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The magnitude of a graph

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Abstract

The magnitude of a graph is one of a family of cardinality-like invariants extending across mathematics; it is a cousin to Euler characteristic and geometric measure. Among its cardinality-like properties are multiplicativity with respect to cartesian product and an inclusion-exclusion formula for the magnitude of a union. Formally, the magnitude of a graph is both a rational function over $\mathbb{Q}$ and a power series over $\mathbb{Z}$. It shares features with one of the most important of all graph invariants, the Tutte polynomial; for instance, magnitude is invariant under Whitney twists when the points of identification are adjacent. Nevertheless, the magnitude of a graph is not determined by its Tutte polynomial, nor even by its cycle matroid, and it therefore carries information that they do not.

1. *Introduction*

The analogy… the two theories, their conflicts and their delicious reciprocal reflections, their furtive caresses, their inexplicable quarrels… Nothing is more fecund than these slightly adulterous relationships. — André Weil [**24**]

In many fields of mathematics, there is a canonical measure of size. Sets have cardinality, vector spaces have dimension, and topological spaces have Euler characteristic (whose status as the topological analogue of cardinality was made explicit by Schanuel [**20**]). Convex subsets of $\mathbb{R}^n$ have, in fact, one cardinality-like invariant of each dimension between 0 and n: the intrinsic volumes [**8**], which when $n = 2$ are the Euler characteristic, perimeter and area.

Many of these cardinality-like invariants arise from a single general definition. This general invariant is called magnitude, and here we investigate its behaviour in the case of graphs.

The full definition of magnitude is framed in the very wide generality of enriched categories [**10**]. Although we will not need that general definition here, it is instructive to look briefly at how it specialises to various branches of mathematics, to give context to what we will do for graphs.

First, one type of enriched category is an ordinary category, and magnitude of categories (also called Euler characteristic) is very closely linked to topological Euler characteristic [**9**, propositions 2·11 and 2·12]. The theory of magnitude of categories also extends the theory of Möbius inversion in posets, made famous by Rota for its applications in enumerative combinatorics [**18**].

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A second type of enriched category arises commonly in algebra, where one often encounters categories that are 'linear' in the sense that their hom-sets are vector spaces. In the representation theory of associative algebras A, an important role is played by the linear category of indecomposable projective A-modules. What is its magnitude? Under suitable hypotheses, it is a recognisable homological invariant of A. Specifically, it is $\chi_A(S, S)$, where χ_A is the Euler form of A and S is the direct sum of the simple A-modules [**2**].

Metric spaces provide a third context for magnitude [**10**, **13**, **14**]. These too can be seen as enriched categories, and metric magnitude is a previously undiscovered invariant that encodes many classical quantities. For example, given a compact subset $X \subseteq \mathbb{R}^n$, write $|tX|$ for the magnitude of the space $tX = \{tx : x \in X\}$. Meckes has shown that that the asymptotic growth of $|tX|$ as $t \to \infty$ is equal to the Minkowski dimension of X [**16**, corollary 7·4]. Work of Barceló and Carbery [**1**], and of Gimperlein and Goffeng [**3**], shows that one can recover from the asymptotic behaviour of $|tX|$ the top few intrinsic volumes of X: the volume, surface area, and so on. The magnitude of metric spaces is also closely related to certain measures of entropy and of biological diversity [**12**], and admits a further interpretation in terms of potential theory [**16**].

Graphs are metric spaces, with distance between vertices measured as the length of a shortest path. Among their special properties is that distances are integers. As we shall see, this has the consequence that for a graph G, the magnitude $|tG|$ is a rational function of $q = e^{-t}$ over $\mathbb{Q}$. It can also be expressed as a power series in q over $\mathbb{Z}$. We write it as $\#G = \#G(q)$ to avoid confusion with the usage of $|G|$ for the number of vertices of G, while still evoking the analogy with cardinality.

Among the cardinality-like properties of magnitude are that

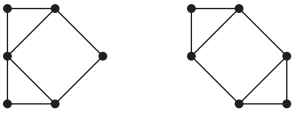
$$\#(G \,\square\, H) = \#G \cdot \#H, \tag{1·1}$$

where $\square$ denotes the cartesian product of graphs (defined below), and that

$$\#(G \cup H) = \#G + \#H - \#(G \cap H) \tag{1·2}$$

under certain hypotheses. The trivial invariant 'number of vertices' also satisfies these equations, and indeed, the number of vertices of G can be recovered from its magnitude as $\#G(0)$; but of course, magnitude is much more informative than that.

The information carried by magnitude appears to be quite different from that carried by existing graph invariants. For instance, the Tutte polynomial [**23**] is perhaps the most important graph invariant of all, and many other graph invariants are specialisations of it, but magnitude is not; it is not even determined by the graph's cycle matroid. This statement is trivial for disconnected graphs, since the graph with n vertices and no edges has magnitude n but Tutte polynomial 1 and trivial cycle matroid. However, magnitude is not a specialisation of the Tutte polynomial even for connected graphs. For example, the graphs



have the same cycle matroid, hence also the same Tutte polynomial, the same number of proper vertex colourings by any given number of colours, the same number of spanning trees, the same connectivity, the same girth, etc.; but their magnitudes are different. Conversely, there are graphs with the same magnitude that are distinguished by well-known graph invariants (Example 3·7). In that sense, magnitude seems to capture genuinely new

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aspects of a graph relative to existing invariants, at the same time as having uniquely good cardinality-like properties.

This paper is principally intended to 'disclose a new world' (in the words of Rota [**19**]) rather than to prove difficult results. It nevertheless contains two nontrivial theorems. The first is the inclusion-exclusion formula (1·2) (Theorem 4·9). For this we must impose some hypotheses. Indeed, there is *no* nontrivial graph invariant that is fully cardinality-like in the sense of satisfying both (1·1) and (1·2) without restriction (by Lemma 4·1). But the hypotheses we impose are mild enough to include, for instance, the case where all the graphs involved are trees, and the case where $G \cap H$ consists of a single vertex.

It follows that when we join a vertex of a graph G to a vertex of a graph H to form a new graph $G \vee H$, the magnitude of $G \vee H$ depends only on the magnitudes of G and H, not the vertices chosen. This is a property that magnitude shares with the Tutte polynomial. Another important invariance property of the Tutte polynomial is invariance under Whitney twists (Figure 2).This means the following: given graphs G and H, each with two chosen, distinct vertices, we may form new graphs X and Y by gluing G to H at the chosen vertices one way round or the other; then the Tutte polynomials of X and Y are equal. Our second main theorem (Theorem 5·2) is that this is also true for magnitude, provided that there is an edge between the chosen vertices of either G or H.

Speyer and Willerton showed that even in the case of connected graphs, this last hypothesis cannot be dropped [**21**, **26**]. It follows that magnitude is not a specialisation of the Tutte polynomial.

This paper is laid out as follows. In Section 2, we define the magnitude of a graph, expressing it as both a rational function and a power series over $\mathbb{Z}$. Section 3 sets out the most basic properties and examples of magnitude, including a simple formula for the magnitude of any graph whose automorphism group acts transitively on vertices. We prove that magnitude has some basic cardinality-like properties. Viewing $\#G$ as a power series over $\mathbb{Z}$, we also answer the question: what do the coefficients count?

The remaining two sections prove the two main results: the inclusion-exclusion theorem (Section 4) and the theorem on invariance under Whitney twists (Section 5). Although both concern the magnitude of the union of two graphs, the latter is not a special case of the former, as noted after the statement of Theorem 5·2.

Recent work of Hepworth and Willerton [**5**] constructs a graded homology theory of graphs, of which magnitude is the Euler characteristic. Their homology theory is a categorification of graph magnitude in the same sense that Khovanov's homology theory of knots is a categorification of the Jones polynomial [**7**]. For example, the multiplicativity property (1·1) for magnitude of graphs can be derived from a Künneth theorem for magnitude homology of graphs, and similarly, the inclusion-exclusion formula for magnitude (Theorem 4·9) lifts to a Mayer–Vietoris theorem in homology. It remains to be seen whether the theorem on invariance under Whitney twists is also the shadow of a homological theorem.

2. *The definition*

Here we define the magnitude of a graph, showing that it can be expressed as either a rational function over $\mathbb{Q}$ or a power series over $\mathbb{Z}$. We also show how to calculate magnitude.

Our conventions are these. A **graph** is a finite, undirected graph with no loops or multiple edges. Graphs may be disconnected or even have isolated vertices. Given a graph G, we write $V(G)$ for the set of vertices, $E(G)$ for the set of edges, $v(G)$ for the order of G (the

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number of vertices), $e(G)$ for its size (the number of edges) and $k(G)$ for the number of connected-components. We write $x \in G$ for $x \in V(G)$.

For vertices x and y of a graph G, let $d_G(x, y)$ or $d(x, y)$ denote the length of a shortest path between x and y, taken to be ∞ if there is no such path. This defines a metric on the set of vertices, provided that we relax the definition of metric space to allow ∞ as a distance.

We now define the magnitude of a graph G. Write $\mathbb{Z}[q]$ for the polynomial ring over the integers in one variable q. Let $Z_G = Z_G(q)$ be the square matrix over $\mathbb{Z}[q]$ whose rows and columns are indexed by the vertices of G, and whose (x, y)-entry is

$$Z_G(q)(x, y) = q^{d(x,y)}$$

$(x, y \in G)$, where by convention $q^\infty = 0$. Since $Z_G(0)$ is the identity matrix, the polynomial $\det(Z_G(q))$ has constant term 1. In particular, $\det(Z_G(q))$ is nonzero in the field $\mathbb{Q}(q)$ of rational functions over $\mathbb{Q}$, and so is invertible there. It follows that $Z_G(q)$ is invertible as a matrix over $\mathbb{Q}(q)$.

Definition 2·1. The **magnitude** of a graph G is

$$\#G(q) = \sum_{x,y \in G} (Z_G(q))^{-1}(x, y) \in \mathbb{Q}(q).$$

We usually abbreviate $\#G(q)$ as $\#G$.

Writing $\mathrm{sum}(M)$ for the sum of all the entries of a matrix M, and $\mathrm{adj}(M)$ for the adjugate of M, we have

$$\#G(q) = \mathrm{sum}\big(Z_G(q)^{-1}\big) = \frac{\mathrm{sum}\big(\mathrm{adj}(Z_G(q))\big)}{\det(Z_G(q))}. \tag{2·1}$$

Both the numerator and the denominator are polynomials in q over $\mathbb{Z}$.

Any rational function over $\mathbb{Q}$ can be expanded as a (formal) Laurent series over $\mathbb{Q}$, but $\#G$ has the special property that it is a power series over $\mathbb{Z}$. This follows from equation (2·1), since the polynomial $\det(Z_G(q))$ has constant term 1 and is therefore invertible in the ring $\mathbb{Z}[\![q]\!]$ of power series.

(Formally, both $\mathbb{Q}(q)$ and $\mathbb{Z}[\![q]\!]$ are subrings of $\mathbb{Q}(\!(q)\!)$, the field of Laurent series over $\mathbb{Q}$. When we speak of a rational function being equal to a power series, this means equality as elements of $\mathbb{Q}(\!(q)\!)$.)

Remarks 2·2.

(i) As explained in the Introduction, this apparently unmotivated definition is a special case of the very general definition of the magnitude of an enriched category [**10**, section 1], which in several other contexts produces classical invariants of size whose importance is extremely well-established.

(ii) The definition of magnitude also makes sense for directed graphs, with distance defined non-symmetrically in terms of directed paths. For simplicity, we confine ourselves to the undirected case.

The magnitude of G is the sum of all the entries of $Z_G(q)^{-1}$, but it is sometimes useful to consider the individual row-sums. We define the **weight** $w_G(x) = w_G(q)(x)$ of a vertex x to be the corresponding row-sum:

$$w_G(x) = \sum_{y \in G} (Z_G(q))^{-1}(x, y) \in \mathbb{Q}(q).$$

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The function $w_G\colon V(G) \to \mathbb{Q}(q)$ is called the **weighting** on G, and satisfies the **weighting equations**

$$\sum_{y \in G} q^{d(x,y)} w_G(y) = 1 \qquad (x \in G). \tag{2·2}$$

(The weighting can alternatively be understood as taking values in $\mathbb{Z}[\![q]\!]$, just as for magnitude itself.) Magnitude is total weight: $\#G = \sum_{x \in G} w_G(x)$. This is loosely analogous to the Gauss–Bonnet formula for the Euler characteristic of a surface, with weight playing the role of curvature [**9**, section 2].

The magnitude of a graph is most efficiently computed by solving the weighting equations (2·2):

LEMMA 2·3. *Let G be a graph and let $\widetilde{w}_G\colon V(G) \to \mathbb{Q}(q)$ be a function satisfying the weighting equations. Then $\widetilde{w}_G = w_G$ and $\#G = \sum_{x \in G} \widetilde{w}_G(x)$. The same is true when $\mathbb{Q}(q)$ is replaced by $\mathbb{Z}[\![q]\!]$.*

Proof. The matrix $Z_G(q)$ is invertible over $\mathbb{Q}(q)$, so $w_G\colon V(G) \to \mathbb{Q}(q)$ is the unique solution to the weighting equations. Hence $\widetilde{w}_G = w_G$, giving the result. The same argument holds over $\mathbb{Z}[\![q]\!]$.

3. *Basic properties and examples*

Here we state the most basic facts about magnitude. We derive formulas for the magnitudes of vertex-transitive and complete bipartite graphs. We also establish some evidence for the claim that magnitude of graphs is analogous to cardinality of sets, proving that magnitude has additivity and multiplicativity properties similar to those enjoyed by cardinality.

When the magnitude of a graph is expressed as a power series, its coefficients are integers. We give a formula for them. From this it will follow that the magnitude of a graph determines its order and size. On the other hand, it determines neither the chromatic number nor the number of connected-components, as we show.

We begin with the simplest of examples.

Example 3·1. Let G be a graph with no edges. Then Z_G is the identity matrix, so $\#G$ is the order $v(G)$. This fits with the conception of magnitude as generalised cardinality: when a graph has no edges, it is essentially just a set, and magnitude then reduces to cardinality.

It follows that magnitude is not a specialisation of the Tutte polynomial, since the Tutte polynomial of any edgeless graph is 1. Less obvious is that magnitude is not a specialisation of the Tutte polynomial for *connected* graphs. We prove this in Section 5.

A graph is **vertex-transitive** if its automorphism group acts transitively on vertices. The following result is a special case of [**10**, proposition 2·1·5].

LEMMA 3·2 (Speyer). *Let G be a vertex-transitive graph. Then*

$$\#G(q) = \frac{v(G)}{\sum_{x \in G} q^{d(g,x)}}$$

for any $g \in G$.

Proof. By transitivity, the sum $s(q) = \sum_{x \in G} q^{d(g,x)}$ is independent of g. The result follows by applying Lemma 2·3 with $\widetilde{w}_G(x) = 1/s$ for all $x \in G$.

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In particular, the diameter of a connected vertex-transitive graph can be recovered as the degree of its magnitude.

The denominator of the expression in Lemma 3·2 closely resembles the weight enumerator of a linear code, a connection discussed in [**10**, example 2·3·7].

Examples 3·3.

(i) By Lemma 3·2, the complete graph K_n on n vertices has magnitude

$$\#K_n = \frac{n}{1 + (n-1)q} = n \sum_{k=0}^{\infty} (1-n)^k q^k.$$

(ii) Similarly, the cycle graph C_n on n vertices has magnitude

$$\#C_n = \frac{n(q-1)}{q^{\lfloor (n+1)/2 \rfloor} + q^{\lceil (n+1)/2 \rceil} - q - 1} = \begin{cases} \dfrac{n(q-1)}{(q^{n/2}-1)(q+1)} & \text{if } n \text{ is even,} \\ \dfrac{n(q-1)}{2q^{(n+1)/2} - q - 1} & \text{if } n \text{ is odd.} \end{cases}$$

These equations hold for all $n \geqslant 1$, interpreting C_1 as the graph with just one vertex and C_2 as the graph with just one edge.

(iii) The Petersen graph (shown) is also vertex-transitive, so has magnitude as follows:



$$\frac{10}{1 + 3q + 6q^2} = 10 - 30q + 30q^2 + 90q^3 - 450q^4 + \cdots.$$

Example 3·4. By direct calculation using Lemma 2·3, the complete bipartite graph $K_{m,n}$ has magnitude

$$\#K_{m,n} = \frac{(m+n) - (2mn - m - n)q}{(1+q)(1 - (m-1)(n-1)q^2)}.$$

The cardinality of a disjoint union of sets is the sum of their individual cardinalities. The same is true of the magnitude of graphs:

LEMMA 3·5. *Let G and H be graphs. The magnitude of their disjoint union $G \sqcup H$ is given by $\#(G \sqcup H) = \#G + \#H$.*

Proof. $Z_{G \sqcup H}$ is the block sum of Z_G and Z_H, and the result follows.

The cardinality of a cartesian product of sets is the product of their cardinalities, and again, there is an analogous result for the magnitude of graphs. Recall that the **cartesian product** $G \,\square\, H$ has $V(G) \times V(H)$ as its vertex-set, with an edge between (x, y) and (x', y') if *either* $x = x'$ and $\{y, y'\} \in E(H)$ *or* $y = y'$ and $\{x, x'\} \in E(G)$.

LEMMA 3·6. *Let G and H be graphs. The magnitude of their cartesian product is given by $\#(G \,\square\, H) = \#G \cdot \#H$.*

Proof. For $x, x' \in G$ and $y, y' \in H$,

$$d_{G \,\square\, H}((x, y), (x', y')) = d_G(x, x') + d_H(y, y')$$

and so

$$Z_{G \,\square\, H}((x, y), (x', y')) = Z_G(x, x') \cdot Z_H(y, y').$$

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Hence $Z_{G\Box H}$ is the Kronecker product of Z_G and Z_H, which implies that $Z_{G\Box H}^{-1}$ is the Kronecker product of Z_G^{-1} and Z_H^{-1}. The result follows.

Example 3·7. By Example 3·3(i) and Lemma 3·6,

$$\#(K_2 \,\Box\, K_3) = \#K_2 \cdot \#K_3 = \frac{2}{1+q} \cdot \frac{3}{1+2q} = \frac{6}{1+3q+2q^2}.$$

So by Example 3·4, $K_2 \,\Box\, K_3$ has the same magnitude as $K_{3,3}$, a graph with a different chromatic number. The chromatic number cannot, therefore, be derived from the magnitude, even for connected graphs; hence the Tutte polynomial cannot be either. We prove the converse in Section 5.

Remarks 3·8.

(i) There is an unfortunate clash of terminology for graph products. For any symmetric monoidal category $\mathscr{V}$, the category of $\mathscr{V}$-enriched categories carries a tensor product [**6**, section 1·4]. Taking $\mathscr{V} = (\mathbb{N}, \geqslant, +, 0)$, this gives a tensor product of graphs, which is what graph theorists call the cartesian product, $\Box$. On the other hand, the category of graphs also has what category theorists call a product, or, for emphasis, a cartesian product. This is what graph theorists sometimes call the tensor product, $\times$ [**4**, section 6·3].

(ii) Neither the magnitude of $G \times H$ nor that of the strong product $G \boxtimes H$ [**4**, section 7·15] is determined by the magnitudes of G and H. Indeed, by Example 3·7, it is enough to show that

$$\#(K_2 \times (K_2 \,\Box\, K_3)) \neq \#(K_2 \times K_{3,3}), \quad \#(K_2 \boxtimes (K_2 \,\Box\, K_3)) \neq \#(K_2 \boxtimes K_{3,3}),$$

and this is easily done using Lemma 3·2.

We saw in Section 2 that the magnitude of a graph can be expressed as a power series with integer coefficients. Those coefficients can be described explicitly:

PROPOSITION 3·9. *For any graph G,*

$$\#G(q) = \sum_{k=0}^{\infty} (-1)^k \sum_{x_0 \neq x_1 \neq \cdots \neq x_k} q^{d(x_0,x_1)+\cdots+d(x_{k-1},x_k)} \in \mathbb{Z}[\![q]\!],$$

where $x_0, \ldots, x_k$ denote vertices of G. That is, writing $\#G(q) = \sum_{n=0}^{\infty} c_n q^n \in \mathbb{Z}[\![q]\!]$,

$$c_n = \sum_{k=0}^{n} (-1)^k \left|\left\{(x_0, \ldots, x_k) : x_0 \neq x_1 \neq \cdots \neq x_k,\ d(x_0, x_1) + \cdots + d(x_{k-1}, x_k) = n\right\}\right|.$$

Proof. The two statements are trivially equivalent; we prove the first. For $x \in G$, define $\widetilde{w}_G(x) \in \mathbb{Z}[\![q]\!]$ by

$$\widetilde{w}_G(x) = \sum_{k=0}^{\infty} (-1)^k \sum_{x = x_0 \neq x_1 \neq \cdots \neq x_k} q^{d(x_0,x_1)+\cdots+d(x_{k-1},x_k)}.$$

We show that $\widetilde{w}_G$ satisfies the weighting equations. The result then follows from Lemma 2·3.

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To verify the weighting equations, let $x \in G$. Then

$$\sum_{y \in G} q^{d(x,y)} \widetilde{w}_G(y) = \widetilde{w}_G(x) + \sum_{y:\, y \neq x} q^{d(x,y)} \widetilde{w}_G(y)$$
$$= \widetilde{w}_G(x) + \sum_{k=0}^{\infty} (-1)^k \sum_{x \neq y_0 \neq \cdots \neq y_k} q^{d(x,y_0)+d(y_0,y_1)+\cdots+d(y_{k-1},y_k)},$$

which cancels to give 1, as required.

Proposition 3·9 bears a formal resemblance to Hall's formula for Möbius inversion in posets ([**18**, proposition 6] and [**22**, proposition 3·8·5]), as well as the classical alternating sum formula for the Euler characteristic of a topological space. The three formulas are connected by the notion of the Euler characteristic of a category [**9**, corollary 1·5 and proposition 2·11], of which magnitude is the graph-theoretic analogue.

COROLLARY 3·10. *Let G be a graph. Then $v(G) = \#G(0)$ and $e(G) = -\frac{1}{2}\frac{d}{dq}\#G(q)\Big|_{q=0}$.*

Proof. In the notation of Proposition 3·9, $c_0 = v(G)$ and $c_1 = -2e(G)$.

In particular, magnitude determines both order and size. Unlike the Tutte polynomial, it even determines order for disconnected graphs.

Remark 3·11. It follows from Proposition 3·9 that $c_0 \geqslant 0$, $c_1 \leqslant 0$, and

$$c_2 = |\{(x, y, z) : d(x, y) = d(y, z) = 1\}| - |\{(x, z) : d(x, z) = 2\}| \geqslant 0,$$

suggesting that the coefficients c_n alternate in sign indefinitely. However, the Petersen graph (Example 3·3(iii)) shows that this is not true in general.

Example 3·12. In all the examples so far, $\#G(1)$ is the number $k(G)$ of connected-components of G. Indeed, this is easily proved in the case where none of the weights of G has a pole at 1: for the weighting equations (2·2) then imply that at $q = 1$, the weights in each component sum to 1. But $\#G(1) \neq k(G)$ in general, by the following example of Willerton [**10**, example 2·2·8]. Let W be the complete graph K_6 with a triangle of edges (but no vertices) removed. By direct calculation,

$$\#W = \frac{6}{1 + 4q},$$

giving $\#W(1) = 6/5 \neq 1 = k(W)$.

In fact, there is *no* way to derive the number of connected-components from the magnitude. For, writing mG for the disjoint union of m copies of a graph G, we have

$$\#5W = \frac{30}{1 + 4q} = \#6K_5$$

(by Example 3·3(i) and Lemma 3·5), but $k(5W) = 5 \neq 6 = k(6K_5)$.

4. *The magnitude of a union*

We now develop the analogy between magnitude of graphs and cardinality of sets. We have already seen several aspects of this: the magnitude of a disjoint union is the sum of the magnitudes (Lemma 3·5), the magnitude of a cartesian product is the product of the magnitudes (Lemma 3·6), and the magnitude of a graph with no edges is simply the cardinality

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of the vertex-set (Example 3·1). It is natural, therefore, to ask whether magnitude obeys the inclusion-exclusion principle.

In fact, it does not, for reasons that have nothing to do with magnitude. As we shall see, no nontrivial graph invariant behaves wholly like cardinality. However, magnitude does satisfy inclusion-exclusion under reasonably generous hypotheses on the subgraphs concerned. This is our first main result, Theorem 4·9.

Let us begin by making precise the claim about cardinality-like invariants. For a ring R, an R-valued **graph invariant** is a function Φ assigning an element $\Phi(G) \in R$ to each graph G, in such a way that $\Phi(G) = \Phi(H)$ whenever $G \cong H$. It is **multiplicative** if $\Phi(K_1) = 1$ and $\Phi(G \,\square\, H) = \Phi(G) \cdot \Phi(H)$ for all G and H. (Here K_1 is the one-vertex graph, the unit for $\square$.) It satisfies **inclusion-exclusion** if $\Phi(\varnothing) = 0$ and

$$\Phi(X) = \Phi(G) + \Phi(H) - \Phi(G \cap H)$$

whenever X is a graph with subgraphs G and H such that $G \cup H = X$.

For example, take any ring R, and let $\Phi(G) = v(G)$ be the order of G, interpreted as the element $v(G) \cdot 1 = 1 + \cdots + 1$ of R. Then Φ is a multiplicative R-valued graph invariant satisfying inclusion-exclusion. The next lemma tells us that under mild assumptions on R, it is the only one.

LEMMA 4·1. *Let R be a ring containing no nonzero nilpotents. Then the only multiplicative R-valued graph invariant satisfying inclusion-exclusion is the order.*

Proof. Let Φ be a multiplicative R-valued graph invariant satisfying inclusion-exclusion. Then $\Phi(G \sqcup H) = \Phi(G) + \Phi(H)$ for all G and H. Writing $\overline{K_n}$ for the edgeless graph on n vertices, we have $\Phi(\overline{K_0}) = \Phi(\varnothing) = 0$ and $\Phi(\overline{K_1}) = \Phi(K_1) = 1$, so by induction, $\Phi(\overline{K_n}) = n$ for all $n \geqslant 0$.

Let X be a graph. Choose an edge e of X, write X' for the subgraph of X containing all the vertices and all the edges except e, and write H for the subgraph of X consisting of just e and its two endpoints. Then by inclusion-exclusion,

$$\begin{aligned}\Phi(X) &= \Phi(X') + \Phi(H) - \Phi(X' \cap H)\\ &= \Phi(X') + \Phi(K_2) - \Phi(\overline{K_2}).\end{aligned}$$

So, writing $\varepsilon = \Phi(K_2) - 2$, we have $\Phi(X) = \Phi(X') + \varepsilon$. Applying this argument repeatedly gives $\Phi(X) = \Phi(\overline{K_{v(X)}}) + \varepsilon \cdot e(X)$, that is, $\Phi(X) = v(X) + \varepsilon \cdot e(X)$.

It remains to show that $\varepsilon = 0$, which we do by computing $\Phi(C_4)$ in two ways. On the one hand, $\Phi(C_4) = 4 + 4\varepsilon$ by the previous paragraph. On the other, $C_4 = K_2 \,\square\, K_2$ and Φ is multiplicative, so $\Phi(C_4) = (2 + \varepsilon)^2$. Comparing the two expressions gives $\varepsilon^2 = 0$. But R has no nonzero nilpotents, so $\varepsilon = 0$, as required.

(For an arbitrary ring R, the graph invariants satisfying multiplicativity and inclusion-exclusion are exactly those of the form $v + \varepsilon e$ where $\varepsilon \in R$ with $\varepsilon^2 = 0$.)

We already know that magnitude is a multiplicative graph invariant (Lemma 3·6) and that it is not simply the order. It cannot, therefore, satisfy inclusion-exclusion.

Nevertheless, we can seek conditions under which the inclusion-exclusion principle does hold. Consider a graph X expressed as the union of subgraphs G and H. Since magnitude is defined in terms of the metric, it is natural to ask that distances between vertices of G are the same no matter whether we measure them in G or in X, and similarly for H and $G \cap H$. We therefore make the following definition.

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Definition 4·2. A subgraph U of a graph X is **convex** if $d_U(u, u') = d_X(u, u')$ for all $u, u' \in U$.

The terminology comes from a useful analogy between graphs and convex sets. A subgraph of a graph is convex if its shortest-path metric is the same as its subspace metric. Analogously, a compact subset of $\mathbb{R}^n$ is convex if its shortest-path metric is the same as the subspace metric. Of course, the two uses of 'path' are different: in the discrete case, a path of length D is a distance-preserving map out of $\{0, 1, \ldots, D\}$, while in the continuous case, it is a distance-preserving map out of $[0, D]$.

When a convex set $X \subseteq \mathbb{R}^n$ is covered by closed subsets G and H, it is a fact that if $G \cap H$ is convex then so are G and H. Here is the graph-theoretic analogue.

LEMMA 4·3. *Let X be a graph, and let G and H be subgraphs with $G \cup H = X$. If $G \cap H$ is convex in X then G and H are also convex in X.*

Proof. We prove it for G. Let $g, g' \in G$. If $d_X(g, g') = \infty$ then certainly $d_G(g, g') = d_X(g, g')$. Otherwise, write $n = d_X(g, g') < \infty$. We may choose a shortest path $g = x_0, x_1, \ldots, x_n = g'$ from g to g' in X containing the greatest possible number of vertices of G. Suppose for a contradiction that $x_j \notin G$ for some j.

By Lemma 4·4 below, we may choose i and k with $0 \leqslant i < j < k \leqslant n$ and $x_i, x_k \in G \cap H$. Then $x_i, x_{i+1}, \ldots, x_k$ is a shortest path from x_i to x_k in X, so $d_X(x_i, x_k) = k - i$. But $G \cap H$ is convex in X, so there is a path $u_i, u_{i+1}, \ldots u_k$ from x_i to x_k in $G \cap H$. Hence

$$g = x_0, x_1, \ldots, x_{i-1}, x_i = u_i, u_{i+1}, \ldots, u_{k-1}, u_k = x_k, x_{k+1}, \ldots, x_{n-1}, x_n = g'$$

is a shortest path from g to g' in X containing more vertices of G than the original path. This is the required contradiction.

This proof used the following lemma, which is a combinatorial counterpart of the fact that when G and H are closed subsets of $\mathbb{R}^n$, any path from a point of G to a point of H passes through some point of $G \cap H$.

LEMMA 4·4. *Let X be a graph, with subgraphs G and H such that $G \cup H = X$. Then every path from a vertex in G to a vertex in H contains at least one vertex in $G \cap H$.*

Proof. Let $x_0, x_1, \ldots, x_n$ be a path with $x_0 \in G$ and $x_n \in H$. Take the largest $i \in \{0, 1, \ldots, n\}$ such that $x_i \in G$. We prove that $x_i \in G \cap H$. If $i = n$, this is immediate. If not, then $x_{i+1} \notin G$, so $\{x_i, x_{i+1}\} \notin E(G)$. But $X = G \cup H$, so $\{x_i, x_{i+1}\} \in E(H)$, so $x_i \in H$, as required.

A wrinkle in the analogy between convex sets and graphs is that in a convex set, there is only one shortest path between each pair of points, but in a graph, there may be many. It is arguably more accurate to say that convex sets are analogous to trees, since shortest paths in a tree are unique. We will see that for trees and subtrees, the inclusion-exclusion formula holds without restriction (Corollary 4·13, due to Meckes). The following example of Willerton [**26**] shows that for convex subgraphs of an arbitrary graph, inclusion-exclusion can fail.

Example 4·5 (Willerton). Let X be the graph formed by gluing two 3-cycles together along an edge. Then

$$\#X = \frac{4 - 2q}{1 + 2q - q^2} \neq \frac{4 + 2q}{1 + 3q + 2q^2} = 2 \cdot \#C_3 - \#C_2,$$

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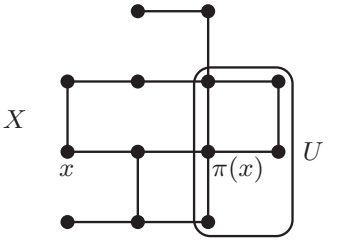


Fig. 1. A graph X, a subgraph U to which it projects, and the effect of the projection map π on a vertex x.

by direct calculation and Example 3·3(ii) respectively. So, magnitude does not satisfy the inclusion-exclusion principle even when all the subgraphs concerned are convex.

Convexity will be one hypothesis in our inclusion-exclusion theorem. We now formulate the other.

Definition 4·6. Let U be a convex subgraph of a graph X. Write

$$V_U(X) = \bigcup_{u \in U} \{x \in X : d(u, x) < \infty\} = \{x \in X : x \text{ is connected to some vertex of } U\}.$$

We say that X **projects** to U (Figure 1) if for all $x \in V_U(X)$, there exists a vertex $\pi(x) \in U$ such that for all $u \in U$,

$$d(x, u) = d(x, \pi(x)) + d(\pi(x), u).$$

If X projects to U then $\pi(x)$ is uniquely determined by x, being the unique vertex of U closest to x. This defines a **projection** map $\pi : V_U(X) \to V(U)$.

Example 4·7. Let e be an edge of a graph X. If the component of X containing e is bipartite, then X projects to the subgraph consisting of e and its endpoints alone.

LEMMA 4·8. *Let X be a graph, and let U be a convex subgraph to which X projects. Then*

$$w_U(u) = \sum_{x \in \pi^{-1}(u)} q^{d(u,x)} w_X(x) \tag{4·1}$$

for each $u \in U$, where π denotes the projection.

Proof. Write $\widetilde{w}_U(u)$ for the right-hand side of (4·1). We verify that $\widetilde{w}_U$ satisfies the weighting equations. It then follows from Lemma 2·3 that $w_U = \widetilde{w}_U$.

Let $u \in U$. Recalling the convention that $q^\infty = 0$, we have

$$\begin{aligned}\sum_{v \in U} q^{d(u,v)} \widetilde{w}_U(v) &= \sum_{v \in U,\ y \in \pi^{-1}(v)} q^{d(u,v)+d(v,y)} w_X(y) \\ &= \sum_{y \in V_U(X)} q^{d(u,\pi(y))+d(\pi(y),y)} w_X(y) \\ &= \sum_{y \in X} q^{d(u,y)} w_X(y) = 1,\end{aligned}$$

as required.

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THEOREM 4·9. *Let X be a graph, with subgraphs G and H such that $G \cup H = X$. Suppose that $G \cap H$ is convex in X and that H projects to $G \cap H$. Then*

$$\#X = \#G + \#H - \#(G \cap H).$$

Proof. We will prove that $w_X = w_G + w_H - w_{G\cap H}$, where on the right-hand side, the function w_G on $V(G)$ is extended by zero to all of $V(X)$, and similarly w_H and $w_{G\cap H}$. The theorem then follows immediately.

We may unambiguously write d for distance, by Lemma 4·3. Also, we write $\pi : V_{G\cap H}(H) \to V(G \cap H)$ for the projection associated with $G \cap H \subseteq H$.

First I claim that for all $g \in G$ and $h \in V_{G\cap H}(H)$,

$$d(g, h) = d(g, \pi(h)) + d(\pi(h), h). \tag{4·2}$$

If $d(g, h) = \infty$, this is immediate from the triangle inequality. Otherwise, by Lemma 4·4, $d(g, h) = d(g, u) + d(u, h)$ for some $u \in G \cap H$. But also

$$\begin{aligned} d(g, u) + d(u, h) &= d(g, u) + d(u, \pi(h)) + d(\pi(h), h) \\ &\geqslant d(g, \pi(h)) + d(\pi(h), h) \geqslant d(g, h), \end{aligned}$$

so equality holds throughout, proving the claim.

We now verify that $w_G + w_H - w_{G\cap H}$ satisfies the weighting equations for X. These state that for all $x \in X$,

$$\sum_{g\in G} q^{d(x,g)} w_G(g) + \sum_{h\in H} q^{d(x,h)} w_H(h) - \sum_{u\in G\cap H} q^{d(x,u)} w_{G\cap H}(u) = 1. \tag{4·3}$$

If $x \in G$ then by Lemma 4·8, the left-hand side of (4·3) is

$$1 + \sum_{h\in H} q^{d(x,h)} w_H(h) - \sum_{u\in G\cap H,\ h\in\pi^{-1}(u)} q^{d(x,u)+d(u,h)} w_H(h),$$

which by Lemma 4·4 is equal to

$$1 + \sum_{h\in V_{G\cap H}(H)} q^{d(x,h)} w_H(h) - \sum_{h\in V_{G\cap H}(H)} q^{d(x,\pi(h))+d(\pi(h),h)} w_H(h),$$

and equation (4·2) implies that this is equal to 1. If $x \in V_{G\cap H}(H)$ then, by equation (4·2), the left-hand side of (4·3) is

$$\begin{aligned} q^{d(x,\pi(x))} \sum_{g\in G} q^{d(\pi(x),g)} w_G(g) + 1 - q^{d(x,\pi(x))} \sum_{u\in G\cap H} q^{d(\pi(x),u)} w_{G\cap H}(u) \\ = q^{d(x,\pi(x))} + 1 - q^{d(x,\pi(x))} = 1. \end{aligned}$$

Finally, if $x \in V(H) \setminus V_{G\cap H}(H)$ then by Lemma 4·4, the left-hand side of (4·3) is $0+1-0 = 1$. So equation (4·3) holds in all cases, giving $w_X = w_G + w_H - w_{G\cap H}$ by Lemma 2·3, as required.

We record three corollaries. First, given graphs G and H, we may form their one-point join $G \vee H$, obtained from the disjoint union of G and H by identifying one vertex of G with one vertex of H. In principle, the magnitude of $G \vee H$ could depend on the vertices chosen; but, like the Tutte polynomial, it does not.

COROLLARY 4·10. *Let G and H be graphs. Then $\#(G \vee H) = \#G + \#H - 1$.*

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The Tutte polynomial does not distinguish between the one-point join of two graphs and their disjoint union: $T_{G\vee H} = T_{G\sqcup H}$. Magnitude does: by Corollary 4·10 and Lemma 3·5, $\#(G \vee H) = \#(G \sqcup H) - 1$.

Example 4·11. Consider the following three graphs:



Using the one-point join operation twice, we can build each of them from the same pieces, one copy of K_3 and two of K_2. So all three have the same magnitude (as well as the same Tutte polynomial), namely

$$\#K_3 + 2 \cdot \#K_2 - 2 = \frac{5 + 5q - 4q^2}{(1+q)(1+2q)}.$$

Example 4·12. Any forest G can be obtained by successively joining edges to the edgeless graph with one vertex for each component of G. Repeated application of Corollary 4·10 gives

$$\begin{aligned}\#G &= k(G) + e(G)\frac{1-q}{1+q} = v(G) - 2e(G)\frac{q}{1+q} \\ &= v(G) - 2e(G)q + 2e(G)q^2 - 2e(G)q^3 + \cdots .\end{aligned}$$

In particular, the magnitude of a tree depends only on the number of edges.

Our second corollary, due to Meckes [**15**], follows from Example 4·12 or can be proved directly from Theorem 4·9.

COROLLARY 4·13 (Meckes). *Let X be a tree, with subtrees G and H such that $G \cup H = X$. Then $\#X = \#G + \#H - \#(G \cap H)$.*

COROLLARY 4·14. *Let G be a graph and H a bipartite graph. Let X be a graph obtained by identifying some edge of G with some edge of H. Then*

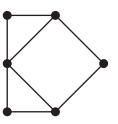
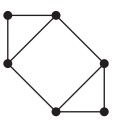
$$\#X = \#G + \#H - \frac{2}{1+q}.$$

Proof. This follows from Theorem 4·9 and Example 4·7, using the formula for $\#K_2$ in Example 3·3(i).

Example 4·15. Corollary 4·14 implies that when an arbitrary graph G has an even cycle glued onto it by an edge, the magnitude of the resulting graph does not depend on which edge of G the cycle was glued onto. This is false in general for odd cycles, as the next example shows.

Example 4·16. Let B be the graph formed by gluing a 3-cycle to a 4-cycle along an edge. By Corollary 4·14, $\#B = \#C_3 + \#C_4 - \#C_2$.

Now consider gluing a 3-cycle to B along another edge of the 4-cycle. Depending on which edge of B we glue along, this could produce either of the two graphs

$X =$  , $Y =$  .

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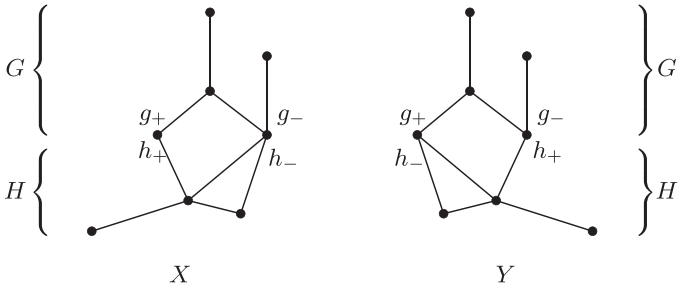


Fig. 2. Two graphs X and Y differing by a Whitney twist.

Neither B nor C_3 is bipartite, so Corollary 4·14 does not apply to either X or Y. However, Theorem 4·9 does apply to X, taking $G = C_3$ and $H = B$. Thus,

$$\#X = \#C_3 + \#B - \#C_2 = 2 \cdot \#C_3 + \#C_4 - 2 \cdot \#C_2 = \frac{6 + 8q - 2q^2}{1 + 4q + 5q^2 + 2q^3}.$$

On the other hand, the hypotheses of Theorem 4·9 do not hold for $Y = C_3 \cup B$. Nor does the conclusion, since a direct calculation shows that

$$\#Y = \frac{6 - 4q}{1 + 2q - q^3} \neq \#X.$$

5. *Whitney twists*

Much information about a graph is contained in its cycle matroid. (See [**17**], for instance.) Essentially by definition, two graphs G and H have isomorphic cycle matroids if and only if there is a bijection between their edge-sets with the property that a sequence of edges in G is a cycle exactly when the corresponding sequence in H is a cycle. In 1933, Whitney [**25**] showed that two graphs have isomorphic cycle matroids if and only if one can be transformed into the other by a finite sequence of moves of the following three types.

The first is **vertex identification**: whenever a graph X can be decomposed as a disjoint union $G \sqcup H$, and g and h are vertices of G and H respectively, change X to the graph $G \vee H$ formed by identifying g with h. The second is the reverse of the first.

The third is the Whitney twist, defined as follows (Figure 2). Take a graph G with two distinct distinguished vertices, g_+ and g_-, and take H, h_+ and h_- similarly. Form a new graph X by taking the disjoint union of G and H then identifying g_+ with h_+ and g_- with h_- (and, if this creates a double edge between the points of identification, identifying those edges). Define Y similarly, but identifying g_+ with h_- and g_- with h_+. The graphs X and Y are said to differ by a **Whitney twist**.

By the theorem of Whitney, a graph invariant assigns the same value to graphs with isomorphic cycle matroids if and only if it is invariant under vertex identification and Whitney twists. Now, magnitude is not invariant under vertex identification, as by Lemma 3·5 and Corollary 4·10,

$$\#(G \sqcup H) = \#G + \#H, \qquad \#(G \vee H) = \#G + \#H - 1.$$

However, these equations imply that $\Phi(G) = \#G - k(G)$ *is* invariant under vertex identification, where k is the number of connected-components. Moreover, k is invariant under Whitney twists. Hence Φ depends only on the cycle matroid if and only if magnitude is invariant under Whitney twists.

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We show here that, in fact, magnitude is not invariant under Whitney twists; so Φ does not depend only on the cycle matroid. Moreover, since the Tutte polynomial can be defined in terms of the cycle matroid and is therefore invariant under Whitney twists, magnitude is not a specialisation of the Tutte polynomial. This is trivially true for disconnected graphs (by Example 3·1), but the graphs in our counterexample are connected.

But there is also a positive result. We prove that magnitude *is* invariant under Whitney twists when the two points of identification are adjacent (Theorem 5·2). In this sense, $\#G - k(G)$ comes close to depending only on the cycle matroid of G.

We begin by exhibiting two graphs that differ by a Whitney twist but do not have the same magnitude. This strategy for showing that magnitude is not a specialisation of the Tutte polynomial was suggested by Speyer [**21**], and the first example of such a pair was found by Willerton [**26**]. The following proof uses a smaller example.

PROPOSITION 5·1 (Speyer and Willerton). *There exists a pair of connected graphs with isomorphic cycle matroids (hence the same Tutte polynomial) but different magnitudes.*

Proof. The graphs X and Y of Example 4·16 differ by a Whitney twist, but their magnitudes are not equal.

Before we prove our main result on Whitney twists, let us fix some notation. We work with graphs X and Y obtained from (G, g_+, g_-) and (H, h_+, h_-), as in the definition of Whitney twist above. The vertex of X formed by identifying g_+ with h_+ will be denoted by either g_+ or h_+; thus, $g_+ = h_+$ as vertices of X. We refer to $g_+ = h_+$ and $g_- = h_-$ as the **gluing points** of X, and similarly for Y.

The vertices of X that are not gluing points are in canonical bijection with the vertices of Y that are not gluing points. The two gluing points are adjacent in X if and only if either g_+ is adjacent to g_- in G or h_+ is adjacent to h_- in H. This in turn is equivalent to the gluing points being adjacent in Y.

THEOREM 5·2. *Let X and Y be graphs differing by a Whitney twist, and suppose that the two gluing points are adjacent in X (or equivalently Y). Then $\#X = \#Y$.*

This was conjectured by Willerton [**26**]. In the proof, we do not attempt to derive any expression for $\#X$ or $\#Y$ in terms of $\#G$ and $\#H$. (Example 4·5 shows that it is not given by the inclusion-exclusion formula.) Instead, we find a direct relationship between the weightings on X and Y.

Proof. We use the same notation as above, and assume without loss of generality that $\{g_+, g_-\} \in E(G)$ and $\{h_+, h_-\} \in E(H)$.

Both G and H are convex in both X and Y, so we may unambiguously use the unsubscripted notation $d(a, b)$ when a and b both belong to G or both belong to H. To describe the other distances in X and Y, it is convenient to introduce some further notation. For $g \in G$, write

$$\delta(g) = \min\{d(g, g_-), d(g, g_+)\},$$

and similarly $\delta(h)$ for $h \in H$. Partition $V(G)$ as $G_+ \cup G_0 \cup G_-$, where

$$\begin{aligned} G_+ &= \{g \in G : d(g, g_+) < d(g, g_-)\}, \\ G_0 &= \{g \in G : d(g, g_+) = d(g, g_-)\}, \\ G_- &= \{g \in G : d(g, g_+) > d(g, g_-)\}, \end{aligned}$$

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and similarly for H. Then for $g \in G$ and $h \in H$, we have

$$d_X(g,h) = \begin{cases} \delta(g) + \delta(h) + 1 & \text{if } (g \in G_+ \text{ and } h \in H_-) \text{ or } (g \in G_- \text{ and } h \in H_+) \\ \delta(g) + \delta(h) & \text{otherwise,} \end{cases}$$

$$d_Y(g,h) = \begin{cases} \delta(g) + \delta(h) + 1 & \text{if } (g \in G_+ \text{ and } h \in H_+) \text{ or } (g \in G_- \text{ and } h \in H_-) \\ \delta(g) + \delta(h) & \text{otherwise.} \end{cases}$$

We now describe the weighting on Y. Put

$$u_+^G = \sum_{g \in G_+} q^{\delta(g)} w_X(g), \quad u_0^G = \sum_{g \in G_0} q^{\delta(g)} w_X(g), \quad u_-^G = \sum_{g \in G_-} q^{\delta(g)} w_X(g),$$

and similarly u_+^H, u_0^H and u_-^H. Define $\widetilde{w}_Y \colon V(Y) \to \mathbb{Q}(q)$ by $\widetilde{w}_Y(y) = w_X(y)$ whenever y is not a gluing point, and

$$\widetilde{w}_Y(g_+) = w_X(g_+) - u_+^G + u_-^G, \tag{5·1}$$

$$\widetilde{w}_Y(g_-) = w_X(g_-) - u_-^G + u_+^G. \tag{5·2}$$

We will show that $\widetilde{w}$ satisfies the weighting equations for Y, which by Lemma 2·3 implies that $\widetilde{w}_Y = w_Y$, hence $\#Y = \#X$.

First I claim that the defining equations (5·1) and (5·2) for $\widetilde{w}_Y$ are unchanged if we replace G by H and g by h throughout. Because of the identifications between $g_\pm$ and $h_\pm$ in X and in Y, this reduces to the claim that

$$w_X(g_+) - u_+^G + u_-^G = w_X(h_-) - u_-^H + u_+^H. \tag{5·3}$$

To prove this, note that

$$V(X) = (G_+ \cup G_0 \cup G_-) \cup (H_+ \cup H_0 \cup H_-),$$

this union being disjoint except that $G_+ \cap H_+ = \{g_+\}$ and $G_- \cap H_- = \{g_-\}$. The weighting equation $\sum_{x \in X} q^{d_X(g_+,x)} w_X(x) = 1$ therefore gives

$$(u_+^G + u_0^G + q u_-^G) + (u_+^H + u_0^H + q u_-^H) - (w_X(g_+) + q w_X(g_-)) = 1. \tag{5·4}$$

The same is true when $+$ and $-$ are interchanged:

$$(q u_+^G + u_0^G + u_-^G) + (q u_+^H + u_0^H + u_-^H) - (q w_X(g_+) + w_X(g_-)) = 1. \tag{5·5}$$

Subtracting (5·5) from (5·4) gives (5·3), proving the claim.

We now show that $\widetilde{w}_Y$ satisfies the weighting equations. By the symmetry just established, it is enough to show that $\sum_{y \in Y} q^{d_Y(g,y)} \widetilde{w}_Y(y) = 1$ whenever $g \in G$. Let $g \in G$. We have

$$\sum_{y \in Y} q^{d_Y(g,y)} \widetilde{w}_Y(y) - 1 = \sum_{y \in Y} q^{d_Y(g,y)} \widetilde{w}_Y(y) - \sum_{x \in X} q^{d_X(g,x)} w_X(x), \tag{5·6}$$

and we want to prove that the left-hand side of (5·6) is zero. When $x = y \in G \setminus \{g_+, g_-\}$, we have $d_Y(g,y) = d_X(g,x)$ and $\widetilde{w}_Y(y) = w_X(x)$, so the x- and y-summands on the right-hand side cancel out. The same is true when $x = y \in H_0$. The right-hand side is therefore unchanged if each sum is restricted to run over only $H_+ \cup H_-$. So by definition of $\widetilde{w}_Y$, the right-hand side is equal to

$$\sum_{h \in (H_+ \cup H_-) \setminus \{h_+, h_-\}} \left(q^{d_Y(g,h)} - q^{d_X(g,h)}\right) w_X(h) + \left(q^{d(g,g_+)} - q^{d(g,g_-)}\right)\left(u_-^G - u_+^G\right). \tag{5·7}$$

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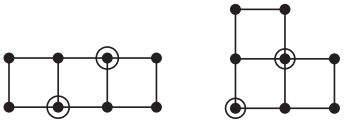
We must show that this is zero. If $g \in G_0$ then every summand in (5·7) vanishes. If $g \in G_+$ then (5·7) is equal to

$$\begin{aligned}\sum_{h \in H_+ \setminus \{h_+\}} \big(q^{\delta(g)+\delta(h)+1} - q^{\delta(g)+\delta(h)}\big) w_X(h) \\ + \sum_{h \in H_- \setminus \{h_-\}} \big(q^{\delta(g)+\delta(h)} - q^{\delta(g)+\delta(h)+1}\big) w_X(h) + \big(q^{\delta(g)} - q^{\delta(g)+1}\big)\big(u_-^G - u_+^G\big) \\ = q^{\delta(g)}(q-1)\Big\{\big(u_+^H - w_X(h_+)\big) - \big(u_-^H - w_X(h_-)\big) - \big(u_-^G - u_+^G\big)\Big\} \\ = 0,\end{aligned}$$

using (5·3) in the last step. By symmetry, if $g \in G_-$ then (5·7) is also zero. Hence (5·7) is zero in all cases, completing the proof.

Example 5·3. Randomly generate graphs G and H, making each pair of vertices adjacent with probability p. Choose at random a pair of distinct vertices in each of G and H, and glue G and H together at these vertices to form graphs X and Y differing by a Whitney twist. The probability that the gluing points are adjacent in X is $p(2-p)$, so by Theorem 5·2, the probability that $\#X = \#Y$ is at least $p(2-p)$. For example, when $p = 1/2$, graphs differing by a Whitney twist have equal magnitude with probability at least $3/4$.

It may happen that graphs differing by a Whitney twist have the same magnitude even if the gluing points are not adjacent. This can occur for trivial reasons of symmetry, or for other reasons. For example, the graphs



differ by a Whitney twist, but the gluing points (circled) are not adjacent, so the hypotheses of Theorem 5·2 are not satisfied. Nevertheless, Example 4·15 guarantees that they have the same magnitude.

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