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The Harris sheet in a dusty plasma

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Abstract. In 1962 E. G. Harris published a solution to the problem of a current sheet separating regions of oppositely directed magnetic field in a fully ionized plasma. The resulting solution has become known as the 'Harris sheet' and has been of great utility to the plasma physics community. In this paper, the footsteps of Harris are retraced with the addition of a multiply charged massive dust component. A set of highly nonlinear differential equations for a current sheet in a dusty plasma are presented. An analytic solution, similar to that of Harris, is found for the depleted electron regime. This solution is of great relevance to many astrophysical and laboratory dusty plasmas. Current sheet thickness and asymptotic field strength are calculated for various dusty plasma environments.

1. Introduction

There have been many attempts to explain processes in both the nebular (Joung et al. 2004) and cometary environments (Jovanović et al. 2005) through current sheets and their resulting dynamics. These studies have for the most part neglected the charged dust as playing a role in the plasma dynamics of the system under study, despite its presence. In recent years the role of charged dust (or dusty plasmas) has come under consideration in these domains. The presence of charged dust in these domains present several novel phenomena not present in ordinary ion–electron plasmas (Shukla and Eliasson 2009). Multi-fluid magnetized dusty plasma simulations have shown that current sheets can form in dusty plasmas undergoing shear-flow instabilities (Birk and Wiechen 2002). It has also been shown that via a Harris sheet configuration magnetic reconnection can occur in a dusty plasma (Birk and Wiechen 2001). Further studies of magnetic reconnection attempted to find an equilibrium configuration through a ballistic relaxation technique (Lazerson and Wiechen 2008). The development of a basic model for a current sheet in a dusty plasma would be of manifest relevance to the study of various dusty plasma processes. The path set by E. G. Harris (Harris 1962) in developing a model for a current sheet is followed, with the addition of a charged dust component. It is assumed an equilibrium charge is maintained on the dust grains, allowing the neglect of dust charge fluctuations. In Sec. 2, Vlasov's equation is solved for each species. Maxwell's equations then allow for the reduction to a pair of coupled differential equations for the vector and electrostatic scalar potentials. These equations are then solved under the assumption of a depleted electron regime in Sec. 3. Section 4 provides concluding remarks on this solution and its applicability.

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2. Analytic solution of the Vlasov equation

Following in the footsteps of E. G. Harris, Vlasov's equation is solved for each species (charged dust, ions and electrons; subscript k used to denote species)

$$\mathbf{v}_k \cdot \frac{\partial f_k}{\partial \mathbf{r}} + \frac{q_k}{m_k}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}_k \times \mathbf{B}\right) \cdot \frac{\partial f_k}{\partial \mathbf{v}} = 0, \tag{2.1}$$

where $\mathbf{v}$, f, $\mathbf{r}$, q, m, $\mathbf{E}$, c, and $\mathbf{B}$ are the species velocity, distribution function, position vector, species charge number, species mass, electric field, speed of light and magnetic field, respectively. A one-dimensional solution is sought allowing the assumptions that $\mathbf{E}$, $\mathbf{B}$ and f are functions of x only. The energy and momenta conjugate to y and z then become constants of motion for each species (subscript k dropped for clarity):

$$W = \frac{m}{2}\left(v_x^2 + v_y^2 + v_z^2\right) + q\Phi(x), \tag{2.2a}$$

$$p_y = mv_y + qA_y(x), \tag{2.2b}$$

$$p_z = mv_z + qA_z(x), \tag{2.2c}$$

where Φ is the electrostatic potential. Here A_y and A_z represent components of the vector potential. The problem is constrained through the assumptions that $\mathbf{E} = E(x)\hat{x}$ and $\mathbf{B} = B(x)\hat{z}$. The vector potential may then be written $\mathbf{A} = A(x)\hat{y}$ (where $\mathbf{B} = \nabla \times \mathbf{A}$). Given these constraints, the constants of motion become

$$W = \frac{m}{2}\left(v_x^2 + v_y^2 + v_z^2\right) + q\Phi(x), \tag{2.3a}$$

$$p_y = mv_y + qA_y(x), \tag{2.3b}$$

$$p_z = mv_z. \tag{2.3c}$$

Through inspection of (2.3c) the velocity in the z direction is a constant of motion ($\alpha_3 = v_z$). Equation (2.3b) provides another constant of motion ($\alpha_2 = v_y + \frac{q}{m}A_y(x)$). The third constant of motion is derived from the energy equation (2.3a):

$$\frac{2W}{m} - v_z^2 = v_x^2 + v_y^2 + \frac{2q}{m}\Phi. \tag{2.4}$$

Eliminating v_y and v_z via the constants of motion it may be written that

$$\alpha_1^2 = \frac{2W}{m} - \alpha_3^2 - \alpha_2^2 = v_x^2 - \frac{2q}{m}v_y A_y - \frac{q^2}{m}A_y^2 + \frac{2q}{m}\Phi. \tag{2.5}$$

With these three constants of motion, Vlasov's equation solves to the familiar 'shifted' Maxwellian distribution function

$$f_k = \left(\frac{m_k}{2\pi k_B T_k}\right)^{3/2} n_k \exp - \frac{m_k\left[\alpha_1^2 + \left(\alpha_2 - v_{ky}\right)^2 + \alpha_3^2\right]}{2k_B T_k} \tag{2.6}$$

where n, k_B, and T are the number density, Boltzmann's constant and temperature, respectively. This is the same distribution function as that found by Harris. The electrostatic and vector potentials may be determined from Maxwell's equations and

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the definition of both total charge and current:

$$\nabla \times \mathbf{E} = \frac{1}{\epsilon_o} \sum_{k=e,i,d} q_k \int f_k d^3v, \tag{2.7a}$$

$$\nabla \times \mathbf{B} = \mu_o \sum_{k=e,i,d} q_k \int f_k \mathbf{v} d^3v, \tag{2.7b}$$

where summations are over all three species (electrons, ions and dust). Here μ_0 is the permeability of free space and ϵ_o is the permittivity of free space. The integrals have general solutions:

$$\int f_k d^3v = n_k \exp \frac{q_k v_{ky} A_y - q_k \Phi}{k_B T_k}, \tag{2.8a}$$

$$\int f_k \mathbf{v} d^3v = v_{ky} n_k \exp \frac{q_k v_{ky} A_y - q_k \Phi}{k_B T_k} \hat{y}. \tag{2.8b}$$

This provides the system of equations that must be solved:

$$\frac{d^2\Phi}{dx^2} = -\frac{e}{\epsilon_o} \begin{bmatrix} -n_e \exp\left(\dfrac{-e}{k_B T_e} v_{ey} A_y + \dfrac{e}{k_B T_e}\Phi\right) + n_i \exp\left(\dfrac{e}{k_B T_i} v_{iy} A_y - \dfrac{e}{k_B T_i}\Phi\right) \\ -Z n_d \exp\left(\dfrac{-Ze}{k_B T_d} v_{dy} A_y + \dfrac{Ze}{k_B T_d}\Phi\right) \end{bmatrix}, \tag{2.9}$$

$$\frac{d^2 A_y}{dx^2} = \mu_0 e \begin{bmatrix} -v_{ey} n_e \exp\left(\dfrac{-e}{k_B T_e} v_{ey} A_y + \dfrac{e}{k_B T_e}\Phi\right) + v_{iy} n_i \exp\left(\dfrac{e}{k_B T_i} v_{iy} A_y - \dfrac{e}{k_B T_i}\Phi\right) \\ -Z v_{dy} n_d \exp\left(\dfrac{-Ze}{k_B T_d} v_{dy} A_y + \dfrac{Ze}{k_B T_d}\Phi\right) \end{bmatrix}. \tag{2.10}$$

This system of equations has been developed under the assumption that the dust has charged negatively. For positively charged dust only the sign of Z must be changed. These equations differ from those found by Harris through the addition of a third 'dust' term. This system of equations may be written in a more compact form:

$$\frac{d^2\Phi}{dx^2} = \sum_{k=e,i,d} -C_k \exp\left(-\alpha_k \Phi + \beta_k A_y\right), \tag{2.11}$$

$$\frac{d^2 A_y}{dx^2} = \sum_{k=e,i,d} D_k \exp\left(-\alpha_k \Phi + \beta_k A_y\right), \tag{2.12}$$

where $C_k = q_k n_k/\epsilon$, $D_k = \mu_0 q_k n_k v_k$, $\alpha_k = q_k/k_B T_k$ and $\beta_k = q_k v_k/k_B T_k$. These set of equations lack a general analytic solution. Harris employed the Weyl gauge ($\Phi = 0$) and boundary conditions (namely at $x = 0$, $A_y = 0$ and $\partial A_y/\partial x = 0$) to find a solution. In the next section, the equations are solved under the assumption of a depleted electron regime. Such a regime is applicable to a wide array of dusty plasma environments.

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3. Depleted electron field

In many dusty plasma environments, the electron field is heavily depleted ($n_e \sim 0$). In this situation n_e may be set equal to zero. This sets the coefficients in front of the electron term in (2.11) and (2.12) to zero. In condensed notation, the set of equations reduces to

$$\frac{d^2\Phi}{dx^2} = -C_i \exp\left(-\alpha_i\Phi + \beta_i A_y\right) - C_d \exp\left(-\alpha_d\Phi + \beta_d A_y\right), \tag{3.1a}$$

$$\frac{d^2 A_y}{dx^2} = D_i \exp\left(-\alpha_i\Phi + \beta_i A_y\right) + D_d \exp\left(-\alpha_d\Phi + \beta_d A_y\right). \tag{3.1b}$$

Solutions are sought such that the plasma is exactly neutral. The scalar potential then vanishes ($\Phi = 0$ otherwise known as the Weyl gauge). The first equation becomes identically zero and the vector potential equation becomes

$$\frac{d^2 A_y}{dx^2} = D_i \exp\left(\beta_i A_y\right) + D_d \exp\left(\beta_d A_y\right). \tag{3.2}$$

This equation simplifies through four assumptions. First, as already stated, the plasma is taken to be neutral and depleted of electrons. This allows the ion number density to be written in terms of the dust number density and charge number ($n_i = Z_d n_d$). Second, the temperature of the dust and ions are taken to be constant across the current sheet. This assumption was made in the original paper by Harris (applicable in the ideal regime). Third, the dust and ion velocities in the current sheet are taken to be equal and opposite ($v_{iy} = -v_{dy}$). Finally, in order to reduce the equation to a single exponential the dust temperature must be taken to be the product of the ion temperature and the dust charge ($T_d = Z_d T_i$). This is an idealization made in order to achieve an analytically tractable form of the equation. Experimental evidence indicates that dust kinetic temperatures are orders of magnitude larger than that of the ions in the weakly coupled regime (Williams and Thomas 2006, 2007). This idealization may be more realistic than that made in the derivation of Harris (specifically $T_i = T_e$, which is rarely true in a plasma). It may then be written as

$$\frac{d^2 A_y}{dx^2} = 2\mu_o e Z_d n_d v_d \exp\left(-\frac{Z_d e}{k_B T_d} v_{dy} A_y\right). \tag{3.3}$$

Employing the same boundary conditions on the vector potential as Harris (namely at $x = 0$, $A_y = 0$ and $\partial A_y/\partial x = 0$) the vector potential becomes

$$A_y(x) = \frac{2k_B T_i}{e v_i} \log\cosh^2\left(\frac{v_{iy}}{c}\frac{x}{\lambda_{Di}}\right). \tag{3.4}$$

The ion Debye length ($\lambda_{Di} = \sqrt{\epsilon_o k_B T_i / q_i^2 n_i}$) is utilized to simplify the argument of the hyperbolic function. The magnetic field may then be written as

$$B_z(x) = \sqrt{4\mu_o n_i k_B T_i}\tanh\left(\frac{v_{iy}}{c}\frac{x}{\lambda_{Di}}\right). \tag{3.5}$$

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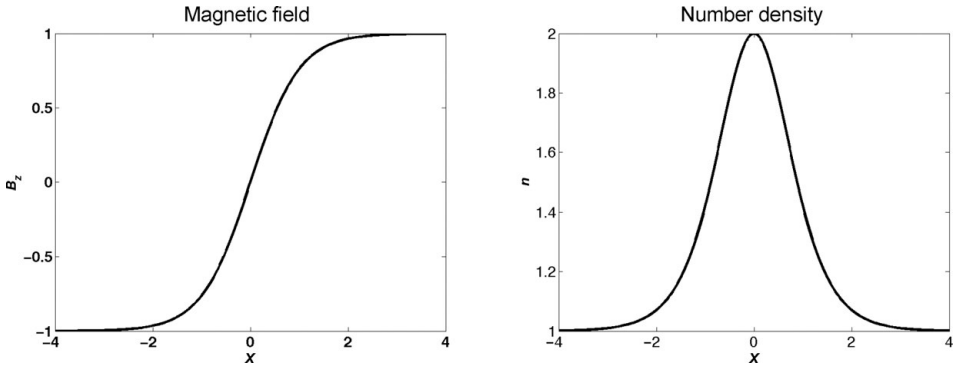


Figure 1. Dusty Harris sheet profiles for magnetic field and number densities. Values are shown in normalized units. ($B = \sqrt{4\mu_o n_d k_B T_d}$,$n = n_{do}$ and $x = c\lambda_d/v_{dy}$).

The asymptotic magnetic field strength is now recognized as $B_o = \sqrt{4\mu_o n_i k_B T_i}$. The dust and ion densities then become

$$n_d(x) = n_d \cosh^{-2}\left(\frac{v_{dy}}{c}\frac{x}{\lambda_D}\right) + n_d, \tag{3.6a}$$

$$n_i(x) = n_i \cosh^{-2}\left(\frac{v_{dy}}{c}\frac{x}{\lambda_D}\right) + n_i, \tag{3.6b}$$

where n_d and n_i are the asymptotic dust and ion number densities. Here it is assumed that the number densities do not vanish at infinity, unlike Harris. The profiles can be seen in Fig. 1. These profiles have thicknesses that scale as the drifting velocity ($v_y = -v_{dy} = v_{iy}$) and are inversely proportional to the dust debye length and speed of light. These current sheets must then have thicknesses many hundreds to thousands of times the dust Debye length, given our non-relativistic treatment ($v_{dy}/c \ll 1$). It is a trivial matter to show that these profiles result in a pressure balance between the magnetic field and plasma species pressures.

4. Discussion of the solution

It has been shown, through a Harris-like formulation, an analytic solution to a current sheet in a dusty plasma may be found through a kinetic approach. A set of generally applicable coupled nonlinear differential equations with boundary conditions have been presented. An analytic solution is found through a series of assumptions, while not applicable to all dusty plasmas, that is of manifest relevance for many astrophysical (Shukla et al. 1997; Verheest 2001; Shukla and Mamun 2002) and laboratory dusty plasmas (Chen 1995). These assumptions include a depleted electron density, isothermal temperature profiles, counter streaming species and dust kinetic temperatures much larger than that of the ions. This last assumption tracks well with experimental evidence. The set of generally applicable differential equations motivate further numerical investigation.

The analytic solution presented here provides a useful starting point for investigations of current sheet dynamics in dusty plasmas. Table 1 shows stereotypical values for various dusty plasma environments. The ion Debye length places a lower limit on

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Table 1. Stereotypical values for the ion number density (n_i), ion temperature (T_i), ion Debye length (λ_{Di}) and asymptotic magnetic field strength (B_0) in various dusty plasma environments. All values shown are approximations based on cited values. Nebulae, noctilucent cloud and spoke values taken from Mendis and Rosenberg (1994). Values for 3DPX taken from Thomas (2006).

	Nebulae	Noctilucent clouds	Saturns spokes	3DPX experiment
n_i $[m^{-3}]$	1000	1×10^6	100	1×10^{15}
T_i $[K]$	10	150	20 000	600
λ_{Di} $[m]$	10	1	1000	1×10^{-5}
B_0 $[T]$	1×10^{-12}	1×10^{-10}	1×10^{-11}	1×10^{-5}

the scale thickness of a current sheet. The non-relatavistic nature of these plasmas implies a current sheet thickness a few orders of magnitude larger than the ion Debye length. The current sheets have appropriate scale thickness for each environment. The asymptotic magnetic field strengths also scales well with acceptable values for the magnetic field in each environment.

Questions regarding the effects of dust charging and applicability of a shifted Maxwellian still remain. The work presented here indicates the added level of complexity present when dusty plasma interactions are added to a previously solved problem in plasma physics. These results provide a solid theoretical basis onto which future analysis may be conducted into current sheets and magnetic reconnection in dusty plasmas.

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