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Almost transitivity of some function spaces

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Abstract

The almost transitive norm problem is studied for $L^p(\mu, X)$, $C(K, X)$ and for certain Orlicz and Musielak–Orlicz spaces. For example if $p \neq 2 < \infty$ then $L^p(\mu)$ has almost transitive norm if and only if the measure μ is homogeneous. It is shown that the only Musielak–Orlicz space with almost transitive norm is the L^p-space. Furthermore, an Orlicz space has an almost transitive norm if and only if the norm is maximal. $L^p(\mu, X)$ has almost transitive norm if $L^p(\mu)$ and X have. Separable spaces with nontrivial L^p-structure fail to have transitive norms. Spaces with nontrivial centralizers and extreme points in the unit ball also fail to have almost transitive norms.

0. *Preliminaries*

It is known that the norm in $L^p(\lambda)$ $(1 \leqslant p < \infty)$, where λ is the Lebesgue measure on the real line, is almost transitive. Also there are examples of non-separable $L^p(\mu)$-spaces that are transitive [**23**]. In Section 1, we extend these results to $L^p(\mu)$-spaces over any positive measure μ. We show that the norm in $L^p(\mu)$ is almost transitive (resp. transitive) if and only if μ is homogeneous (resp. μ is homogeneous and not σ-finite). The main result of Section 2 is that almost transitivity of the norm in a Banach space X is inherited by the norm in the Bochner space $L^p(\mu, X)$. The converse of this theorem is obtained under certain assumptions on X. We give an example showing that the full converse is not possible. Almost transitivity and L^p-structure in Banach spaces are studied in Section 3. We adapt there the results of Section 1 to a general setting of Banach spaces. In Section 4 we show that several properties like almost transitivity, maximality, and convex transitivity of the norm in an Orlicz space L_ϕ are equivalent and L_ϕ possesses any of those properties if and only if L_ϕ is isometric to L^p. A similar result (but only for almost transitivity) is obtained for Musielak–Orlicz spaces. The spaces of continuous and bounded functions are subjects of study in Section 5. It is well known that the supremum norm in CK is not almost transitive. We show that for a wide class of Banach spaces X (those with extreme points, or those with trivial centralizers) the space $C(K, X)$ of vector-valued continuous functions also lacks almost transitive norm. We discuss Wood's conjecture on almost transitivity of the norm in $C_0 L$ and put it in the general context of Banach spaces with M-structure.

Terminology and notations used further in the paper will be as follows. Let

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$(X, \|\ \|)$ be a Banach space over the field $\mathbb{K}$ of scalars $\mathbb{R}$ or $\mathbb{C}$. In this paper isometry of X will always denote a linear operation of X onto itself, which preserves the norm. If two Banach spaces X and Y are isometric, we will write $X \cong Y$. By $G(X)$ denote the set of all isometries of X. A Banach space $(X, \|\ \|)$ is said to have an almost transitive norm if for any $\epsilon > 0$ and any $x, y \in X$ with $\|x\| = \|y\| = 1$ there exists an isometry U: $X \to X$ such that $\|y - Ux\| \leqslant \epsilon$. If ϵ can be chosen equal to zero, then the space is said to have transitive norm. We shall also use the notion of L^p-summand. If X is an l^p-direct sum of two subspaces, $X = U \oplus_p V$, then U and V are called (complementary) L^p-summands and the corresponding projections are called L^p-projections. The L^p-structure of X, denoted by $\mathbb{P}_p(X)$, is the set of all L^p-projections (the term L-structure is used for $p = 1$). It is called trivial if it consists only of the identity and the zero operators. For $p \neq 2$, L^p-projections commute [2] and $\mathbb{P}_p(X)$ is a Boolean algebra.

Let (T, Σ, μ) be a measure space, where μ is a positive measure on a σ-algebra Σ of subsets of T. We denote by Σ/μ the quotient of the Boolean algebra Σ modulo the ideal of μ-null sets, and we will call it its measure algebra (slightly deviating from the notion of a measure algebra in [16]). We will refer to an isomorphism between Boolean algebras as a Boolean isomorphism. Let $\Sigma|A$ and $\mu|A$ denote the restriction to A of Σ and μ respectively. Recall that for a finite measure space (T, Σ, μ), μ is called homogeneous if for every $A \in \Sigma$, the measure algebra $(\Sigma|A)/(\mu|A)$ is Boolean isomorphic to Σ/μ [16]. We extend this notion to arbitrary positive measures: we will say that μ is homogeneous, if its restriction to every set of finite measure has a homogeneous measure algebra. For (T_i, Σ_i, μ_i), $i = 1, 2$, we use the notation $\mu_1 \sim \mu_2$ if the induced measure algebras Σ_i/μ_i $(i = 1, 2)$ are Boolean isomorphic. If (T_i, Σ_i, μ_i) $(i = 1, 2)$ are measure spaces, then for any Boolean isomorphism $\tau: \Sigma_1/\mu_1 \to \Sigma_2/\mu_2$ there is a unique linear transformation (denoted also by τ) between the spaces of μ_1- and μ_2- measurable functions (modulo equivalence), characterized by $\tau\chi_A = \chi_{\tau A}$.

By $L^p(\mu)$ and $L^p(\mu, X)$, $1 \leqslant p < \infty$, we denote as usual the Lebesgue and the Lebesgue–Bochner space over the measure space (T, Σ, μ). We will use the fact that, like every abstract L^p-space, $L^p(\mu)$ has a Banach lattice representation

$$L^p(\mu) \cong l^p(\Gamma) \oplus_p \sum_{i \in I}^{p} L^p(\lambda^{m_i}), \tag{0·1}$$

where $\oplus_p$ and $\sum^p$ denote l^p-direct sums, Γ and I are (possibly empty) sets, the m_i are infinite cardinals, and λ^{m_i} is the product measure on $S_i = [0, 1]^{m_i}$ of the Lebesgue measure λ on $[0, 1]$ ([18], [21]). For any set A, let $|A|$ be a cardinality of A. The measure μ is σ-finite, atomic, atomless, if and only if $\Gamma \cup I$ is countable, $I = \varnothing$, $\Gamma = \varnothing$, respectively. The measure μ is homogeneous if and only if $\Gamma = \varnothing$ and all m_i are equal, or $I = \varnothing$ and $|\Gamma| = 1$. For every non-zero m, a countable l^p-direct sum of copies of $L^p(\lambda^m)$ is linearly isometric and order isomorphic to $L^p(\lambda^m)$.

Recall some results on isometries of $L^p(\mu, X)$ essential for our purpose.

THEOREM 0·1. ([11], [19], [24]) *Let (T_i, Σ_i, μ_i), $i = 1, 2$, be arbitrary positive measure spaces. Any Boolean isomorphism*

$$\tau: \Sigma_1/\mu_1 \leftrightarrow \Sigma_2/\mu_2$$

induces, via simple functions, an isometry from $L^p(\mu_1, X)$ onto $L^p(\mu_1 \circ \tau^{-1}, X)$, which will be still denoted by τ. If in addition $U_\cdot: T_2 \to G(X)$ is a strongly measurable family of isometries of X (i.e. $U_\cdot x$ is measurable for all x), then

$$Uf(t) = h(t)\, U_t(\tau f(t)) \tag{0·2}$$

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defines an isometry

$$U: L^p(\mu_1, X) \leftrightarrow L^p(\mu_2, X)$$

where $h > 0$ *a.e. and* h^p *is a density of* $\mu_1 \circ \tau^{-1}$ *with respect to* μ_2, *that is* $h^p = d(\mu_1 \circ \tau^{-1})/d\mu_2$.

The following result is a corollary of Theorem 4 in [**20**].

THEOREM 0·2. *Let* μ_i *be* σ*-finite* $(i = 1, 2)$, $\mathbb{K} = \mathbb{C}$, $\mathbb{P}_2(X)$ *be trivial,* X *separable, and* $\dim X > 1$. *Then every isometry between* $L^2(\mu_1, X)$ *and* $L^2(\mu_2, X)$ *has the form* (0·2).

THEOREM 0·3. *Let* μ_i *be* σ*-finite* $(i = 1, 2)$, $2 \neq p < \infty$, *and* $\mathbb{P}_p(X)$ *be trivial. Assume that one of the following conditions holds*: (i) X *is separable,* (ii) X *is Hilbert,* (iii) X *is an* L^q*-space with* $p \neq q$, (iv) $p = 1$. *Then* (*a*) *the Boolean algebras* $\mathbb{P}_p(L^p(\mu_i, X))$ *and* Σ/μ_i *are isomorphic* $(i = 1, 2)$; (*b*) *for every isometry* $U: L^p(\mu_1, X) \leftrightarrow L^p(\mu_2, X)$ *there is a Boolean isomorphism*

$$\tau: \Sigma/\mu_1 \leftrightarrow \Sigma_2/\mu_2$$

reducing U, *i.e.*

$$U(\chi_A F) = \chi_{\tau A} UF \quad (A \in \Sigma_1, F \in L^p(\mu_1, X));$$

(*c*) *in the cases* (i) *and* (ii), *every such isometry* U *has the form* (0·2).

The proofs are in [**11**] and [**26**] for (i), [**10**] and [**13**] for (ii), [**10**] for (iii), and [**12**] for (iv).

In general, this characterization of isometries does not hold for non-σ-finite measures. Let $T_1 = T_2$ be an uncountable set, Σ_1 the power set, Σ_2 the σ-algebra generated by the singletons, and μ_i the counting measure on Σ_i $(i = 1, 2)$. Then $L^p(\mu_1)$ and $L^p(\mu_2)$ contain the same elements and the identity is an isometry, but the two measure algebras are not isomorphic. Obviously, this failure is due to the lack of sufficiently many measurable sets in Σ_2.

There is a simple remedy to avoid this situation. In virtue of representation (0·1), with any space $L^p(\mu)$ over (T, Σ, μ) we can associate $L^p(\mu')$ over (T', Σ', μ') where

$$T' = \Gamma \cup \bigcup_{i \in I} S_i, \quad \text{with} \quad S_i = [0, 1]^{m_i},$$

$$\Sigma' = \{A \subset T' : A \cap S_i \text{ is measurable for each } i \in I\},$$

$$\mu'(A) = |A \cap \Gamma| + \sum_{i \in I} \lambda^{m_i}(A \cap S_i).$$

The measure μ' is strictly localizable, i.e. there is a partition $T' = \bigcup_{j \in J} T'_j$ into sets T'_j of finite measure, such that $A \in \Sigma'$ and $\mu'(A) = \sum_{j \in J} \mu'(A \cap T'_j)$ whenever $A \cap T'_j \in \Sigma'$ for all $j \in J$. The Banach lattice isomorphism

$$L^p(\mu) \leftrightarrow l^p(\Gamma) \oplus_p \sum_{i \in I}^{\cdot p} L^p(\lambda^{m_i}) \leftrightarrow L^p(\mu')$$

induces, via simple functions, an isometry of $L^p(\mu, X)$ onto $L^p(\mu', X)$. (Observe that $p < \infty$.)

Therefore the following theorem is useful for the description of isometries between $L^p(\mu_i, X)$ with arbitrary positive measures μ_i.

THEOREM 0·4. *The conclusions of Theorem* 0·3 *hold if '*μ_i σ*-finite' is replaced by '*μ_i *strictly localizable'*.

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Proof. If μ is strictly localizable, then the existence of the isomorphism between $\mathbb{P}_p(L^p(\mu,X))$ and Σ/μ is reduced to the case of finite measures $\mu|T_j$ by means of Proposition 2·6 in [3]. Part (b) is an immediate consequence of (a) as the L^p-structure is an isometric invariant. Part (c) is reduced to the case of σ-finite measures using the fact that because of (b), the restriction of U to the subspace $L^p(\mu_1|T_j,X)$ is an isometry onto $L^p(\mu_2|\tau T_j,X)$ $(j\in J)$. ∎

1. L^p-spaces

After these preliminaries, let us begin with a look at almost transitivity in $L^p(\mu)$, where μ is an arbitrary, positive measure. The norm in $L^p(\lambda)$ is almost transitive [23] and for $p\neq 2$, the norm is not transitive. The latter is caused by the fact that every isometry leaves the set of nowhere vanishing functions invariant (we have $h\neq 0$ a.e. in (0·2)). This argument applies to any σ-finite measure, not consisting of one single atom (Theorems 0·1 and 0·3).

As a first observation, non-trivial L^p-spaces with almost transitive norm have no one-dimensional L^p-summands.

PROPOSITION 1·1. *If $L^p(\mu)$ has almost transitive norm, $p\neq 2$, and μ has an atom, then $L^p(\mu)=\mathbb{K}$.*

Proof. Assume that A is a μ-atom and $L^p(\mu)\neq\mathbb{K}$, i.e. there is a $B\in\Sigma$ such that $0<\mu(B\backslash A)<\infty$. Set $C=A\cup B$, $f=\chi_A/\|\chi_A\|$, and $g=\chi_C/\|\chi_C\|$. Under every isometry U, f is mapped to a function supported by an atom D. Indeed, 1-dimensional L^p-summands in $L^p(\mu)$ (Uf generates such a subspace) are exactly subspaces supported by an atom [3]. If $D\neq A$, then $\|Uf-g\|\geqslant\|\chi_A(Uf-g)\|=\|\chi_A g\|$. If $D=A$, then $\|Uf-g\|\geqslant\|\chi_{B\backslash A}(Uf-g)\|=\|\chi_{B\backslash A}g\|$. Thus for every isometry U,

$$\|Uf-g\|\geqslant\min\{\|\chi_A g\|,\ \|\chi_{B\backslash A}g\|\}>0,$$

i.e. $L^p(\mu)$ fails to have almost transitive norm. ∎

THEOREM 1·2. *$L^p(\mu)$, $1\leqslant p<\infty$, $p\neq 2$, has almost transitive norm if and only if μ is homogeneous.*

Proof. By the previous proposition, assume μ has no atoms. Assume also w.l.o.g. that $(T,\Sigma,\mu)=(T',\Sigma',\mu')$. If μ is not homogeneous, then there are $i,j\in I$ such that $m_i\neq m_j$. Set

$$f=\chi_{S_i}\quad\text{and}\quad g=\chi_{S_j}.$$

We will show that, for every isometry U of $L^p(\mu)$, $C=\operatorname{supp}Uf\cap S_j$ is a null set and therefore

$$\|Uf-g\|=2^{1/p}.$$

Otherwise, the Boolean isomorphism τ^{-1} induced by U (Theorems 0·1, 0·3 and 0·4) would map C onto a subset D of $\operatorname{supp}f$ and $\mu|C\sim\mu|D$. On the other hand

$$\mu|C\sim\mu|S_j\sim\lambda^{m_j}+\lambda^{m_i}\sim\mu|S_i\sim\mu|D,$$

a contradiction.

If μ is homogeneous, then $L^p(\mu)\cong\sum^p_{i\in I}L^p(\lambda^m)$ for some infinite cardinal m. Let $f,g\in L^p(\mu)$ with $\|f\|=\|g\|=1$. Since f and g have σ-finite supports, we may assume w.l.o.g. that I is countable and consequently $L^p(\mu)\cong\sum^p_{i\in I}L^p(\lambda^m)\cong L^p(\lambda^m)$. As measurable functions, f and g depend only on countably many coordinates of

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$[0,1]^m = [0,1]^J$, say, on the coordinates in $N \subset J$. In other words, as elements of $L^p(\lambda^m) = L^p(\lambda^{|N|}, L^p(\lambda^{|J\setminus N|}))$, both f and g take only constant functions on $[0,1]^{J\setminus N}$ as values, i.e. they are elements of the naturally embedded copy of $L^p(\lambda^{|N|})$ in $L^p(\lambda^{|N|}, L^p(\lambda^{|J\setminus N|}))$. Since $L^p(\lambda^{|N|}) \cong L^p(\lambda)$ and the latter space has almost transitive norm, there is, for every $\epsilon > 0$, an isometry U of $L^p(\lambda^{|N|})$ such that $\|Uf - g\| \leqslant \epsilon$. This isometry has the form (0·2) (with identities for U_t) and it is easily extended to an isometry of $L^p(\lambda^{|N|}, L^p(\lambda^{|J\setminus N|}))$. ∎

Remark. If $\mathfrak{U}$ is a free ultrafilter on $\mathbb{N}$, then it is straightforward that a $\mathfrak{U}$-ultrapower (see [**25**]) of a space with almost transitive norm has transitive norm. An ultrapower of L^p-spaces is again an (abstract) L^p-space [**25**]. Consequently, a $\mathfrak{U}$-ultrapower of $L^p(\mu)$, with a homogeneous measure μ, is an $L^p(\nu)$ with transitive norm. As we have seen, ν cannot be σ-finite (unless $L^p(\nu) = L^p(\mu) = \mathbb{K}$ or $p = 2$).

For which measures ν, then, does $L^p(\nu)$ have transitive norm?

THEOREM 1·3. *Let $1 \leqslant p < \infty$, $p \neq 2$ and $\dim L^p(\mu) > 1$. $L^p(\mu)$ has transitive norm if and only if μ is homogeneous and not σ-finite.*

Proof. The necessity is a result of the previous comments and Theorem 1·2. In order to show sufficiency we need a lemma establishing the existence of certain isometries.

LEMMA 1·4. *Let $f \in L^p(\lambda^m)$ such that $\|f\| = 1$ and $f(t) \neq 0$ a.e. Then there is an isometry U of $L^p(\lambda^m)$ such that $U1 = f$.*

Proof of the lemma. As in the proof of Theorem 1·2, we may assume that f (and 1) are in the naturally embedded copy of $L^p(\lambda^{|N|})$ in $L^p(\lambda^{|N|}, L^p(\lambda^{|J\setminus N|}))$. Since every isometry of $L^p(\lambda^{|N|})$ can be extended to an isometry of the Bochner space, we may assume w.l.o.g. that $f \in L^p(\lambda)$ ($\cong L^p(\lambda^{|N|})$). Following the technique in [**23**, theorem 9·6·3], define

$$\psi(t) = \int_0^t |f(s)|^p \, ds.$$

Then $\psi: [0,1] \to [0,1]$ is continuous, strictly increasing, and onto. The mapping $\tau(A) = \psi^{-1}(A)$ is therefore a Boolean isomorphism of λ's measure algebra, $|f(t)|^p$ is a density of $\lambda \circ \tau^{-1}$ with respect to λ, and the isometry U of $L^p(\lambda)$, defined by

$$Ux(t) = \operatorname{sgn} f(t)\, |f(t)|\, x(\psi(t))$$

maps 1 into f.

Proof of the theorem. Again, like in the proof of Theorem 1·2 we can assume w.l.o.g. that $(T, \Sigma, \mu) = (T', \Sigma', \mu')$. Since μ is homogeneous and not σ-finite, we have that $T = \bigcup_{i\in I} S_i$ with $S_i = [0,1]^m$, $\mu | S_i = \lambda^m$, m an infinite cardinal, and I uncountable. Let $f \in L^p(\mu)$, $\|f\| = 1$. The set $A = \operatorname{supp} f$ is σ-finite, so there is a countable set $N \subset I$ such that $A \subset \bigcup_{i\in N} S_i$. Since $\sum^p_{i\in N} L^p(\lambda^m) \cong L^p(\lambda^m)$, and I is uncountable, we may assume that $A \subset S_k$ for one $k \in I$, and then $\bigcup_{i\in I\setminus\{k\}} S_i$ is not trivial. Let $B = S_k \setminus A$. By homogeneity of μ and Theorem 0·1, we have $L^p(\mu | A) \cong L_p(\lambda^m)$ and either $L^p(\mu | B) \cong L^p(\lambda^m)$ or $L^p(\mu | B) = \{0\}$, when $\mu B > 0$ or $\mu B = 0$, respectively. Thus, there is a chain of isometries

$$L^p(\mu) \cong L^p(\mu | A) \oplus_p L^p(\mu | B) \oplus_p \sum^p_{i\in I\setminus\{k\}} L^p(\lambda^m)$$

$$\cong L^p(\lambda^m) \oplus_p \sum^p_{i\in I} L^p(\lambda^m) \cong L^p(\lambda^m) \oplus_p L^p(\mu),$$

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under which f is mapped into a pair $(f_1, 0)$ of functions with $f_1 \neq 0$ a.e. Indeed, there exists a Boolean isomorphism between $(\Sigma|A)/(\mu|A)$ and $([0,1]^m)/\lambda^m$, which generates (Theorem 0·1) an isometry between the spaces $L^p(\mu|A)$ and $L^p(\lambda^m)$ preserving the set of nowhere vanishing functions. Thus, if f_1 is an image of $f|A$ by this isometry, then $f_1 \neq 0$ a.e. The lemma then shows the existence of an isometry $V_f: L^p(\mu) \leftrightarrow L^p(\lambda^m) \oplus_p L^p(\mu)$ with $V_f(f) = (1, 0)$. Since f was arbitrary with $\|f\| = 1$, the rest is obvious. ∎

2. *Bochner L^p-spaces*

THEOREM 2·1. *If a Banach space X has an almost transitive norm, and μ is a homogeneous measure, then $L^p(\mu, X)$, $1 \leqslant p < \infty$, has almost transitive norm.*

Proof. Let $F, G \in L^p(\mu, X)$, $\|F\| = \|G\| = 1$, and $\epsilon > 0$. We write $|Y|(t) = \|Y(t)\|$ for $Y \in L^p(\mu, X)$. Since $\operatorname{supp}|F|$ and $\operatorname{supp}|G|$ are σ-finite, we may assume as in the proof of Theorem 1·2 that $\mu = \lambda^m$ for some cardinal m. We may also assume that F and G are nowhere vanishing simple functions, because this set of functions is dense in $L^p(\lambda^m, X)$. Choose an isometry $U: L^p(\mu) \leftrightarrow L^p(\mu)$ of the form

$$Uy(t) = h(t)(\tau y)(t)$$

such that $U|F| = |G|$ (apply Lemma 1·4 twice). Extend U to an isometry of $L^p(\mu, X)$ via simple functions,

$$UY(t) = h(t)(\tau Y)(t).$$

We have $|UY| = U|Y|$ for all simple functions Y, by construction, and for all $Y \in L^p(\mu, X)$ by continuity. We also have $|UF| = U|F| = |G|$, and therefore we can write

$$UF(t) = \sum_{i=1}^{n} x_i \chi_{A_i}(t), \quad G(t) = \sum_{i=1}^{n} y_i \chi_{A_i}(t)$$

with $\|x_i\| = \|y_i\| \neq 0$ $(1 \leqslant i \leqslant n)$ and $T = \bigcup_{i=1}^{n} A_i$ with A_i disjoint. Choose isometries U_i of X such that $\|U_i x_i - y_i\| \leqslant \epsilon \|x_i\|$ and set $U_t = \sum_{i=1}^{n} U_i \chi_{A_i}(t)$. Now consider the isometry V of $L^p(\mu, X)$, defined by

$$VY(t) = U_t UY(t) = U_t(h(t)(\tau Y)(t)).$$

Then $\|VF - G\|^p = \sum_{i=1}^{n} \|U_i x_i - y_i\|^p \mu(A_i) \leqslant \sum_{i=1}^{n} \epsilon^p \mu(A_i) = \epsilon^p$. ∎

The following example shows that a full converse of the above theorem is not possible.

Example. Let μ be the Haar measure on $\{0, 1\}$ and $X = \sum_{i \in I}^{p} L^p(\lambda) = L^p(m)$ where I is an uncountable set. Then m is homogeneous and not σ-finite, and therefore the Bochner spaces $L^p(m, l_2^p)$ and $L^p(\mu, X)$, which are both isometric to $L^p(m)$, have transitive norms. However, the range space of the former and the scalar function space associated with the latter fail to have almost transitive norms. We have not been able to find a space $L^p(\mu, X)$ with (almost) transitive norm where both $L^p(\mu)$ and X lack (almost) transitive norms.

We can establish the full converse of Theorem 2·1 in those cases where the isometries of $L^p(\mu, X)$ are represented as (0·2).

THEOREM 2·2. *Let $1 \leqslant p < \infty$, $L^p(\mu, X)$ have almost transitive norm, and X have trivial L^p-structure.*

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(*a*) *If one of the following conditions holds, then* $L^p(\mu)$ *has almost transitive norm* (*i.e.*, $p = 2$ *or* μ *is homogeneous*): (i) X *is separable*, (ii) X *is Hilbert*, (iii) X *is* L^q*-space*, $q < \infty$, (iv) $p = 1$.

(*b*) *If one of the following conditions holds, then* X *has almost transitive norm*: (v) X *is separable*, $p \neq 2$, (vi) X *is separable*, $p = 2$, $\mathbb{K} = \mathbb{C}$, μ *is* σ*-finite.*

Proof. Assume, w.l.o.g., that $(T, \Sigma, \mu) = (T', \Sigma', \mu')$.

For (*a*), use the Boolean isomorphism furnished by Theorem 0·2 and Theorem 0·3(*b*) for a vector-valued analogon to the first part of the proof of Theorem 1·2.

For (*b*), observe that every isometry of $L^p(\mu, X)$ has the form (0·2) (Theorems 0·3(*c*) and 0·4). Now let $x, y \in X$ with $\|x\| = \|y\| = 1$, choose A such that $0 < \mu(A) < \infty$, and put $F = \chi_A x$, $G = \chi_A y$. Then there is a sequence of isometries U_n of $L^p(\mu, X)$ such that $\|U_n F - G\| \to 0$. Since convergence in $L^p(\mu, X)$ implies convergence a.e. of a suitable subsequence we may assume that $(U_n F)(t) \to G(t)$ a.e. Each U_n has a representation

$$(U_n Y)(t) = h_n(t)\, U_{n,t}(\tau_n Y(t)).$$

Since $\tau_n F(t) = \chi_{\tau_n A}(t)\, x$ we have

$$g_n(t)\, U_{n,t}(x) \to y$$

a.e. on A, with $g_n(t) = h_n(t)\, \chi_{\tau_n A}(t)$. Also, $g_n(t) \geqslant 0$ and $\|U_{n,t}(x)\| = 1$, therefore $g_n(t) \to \|y\| = 1$, and thus $U_{n,t}(x) \to y$ for a suitable $t \in A$. Since x and y were arbitrary, X has almost transitive norm. ∎

3. *Almost transitivity and* L^p*-structure*

In this short section we adapt some results from Section 1 to the setting of general Banach spaces. The following representation of a Banach space is described in [**3**] for the case $\mathbb{K} = \mathbb{R}$. It holds in the complex case as well ([**12**, p. 822]). The L^p-projections of a Banach space X form a complete Boolean algebra $\mathbb{P}_p$ ($1 \leqslant p < \infty$, $p \neq 2$). Its Stonean space Ω_p is hyperstonean – it admits a Borel measure m whose null sets are the nowhere dense sets. $C\Omega_p$ is Banach-algebra isomorphic to the closed operator space C_p generated by $\mathbb{P}_p$, so X is a $C\Omega_p$-module. There is a family of Banach spaces X_k ($k \in \Omega_p$) and a representation $x \mapsto \langle x \rangle$ of X as a $C\Omega_p$-module of functions $\langle x \rangle$ on Ω_p such that $\langle x \rangle\,(k) \in X_k \cup \{\infty\}$ ($k \in \Omega_p$), the scalar function $\|\langle x \rangle\,(\cdot)\|$ is in $L^p(m)$, and its norm is $\|x\|$. Under that representation, the L^p-projections of X are exactly the multiplications by functions χ_C, $C \in \operatorname{Clop} \Omega_p$ (the clopen subsets of Ω_p).

The key to adapting the proofs of Proposition 1·1 and Theorem 1·2 to Theorems 3·1 and 3·2 are the following two facts:

(1) every isometry U of X is reduced by a Boolean isomorphism τ of $\operatorname{Clop} \Omega_p$, i.e.

$$U(\chi_C x) = \chi_{\tau C}\, Ux \quad (x \in X, C \text{ clopen});$$

(2) for every clopen set C of finite measure, there is an $x \in X$ such that

$$\|\langle x \rangle(\cdot)\| = \chi_C.$$

THEOREM 3·1. *Let* X *have a minimal non-trivial* L^p*-summand for some* $p \neq 2$, $1 \leqslant p < \infty$. *Then* X *fails to have almost transitive norm.*

We call the Boolean algebra $\mathbb{P}_p$ homogeneous if the measure m is homogeneous, i.e.

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if the measure algebras induced on any two sets of finite (equivalently: σ-finite) measure are isomorphic – in other words, if for any two elements of $\mathbb{P}_p$, satisfying the countable chain condition, the induced Boolean algebras are isomorphic.

Observe that, with this definition, the trivial Boolean algebra $\{0,1\}$ is homogeneous. No Banach space, except real l_2^1, has non-trivial L^p-structure for more than one p [2].

THEOREM 3·2. *Banach spaces with almost transitive norm have homogeneous L^p-structure* ($1 \leqslant p < \infty$, $p \neq 2$).

The same technique allows us to give a partial answer to the question, posed in [23], whether there is a separable Banach space with transitive norm, other than l^2. If X is separable, then there are at most countably many disjoint non-void clopen sets in Ω_p; therefore the measure m can be chosen as a probability measure. Assume that Ω_p is not a singleton. Choose a non-void clopen $C \neq \Omega_p$, $x \in X$ with $\|x\| = 1$ and supp $\langle x \rangle = C$, and $y \in X$ with $\|\langle y \rangle(\cdot)\| \equiv 1$. No isometry maps x to y, because no Boolean isomorphism τ maps C to Ω_p. So we have the following theorem.

THEOREM 3·3. *Separable Banach spaces with non-trivial L^p-structure lack transitive norm* ($1 \leqslant p < \infty$, $p \neq 2$).

4. *Orlicz and Musielak–Orlicz spaces*

In this section, let (T, Σ, μ) be the Lebesgue measure space on the interval $T = [0,1]$ or on the set of all real numbers $T = \mathbb{R}$. All results contained here remain true for any measure space isomorphic to the Lebesgue measure space in the sense that there exists a measure preserving Boolean isomorphism between their measure algebras. A function $\Phi: \mathbb{R}_+ \times T \to \mathbb{R}_+$ is said to be a Young function with parameter, if it is convex with respect to the first variable, measurable with respect to the second variable and $\Phi(0,t) = 0$ and $\Phi(u,t) > 0$ for $u > 0$ and $t \in T$. The Musielak–Orlicz space L_Φ associated with Φ is the space of all equivalence classes of (complex or real valued) measurable functions such that

$$I_\Phi(af) = \int_T \Phi(a|f(t)|, t)\, d\mu(t) < \infty$$

for some $a > 0$. The norm of $f \in L_\Phi$ is given by $\|f\| = \inf\{\epsilon > 0 : I_\Phi(f/\epsilon) \leqslant 1\}$ [22]. If Φ does not depend on parameter t, i.e. there exists $\phi: \mathbb{R}_+ \to \mathbb{R}_+$ such that $\Phi(u,t) = \phi(u)$ for all $u \in \mathbb{R}$ and almost all $t \in T$, then L_Φ becomes an Orlicz space L_ϕ [17]. For $\phi(u) = u^p$, $L_\phi = L^p$. In the rest of this section assume w.l.o.g. that $\phi(1) = 1$. Let us start the discussion of almost transitivity and other related properties of the norm in Orlicz spaces with the results on isometries in these spaces. For long time only a characterization of isometries in complex Orlicz spaces (consisting of complex-valued functions) has been known [8]. The recent result by Kalton and Radriantoanina [15] extends this characterization to the real spaces of functions on the interval $[0,1]$ with the Lebesgue measure. Thus the classification of isometries given by Zaidenberg [28] for complex spaces remains true also for real spaces on the interval $[0,1]$.

Let M_ϕ be the set of multipliers of ϕ, that is

$$M_\phi = \{a > 0 : \phi(au) = \phi(a)\,\phi(u) \quad \text{for all} \quad u \geqslant 0\}.$$

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J. Lamperti [19] showed that either $M_\phi = \mathbb{R}_+$ and then $\phi(u) = u^p$ for some p, or M_ϕ is an infinite cyclic group with a generator $k \neq 1$ or $M_\phi = \{1\}$.

THEOREM 4·1. ([15], [28]) *Let L_ϕ be either a complex Orlicz space (on $[0,1]$ or $\mathbb{R}$), or a real space on $[0,1]$. Then*

(a) *If $M_\phi = \{1\}$, then any isometry U of L_ϕ is of the form*

$$(Uf)(t) = h(t)(\tau f)(t), \tag{4·1}$$

where τ is a measure preserving Boolean isomorphism, i.e. $\mu(\tau^{-1}A) = \mu A$, and $|h(t)| \equiv 1$;

(b) *If M_ϕ is a cyclic group generated by $k \neq 1$, then any isometry of L_ϕ is of the form (4·1), where $\phi(|h(t)|) = [d(\mu \circ \tau^{-1})/d\mu](t)$ and $|h(t)| \in M_\phi$, that is $|h(t)| = k^n$ for some integer n;*

(c) *If $M_\phi = \mathbb{R}_+$, then $\phi(u) = u^p$ for some $1 \leqslant p < \infty$ and $L_\phi = L^p$.*

We say that a norm $\| \ \|$ is called maximal if, for any equivalent norm $\| \ \|_1$ such that $G(X, \| \ \|) \subset G(X, \| \ \|_1)$, we have equality $G(X, \| \ \|) = G(X, \| \ \|_1)$. An almost transitive norm must be maximal [23].

THEOREM 4·2. *Let the Orlicz space L_ϕ satisfy the conditions of Theorem 4·1. The norm in L_ϕ is almost transitive if and only if it is maximal. Moreover, it possesses either of these properties if and only if there exists $p \geqslant 1$ such that $\phi(u) = u^p$ and then L_ϕ is the space L^p.*

Proof. Suppose ϕ is not a power function. Then either ϕ has no non-trivial multipliers or ϕ has an infinite cyclic group of multipliers generated by some $k \neq 1$.

If ϕ has no multipliers, then, by Theorem 4·1(a), the only isometries of L_ϕ are generated by measure preserving transformations. It is easy to find a function θ: $T \to T$, one-to-one, onto and such that $\theta^2(x) = x$ and that the set function τ generated by θ, i.e. $\tau(A) = \theta^{-1}(A)$, is not measure preserving. For instance for $T = [0,1]$, set $\theta(x) = -3x + 1$ if $x \in [0, \frac{1}{4}]$ and $\theta(x) = -\frac{1}{3}x + \frac{1}{3}$, if $x \in [\frac{1}{4}, 1]$. Then, since $(\tau f)(x) = f(\theta(x))$, for $f \in L_\phi$

$$\tfrac{1}{3} I_\phi(f) \leqslant I_\phi(\tau f) \leqslant 3 I_\phi(f).$$

Hence $\|\tau f\|$ is a new norm in L_ϕ equivalent to the original one. For $f \in L_\phi$ define

$$\|f\|_1 = \max\{\|\tau f\|, \|f\|\}.$$

The norm $\| \ \|_1$ is equivalent to $\| \ \|$ and $\|\tau f\|_1 = \|f\|_1$. Since τ is not measure preserving, the transformation $Uf = \tau f$ is not an isometry of L_ϕ with the original norm, but it is an isometry with respect to $\| \ \|_1$. Thus $(L_\phi, \| \ \|_1)$ has substantially more isometries than $(L_\phi, \| \ \|)$. Hence the norm $\| \ \|$ is not maximal.

Now let ϕ not be a power function, but have a cyclic group of multipliers generated by $k \neq 1$. Defining $p = \ln \phi(k)/\ln k$, the spaces L_ϕ and L^p contain the same elements and their norms are equivalent (see [28]). Moreover, any isometry of L_ϕ is also an isometry of L^p. Indeed, if U is an isometry of L_ϕ, then $Uf = h(\tau f)$, where $|h(t)| = k^n$ for some integer $n \neq 0$, and $[d(\mu \circ \tau^{-1})/d\mu] = \phi(|h(t)|)$. However, by the definition of p, $\phi(|h(t)|) = |h(t)|^p$, which shows that U is also an isometry of L^p. L^p has essentially more isometries than L_ϕ. Indeed, since ϕ is different from u^p, there exists $a \neq k^n$ for all integers n, and such that $\phi(a) \neq a^p$. Now, if for instance $T = [0,1]$, then setting

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$\theta(t) = a^p t$ for $t \in [0, 1/2a]$ and $\theta(t) = [(2a - a^p)/(2a-1)]t + (a^p - 1)/(2a-1)$ for $t \in [1/2a, 1]$, θ is a one-to-one transformation of $[0,1]$ onto itself. Thus $Uf = h(t) f(\theta(t))$ is an isometry of L^p for $h(t) = \theta'(t)^{1/p}$. However, it is not an isometry of L_ϕ, since $h(t) = a$ for $t \in [0, 1/(2a)]$ and thus the conditions (b) of Theorem 4·1 are not satisfied. Hence the norm $\| \ \|$ is maximal and the proof is complete. ∎

Remark. Since the property of convex transitivity [23] (equivalent to the uniquely maximal property [7]) of a norm is stronger than almost transitivity and weaker than maximality, L_ϕ has convex transitive (or uniquely maximal) norm if and only if L_ϕ is isometric to L^p.

Now we shall show that a complex Musielak–Orlicz space with almost transitive norm must be isometric to a Lebesgue space. In fact we will prove that $\Phi(u,t) = c(t)\, u^p$ for some $p \geqslant 1$ and some measurable function $c(t)$, if L_Φ has almost transitive norm.

Let $D_u \Phi$ denote the partial derivative of Φ with respect to u. In the following, let

$$T_0 = \left\{ t : \frac{D_u \Phi(u,t)}{u} \text{ is a constant function with respect to } u \right\}.$$

LEMMA 4·3. *Assume that $\mu T_0 = 0$ and $\Phi(1,t) \equiv 1$. If a complex space L_Φ has almost transitive norm, then Φ does not depend on t, that is $\Phi(u,t_1) = \Phi(u,t_2)$ for all $u \in \mathbb{R}_+$ and any couple $t_1, t_2 \in T$, except for a null set.*

Proof. For a contrary suppose that there exist $\epsilon > 0$, $\alpha > 0$ and $a > 0$ such that $\mu T_1 > 0$ and $\mu T_2 > 0$, where

$$T_1 = \{t : \Phi(a,t) > \alpha + \epsilon\},$$
$$T_2 = \{t : \Phi(a,t) \leqslant \alpha\}.$$

We can find two measurable sets $A \subset T_1$ and $B \subset T_2$ of finite and positive measure with

$$\int_A \Phi(a,t) = \int_B \Phi(a,t).$$

Setting $k(\beta) = \int_A \Phi(\beta a, t)$ and $l(\beta) = \int_B \Phi(\beta a, t)$ for $\beta \geqslant 0$, both functions k and l are strictly increasing, convex and $\lim_{\beta \to \infty} k(\beta) = \infty = \lim_{\beta \to \infty} l(\beta)$. Thus there exist $\beta_1 > 0$ and $\beta_2 > 0$ such that $k(\beta_1) = 1 = l(\beta_2)$.

Assume first that $\beta_1 \neq \beta_2$, so for instance let $\beta_1 > \beta_2$. Then $k(\beta_1) = 1$ and $l(\beta_1) > 1$. Therefore, there exists a set $B_1 \subset B$ such that $\mu(B \setminus B_1) > 0$ and $\int_{B_1} \Phi(\beta_1 a, t) = 1$. Let

$$f = \beta_1 a \chi_{B_1}, \quad g = \beta_1 a \chi_A.$$

Then $\|f\| = \|g\| = 1$. By almost transitivity, there is a sequence $\{U_n\}$ of isometries such that $\|U_n f - g\| \to 0$. Thus there is a subsequence $\{n_k\}$ such that $U_{n_k} f \to g$ a.e. Hence $\Phi(|U_{n_k} \beta^{-1} f(t)|, t) \to \Phi(\beta^{-1}|g(t)|, t)$ a.e. and by the Fatou Lemma

$$I_\Phi(\beta^{-1} g) \leqslant \underline{\lim} I_\Phi(U_{n_k} \beta^{-1} f).$$

This and $I_\Phi(U_{n_k} \beta^{-1} f) = I_\Phi(\beta^{-1} \phi)$ imply that $I_\Phi(\beta^{-1} g) \leqslant I_\Phi(\beta^{-1} f)$, for any $\beta > 0$. However,

$$I_\Phi(\beta_1^{-1} f) = \int_{B_1} \phi(a,t) < \int_B \Phi(a,t) = \int_A \Phi(a,t) = I_\Phi(\beta_1^{-1} g),$$

a contradiction.

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Now let $\beta_1 = \beta_2$. Setting $\beta = \beta_i$, $i = 1, 2$ and

$$f = \beta a \chi_A, \quad g = \beta a \chi_B,$$

$\|f\| = \|g\| = 1$. There exists a sequence $\{U_n\}$ of isometries such that $U_n f \to g$ a.e. on T. Now, recall that any isometry U of L_Φ (under the assumptions of the lemma) has the form (4·1) with

$$\frac{d(\mu \circ \tau^{-1})}{d\mu}(t) = \Phi(|h(t)|, t)$$

and $I_\Phi(Uf) = I_\Phi(f)$ [**14**]. Thus

$$U_n f = h_n \beta a \chi_{\tau_n A},$$

where h_n are measurable functions and τ_n Boolean isomorphisms. Therefore and by the assumption $\Phi(1, t) \equiv 1$, $\Phi(|h_n(t)|\chi_{\tau_n A \cap B}(t), t) \to \Phi(\chi_B(t), t) = \chi_B(t)$. Moreover, $d(\mu \circ \tau_n^{-1})/d\mu\,(t) = \phi(|h_n(t)|, t)$, which implies that

$$\int_{B \cap \tau_n A} \Phi(|h_n(t)|, t) = \mu(\tau_n^{-1}(B \cap \tau_n A)) \leqslant \mu A.$$

Thus, by the Fatou Lemma,

$$\mu B \leqslant \underline{\lim} \int_{B \cap \tau_n A} \Phi(|h_n(t)|, t) \leqslant \mu A.$$

However, $\Phi(a, t) > \alpha + \epsilon$ for $t \in A$ and $\Phi(a, t) \leqslant \alpha$ for $t \in B$, since $A \subset T_1$ and $B \subset T_2$. This implies that

$$\mu A \leqslant \frac{1}{\alpha + \epsilon} \int_A \Phi(a, t) < \frac{1}{\alpha} \int_B \Phi(a, t) \leqslant \mu B,$$

which contradicts the previous inequality. ∎

For $A \in \Sigma$, let $L_\Phi(A)$ denote the Musielak–Orlicz space of measurable functions restricted to the set A.

LEMMA 4·4. *If a complex space L_Φ has an almost transitive norm, then either $\mu T_0 = 0$ or $\mu(T \setminus T_0) = 0$.*

Proof. It is known that a restriction of an isometry U of L_Φ to $L_\Phi(T_0)$ [resp. $L_\Phi(T \setminus T_0)$] is an isometry of $L_\Phi(T_0)$ [resp. $L_\Phi(T \setminus T_0)$] onto itself [**14**].

Suppose $\mu T_0 > 0$ and $\mu(T \setminus T_0) > 0$. Let f, g be such that $\|f\| = \|g\| = 1$ and $\operatorname{supp} f \subset T_0$ and $\operatorname{supp} g \subset T \setminus T_0$. Then $I_\Phi(f) = 1$ and $I_\Phi(g) > 0$. Since the function I_Φ is preserved by an isometry, $I_\Phi(Uf - g) = I_\Phi(Uf) + I_\Phi(g) = I_\Phi(f) + I_\Phi(g) > 1$. Hence the norm is not almost transitive. ∎

THEOREM 4·5. *Let L_Φ be a complex Musielak–Orlicz space. If L_Φ has almost transitive norm, then $\Phi(u, t) = c(t)\, u^p$ for some $p \geqslant 1$ and some positive measurable function $c(t)$. Thus L_Φ is isometric to L^p.*

Proof. Let $\bar{\Phi}(u, t) = \Phi(a(t)\, u, t)$, where $a(t)$ is a measurable positive function such that $\Phi(a(t), t) = 1$. Then $L_{\bar{\Phi}}$ is isometric to L_Φ and $\bar{\Phi}(1, t) \equiv 1$. By the previous lemma let $\mu T_0 = 0$. Now since $\bar{\Phi}(u, t) \equiv \phi$ (Lemma 4·3) and L_ϕ has almost transitive norm

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only if ϕ is a power function (Theorem 4·2), $\bar{\phi}(u,t) = u^p$, for some $1 \leqslant p < \infty$. Hence $\Phi(u,t) = c(t)\,u^p$, where $c(t) = 1/a(t)$. ∎

5. *Supremum norm spaces*

For a locally compact Hausdorff space K, $C_0(K,X)$ will be the space of X-valued continuous functions on K which vanish at infinity, provided with the supremum norm. $C_0(K,\mathbb{K})$ will be denoted by C_0K, and we will write $C(K,X)$ (and CK) instead of $C_0(K,X)$ (and C_0K) if K is compact.

In this section we try to see how much of CK's failure to have almost transitive norm can be passed on to general Banach spaces X.

We denote the set of extreme points of the unit ball of X by $\operatorname{ex}X$. Since extreme points are isometric invariants, in a space X with almost transitive norm the set $\operatorname{ex}X$ is either empty or dense in the unit sphere.

CK-spaces with $|K| > 1$ fail almost transitivity of the norm since all extreme points have modulus $\equiv 1$ and thus $\operatorname{ex}CK$ is not dense in the sphere.

The centralizer $Z(X)$ of a Banach space X has been used to describe how much X resembles a CK [**4**]. $Z(X)$ is a closed subalgebra of the operator algebra $L(X)$; it is isometrically isomorphic to a space CK. The space X has a function module representation $\langle . \rangle$ as a CK-module of vector-valued functions on $\mathbb{K}$ with the supremum norm [**4**, theorem 4·14]. Since X is a CK-module we have, as in the scalar case, that

(a) every extreme point e of X has modulus $\equiv 1$,

(b) if $|K| > 1$, then $\operatorname{ex}X$ is not dense in the sphere.

THEOREM 5·1. *Let* $\operatorname{ex}X \neq 0$ *and* X *have non-trivial centralizer. Then* X *lacks almost transitive norm.*

Examples of such spaces are $C(K,(X^*,\sigma^*))$ and $C(K,X^*)$ (the spaces of all bounded, respectively compact, operators from X to CK), and $C(K,X)$ and $L^\infty(\mu,X)$ if $\operatorname{ex}X \neq \varnothing$. (We assume of course $CK \neq \mathbb{K} \neq L^\infty(\mu)$; the centralizers of these function spaces contain CK and $L^\infty(\mu)$, respectively, and constant functions taking an extreme point as value are extreme.)

Similarly, in an l^1-direct sum $X = Y \oplus_1 Z$ of two non-zero Banach spaces, $\operatorname{ex}X$ is contained in $Y \cup Z$ and therefore not dense in the sphere.

THEOREM 5·2. *Let* $\operatorname{ex}X \neq \varnothing$ *and* X *have non-trivial L-structure. Then* X *lacks almost transitive norm.*

If $\operatorname{ex}X = \varnothing$, then we need to control the isometries in some other way to guarantee that the norm is not almost transitive. We can use the function module representation of X over K where $CK = Z(X)$. Every isometry U of X induces a homeomorphism $\rho: K \leftrightarrow K$ such that the moduli of Ux and $x \circ \rho$ are equal [**4**, corollary 4·17]. This fact allows us to prove the following theorem using the same technique as in the scalar case – compare the image of a modulus 1 function to a norm 1 function vanishing in a point – provided there is such an $x \in X$ with $\|\langle x \rangle(\cdot)\| \equiv 1$. X has this latter property if and only if it has a centralizer norming element, [**4**, proposition 9·4] i.e. there is an $x \in X$ such that

$$\inf\{\|Tx\| : T \in Z(X), \|T\| = 1\} > 0$$

(See [**4**] for examples of such spaces.)

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An example of such a space without extreme points is $L^1(\lambda) \oplus_\infty L^1(\lambda)$. More generally, any l^∞-direct product

$$\sum_{i \in I}^{\infty} X_i$$

of two or more spaces X_i without extreme points and with trivial centralizers has that property – any (x_i) with $\|x_i\| = 1$ $(i \in I)$ will do.

THEOREM 5·3. *Let X have a centralizer-norming element and $Z(X) \neq \mathbb{K}$. Then X lacks almost transitive norm.*

For some vector-valued function spaces we know that the constant functions $(\neq 0)$ are centralizer-norming elements:

THEOREM 5·4. *Let $Z(X)$ be trivial and $CK \neq \mathbb{K} \neq L^\infty(\mu)$. Then $C(K,X)$ and, if X is separable, $L^\infty(\mu,X)$ lack almost transitive norm.*

Proof. $Z(C(K,X))$ is (a naturally embedded copy of) CK, and $C(K,X)$ coincides with its own function module representation [**4**, corollary 10·12]. In the case of σ-finite μ, Theorem 16 in [**9**] shows that $Z(L^\infty(\mu,X))$ is the naturally embedded copy of $L^\infty(\mu)$. The proof does not use the σ-finiteness of μ. Again, it is clear that any constant norm one function in $L^\infty(\mu,X)$ norms the centralizer $L^\infty(\mu)$. ∎

The requirement of a centralizer-norming element appears to be of a technical nature. It is therefore natural to ask whether Theorem 5·3 can be improved.

Problem 5·5. Do all spaces with almost transitive norm have trivial centralizer?

If L is locally compact, $|L| > 1$, then the centralizer of C_0L, namely $C\beta L$ [**4**, p. 63] is not trivial. Therefore a positive answer to 5·5 would also settle G. V. Wood's conjecture:

Conjecture 5·6. [**27**] All spaces C_0L $(|L| > 1)$ lack almost transitive norm.

Wood gives some partial results: if L contains a non-void compact open set, or a non-meagre compact connected set, then C_0L lacks almost transitive norm. If L is metrizable or, more generally, if there are two non-zero elements of C_0L whose moduli attain their maximal values on two sets of different cardinalities, then C_0L lacks almost transitive norm. Since the class of C_0L-spaces is closed under ultrapowers, Wood's conjecture is equivalent to the following weaker one:

Conjecture 5·6′. All spaces C_0L $(|L| > 1)$ lack transitive norm.

Similarly, positive answers to 5·5 above and 5·5′ below, formally weaker, and 5·5″ below, formally stronger, are all equivalent. The near-isometric variant of trivial centralizers, namely genealogically trivial centralizer (g.t.c.) goes back to Behrends[**5**]. Cambern and Jarosz[**6**] show that ultrapowers of spaces failing g.t.c. have non-trivial centralizers.

Problem 5·5′. Do all spaces with transitive norm have trivial centralizers?

Problem 5·5″. Do all spaces with almost transitive norm have genealogically trivial centralizers?

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REFERENCES

[1] S. BANACH. *Theory of Linear Operations* (North-Holland, 1987).
[2] E. BEHRENDS. L^p-Struktur in Banachräumen. *Studia Math.* **55** (1976), 71–85.
[3] E. BEHRENDS *et al.* *L^p-Structure in Real Banach Spaces.* Lecture Notes in Mathematics **613** (Springer-Verlag, 1977).
[4] E. BEHRENDS. *M-Structure and the Banach–Stone Theorem.* Lecture Notes in Mathematics **736** (Springer-Verlag, 1979).
[5] E. BEHRENDS. Isomorphic Banach–Stone theorems and isomorphisms which are close to isometries. *Pacific J. Math.* **133**, no. 2 (1988), 229–250.
[6] M. CAMBERN and K. JAROSZ. Ultraproducts, ϵ-multipliers, and isomorphisms. *Proc. Amer. Math. Soc.* **105** (1989), 929–937.
[7] R. E. COWIE. A note on uniquely maximal Banach spaces. *Proc. Edinburgh Math. Soc.* **26** (1983), 85–87.
[8] R. G. FLEMING and J. E. JAMISON. Isometries of Banach spaces, a survey in *The Riemann Legacy Volume* (Hadronic Press), (to appear).
[9] P. GREIM. The centralizer of Bochner L^∞-spaces. *Math. Ann.* **260** (1982), 463–468.
[10] P. GREIM. Hilbert spaces have the Banach–Stone property for Bochner spaces. *Bull. Austr. Math. Soc.* **27** (1983), 121–128.
[11] P. GREIM. Isometries and L^p-structure of separably valued Bochner L^p-spaces, in *Measure theory and its applications.* Lecture Notes in Mathematics **1033** (Springer-Verlag, 1983).
[12] P. GREIM. Banach spaces with the L^1-Banach–Stone property. *Trans. Amer. Math. Soc.* **287** (1985), 819–828.
[13] P. GREIM and J. E. JAMISON. Hilbert spaces have the strong Banach–Stone property for Bochner spaces. *Math. Z.* **196** (1987), 511–515.
[14] J. E. JAMISON, A. KAMIŃSKA and PEI-KEE LIN. Isometries of Musielak–Orlicz spaces II, *Studia Math.* **104** (1) (1993), 75–89.
[15] N. J. KALTON and B. RADRIANANTOANINA. Surjective isometries on rearrangement-invariant spaces, *Quart. J. Math. Oxford* (to appear).
[16] S. KOPPELBERG. *Handbook of Boolean Algebras, vol. I* (North-Holland, 1989).
[17] M. A. KRASNOSELSKII and YA. B. RUTICKII. *Convex Functions and Orlicz Spaces* (Noordhoff Ltd. 1961).
[18] H. ELTON LACY. *The Isometric Theory of Classical Banach Spaces* (Springer-Verlag, 1974).
[19] J. LAMPERTI. On the isometries of certain function spaces. *Pacific J. Math.* **8** (1958), 459–466.
[20] PEI-KEE LIN. The isometries of $L^2(\Omega, X)$. *Ill. J. Math.* **33** no. 4 (1989), 621–630.
[21] J. LINDENSTRAUSS and L. TZAFRIRI. *Classical Banach spaces II* (Springer-Verlag, 1979).
[22] J. MUSIELAK. *Orlicz Spaces and Modular Spaces.* Lecture Notes in Mathematics **1034** (Springer-Verlag, 1983).
[23] S. ROLEWICZ. *Metric Linear Spaces.* (Polish Scientific Publishers and D. Reidel Publishing Company, 1984).
[24] H. L. ROYDEN. *Real Analysis* (Macmillan, 1988).
[25] B. SIMS. *Ultra-techniques in Banach space theory.* Queens Papers in Pure and Applied Mathematics, no. 60 (Queen's University, Kingston, Ontario, Canada, 1982).
[26] A. R. SOUROUR. The isometries of $L^p(\Omega, X)$. *J. Funct. Anal.* **30** (1978), 276–285.
[27] G. V. WOOD. Maximal symmetry in Banach spaces. *Proc. Royal Irish Acad.* **82A**, no. 2 (1982), 177–186.
[28] M. G. ZAIDENBERG. Groups of isometries of Orlicz spaces. *Soviet Math. Dokl.* **17** (1976), 432–436.

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