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# Potential vorticity in magnetohydrodynamics

**G. M. Webb**[1]† **and R. L. Mace**[2]

[1]Center for Space Plasma and Aeronomic Research, The University of Alabama in Huntsville, Huntsville AL 35805, USA
[2]School of Chemistry and Physics, The University of KwaZulu-Natal, Durban Westville, Durban, Natal 4000, South Africa

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A version of Noether's second theorem using Lagrange multipliers is used to investigate fluid relabelling symmetries conservation laws in magnetohydrodynamics (MHD). We obtain a new generalized potential vorticity type conservation equation for MHD which takes into account entropy gradients and the $\mathbf{J}\times\mathbf{B}$ force on the plasma due to the current $\mathbf{J}$ and magnetic induction $\mathbf{B}$. This new conservation law for MHD is derived by using Noether's second theorem in conjunction with a class of fluid relabelling symmetries in which the symmetry generator for the Lagrange label transformations is non-parallel to the magnetic field induction in Lagrange label space. This is associated with an Abelian Lie pseudo algebra and a foliated phase space in Lagrange label space. It contains as a special case Ertel's theorem in ideal fluid mechanics. An independent derivation shows that the new conservation law is also valid for more general physical situations.

## 1. Introduction

Variational methods are widely used in physics, engineering and applied mathematics. Noether's theorems provide a route to deriving conservation laws for systems of differential equations governed by an action principle. Noether's theorem applies to systems of Euler-Lagrange equations that are in Kovalevskaya form (e.g Olver 1993). For other Euler-Lagrange systems, each non-trivial variational symmetry leads to a conservation law, but there is no guarantee that it is non-trivial.

The main aim of the present paper is to derive a generalized potential vorticity equation in MHD by using a version of Noether's second theorem obtained by Hydon and Mansfield (2011). Cheviakov (2014) has derived new conservation laws for fluid systems involving vorticity and vorticity related equations (potential type systems) including MHD and Maxwell's equations, which may be used to derive the potential vorticity equation (the actual potential vorticity equation we obtain is not explicitly given in Cheviakov 2014).

In ideal fluid mechanics, Ertel's theorem is usually written in the form:

$$\frac{d}{dt}\left(\frac{\boldsymbol{\omega}\cdot\nabla\psi}{\rho}\right)=0 \quad \text{where} \quad \boldsymbol{\omega}=\nabla\times\mathbf{u}, \tag{1.1}$$

is the fluid vorticity and $d/dt=\partial/\partial t+\mathbf{u}\cdot\nabla$ is the Lagrangian time derivative moving with the flow, $\mathbf{u}$ is the fluid velocity and $\psi$ is a scalar advected with the flow, i.e.

† Email address for correspondence: gmw0002@uah.edu

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$d\psi/dt = 0$. In addition it is required that $\nabla\psi \times \nabla S = 0$ for (1.1) to apply. The quantity $I_e = \boldsymbol{\omega}\cdot\nabla\psi/\rho$ is known as the potential vorticity or Ertel's invariant (e.g. Salmon 1982; Pedlosky 1987; Salmon 1988; Padhye and Morrison 1996a,b; Padhye 1998). In particular, if $\psi = S$ then $dI_e/dt = 0$ and $I_e = \boldsymbol{\omega}\cdot\nabla S/\rho$ is conserved following the flow. Ertel's theorem is important in geophysical fluid dynamics, in describing Rossby waves (i.e. planetary scale waves in the atmosphere for which the Coriolis force on the fluid element plays an important role). In the quasi-geostrophic approximation, Rossby waves are governed by the conservation of potential vorticity, in which the fluid vorticity is split up into the planetary vorticity $2\boldsymbol{\Omega}$, where $\boldsymbol{\Omega}$ is the angular velocity of the planet and the local fluid vorticity $\boldsymbol{\omega}$ (e.g. Pedlosky 1987). The Eulerian version of (1.1) is:

$$\frac{\partial}{\partial t}(\boldsymbol{\omega}\cdot\nabla\psi) + \nabla\cdot[(\boldsymbol{\omega}\cdot\nabla\psi)\mathbf{u}] = 0. \tag{1.2}$$

The generalized potential vorticity equation for MHD obtained in the present paper has the form:

$$\frac{\partial}{\partial t}(\boldsymbol{\omega}\cdot\nabla\psi) + \nabla\cdot\left[(\boldsymbol{\omega}\cdot\nabla\psi)\mathbf{u} - \left(T\nabla S + \frac{\mathbf{J}\times\mathbf{B}}{\rho}\right)\times\nabla\psi\right] = 0, \tag{1.3}$$

where $T$ is the temperature of the gas, $S$ is the entropy and $\mathbf{J} = \nabla\times\mathbf{B}/\mu_0$ is the MHD current density. There are an infinite number of conserved quantities $\psi$ advected with the flow in ideal MHD and fluid dynamics, which are Casimirs, which have zero non-canonical Poisson bracket with other functionals of the physical variables (e.g. Padhye and Morrison 1996a,b; Webb et al. 2014b, see also Sec. 5). Thus, each independent $\psi$ gives rise to an independent conservation law, i.e. there are an infinite number of conservation laws of the form (1.3).

Cheviakov's generalized vorticity conservation law applies to systems of partial differential equations of the form:

$$\nabla\cdot\mathbf{N} = 0, \quad \frac{\partial\mathbf{N}}{\partial t} + \nabla\times\mathbf{M} = 0. \tag{1.4}$$

The system (1.4) has potential vorticity like conservation laws of the form:

$$\frac{\partial}{\partial t}(\mathbf{N}\cdot\nabla F) + \nabla\cdot(\mathbf{M}\times\nabla F - F_t\mathbf{N}) = 0, \tag{1.5}$$

where $F(t, x, y, z)$ is an arbitrary function of $(t, x, y, z)$. The choices:

$$\mathbf{N} = \boldsymbol{\omega} = \nabla\times\mathbf{u}, \quad F = \psi(\mathbf{x}, t),$$
$$\mathbf{M} = -\mathbf{u}\times\boldsymbol{\omega} - \left(T\nabla S + \frac{\mathbf{J}\times\mathbf{B}}{\rho}\right), \tag{1.6}$$

in Cheviakov's conservation law (1.5) gives the potential vorticity type conservation law (1.3) obtained in our analysis, where $\psi$ is advected with the flow.

We derive the new conservation law (1.3) using the non-field aligned MHD fluid relabelling symmetries and Noether's second theorem. Cheviakov (2014) obtained conservation laws for potential and vorticity equations related to non-local symmetries of partial differential equation (pde) systems (e.g. Bluman et al. 2010). Kelbin et al. (2014) obtain new conservation laws for helically symmetric, plane, and rotationally symmetric flows. Webb et al. (2014a) derived MHD conservation laws using Lie dragging techniques (see also Tur and Yanovsky 1993). Webb et al. (2014b), obtained advected invariants (i.e. advected invariant geometrical objects) in MHD and gas dynamics by using Noether's first and second theorems and the Euler Poincaré

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approach to variational problems in fluids and MHD (e.g. Holm et al. 1998; Cotter et al. 2007; Cotter and Holm 2013).

The MHD equations admit the Galilei Lie group, including the space and time translation symmetries, the space rotations and the Galilean boosts (e.g. Fuchs 1991; Grundland and Lalague 1995; Webb and Zank 2007). For a polytropic gas with $p \propto \rho^{\gamma}$ there are three extra scaling symmetries of the MHD and gas dynamic equations (Fuchs 1991; Webb et al. 2005; Webb and Zank 2007). For $\gamma = (n+2)/n$ (n a positive integer) the equations also admit the projective symmetry (e.g. Ibragimov 1985).

The infinite dimensional fluid relabelling symmetries conservation laws are associated with Noether's second theorem (e.g. Salmon 1982, 1988; Padhye and Morrison 1996a,b; Padhye 1998; Ilgisnos and Pastukhov 2000; Kats 2003; Webb et al. 2005; Webb and Zank 2007; Cotter and Holm 2013).

In Sec. 2, the MHD equations and the Lagrangian map are introduced. In Sec. 3 we describe Noether's first and second theorems. We outline the formulation of Noether's second theorem using Lagrangian constraints, developed by Hydon and Mansfield (2011). In Sec. 4 we formulate Noether's theorems in MHD using the Lagrangian MHD approach of Newcomb (1962). We apply the Lagrange multiplier analysis of Hydon and Mansfield (2011) to obtain Noether's second theorem in MHD, which is subsequently used to derive the new potential vorticity type conservation law for MHD. Section 5 concludes with a summary and discussion.

## 2. The MHD equations and the Lagrangian map

The time dependent MHD equations consist of the mass, momentum and entropy advection equations:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0, \tag{2.1}$$

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \mathbf{J}\times\mathbf{B} - \rho\nabla\Phi, \tag{2.2}$$

$$\frac{\partial S}{\partial t} + \mathbf{u}\cdot\nabla S = 0, \tag{2.3}$$

coupled with Maxwell's equations in the MHD limit:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{u}\times\mathbf{B}), \tag{2.4}$$

$$\mathbf{J} = \frac{\nabla\times\mathbf{B}}{\mu_0}, \quad \nabla\cdot\mathbf{B} = 0. \tag{2.5}$$

Equations (2.4)–(2.5) are Faraday's induction equation, Ampere's law for the current $\mathbf{J}$ and Gauss's equation $\nabla\cdot\mathbf{B} = 0$. $\mathbf{F}_g = -\rho\nabla\Phi$ is the force due to an external gravitational potential $\Phi$ . The gas internal energy density $\varepsilon = \varepsilon(\rho, S)$. For an ideal gas, the first law of thermodynamics gives:

$$p = \rho\frac{\partial \varepsilon}{\partial \rho} - \varepsilon, \quad \rho T = \frac{\partial \varepsilon}{\partial S}, \tag{2.6}$$

for the pressure $p$ and the temperature $T$ of the gas.

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### 2.1. *The Lagrangian map and variational formulation*

The Lagrangian map: $\mathbf{x} = \mathbf{X}(\mathbf{x}_0, t)$ is obtained by integrating the fluid velocity equation $d\mathbf{x}/dt = \mathbf{u}(\mathbf{x}, t)$, subject to the initial condition $\mathbf{x} = \mathbf{x}_0$ at time $t = 0$. The mass continuity equation and entropy advection equation are replaced by the equivalent algebraic equations:

$$\rho = \frac{\rho_0(\mathbf{x}_0)}{J}, \quad S = S(\mathbf{x}_0), \tag{2.7}$$

where

$$J = \det(x_{ij}) \quad \text{and} \quad x_{ij} = \frac{\partial x^i(\mathbf{x}_0, t)}{\partial x_0^j}. \tag{2.8}$$

Similarly, Faraday's equation (2.4) has the formal solution for the magnetic field induction $\mathbf{B}$ of the form:

$$B^i = \frac{x_{ij} B_0^j}{J}, \quad \nabla_0 \cdot \mathbf{B}_0 = 0. \tag{2.9}$$

The solution (2.9) for $B^i$ is equivalent to the frozen in field theorem in MHD (e.g. Stern 1966; Parker 1979), and the initial condition $\nabla_0 \cdot \mathbf{B}_0 = 0$ ensures that Gauss's law $\nabla \cdot \mathbf{B} = 0$ is satisfied.

The action for the MHD system is:

$$A = \iint \ell \; d^3x \, dt \equiv \iint \ell^0 \; d^3x_0 dt, \tag{2.10}$$

where

$$\ell = \frac{1}{2}\rho |\mathbf{u}|^2 - \varepsilon(\rho, S) - \frac{B^2}{2\mu} - \rho\Phi, \quad \ell^0 = \ell J, \tag{2.11}$$

are the Eulerian ($\ell$) and Lagrangian ($\ell^0$) Lagrange densities respectively. Using (2.7)–(2.9), and (2.11) we obtain:

$$\ell^0 = \frac{1}{2}\rho_0 |\mathbf{x}_t|^2 - J\varepsilon\left(\frac{\rho_0}{J}, S\right) - \frac{x_{ij} x_{is} B_0^j B_0^s}{2\mu J} - \rho_0 \Phi, \tag{2.12}$$

for $\ell^0$. Note that in the Lagrange density $\ell^0 = \ell^0(\mathbf{x}_0, t; \mathbf{x}, \mathbf{x}_t, x_{ij})$, $\mathbf{x}_0$ and $t$ are the independent variables, and $\mathbf{x}$ and its derivatives with respect to $\mathbf{x}_0$ and $t$ are dependent variables.

Setting $\delta A/\delta x^i = 0$ gives the Lagrangian momentum equation for the system in the form (Newcomb 1962):

$$\rho_0 \left(\frac{\partial^2 x^i}{\partial t^2} + \frac{\partial \Phi}{\partial x^i}\right) + \frac{\partial}{\partial x_0^j}\left\{A_{kj}\left[\left(p + \frac{B^2}{2\mu}\right)\delta^{ik} - \frac{B^i B^k}{\mu_0}\right]\right\} = 0, \tag{2.13}$$

where $A_{kj} = \mathrm{cofac}(x_{kj})$. Dividing (2.13) by $J$, and using the fact that $\partial A_{kj}/\partial x_0^j = 0$, gives the Eulerian form of the momentum equation (2.2).

## 3. Noether's theorems

Noether's theorems describes the relationship between conservation laws, and Lie symmetries of differential equation systems represented by a variational principle. In Sec. 3.1 we outline Noether's first theorem, which applies to conservation laws associated with a variational principle where the action is invariant under a finite Lie group. Section 3.2 describes Noether's second theorem. Noether's second theorem

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applies when the action is invariant under an infinite dimensional pseudo-Lie group, in which case there are in general dependencies between the Euler Lagrange equations for the system (Noether 1918; Padhye and Morrison 1996a,b; Hydon and Mansfield 2011; Cotter and Holm 2013). In this case, the generators of the Lie pseudo algebra depend on arbitrary functions of the independent variables.

### 3.1. *Noether's first theorem*

Noether's first theorem gives a conservation law for a system of partial differential equations described by an action principle when there are a finite number of symmetries, or for a finite Lie group of symmetries that leave the action invariant under variational or divergence symmetries.

Consider the variational and divergence symmetries of the action

$$J = \int L(\mathbf{x}, [\mathbf{u}])\, d\mathbf{x}, \tag{3.1}$$

where $[\mathbf{u}]$ denotes the dependent variables and their derivatives with respect to the independent variables $\mathbf{x}$. In Sec. 4, we use $\mathbf{u}$ to denote the fluid velocity. The Euler Lagrange equations describing the system are obtained by varying the action $J$:

$$\delta_u J([\mathbf{u}]) = \int \frac{\partial L}{\partial u^\alpha_{,I}} \delta u^\alpha_{,I}\, d\mathbf{x} = \int \frac{\partial L}{\partial u^\alpha_{,I}} \mathbf{D}_I(\delta u^\alpha)\, d\mathbf{x} = \int \mathbf{E}_\alpha(L)\delta u^\alpha\, d\mathbf{x} = 0, \tag{3.2}$$

where we have dropped divergence surface terms, that are assumed to vanish on the boundary $\partial R$ of the integration region $R$ of interest. The operator $E_\alpha(L)$ in (3.2) is known as the Euler operator (e.g. Bluman and Kumei 1989). Here we use the usual multi-index notation where $u^\alpha_{,I} = \partial u^\alpha / \partial x^{i_1} \partial x^{i_2} \dots \partial x^{i_n}$, i.e. $I = i_1 i_2 i_3 \dots i_n$ for some arbitrary given $n$. From (3.2) we obtain the Euler-Lagrange equations for the system:

$$E_\alpha(L) \equiv (-D)_I \left( \frac{\partial L}{\partial u^\alpha_{,I}} \right) = 0, \quad \alpha = 1, 2, \dots, q, \tag{3.3}$$

which are equivalent to the differential equation system of interest. In (3.3) for $I = i_1 i_2 i_3 \dots j_k$, $(-D)_I$ is defined as:

$$(-D)_I = (-1)^k D_I = (-D_{i_1})(-D_{i_2}) \dots (-D_{i_k}), \tag{3.4}$$

(see Olver 1993, p. 245). Here $D_i$ is the total partial derivative with respect to $x^i$.

In the general case, we are interested in variations that correspond to infinitesimal Lie transformations of the form:

$$x'^i = x^i + \epsilon V^{x^i}, \quad u'^\alpha = u^\alpha + \epsilon V^{u^\alpha}, \tag{3.5}$$

and under the divergence transformation:

$$L' = L + \epsilon D_i \Lambda^i + O(\epsilon^2). \tag{3.6}$$

The $\Lambda^i$ are gauge potentials. In (3.5) $V^{x^i}$ and the $V^{u^\alpha}$ are the infinitesimal generators of the Lie transformation group in the analysis. The Euler-Lagrange equations $E_\alpha(L) = 0$ obtained from the stationary point solutions of $\delta_{\mathbf{u}} J = 0$ are invariant under a divergence transformation of the form (3.6) (e.g. Bluman and Kumei 1989). The canonical Lie transformations equivalent to (3.5) are:

$$\mathbf{x}' = \mathbf{x}, \quad u'^\alpha = u^\alpha + \epsilon \hat{V}^{u^\alpha}, \quad \hat{V}^{u^\alpha} = V^{u^\alpha} - V^{x^i} u^\alpha_{,i}, \tag{3.7}$$

in which the independent variables $\mathbf{x}$ are fixed (e.g. Ibragimov 1985).

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The condition for the action (3.1) to be stationary ($\delta J = 0$) under the transformations (3.5)-(3.7) reduces to (e.g Webb et al. 2005) :

$$\hat{X}L + D_i(LV^{x^i} + \Lambda^i) = 0. \tag{3.8}$$

where

$$\hat{X} = \hat{V}^{u^\alpha}\frac{\partial}{\partial u^\alpha} + D_i(\hat{V}^{u^\alpha})\frac{\partial}{\partial u_i^\alpha} + \ldots \equiv D_I(\hat{V}^{u^\alpha})\frac{\partial}{\partial u_I^\alpha} \tag{3.9}$$

is the extended operator corresponding to the characteristic Lie transformations (3.7). Using integration by parts to calculate $\hat{X}L$ gives:

$$\hat{X}(L) = \hat{V}^{u^\alpha}E_\alpha(L) + D_i(W^i[\mathbf{u}, \hat{V}^{\mathbf{u}}]), \tag{3.10}$$

where the surface integral term $W^i$ obtained by integration by parts is listed in Webb et al. (2005). Using (3.8)–(3.9) the condition for the action to be invariant under the Lie transformation (3.6)–(3.7) may be written in the form (e.g. Webb et al. 2005; Webb and Zank 2007)

$$\hat{V}^{u^\alpha}E_\alpha(L) + D_i(W^i + LV^{x^i} + \Lambda^i) = 0. \tag{3.11}$$

For a finite number of Lie symmetries, the $\hat{V}^{u^\alpha}$ and Euler Lagrange equations $E_\alpha(L) = 0$ ($\alpha = 1, 2 \ldots q$), are independent. (3.11) then reduces to the conservation law:

$$D_i(W^i + LV^{x^i} + \Lambda^i) = 0. \tag{3.12}$$

Equation (3.12) is the conservation law for Noether's first theorem.

### 3.2. *Noether's second theorem*

Noether's second theorem (Olver 1993; Hydon and Mansfield 2011) expresses the idea that there must exist a relation between the Euler-Lagrange equations if the symmetry operator $\hat{V}^{\mathbf{u}}(\mathbf{x}, [\mathbf{u}, \mathbf{g}])$ depends on an arbitrary function $\mathbf{g}(\mathbf{x})$.

*Theorem: Olver (1993)*

The variational problem (3.2) admits an infinite dimensional group of variational symmetries whose characteristics $\hat{V}^{\mathbf{u}}(\mathbf{x}, [\mathbf{u}, \mathbf{g}])$ depend on an arbitrary function $\mathbf{g}(\mathbf{x})$ (and its derivatives) if and only if there exist differential operators $\mathscr{D}^1, \mathscr{D}^2, \ldots \mathscr{D}^q$, not all zero, such that:

$$\mathscr{D}^1E_1(L) + \mathscr{D}^2E_2(L) + \cdots + \mathscr{D}^qE_q(L) = 0, \tag{3.13}$$

for all $\mathbf{x}$ and $\mathbf{u}$.

A proof of the theorem is given by Olver (1993). Hydon and Mansfield (2011) give a simpler proof of the *only if* part of the proof, (based on Noether 1918) and identify the operators $\{\mathscr{D}^s : 1 \leqslant s \leqslant q\}$.

Hydon and Mansfield (2011), apply the operator $E_g$ to (3.11) to obtain:

$$E_g\{\hat{V}^{u^\alpha}(\mathbf{x}, [\mathbf{u}; g])E_\alpha(L)\} = 0, \tag{3.14}$$

as the required differential relation (3.13) between the Euler-Lagrange equations. The theorem extends immediately to variational symmetries whose characteristics $\hat{V}^{u^\alpha}$ depend on $R$ independent arbitrary functions $\mathbf{g} = (g^1(\mathbf{x}), \ldots, g^R(\mathbf{x}))$ and their derivatives. This gives $R$ differential relations between the Euler Lagrange equations:

$$E_{g^r}\{\hat{V}^{u^\alpha}(\mathbf{x}, [\mathbf{u}; \mathbf{g}])E_\alpha(L)\} = (-D)_I\left(\frac{\partial \hat{V}^{u^\alpha}(\mathbf{x}, [\mathbf{u}; \mathbf{g}])}{\partial g^r_{,I}}E_\alpha(L)\right) = 0, \quad r = 1, \ldots, R. \tag{3.15}$$

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It is useful to consider (3.14) as Euler Lagrange equations for the action

$$\hat{J}[\mathbf{u};\mathbf{g}] = \int \hat{L}(\mathbf{x};[\mathbf{u};\mathbf{g}])\, d\mathbf{x}, \tag{3.16}$$

where

$$\hat{L}(\mathbf{x},[\mathbf{u};\mathbf{g}]) = \hat{V}^{u^\alpha}(\mathbf{x},[\mathbf{u};g])E_\alpha(L(\mathbf{x},[\mathbf{u}])). \tag{3.17}$$

If the functions $\mathbf{g} = (g^1, g^2, \ldots, g^R)$ are subject to $S$ constraints, of the form:

$$\mathscr{D}_{sr}(g^r) = 0, \quad s = 1, \ldots, S, \tag{3.18}$$

where the $\mathscr{D}_{sr}$ are differential operators, then the constraints can be incorporated in the Lagrangian $\hat{L}$, by using the modified Lagrangian:

$$\hat{L}(\mathbf{x},[\mathbf{u};\mathbf{g}]) = \hat{V}^{u^\alpha}(\mathbf{x},[\mathbf{u};\mathbf{g}])E_\alpha(L(\mathbf{x},[\mathbf{u}])) - \nu^s \mathscr{D}_{sr}(g^r), \tag{3.19}$$

in which the $\nu^s$ are Lagrange multipliers.

By varying the action (3.16) and (3.19) with respect to $g^r$, we obtain

$$\delta\left(\int \nu^s \mathscr{D}_{sr}(g^r)\, d\mathbf{x}\right) = \langle \nu^s, \mathscr{D}_{sr}(\delta g^r)\rangle = \langle \mathscr{D}^\dagger_{sr}(\nu^s), \delta g^r\rangle, \tag{3.20}$$

where the surface terms have been dropped. The angle brackets define the usual inner product for functions. Taking variations of (3.16) and (3.19) with respect to $g^r$ yields

$$\frac{\delta \hat{J}}{\delta g^r} = (-D)_I \left(\frac{\partial \hat{V}^{u^\alpha}(\mathbf{x},[\mathbf{u},\mathbf{g}])}{\partial g^r_{,I}} E_\alpha(L)\right) - \mathscr{D}^\dagger_{sr}(\nu^s) = 0, \quad r = 1, \ldots, R. \tag{3.21}$$

Thus, if $S < R$, it may be possible to eliminate the Lagrange multipliers in (3.21). In any event, (3.21) relates the Lagrange multipliers $\nu^s$ to the solutions of the original Euler Lagrange (3.3).

## 4. MHD conservation laws and symmetries

We consider Noether's theorems in Lagrangian MHD (e.g. Webb et al. 2005; Webb and Zank 2007), which may be used to derive MHD conservation laws.

PROPOSITION 4.1 (NOETHER'S THEOREM). *If the action (2.10) is invariant to $O(\epsilon)$ under the infinitesimal Lie transformations:*

$$x'^i = x^i + \epsilon V^{x^i}, \quad x'^j_0 = x^j_0 + \epsilon V^{x^j_0}, \quad t' = t + \epsilon V^t, \tag{4.1}$$

*and the divergence transformation:*

$$\ell^{0'} = \ell^0 + \epsilon D_\alpha \Lambda^\alpha_0 + O(\epsilon^2), \tag{4.2}$$

*(here $D_0 \equiv \partial/\partial t$ and $D_i \equiv \partial/\partial x^i_0$ are the total derivative operators in the jet-space consisting of the derivatives of $x^k(\mathbf{x}_0, t)$ and physical quantities that depend on $\mathbf{x}_0$ and $t$) then the MHD system admits the Lagrangian identity:*

$$\hat{V}^{x^k} E_{x^k}(\ell^0) + \frac{\partial I^0}{\partial t} + \frac{\partial I^j}{\partial x^j_0} = 0, \tag{4.3}$$

*where*

$$I^0 = \rho_0 u^k \hat{V}^{x^k} + V^t \ell^0 + \Lambda^0_0, \tag{4.4}$$

$$I^j = \hat{V}^{x^k}\left[\left(p + \frac{B^2}{2\mu}\right)\delta^{ks} - \frac{B^k B^s}{\mu}\right] A_{sj} + V^{x^j_0}\ell^0 + \Lambda^j_0, \tag{4.5}$$

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*In (4.4)-(4.5) we use the notation:* $x_{kj} = \partial x^k/\partial x_0^j$, $A_{sj} = cofac(x_{sj})$ *is the co-factor matrix corresponding to* $x_{sj}$ *and*

$$\hat{V}^{x^k(\mathbf{x}_0,t)} = V^{x^k(\mathbf{x}_0,t)} - \left(V^t \frac{\partial}{\partial t} + V^{x_0^s} \frac{\partial}{\partial x_0^s}\right) x^k(\mathbf{x}_0, t), \tag{4.6}$$

*is the canonical Lie symmetry transformation generator corresponding to the Lie transformation (4.1) (i.e.* $x'^k = x^k + \epsilon \hat{V}^{x^k}$, $t' = t$, $x_0'^j = x_0^j$*). If the Euler Lagrange equations* $E_{x^k}(\ell^0) = 0$ *are independent one obtains the Lagrangian conservation law:*

$$\frac{\partial I^0}{\partial t} + \frac{\partial I^j}{\partial x_0^j} = 0, \tag{4.7}$$

*The conservation law (4.7) is satisfied for the case of a finite group of transformations. For the case of an infinite Lie pseudo group, the Euler Lagrange equations are not all independent. This latter case corresponds to Noether's second theorem.*

Proofs of Noether's first theorem are given in Bluman and Kumei (1989) and Olver (1993) (see Webb et al. 2005 for the MHD system, including fully nonlinear waves).

Proposition 4.2. *The Lagrangian conservation law (4.7) can be written as an Eulerian conservation law of the form (Padhye 1998):*

$$\frac{\partial F^0}{\partial t} + \frac{\partial F^j}{\partial x^j} = 0, \tag{4.8}$$

*where*

$$F^0 = \frac{I^0}{J}, \quad F^j = \frac{u^j I^0 + x_{jk} I^k}{J}, \quad (j = 1, 2, 3), \tag{4.9}$$

*are the conserved density* $F^0$ *and flux components* $F^j$, $J = \det(x_{ij})$ *and* $x_{ij} = \partial x^i/\partial x_0^j$.

The Lagrangian conservation laws described in Proposition 4.1 can be converted to Eulerian conservation laws using Proposition 4.2. The Eulerian version of Noether's first theorem for MHD is given in Webb et al. (2005).

### 4.1. *Fluid relabelling symmetries*

Consider infinitesimal Lie transformations of the form (4.1)-(4.2), with

$$V^t = 0, \quad V^{\mathbf{x}} = 0, \quad V^{\mathbf{x}_0} \neq 0, \tag{4.10}$$

which leave the action (2.10) invariant. The extended Lie transformation operator $\tilde{X}$ for the case (4.10) has generators:

$$\begin{aligned} &\hat{V}^{\mathbf{x}} = -V^{\mathbf{x}_0} \cdot \nabla_0 \mathbf{x}, \quad V^{\mathbf{x}_t} = -D_t(V^{\mathbf{x}_0}) \cdot \nabla_0 \mathbf{x}, \\ &V^{\nabla_0 \mathbf{x}} = -\nabla_0(V^{\mathbf{x}_0}) \cdot \nabla_0 \mathbf{x}. \end{aligned} \tag{4.11}$$

The condition for a divergence symmetry of the action (e.g. Webb et al. 2005; Webb and Zank 2007) reduces to:

$$\begin{aligned} &\nabla_0 \cdot (\rho_0 V^{\mathbf{x}_0}) \left(\frac{1}{2}|\mathbf{u}|^2 - \Phi(\mathbf{x}) - \frac{\varepsilon + p}{\rho}\right) - J \frac{\partial \varepsilon(\rho, S)}{\partial S} V^{\mathbf{x}_0} \cdot \nabla_0 S \\ &\quad - D_t(\rho_0 V^{\mathbf{x}_0}) \mathbf{u} \cdot (\nabla_0 \mathbf{x})^T \\ &\quad + \frac{1}{\mu J} (\nabla_0 \mathbf{x}) \cdot (\nabla_0 \mathbf{x})^T : \mathbf{B}_0 \left[\nabla_0 \times (V^{\mathbf{x}_0} \times \mathbf{B}_0) - \nabla \cdot \mathbf{B}_0 V^{\mathbf{x}_0}\right] \\ &= -\partial \Lambda_0^\alpha / \partial x_0^\alpha. \end{aligned} \tag{4.12}$$

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A relatively simple class of solutions of the Lie determining equation (4.12) for the fluid relabelling symmetries are obtained by solving the equations:

$$\nabla_0\cdot(\rho_0 V^{\mathbf{x}_0})=0,\quad V^{\mathbf{x}_0}\cdot\nabla_0 S=0,\quad D_t(\rho_0 V^{\mathbf{x}_0})=0,$$
$$\nabla_0\times(V^{\mathbf{x}_0}\times\mathbf{B}_0)=0,\quad \nabla_0\cdot\mathbf{B}_0=0,\quad \Lambda_0^\alpha=0. \tag{4.13}$$

Equations (4.13) are the Lie determining equations for the fluid relabelling symmetries obtained by Padhye (1998) and Webb et al. (2005). The condition $\nabla_0\cdot\mathbf{B}_0=0$ is required in order that $\nabla\cdot\mathbf{B}=0$.

PROPOSITION 4.3. *The condition (4.12) for a divergence symmetry of the action converted to Eulerian form is:*

$$\nabla\cdot(\rho\hat{V}^{\mathbf{x}})\left(h+\Phi(\mathbf{x})-\frac{1}{2}|\mathbf{u}|^2\right)+\rho T\hat{V}^{\mathbf{x}}\cdot\nabla S+\rho\mathbf{u}\cdot\left(\frac{d\hat{V}^{\mathbf{x}}}{dt}-\hat{V}^{\mathbf{x}}\cdot\nabla\mathbf{u}\right)$$
$$+\frac{\mathbf{B}}{\mu_0}\cdot\left[-\nabla\times(\hat{V}^{\mathbf{x}}\times\mathbf{B})+\hat{V}^{\mathbf{x}}\nabla\cdot\mathbf{B}\right]=-\nabla_\alpha\Lambda^\alpha, \tag{4.14}$$

*where*

$$\nabla_\alpha\Lambda^\alpha=\frac{\partial\Lambda^0}{\partial t}+\frac{\partial\Lambda^i}{\partial x^i}, \tag{4.15}$$

*is the 4-divergence of* $\boldsymbol{\Lambda}=(\Lambda^0,\Lambda^1,\Lambda^2,\Lambda^3)$. $\boldsymbol{\Lambda}$ *is related the the Lagrange label space vector* $\Lambda_0^\alpha$ *by the transformations:*

$$\Lambda^\alpha=\Lambda_0^\beta B_{\beta\alpha}\equiv\Lambda_0^\beta\frac{x_{\alpha\beta}}{J}, \tag{4.16}$$

*where* $x_{\alpha\beta}=\partial x^\alpha/\partial x_0^\beta$, $J=\det(x_{ij})$ *and* $B_{\alpha\beta}=cofac(\partial x_0^\alpha/\partial x^\beta)$

In Appendix A, (4.14) is derived by using Lie dragging methods (e.g. Holm et al. 1998; Cotter and Holm 2013 and Flanders 1963; Harrison and Estabrook 1971; Ivey and Landsberg 2003; Webb et al. 2014a).

### 4.2. *The Lagrange multiplier approach*

Following Hydon and Mansfield (2011) we regard the fluid relabelling symmetry equations as auxiliary constraints. Thus, for Noether's second theorem, we consider the action (3.16) with Lagrange density:

$$\hat{J}[\mathbf{x};\mathbf{g}]=\int\hat{L}(\mathbf{y};[\mathbf{x};\mathbf{g}])\,d\mathbf{y}, \tag{4.17}$$

where $\mathbf{x}=(x,y,z)$ is the Eulerian position coordinate, $\mathbf{y}=(t,x_0,y_0.z_0)$ are the independent variables, and $\mathbf{g}$ corresponds to a relabelling symmetry generated by $V^{\mathbf{x}_0}$. The Lagrangian: $\hat{L}$ has the form:

$$\hat{L}(\mathbf{y},[\mathbf{x};\mathbf{g}])=-V^{x_0^j}x_{ij}E_{x^i}\left(\ell^0\right)-\nu^1\nabla_0\cdot(\rho_0V^{\mathbf{x}_0})-\nu^2V^{\mathbf{x}_0}\cdot\nabla_0S$$
$$-\nu_a^3\nabla_0\times(V^{\mathbf{x}_0}\times\mathbf{B}_0)^a-\nu_a^4D_t\left(\rho_0V^{x_0^a}\right), \tag{4.18}$$

where $\nu^1$, $\nu^2$, $\nu_a^3$ and $\nu_a^4$ are Lagrange multipliers.

Setting the variational derivative $\delta\hat{J}/\delta V^{x_0^a}=0$ we obtain the relations between the Euler-Lagrange equations of the form:

$$\frac{\delta\hat{J}}{\delta V^{x_0^a}}=-x_{ia}E_{x^i}(\ell^0)+\rho_0\frac{\partial\nu^1}{\partial x_0^a}-\nu^2\frac{\partial S}{\partial x_0^a}+B_0^j\left(\frac{\partial\nu_a^3}{\partial x_0^j}-\frac{\partial\nu_j^3}{\partial x_0^a}\right)+\rho_0\frac{\partial\nu_a^4}{\partial\tau}=0, \tag{4.19}$$

where $\partial/\partial\tau=\partial_t+\mathbf{u}\cdot\nabla$ is the Lagrangian time derivative moving with the flow.

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Next note from (2.13):

$$E_{x^i}\left(\ell^0\right) = -\left(\rho_0 \frac{\partial u^i}{\partial \tau} + \frac{\partial}{\partial x_0^j}\left\{A_{kj}\left[\left(p + \frac{B^2}{2\mu}\right)\delta^{ik} - \frac{B^i B^k}{\mu_0}\right]\right\}\right), \tag{4.20}$$

(we neglect gravity, i.e. we set $\Phi = 0$). Using the first law of thermodynamics, we obtain:

$$T\nabla_0 S = \nabla_0 h - \frac{1}{\rho}\nabla_0 p, \tag{4.21}$$

where $T$ is the temperature and $h = (\varepsilon + p)/\rho$ is the enthalpy of the gas. Using (4.20) and (4.21) we find solutions of (4.19) for $\nu^1$, $\nu^2$, $\boldsymbol{\nu}^3$ and $\boldsymbol{\nu}^4$ given by:

$$\begin{aligned} &\nu^1 = \frac{1}{2}u^2 - h, \quad h = \frac{\varepsilon + p}{\rho}, \quad \nu^2 = -\rho_0 T, \\ &\boldsymbol{\nu}^3 = \frac{\nabla_0 \mathbf{x}\cdot\mathbf{B}}{\mu}, \quad \boldsymbol{\nu}^4 = -(\nabla_0 \mathbf{x})\cdot\mathbf{u}. \end{aligned} \tag{4.22}$$

### 4.3. *Non-field aligned* $V^{\mathbf{x}_0}$

Webb et al. (2005) investigated solutions of the fluid relabelling symmetry determining equation (4.13) for the cases: (a) $V^{\mathbf{x}_0}$ parallel to $\mathbf{B}_0$ and (b) $V^{\mathbf{x}_0}$ not parallel to $\mathbf{B}_0$. In case (b) equation (4.13) implies the existence of a potential $\psi(\mathbf{x}_0)$ such that:

$$V^{\mathbf{x}_0} \times \mathbf{B}_0 = \nabla_0 \psi, \quad \nabla_0\cdot(\rho_0 V^{\mathbf{x}_0}) = 0, \quad V^{\mathbf{x}_0}\cdot\nabla_0 S = 0, \quad \nabla_0\cdot\mathbf{B}_0 = 0. \tag{4.23}$$

Note that (4.23) satisfies the Lie determining equation:

$$\nabla_0 \times (V^{\mathbf{x}_0} \times \mathbf{B}_0) = 0. \tag{4.24}$$

Using $\nabla_0\cdot(\rho_0 V^{\mathbf{x}_0}) = 0$, (4.24) can be written in the form:

$$\nabla_0 \times (V^{\mathbf{x}_0} \times \mathbf{B}_0) = \rho_0\,[\mathbf{b}_0, V^{\mathbf{x}_0}] = 0, \tag{4.25}$$

where $\mathbf{b}_0 = \mathbf{B}_0/\rho_0$ and $[\mathbf{b}_0, V^{\mathbf{x}_0}]^i \partial/\partial x_0^i$ is the Lie bracket of the tangent vector fields:

$$\mathbf{b}_0 = b_0^i \frac{\partial}{\partial x_0^i}, \quad V^{\mathbf{x}_0} = V^{x_0^i}\frac{\partial}{\partial x_0^i}. \tag{4.26}$$

From (4.25), the vector fields $\{\mathbf{b}_0, V^{\mathbf{x}_0}\}$ describes a two dimensional Abelian Lie algebra. From Frobenius theorem (e.g. Olver 1993, p. 39), the integral trajectories of $\mathbf{b}_0$ and $V^{\mathbf{x}_0}$ generate the the integral foliation $\psi(\mathbf{x}_0) = c$ ($c = const.$). A parametric representation of the surface $\psi = c$ is $\mathbf{x}_0 = \mathbf{x}_0(\phi, \chi, c)$ where

$$V^{\mathbf{x}_0}\cdot\nabla_0 = \frac{\partial}{\partial \phi} \quad \text{and} \quad \mathbf{b}_0\cdot\nabla_0 = \frac{\partial}{\partial \chi} \tag{4.27}$$

are directional derivatives (tangent vectors) in the surface $\psi = c$. We set:

$$\begin{aligned} &\omega^1 = \nabla_0\phi, \quad \omega^2 = \nabla_0\chi, \quad \omega^3 = \nabla_0\psi, \\ &\mathbf{e}_1 = \frac{\partial \mathbf{x}_0}{\partial \phi}, \quad \mathbf{e}_2 = \frac{\partial \mathbf{x}_0}{\partial \chi}, \quad \mathbf{e}_3 = \frac{\partial \mathbf{x}_0}{\partial \psi}. \end{aligned} \tag{4.28}$$

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The bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\omega^1, \omega^2, \omega^3\}$ are dual bases, i.e. $\langle \omega^i, \mathbf{e}_j \rangle = \delta^i_j$. Set $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$ and $g^{ij} = \omega^i \cdot \omega^j$ as the metric tensors. The bases are related by the equations:

$$\omega^i = g^{ij}\mathbf{e}_j, \quad \mathbf{e}_i = g_{ij}\omega^j,$$
$$\mathbf{e}_a \times \mathbf{e}_b = \sqrt{|g|}\varepsilon_{abc}\omega^c, \quad \omega^a \times \omega^b = \frac{\varepsilon_{abc}}{\sqrt{|g|}}\mathbf{e}_c, \tag{4.29}$$

where $\varepsilon_{abc}$ is the anti-symmetric Levi-Civita tensor density in three dimensions. The Lie determining equation (4.13) or equation (4.23) have solutions:

$$\rho_0 V^{\mathbf{x}_0} = \omega^2 \times \omega^3 = \nabla_0\chi \times \nabla_0\psi = \nabla_0 \times (\chi\nabla_0\psi),$$
$$\mathbf{B}_0 = \omega^3 \times \omega^1 = \nabla_0\psi \times \nabla_0\phi = \nabla_0 \times (\psi\nabla_0\phi),$$
$$\rho_0 = \frac{1}{\sqrt{|g|}} = \sqrt{|G|}, \quad S = S(\chi, \psi), \tag{4.30}$$

where $g = \det(g_{ij})$ and $G = \det(g^{ij})$ are the determinants of the metric tensors $g_{ij}$ and $g^{ij}$ respectively. The potentials $\phi$ $\psi$ and $\chi$ are functions only of $\mathbf{x}_0$, i.e. they are scalar invariants advected with the flow.

The Eulerian symmetry generator $\hat{V}^{\mathbf{x}}$ and $\mathbf{B}$ are given by:

$$\hat{V}^{\mathbf{x}} = \frac{\nabla\psi \times \nabla\chi}{\rho}, \quad \mathbf{B} = \nabla\psi \times \nabla\phi. \tag{4.31}$$

Because $\chi$, $\phi$ and $\psi$ are functions only of the Lagrange labels $\mathbf{x}_0$ it follows that:

$$\frac{d\phi}{dt} = \frac{d\chi}{dt} = \frac{d\psi}{dt} = 0. \tag{4.32}$$

Note that $\psi$ and $\phi$ are Euler potentials for the magnetic field $\mathbf{B}$.

The basic Noether identity (4.3) , namely:

$$\hat{V}^{x^k} E_{x^k}\left(\ell^0\right) + \frac{\partial I^0}{\partial t} + \nabla_0 \cdot \mathbf{I} = 0, \tag{4.33}$$

describes the conservation laws for the symmetries (4.30)–(4.31), for which

$$I^0 = (\nabla_0\psi \times \nabla_0\chi)\cdot\nabla_0\mathbf{x}\cdot\mathbf{u}, \tag{4.34}$$
$$\mathbf{I} = (\nabla_0\psi \times \nabla_0\chi)\cdot\left\{\left[\frac{p + B^2/(2\mu_0) - \ell}{\rho}\right]\mathsf{I} - \frac{(\nabla_0\mathbf{x}\cdot\mathbf{B})\mathbf{B}_0}{\mu_0\rho_0}\right\}, \tag{4.35}$$

where $\mathsf{I}$ is the unit $3 \times 3$ dyadic.

Webb et al. (2005) used (4.33) and Noether's second theorem to study the conservation law associated with the non-field aligned solutions (4.30)–(4.31) of the fluid relabelling Lie determining equations. Unfortunately, there is an error in their analysis. Below we obtain the correct conservation law associated with the symmetry (4.30)-(4.31) where $\chi(\mathbf{x}_0)$ is an arbitrary function defining a pseudo Lie algebra, by using the Hydon and Mansfield approach.

PROPOSITION 4.4. *The fluid relabelling symmetry (4.30) with generators:*

$$V^{\mathbf{x}_0} = \frac{\nabla_0\chi \times \nabla_0\psi}{\rho_0}, \quad \hat{V}^{\mathbf{x}} = \frac{\nabla\psi \times \nabla\chi}{\rho},$$
$$\mathbf{B}_0 = \nabla_0\psi \times \nabla_0\phi, \quad \mathbf{B} = \nabla\psi \times \nabla\phi,$$
$$\rho_0 = \frac{1}{\sqrt{|g|}}, \quad S = S(\chi, \psi), \tag{4.36}$$

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*used with Noether's second theorem gives rise to the generalized Bianchi identity:*

$$\frac{d}{dt}\left(\rho_0\frac{\boldsymbol{\omega}\cdot\nabla\psi}{\rho}\right)-\nabla_0\cdot(\mathbf{G}\times\nabla_0\psi)+\nabla_0\psi\cdot\nabla_0\times\left(\frac{E_{\mathbf{x}}(\ell^0)\cdot(\nabla_0\mathbf{x})^T}{\rho_0}\right)=0, \tag{4.37}$$

*where* $\boldsymbol{\omega}=\nabla\times\mathbf{u}$ *is the fluid vorticity and*

$$\mathbf{G}=\mathbf{F}\cdot(\nabla_0\mathbf{x})^T, \quad \mathbf{F}=T\nabla S+\nabla\left(\frac{1}{2}u^2-h\right)+\frac{\mathbf{J}\times\mathbf{B}}{\rho}, \tag{4.38}$$

*and* $\mathbf{J}=\nabla\times\mathbf{B}/\mu_0$ *is the MHD current density. The Eulerian form of (4.37) is:*

$$\frac{\partial}{\partial t}(\boldsymbol{\omega}\cdot\nabla\psi)+\nabla\cdot[(\boldsymbol{\omega}\cdot\nabla\psi)\mathbf{u}-\mathbf{F}\times\nabla\psi]+\nabla\psi\cdot\nabla\times\left(\frac{E_{\mathbf{x}}(\ell^0)}{\rho_0}\right)=0. \tag{4.39}$$

*For the cases where the Euler Lagrange equations are satisfied, i.e.* $E_{\mathbf{x}}(\ell^0)=0$*, (4.37) and (4.39) reduce to the conservation laws:*

$$\frac{d}{dt}\left(\rho_0\frac{\boldsymbol{\omega}\cdot\nabla\psi}{\rho}\right)-\nabla_0\cdot(\mathbf{G}\times\nabla_0\psi)=0, \tag{4.40}$$

$$\frac{\partial}{\partial t}(\boldsymbol{\omega}\cdot\nabla\psi)+\nabla\cdot[(\boldsymbol{\omega}\cdot\nabla\psi)\mathbf{u}-\mathbf{F}\times\nabla\psi]=0. \tag{4.41}$$

*Equation (4.41) can be written in the form:*

$$\frac{\partial}{\partial t}(\boldsymbol{\omega}\cdot\nabla\psi)+\nabla\cdot\left[(\boldsymbol{\omega}\cdot\nabla\psi)\mathbf{u}-\left(T\nabla S+\frac{\mathbf{J}\times\mathbf{B}}{\rho}\right)\times\nabla\psi\right]=0. \tag{4.42}$$

Remark. More generally, conservation laws (4.40) and (4.41) hold for the MHD equations, provided $d\psi/dt=(\partial/\partial t+\mathbf{u}\cdot\nabla)\psi=0$.

*Proof.* First note that:

$$\delta\hat{J}=\int\frac{\delta\hat{J}}{\delta V^{\mathbf{x}_0}}\cdot\delta V^{\mathbf{x}_0}\,d^3\mathbf{x}_0dt=\int\frac{\delta\hat{J}}{\delta\chi}\delta\chi\,d^3\mathbf{x}_0dt. \tag{4.43}$$

Using $V^{\mathbf{x}_0}$ from (4.30), i.e. $V^{\mathbf{x}_0}=\nabla_0\chi\times\nabla_0\psi/\rho_0$, integrating by parts, and dropping the surface term , we obtain the generalized Bianchi identity:

$$\frac{\delta\hat{J}}{\delta\chi}=\nabla_0\cdot\left[\left(\frac{1}{\rho_0}\frac{\delta\hat{J}}{\delta V^{\mathbf{x}_0}}\right)\times\nabla_0\psi\right]=0. \tag{4.44}$$

Using the expression (4.19) for $\delta\hat{J}/\delta V^{\mathbf{x}_0}$ in (4.44) we obtain:

$$\frac{\delta\hat{J}}{\delta\chi}=\nabla_0\psi\cdot\nabla_0\times\left(-\frac{E_{\mathbf{x}}(\ell^0)\cdot(\nabla_0\mathbf{x})^T}{\rho_0}+\frac{\partial\boldsymbol{\nu}^4}{\partial\tau}+\mathbf{G}\right), \tag{4.45}$$

where $\partial/\partial\tau\equiv d/dt$ is the Lagrangian time derivative following the flow and

$$\mathbf{G}=\nabla_0\nu^1-\frac{\nu^2}{\rho_0}\nabla_0S+\mathbf{b}_0\cdot\left[\nabla_0\boldsymbol{\nu}^3-(\nabla_0\boldsymbol{\nu}^3)^T\right], \tag{4.46}$$

and $\mathbf{b}_0=\mathbf{B}_0/\rho_0$.

Using the vector identity

$$\nabla_0\cdot(\mathbf{A}\times\mathbf{B})=(\nabla_0\times\mathbf{A})\cdot\mathbf{B}-(\nabla_0\times\mathbf{B})\cdot\mathbf{A}, \tag{4.47}$$

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and (4.22) for $\nu^1$, $\nu^2$, $\boldsymbol{\nu}^3$ and $\boldsymbol{\nu}^4$, (4.45) can be written in the form:

$$\frac{\delta \hat{J}}{\delta \chi} = \nabla_0\psi\cdot\nabla_0 \times \left(-\frac{E_{\mathbf{x}}\left(\ell^0\right)\cdot(\nabla_0\mathbf{x})^T}{\rho_0}\right) + \frac{\partial Q}{\partial \tau} + \nabla_0\cdot(\mathbf{G}\times\nabla_0\psi), \tag{4.48}$$

where

$$Q = \nabla_0\psi\cdot\nabla_0\times\boldsymbol{\nu}^4 = -\nabla_0\psi\cdot\nabla_0\times\left[\mathbf{u}\cdot(\nabla_0\mathbf{x})^T\right]. \tag{4.49}$$

By converting from $\mathbf{x}_0$ to $\mathbf{x}$ as independent variables in (4.49) gives:

$$Q = -\varepsilon_{ijk}\frac{\partial\psi}{\partial x_0^i}\frac{\partial u^s}{\partial x_0^j}x_{sk} \equiv -\frac{(\boldsymbol{\omega}\cdot\nabla\psi)\rho_0}{\rho}, \tag{4.50}$$

where $\boldsymbol{\omega} = \nabla\times\mathbf{u}$ is the fluid vorticity and $x_{sk} = \partial x^s/\partial x_0^k$.

Similarly, using (4.22) for $\nu^1$, $\nu^2$, $\boldsymbol{\nu}^3$, the expression for $\mathbf{G}$ in (4.46) reduces to (4.38). Using (4.38) for $\mathbf{G}$ and (4.50) for $Q$ gives:

$$\frac{\delta \hat{J}}{\delta \chi} = -\left\{\frac{d}{dt}\left(\rho_0\frac{\boldsymbol{\omega}\cdot\nabla\psi}{\rho}\right) - \nabla_0\cdot(\mathbf{G}\times\nabla_0\psi) + \nabla_0\psi\cdot\nabla_0\times\left(\frac{E_{\mathbf{x}}\left(\ell^0\right)\cdot(\nabla_0\mathbf{x})^T}{\rho_0}\right)\right\} = 0. \tag{4.51}$$

Equation (4.51) is equivalent to (4.37). Converting (4.51) to its Eulerian form gives (4.39). The conservation laws (4.40) and (4.41) follow from (4.37) and (4.39) and by setting $E_{\mathbf{x}}(\ell^0) = 0$. This completes the proof. □

Below we provide an independent verification of the conservation law (4.41) or (4.42). This derivation shows that the potential $\psi$ only needs to be an advected invariant. Thus, for example, $\psi$ could be any of the advected scalar Lie dragged invariants, e.g.:

$$\psi_1 = \frac{\mathbf{B}\cdot\nabla S}{\rho}, \quad \psi_2 = \frac{\mathbf{A}\cdot\mathbf{B}}{\rho}, \quad \psi_3 = \frac{\mathbf{B}\cdot\nabla}{\rho}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{\rho}\right), \tag{4.52}$$

could be used for $\psi$. In the above examples, we choose the gauge for $\mathbf{A}$ so that $\mathbf{A}\cdot d\mathbf{x}$ is a Lie dragged 1-form (see e.g. Webb et al. 2014a for more detail).

To derive (4.41) or (4.42) write the MHD momentum equation in the form:

$$\frac{d}{dt}(\mathbf{u}\cdot d\mathbf{x}) = \left[\mathbf{u}_t - \mathbf{u}\times\boldsymbol{\omega} + \nabla|\mathbf{u}|^2\right]\cdot d\mathbf{x} = \mathbf{F}\cdot d\mathbf{x}, \tag{4.53}$$

where $\mathbf{F}$ is given by (4.38).

A more recognizable form of (4.53) is:

$$\mathbf{u}_t - \mathbf{u}\times\boldsymbol{\omega} + \nabla|\mathbf{u}|^2 = T\nabla S + \nabla\left(\frac{1}{2}u^2 - h\right) + \frac{\mathbf{J}\times\mathbf{B}}{\rho}. \tag{4.54}$$

Taking the curl of (4.54) gives the fluid vorticity equation for MHD in the form:

$$\boldsymbol{\omega}_t - \nabla\times(\mathbf{u}\times\boldsymbol{\omega}) = \nabla\times\mathbf{F}. \tag{4.55}$$

For $\psi$ a scalar advected invariant $\nabla\psi$ satisfies the equation

$$\frac{\partial}{\partial t}\nabla\psi + \nabla(\mathbf{u}\cdot\nabla\psi) \equiv \nabla\left(\frac{\partial\psi}{\partial t} + \mathbf{u}\cdot\nabla\psi\right) = 0. \tag{4.56}$$

In the Lie dragging formalism, $\alpha = d\psi = \nabla\psi\cdot d\mathbf{x}$ is a Lie dragged 1-form that is invariant moving with the flow (e.g. Tur and Yanovsky 1993; Webb et al. 2014a).

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Taking the scalar product of (4.55) with $\nabla\psi$ and the scalar product of (4.56) with $\boldsymbol{\omega}$ and adding the resultant equations gives:

$$\frac{\partial}{\partial t}(\boldsymbol{\omega}\cdot\nabla\psi) - \nabla\psi\cdot\nabla\times(\mathbf{u}\times\boldsymbol{\omega}) + (\nabla\times\mathbf{u})\cdot\nabla(\mathbf{u}\cdot\nabla\psi) - \nabla\psi\cdot\nabla\times\mathbf{F} = 0. \tag{4.57}$$

Equation (4.57) can be re-written in the conservation form:

$$\frac{\partial}{\partial t}(\boldsymbol{\omega}\cdot\nabla\psi) + \nabla\cdot[\mathbf{u}\times\nabla(\mathbf{u}\cdot\nabla\psi) - (\mathbf{u}\times\boldsymbol{\omega})\times\nabla\psi - \mathbf{F}\times\nabla\psi] = 0. \tag{4.58}$$

Using the identity $\nabla\psi\times(\mathbf{u}\times\boldsymbol{\omega}) = (\nabla\psi\cdot\boldsymbol{\omega})\mathbf{u} - (\nabla\psi\cdot\mathbf{u})\boldsymbol{\omega}$, (4.58) reduces to:

$$\frac{\partial}{\partial t}(\boldsymbol{\omega}\cdot\nabla\psi) + \nabla\cdot[(\boldsymbol{\omega}\cdot\nabla\psi)\mathbf{u} - \nabla\times[(\mathbf{u}\cdot\nabla\psi)\mathbf{u}] - \mathbf{F}\times\nabla\psi] = 0. \tag{4.59}$$

Because the divergence of a curl is zero, (4.59) is equivalent to the conservation law (4.41) or (4.42). This completes our independent derivation of (4.41).

Proposition 4.4 used the Lagrangian version of Noether's second theorem to derive the conservation laws (4.37) and (4.39). An alternative approach is to use an Eulerian version of the variational principle (4.17)–(4.18), which we omit in the present paper.

## 5. Summary and concluding remarks

There are several related methods used to derive conservation laws in MHD and fluid dynamics. Moiseev et al. (1982), Tur and Yanovsky (1993) and Webb et al. (2014a) used the technique of Lie dragging of 0-forms, $p$-forms ($p = 1, 2, 3$) (see also Kuvshinov and Schep 1997). Another method is to use Noether's theorems (e..g. Noether 1918; Salmon 1982, 1988; Olver 1993; Padhye and Morrison 1996a,b; Holm et al. 1998; Padhye 1998; Webb et al. 2005; Hydon and Mansfield 2011; Cotter and Holm 2013; Webb et al. 2014b).

A new conservation law related to Ertel's theorem for MHD was derived in Sec. 4.3 which is associated with a non-field aligned fluid relabelling symmetry derived by Webb et al. (2005). The conservation law for this symmetry is given in (4.42), i.e.

$$\frac{\partial}{\partial t}(\boldsymbol{\omega}\cdot\nabla\psi) + \nabla\cdot\left[(\boldsymbol{\omega}\cdot\nabla\psi)\mathbf{u} - \left(T\nabla S + \frac{\mathbf{J}\times\mathbf{B}}{\rho}\right)\times\nabla\psi\right] = 0, \tag{5.1}$$

where $\boldsymbol{\omega} = \nabla\times\mathbf{u}$ is the fluid vorticity, $\mathbf{J} = \nabla\times\mathbf{B}/\mu_0$ is the MHD current density, and $\psi = \psi(\mathbf{x}_0)$ is a scalar invariant advected with the flow, i.e. $d\psi/dt = (\partial/\partial t + \mathbf{u}\cdot\nabla)\psi = 0$. The conservation law is more general than that expected from its derivation using Noether's second theorem. We expect that there are more general solutions of the fluid relabelling symmetry determining equation (4.12) than the solutions used in the Noether theorem derivation of (5.1), which could help to resolve this issue. Volkov et al. (1995) discuss the Lie dragged invariants of MHD and gas dynamics in terms of supersymmetry and the odd Buttin bracket. The relationship between this approach and Noether's theorems is not clear at present.

It is important to note that there are an infinite number of physical quantities $\psi$ in ideal fluid mechanics and MHD that are conserved following the flow. For example, if the gauge of the magnetic potential $\mathbf{A}$ in ideal MHD is chosen so that the magnetic vector potential one-form $\mathbf{A}\cdot d\mathbf{x}$ is advected with the flow, then $\psi_m = \mathbf{A}\cdot\mathbf{B}/\rho$ is a scalar invariant advected with flow (e.g. Padhye and Morrison 1996a,b; Webb et al. 2014b). The quantity $h_m = \mathbf{A}\cdot\mathbf{B}$ is the magnetic helicity density. The integral form

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of $H_m = \int h_m d^3x$ moving with the flow, is related to the topological structure of the magnetic field (e.g. Moffatt 1978; Berger and Field 1984; Moffatt and Ricca 1992). One can show that the functional

$$\psi = \psi\left(\frac{\mathbf{A}\cdot\mathbf{B}}{\rho}, S, \frac{\mathbf{B}}{\mathbf{A}\cdot\mathbf{B}}, \frac{\mathbf{B}}{\mathbf{A}\cdot\mathbf{B}}\cdot\nabla\left(\frac{\mathbf{A}\cdot\mathbf{B}}{\rho}\right), \frac{\mathbf{B}\cdot\nabla S}{\rho}, \frac{\mathbf{B}}{\rho}\cdot\nabla\left(\frac{\mathbf{B}\cdot\nabla S}{\rho}\right), \ldots\right), \tag{5.2}$$

for ideal MHD is a scalar invariant advected with the flow (e.g. Padhye and Morrison 1996a,b; Webb et al. 2014a,b). Thus, there are an infinite number of conservation laws of the form (5.1) corresponding to each independent scalar invariant $\psi$ that is advected with the flow (other examples of scalar invariants advected by the flow are given in Tur and Yanovsky 1993 and Webb et al. 2014a,b).

The conservation law (5.1) is associated with a non-field aligned fluid relabelling symmetry with generator $V^{\mathbf{x}_0} = \nabla_0\chi \times \nabla_0\psi/\rho_0$ and $\hat{V}^{\mathbf{x}} = \nabla\psi \times \nabla\chi/\rho$. This symmetry is much more complicated than previous fluid relabelling symmetries used in MHD and fluid dynamics, with a non-trivial Abelian, pseudo Lie algebra. For example, the field aligned fluid relabelling symmetry with $\hat{V}^{\mathbf{x}} = \zeta(\mathbf{x}_0)\mathbf{B}/\rho$ in barotropic MHD gives rise to the cross helicity conservation law with conserved density $h_c = \mathbf{u}\cdot\mathbf{B}$ (this conservation law can be extended to the case of a non-barotropic gas, by the use of non-zero gauge potentials $\Lambda^\alpha$ in Noether's theorem and by the use of Clebsch potentials to obtain nonlocal conservation laws (Webb et al. 2014a,b). Similarly, the choice $\hat{V}^{\mathbf{x}} = \zeta(\mathbf{x}_0)\boldsymbol{\omega}/\rho$ in ideal, barotropic fluid dynamics where $\zeta(\mathbf{x}_0)$ is an arbitrary function of $\mathbf{x}_0$, gives rise to the fluid helicity conservation law, with conserved density $h_f = \mathbf{u}\cdot\boldsymbol{\omega}$. A similar analysis, with $\hat{V}^{\mathbf{x}} = \nabla\times\mathbf{C}/\rho$ and $\mathbf{C} = \Phi\nabla S$ gives rise to Ertel's theorem for a barotropic gas, where $\Phi(\mathbf{x}_0)$ is a scalar invariant advected with the flow. These conservation laws were discussed by Padhye and Morrison (1996a,b), Cotter and Holm (2013) and Webb et al. (2005, 2014a,b). The conservation law (5.1) was derived with the use of the Lagrange multiplier approach to Noether's second theorem of Hydon and Mansfield (2011), which allows the incorporation of the fluid relabelling symmetries determining equations into Noether's second theorem by means of Lagrange multipliers. In principle, one should be able to solve the Lie determining equations and obtain the conservation law (5.1) without using Lagrange multipliers. However, as the example (5.1) shows, the Lagrange multiplier approach is very useful for cases with complicated fluid relabelling symmetry determining equations.

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## Appendix A

In this appendix we provide an independent derivation of the Lie invariance condition (4.14) for the MHD action:

$$\mathscr{A} = \int \ell \; d^3x \; dt, \tag{A 1}$$

to admit a divergence symmetry, where $\ell$ is the Eulerian MHD Lagrange density (2.11). We use Eulerian variations of the form:

$$\mathbf{x}' = \mathbf{x} + \epsilon \hat{V}^{\mathbf{x}}, \quad \ell' = \ell + \epsilon D_{x^\alpha} \Lambda^\alpha, \tag{A 2}$$

where

$$\hat{V}^{x^i} = V^{x^i} - V^{x^\delta_0} x_{is} - V^t x_{it}. \tag{A 3}$$

From Holm et al. (1998) and Webb et al. (2014b) $\delta\mathbf{u}$ is given by:

$$\delta\mathbf{u} = \left(\frac{\partial}{\partial t} + \mathscr{L}_{\mathbf{u}}\right)\hat{V}^{\mathbf{x}} = \frac{\partial \hat{V}^{\mathbf{x}}}{\partial t} + [\mathbf{u}, \hat{V}^{\mathbf{x}}]_L \equiv \frac{d\hat{V}^{\mathbf{x}}}{dt} - \hat{V}^{\mathbf{x}}\cdot\nabla\mathbf{u}, \tag{A 4}$$

where $\mathscr{L}_{\mathbf{u}}$ is the Lie derivative with respect $\mathbf{u}$, i.e. $\mathscr{L}_{\mathbf{u}}(\hat{V}^{\mathbf{x}}) = [\mathbf{u}, \hat{V}^{\mathbf{x}}] \equiv \mathrm{ad}_{\mathbf{u}}(\hat{V}^{\mathbf{x}})_L$ is the left Lie bracket of the vector fields $\mathbf{u}$ and $\hat{V}^{\mathbf{x}}$. The variations of the other physical variables (denoted by $\psi$) are given by:

$$\delta\psi = -\mathscr{L}_{\hat{V}^{\mathbf{x}}}(\psi). \tag{A 5}$$

Using the Lie derivative formulae:

$$\begin{aligned}
&\delta(\rho d^3x) = -\mathscr{L}_{\hat{V}^{\mathbf{x}}}(\rho d^3x) = -\nabla\cdot(\rho\hat{V}^{\mathbf{x}})d^3x, \quad \delta S = -\mathscr{L}_{\hat{V}^{\mathbf{x}}}(S) = -\hat{V}^{\mathbf{x}}\cdot\nabla S,\\
&\delta(\mathbf{B}\cdot d\mathbf{S}) = -\mathscr{L}_{\hat{V}^{\mathbf{x}}}(\mathbf{B}\cdot d\mathbf{S}) = [\nabla\times(\hat{V}^{\mathbf{x}}\times\mathbf{B}) - \hat{V}^{\mathbf{x}}\nabla\cdot\mathbf{B}]\cdot d\mathbf{S},
\end{aligned} \tag{A 6}$$

we obtain:

$$\begin{aligned}
\delta\rho &= -\nabla\cdot(\rho\hat{V}^{\mathbf{x}}), \quad \delta S = -\hat{V}^{\mathbf{x}}\cdot\nabla S,\\
\delta\mathbf{B} &= \nabla\times(\hat{V}^{\mathbf{x}}\times\mathbf{B}) - \hat{V}^{\mathbf{x}}\nabla\cdot\mathbf{B}.
\end{aligned} \tag{A 7}$$

(cf. Newcomb 1962; Webb et al. 2005).

The variation of the Lagrangian $\ell$ in (2.11) is given by:

$$\delta\ell = \frac{\delta\ell}{\delta\mathbf{u}}\cdot\delta\mathbf{u} + \frac{\delta\ell}{\delta\rho}\delta\rho + \frac{\delta\ell}{\delta\mathbf{B}}\cdot\delta\mathbf{B} + \frac{\delta\ell}{\delta S}\delta S + D_{x^\alpha}\Lambda^\alpha. \tag{A 8}$$

Using the Lagrangian density (2.11) we obtain:

$$\begin{aligned}
&\frac{\delta\ell}{\delta\rho} = \frac{1}{2}u^2 - \varepsilon_\rho - \Phi \equiv \frac{1}{2}u^2 - h - \Phi, \quad \frac{\delta\ell}{\delta\mathbf{u}} = \rho\mathbf{u},\\
&\frac{\delta\ell}{\delta S} = -\varepsilon_S = -\rho T, \quad \frac{\delta\ell}{\delta\mathbf{B}} = \frac{\mathbf{B}}{\mu_0}.
\end{aligned} \tag{A 9}$$

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Using (A 4), (A 7) and (A 9) in (A 8) gives:

$$\delta\ell = \left(h + \Phi - \frac{1}{2}u^2\right)\nabla\cdot(\rho\hat{V}^{\mathbf{x}}) + \rho T\hat{V}^{\mathbf{x}}\cdot\nabla S + \frac{\mathbf{B}}{\mu_0}\cdot[-\nabla\times(\hat{V}^{\mathbf{x}}\times\mathbf{B}) + \hat{V}^{\mathbf{x}}\nabla\cdot\mathbf{B}] + \rho\mathbf{u}\cdot\left[\frac{d\hat{V}^{\mathbf{x}}}{dt} - \hat{V}^{\mathbf{x}}\cdot\nabla\mathbf{u}\right] + D_{x^\alpha}\Lambda^\alpha. \tag{A 10}$$

Setting $\delta\mathscr{A} = 0$ (i.e. $\delta\ell = 0$) gives the divergence symmetry condition for the action given in (4.14).

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