_{\widetilde{A}_0}(M)$ and $\Lambda_1^+(y) \subset R_{\widetilde{A}_1}(M)$. Hence $p_1(\widetilde{z}_k) < p_1(\widetilde{f}^{n_k}(\widetilde{z}_k))$. If ρ is a limit point of the sequence $(\rho_{n_k}(\widetilde{z}_k))_{k\in\mathbb{N}}$, where

$$\rho_{n_k}(\widetilde{z}_k) := \frac{1}{n_k}(p_1(\widetilde{f}^{n_k}(\widetilde{z}_k)) - p_1(\widetilde{z}_k)),$$

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

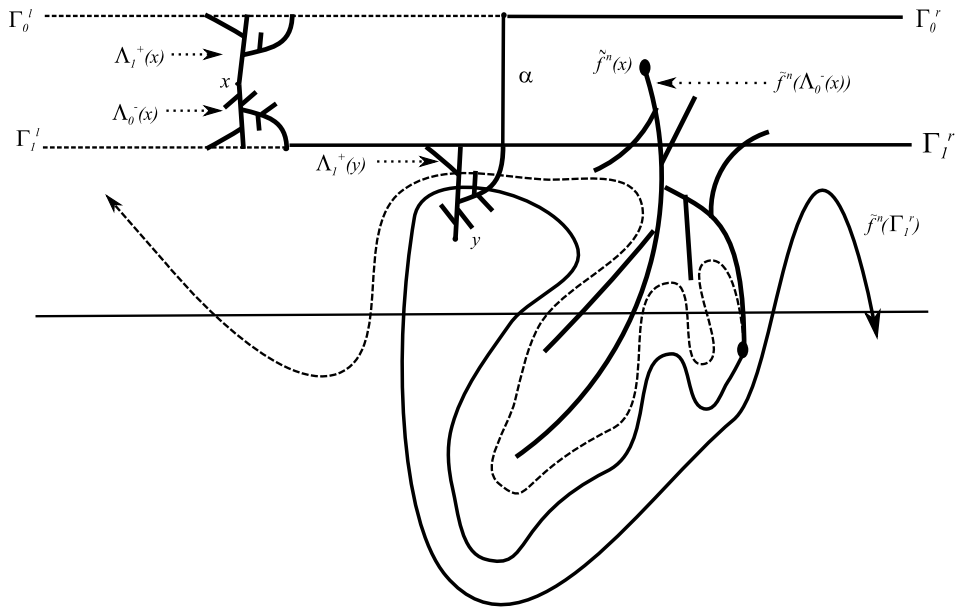


FIGURE 9. Proof of Lemma 6.13.

then $\rho \geq 0$. This implies that $0 \leq \rho \in \rho_{\widetilde{A}_1}(\widetilde{f})$, which contradicts Proposition 6.7. This completes the proof of Proposition 6.11. □

6.5. *End of the proof of Theorem C*.* Let $n_0 \in \mathbb{N}$ and let $\widetilde{z} \in \widetilde{f}^{n_0}(\Lambda_0^-(x)) \cap \Lambda_1^+(y)$ be as given by Proposition 6.11. From the definitions of $\Lambda_0^-(x)$ and $\Lambda_1^+(y)$ (see Remark 10 following Proposition 6.3) and the fact that Γ_0 and Γ_2 are $\widetilde{f}$-free (Hypothesis (H_1)), we deduce that $\widetilde{z} \in \Theta(\widetilde{A})$. Moreover, by Lemma 6.10 and its symmetric property for stable branches (Remark 11 following Lemma 6.10), we know that

$$\lim_{n\to+\infty} p_1(\widetilde{f}^{-n}(\widetilde{z})) = -\infty \quad \text{and} \quad \lim_{n\to+\infty} p_1(\widetilde{f}^{n}(\widetilde{z})) = -\infty. \tag{5}$$

Let us consider

$$M_0 := \sup_{\widetilde{z}' \in \widetilde{A}} |p_1(\widetilde{f}(\widetilde{z}')) - p_1(\widetilde{z}')|.$$

Let k be a positive integer and let us define $M := p_1(\widetilde{z}) - kM_0$. By the claim in Lemma 6.9, we prove that there are two positive integers n^+ and n^- such that

$$M - M_0 \leq p_1(\widetilde{f}^{-n^-}(\widetilde{z})) < M \quad \text{and} \quad M - M_0 \leq p_1(\widetilde{f}^{n^+}(\widetilde{z})) < M.$$

Therefore, by the choice of M and M_0, we deduce that:

- $n^+ + n^- \geq 2k$; and
- $$|p_1(\widetilde{f}^{n^+}(\widetilde{z})) - p_1(\widetilde{f}^{-n^-}(\widetilde{z}))| \leq M_0.$$

Since k can be chosen arbitrarily large, this implies that zero belongs to $\rho_{\Theta(\widetilde{A})}(\widetilde{f})$. This is what we needed to prove.

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press

Acknowledgements. I thank F. Le Roux and S. Crovisier for proposing this problem. I thank them for key comments and ideas that led to the main constructions and proofs and, in particular, for their constant support and encouragement.

REFERENCES

[**Dav13**] P. Dávalos. On torus homeomorphisms whose rotation set is an interval. *Math. Z.* **275**(3–4) (2013), 1005–1045.
[**dMP82**] W. de Melo and J. Palis Jr.. *Geometric Theory of Dynamical Systems: An Introduction*. Springer, New York, 1982.
[**FM90**] J. Franks and M. Misiurewicz. Rotation sets of toral flows. *Proc. Amer. Math. Soc.* **109**(1) (1990), 243–249.
[**Fra88**] J. Franks. Generalizations of the Poincaré–Birkhoff theorem. *Ann. of Math. (2)* **128**(1) (1988), 139–151.
[**Fra96**] J. Franks. Area preserving homeomorphisms of open surfaces of genus zero. *New York J. Math.* **2** (1996), 1–19.
[**GP95**] J.-M. Gambaudo and É. Pécou. A topological invariant for volume preserving diffeomorphisms. *Ergod. Th. & Dynam. Sys.* **15**(3) (1995), 535–541.
[**Hae62**] A. Haefliger. Variétés feuilletées. *Ann. Sc. Norm. Super. Pisa (3)* **16**(4) (1962), 367–397.
[**Ham74**] M.-E. Hamströn. Homotopy in homeomorphism spaces, TOP and PL. *Bull. Amer. Math. Soc. (N.S.)* **80** (1974), 207–230.
[**LeC91**] P. Le Calvez. Propriétés dynamiques des difféomorphismes de l'anneau et du tore. *Astérisque* (204) (1991), 1–131.
[**LeC01**] P. Le Calvez. Rotation numbers in the infinite annulus. *Proc. Amer. Math. Soc.* **129**(11) (2001), 3221–3230.
[**LeC03**] P. Le Calvez. Dynamique des homéomorphismes du plan au voisinage d'un point fixe. *Ann. Sc. Éc. Norm. Supér. (4)* **36**(1) (2003), 139–171.
[**LeC05**] P. Le Calvez. Une version feuilletée équivariante du théorème de translation de Brouwer. *Publ. Math. Inst. Hautes Études Sci.* (102) (2005), 1–98.
[**LeC08**] P. Le Calvez. Pourquoi les points périodiques des homéomorphismes du plan tournent-ils autour de certains points fixes? *Ann. Sc. Éc. Norm. Supér. (4)* **41**(1) (2008), 141–176.
[**LeR00**] F. Le Roux. Migration des points errants d'un homéomorphisme de surface. *C. R. Math. Acad. Sci. Paris, Sér. I* **330**(3) (2000), 225–230.
[**LeR04**] F. Le Roux. Homéomorphismes de surfaces: théorèmes de la fleur de Leau-Fatou et de la variété stable. *Astérisque* (292) (2004), 1–120.
[**LeR13**] F. Le Roux. L'ensemble de rotation autour d'un point fixe. *Astérisque* (350) (2013), 1–109.
[**LM91**] J. Llibre and R. S. MacKay. Rotation vectors and entropy for homeomorphisms of the torus isotopic to the identity. *Ergod. Th. & Dynam. Sys.* **11**(1) (1991), 115–128.
[**LT15**] P. Le Calvez and F. A. Tal. Forcing theory for transverse trajectories of surface homeomorphisms. *Preprint*, 2015, arXiv:1503.09127 (e-print).
[**LY97**] P. Le Calvez and J.-C. Yoccoz. Un théorème d'indice pour les homéomorphismes du plan au voisinage d'un point fixe. *Ann. of Math. (2)* **146**(2) (1997), 241–293.
[**MZ89**] M. Misiurewicz and K. Ziemian. Rotation sets for maps of tori. *J. Lond. Math. Soc.* **40**(2) (1989), 490–506.
[**Naĭ82**] V. A. Naĭshul'. Topological invariants of analytic and area-preserving mappings and their application to analytic differential equations in $\mathbf{C}^2$ and $\mathbf{C}P^2$. *Tr. Mosk. Mat. Obs.* **44** (1982), 235–245.
[**Poi85**] H. Poincaré. Mémoire sur les courbes définies par une équation différentielle, chapitre XV. *J. Math. Pures Appl. Série IV* (1) (1885), 167–244.
[**Wan14**] J. Wang. A generalization of the line translation theorem. *Trans. Amer. Math. Soc.* **366**(11) (2014), 5903–5923.
[**Whi33**] H. Whitney. Regular families of curves. *Ann. of Math. (2)* **34**(2) (1933), 244–270.
[**Whi41**] H. Whitney. On regular families of curves. *Bull. Amer. Math. Soc.* **47** (1941), 145–147.

https://doi.org/10.1017/etds.2016.129 Published online by Cambridge University Press