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IMPULSE CONTROL AND EXPECTED SUPREMA

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Abstract

We consider a class of impulse control problems for general underlying strong Markov processes on the real line, which allows for an explicit solution. The optimal impulse times are shown to be of a threshold type and the optimal threshold is characterised as a solution of a (typically nonlinear) equation. The main ingredient we use is a representation result for excessive functions in terms of expected suprema.

Keywords: Impulse control; Hunt process; threshold rule; excessive function

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Secondary 60G40

1. Introduction

Impulse control problems form an important class of stochastic control problems and find applications in a wide variety of fields ranging from finance, e.g. cash management and portfolio optimisation [5], [18], [19], optimal forest management [3], [34], and control of an exchange rate by the central bank [9], [25]. The general theory for impulse control problems is often based on the seminal work [6], where the problem was treated using quasivariational inequalities; see also [19] for a survey with focus on financial applications. For a treatment based on the use of superharmonic functions in a general Markovian framework, we refer the reader to [10]. However, the class of problems that allow for an explicit solution is very limited. Even for underlying one-dimensional diffusion processes, general solutions are known only for subclasses of problems, see [2], [4], and [16]. In these references, solution methods based on excessive functions were established under assumptions on the reward structure that forces the optimal strategy to be in a certain class. For processes with jumps, the class of explicitly solvable problems is even more scarce. In [28], a general verification theorem for jump diffusions and a connection to sequences of optimal stopping problems was established. Some examples were discussed therein for general classes of jump processes and particular reward structures arising in the control of exchange rates and optimal stream of dividends under transaction costs, which allow for a solution using the guess-and-verify approach.

On the other hand, the optimal stopping theory for general Lévy processes was developed in the last decade starting with a treatment of the perpetual American put optimal stopping problem in [23], see also [1]. Another source was the treatment of the Novikov–Shiryayev optimal stopping problem

$$\sup_{\tau} \mathbb{E}_x(\mathrm{e}^{-r\tau}(X_\tau^+)^m),$$

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where X is a general Lévy process and m is a positive constant. The main tools for the treatment are the Wiener–Hopf factorisation and, based on this, the use of Appell polynomials associated with the running maximum of X up to an independent exponentially distributed time; see [21], [26], [27], [29]. Inspired by these findings, solution techniques were developed in this setting for more general reward functions g, see [15], [32]. In [11], this approach was developed for general underlying Markov processes X on the real line. The starting point is to represent the reward function g in the form

$$g(x) = \mathbb{E}_x(f(M_T)) \tag{1}$$

for some function f; here and in the sequel, M denotes the running maximum of X and T is an independent Exp(r)-distributed time. For example, if $g(x) = x^m$, $m = 1, 2, \ldots$, and X is a Lévy process, then f is the mth Appell polynomial associated with M_T.

The aim of this paper is to treat in a similar manner the impulse control problem for general underlying Markov processes on the real line and general reward function g (fulfilling certain conditions discussed below) with value function

$$v(x) = \sup_{S=(\tau_n,\gamma_n)_{n\in\mathbb{N}}} \mathbb{E}_x^S\left(\sum_{n=1}^{\infty} \mathrm{e}^{-r\tau_n} g(X_{\tau_n,-})\right), \tag{2}$$

where the supremum is taken over all impulse control strategies $S = (\tau_n, \gamma_n)_{n=1}^{\infty}$ such that under the measure $\mathbb{P}_x^S$ with expectation $\mathbb{E}_x^S$ the process is restarted at a fixed point $\gamma_n = x_0$ after an impulse is exercised and $X_{\tau_n,-}$ denotes the value before the nth impulse is exercised. A more detailed description is given in the following section. We show that the optimal strategy is of a threshold type, where the threshold x^* can be described (semi-)explicitly in terms of certain c-values of the function f, i.e. $f(x^*) = c$, in the representation (1).

Value functions of the form (2) arise naturally in many applications. One motivation stems from optimal harvesting problems. Here, the process X is interpreted as, e.g. the random volume of wood in a forest stand. At the intervention points τ_n, the forest is harvested to a base level x_0 (we consider without loss of generality $x_0 = 0$ in the following). Selling the wood yields the gross reward $g(X_{\tau_n,-}) - g(x_0)$. Here, $g(x_0) \leq 0$ can be interpreted as a fixed cost component – the costs of one intervention – so that the net reward is given by $g(X_{\tau_n,-}) - g(x_0) + g(x_0) = g(X_{\tau_n,-})$ and the value function above describes the maximal expected discounted reward in this setting. The associated control problem is known as (a stochastic version of) Faustmann's problem. We refer the reader to [3] for further discussions and references.

Let us mention that in our analysis below, the $g(x_0) = 0$ case is included, i.e. no fixed cost component may be present. This situation is typically not considered in the literature on impulse control problems as these problems are typically formulated as singular control problems. However, as we will see below, also in this case strategies of impulse control-type turn out to be optimal if the gain function g grows slow enough around x_0.

Following the standard literature on Faustmann's problems, we assume a fixed base level x_0. In other classes of problems, it seems to be more natural to allow the decision maker to choose this base level also. As we do not want to overload this paper, we do not go into details here. Nonetheless, let us mention that the theory developed here can be used as a building block for the solution of this class of problems also.

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Furthermore, many results in the standard literature on impulse control theory are formulated for rewards including a running cost/reward component of the form

$$\mathbb{E}_x^S\left(\int_0^\infty \mathrm{e}^{-rs}k(X_s)\,\mathrm{d}s\right).$$

Note that – at least in the case that k fulfils a suitable integrability condition – this is no generalisation at all as the problem can be reduced to the form (2) using the r-resolvent operator for the uncontrolled process; see, e.g. [10, Lemma 2.1].

The structure of the paper is as follows. After summarising some facts about Hunt processes, an exact description of the problem is given in Subsection 2.2. The main theoretical findings of this paper are given in Section 3. In Subsection 3.1 we first characterise situations where no optimal strategies exist (in the class of impulse control strategies) and give ε-optimal strategies in this case. The nondegenerated case is then treated in Subsection 3.2, where the solution is given under general conditions and Assumption 2 introduced therein. The validity of this assumption is then discussed for certain classes of processes in Subsection 3.3. The results are illustrated on different examples for Lévy processes and reflected Brownian motions in Section 4.

2. Preliminaries

2.1. Hunt processes

For this paper we consider a general Markovian framework. More precisely, we assume the underlying process $X = (X_t)_{t\geq 0}$ to be a Hunt process (with infinite lifetime) on a subset E of the real line $\mathbb{R}$. We let $(\mathcal{F}_t)_{t\geq 0}$ denote the natural filtration generated by X, and $\mathbb{P}_x$ and $\mathbb{E}_x$ the probability measure and the expectation, respectively, associated with X when started at $x \in E$. In [11], [13], and [24] this class of processes was discussed in the context of optimal stopping. A more detailed treatment was given in [7], [12], and [30]. Hunt processes have the following important regularity properties: they are quasileft-continuous strong Markov processes with right-continuous sample paths having left limits.

Throughout the paper the notation T is used for an exponentially distributed random variable assumed to be independent of X and having the mean $1/r$, where $r > 0$ is the discounting parameter for the problem at hand. A key result in our approach is the following representation result for excessive functions of X; see, e.g. [17] and [11, Lemma 2.2] for a proof.

Lemma 1. *Let $f\colon E \to \mathbb{R}_+ \cup \{+\infty\}$ be an upper semicontinuous function and define*

$$u(x) := \mathbb{E}_x\Big(\sup_{0\leq t\leq T} f(X_t)\Big).$$

Then the function $u\colon E \to \mathbb{R} \cup \{+\infty\}$ is r-excessive.

The connection between the running maximum of the process X and the first-passage times is given by the following result, which corresponds to the well-known fluctuation identities for Lévy processes as used, e.g. in [20], [21], [26] for special payoffs and in [32] for the general case.

Lemma 2. *Let f and g be real functions such that $g(x) = \mathbb{E}_x(f(M_T))$ for all x, where M denotes the running maximum of X. Furthermore, let $y \in \mathbb{R}$ and $\tau_y = \inf\{t \geq 0\colon X_t \geq y\}$. Then, whenever the expectations exist,*

$$\mathbb{E}_x(f(M_T);\ M_T \geq y) = \mathbb{E}_x(g(X_{\tau_y})\,\mathbf{1}_{\{\tau_y<T\}}) = \mathbb{E}_x(\mathrm{e}^{-r\tau_y}g(X_{\tau_y})\,\mathbf{1}_{\{\tau_y<\infty\}}), \tag{3}$$

where $\mathbf{1}$ is the indicator function.

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Proof. As τ_y is a threshold time, we have $M_T = \sup_{\tau_y \le t \le T} X_t$ on $\{\tau_y \le T\}$. Using this and the memoryless property of the exponential distribution, we obtain

$$\begin{aligned}
\mathbb{E}_x(f(M_T);\ M_T \ge y) &= \mathbb{E}_x(f(M_T);\ \tau_y \le T) \\
&= \mathbb{E}_x\Big(f\Big(\sup_{\tau_y \le t \le T} X_t\Big);\ \tau_y \le T\Big) \\
&= \int_0^\infty \mathbb{E}_x\Big(\mathbb{E}_x\Big(f\Big(\sup_{\tau_y \le t \le s} X_t\Big) \,\Big|\, \mathcal{F}_{\tau_y}\Big)\mathbf{1}_{\{\tau_y \le s\}}\Big)\mathbb{P}(T \in \mathrm{d}s) \\
&= \mathbb{E}_x\left(\mathbb{E}_{X_{\tau_y}}\left(\int_0^\infty f\Big(\sup_{0 \le t \le s} X_t\Big)\mathbb{P}(T \in \mathrm{d}s)\right)\mathrm{e}^{-r\tau_y}\,\mathbf{1}_{\{\tau_y < \infty\}}\right) \\
&= \mathbb{E}_x(\mathrm{e}^{-r\tau_y}\mathbb{E}_{X_{\tau_y}}(f(M_T))\,\mathbf{1}_{\{\tau_y < \infty\}}) \\
&= \mathbb{E}_x(\mathrm{e}^{-r\tau_y} g(X_{\tau_y})\,\mathbf{1}_{\{\tau_y < \infty\}}) \\
&= \mathbb{E}_x(g(X_{\tau_y})\,\mathbf{1}_{\{\tau_y < T\}}).
\end{aligned}$$

□

For ease of notation, we always omit the indicator $\mathbf{1}_{\{\tau_y < \infty\}}$ in the following.

2.2. General impulse control problem

For general Hunt processes on $E \subseteq \mathbb{R}$, the exact definition of the correct setting for treating impulse control problems including all technicalities is lengthy. For our considerations in this paper, we explain the objects only on an intuitive level, which is sufficient for our further considerations. For an exact treatment, we refer the interested reader to the appendix in [10] and the references therein.

The object is to maximise over impulse control strategies, which are sequences $S = (\tau_n, \gamma_n)_{n=1}^\infty$ of times and impulses, respectively. The associated family of measures $(\mathbb{P}_x^S)_{x \in E}$ is taken so that the process X is still a strong Markov process with respect to $(\mathbb{P}_x^S)_{x \in E}$, where between each two random times $\tau_{n-1} < \tau_n$, the process runs uncontrolled with the same dynamics as the original Hunt process. At each random time τ_n, an impulse is exercised and the process is restarted at the new random state $X_{\tau_n} = \gamma_n$. It is assumed that τ_n is a stopping time for the process with only $n-1$ controls and γ_n is a random variable measurable with respect to the corresponding pre-τ_n σ-algebra. Furthermore, we consider only admissible impulse control strategies, i.e. strategies with the properties given above such that $\tau_n \nearrow \infty$ as $n \nearrow \infty$. For a formal construction of $(\mathbb{P}_x^S)_{x \in E}$, we refer the reader to [31].

Since the underlying process may have jumps, we have to distinguish jumps coming from the dynamics of the process from jumps that arise due to a control, which may take place at the same time. Hence, we have $X_{\tau_n}^{(n)} \neq X_{\tau_n -}$ in general, where $X^{(n)}$ denotes the process with only $n-1$ controls. For our further developments, it will be important to consider the process $X^{(n)}$ also at time point τ_n. Therefore, we write $X_{\tau_n,-} := X_{\tau_n}^{(n)}$ to denote the value of the process at τ_n if no control had been exercised. For continuous underlying processes as diffusion processes, we have $X_{\tau_n,-} = X_{\tau_n -}$, which motivates this notation.

To state the class of problems we are interested in and to fix ideas we assume that $0 \in E$. The decision maker can restart the process at $x_0 = 0$ coming from a positive state and no other actions are allowed. This means that $X_{\tau_n} = \gamma_n = 0$ and $X_{\tau_n,-} > 0$ for all n. Furthermore, we fix a continuous function $g\colon [0, \infty) \cap E \to \mathbb{R}$ which is bounded from below. For each control from x to 0, we have a reward $g(x)$, i.e. we consider the impulse control problem with value

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function

$$v(x) = \sup_{S=(\tau_n,\gamma_n)_{n\in\mathbb{N}}} \mathbb{E}_x^S\left(\sum_{n=1}^{\infty} \mathrm{e}^{-r\tau_n} g(X_{\tau_n,-})\right), \tag{4}$$

under the assumption that, for all S and $x \in E$,

$$\mathbb{E}_x^S\left(\sum_{n=1}^{\infty} \mathrm{e}^{-r\tau_n} g(X_{\tau_n,-})\right) < \infty. \tag{5}$$

In an even much more general setting, a verification theorem for treating these problems is given in [10, Proposition 2.2], which in our situation reads as follows. For the sake of completeness, we also include the proof.

Theorem 1. *Let $\hat{v}\colon E \to \mathbb{R}$ be measurable and define the maximum operator $\boldsymbol{M}$ by*

$$\boldsymbol{M}\hat{v}(x) = g(x) + \hat{v}(0) \quad \textit{for } x \geq 0, \qquad \boldsymbol{M}\hat{v}(x) = -\infty \quad \textit{for } x < 0.$$

(i) *If $\hat{v}$ is nonnegative, r-excessive for the underlying uncontrolled process, and $\boldsymbol{M}\hat{v}(x) \leq \hat{v}(x)$ for all x, then it holds that $v(x) \leq \hat{v}(x)$ for all x.*

(ii) *If $x \in E$ and $S = (\tau_n, \gamma_n)_{n=1}^{\infty}$ is an admissible impulse control strategy such that*

$$\mathbb{E}_x^S(\mathrm{e}^{-r\tau_n}\hat{v}(X_{\tau_n,-})) = \mathbb{E}_x^S(\mathrm{e}^{-r\tau_{n-1}})\hat{v}(0) \quad \textit{for all } n \in \mathbb{N}, \tag{6}$$
$$\hat{v}(0) + g(X_{\tau_n,-}) \geq \hat{v}(X_{\tau_n,-}), \quad \mathbb{P}_x^S\textit{-a.s.},$$

then $\hat{v}(x) \leq v(x)$. We abbreviate 'almost surely' to 'a.s.'

Proof. Let $S = (\tau_n, \gamma_n)_{n=1}^{\infty}$ be an arbitrary admissible impulse control strategy and define $\tau_0 = 0$. If $\hat{v}$ is r-excessive, by the optional sampling theorem for nonnegative supermartingales, we obtain (keeping in mind that X runs uncontrolled between τ_{n-1} and τ_n under $\mathbb{P}^S$)

$$\mathbb{E}_x^S(\mathrm{e}^{-r\tau_n}\hat{v}(X_{\tau_n,-}) - \mathrm{e}^{-r\tau_{n-1}}\hat{v}(X_{\tau_{n-1}})) \leq 0,$$

hence

$$\begin{aligned}
&\mathbb{E}_x^S\left(\sum_{n=1}^{\infty} \mathrm{e}^{-r\tau_n}(\hat{v}(X_{\tau_n}) - \hat{v}(X_{\tau_n,-}))\right) + \hat{v}(x) \\
&\quad = -\mathbb{E}_x^S\left(\sum_{n=1}^{\infty}(\mathrm{e}^{-r\tau_n}\hat{v}(X_{\tau_n,-}) - \mathrm{e}^{-r\tau_{n-1}}\hat{v}(X_{\tau_{n-1}}))\right) \\
&\quad \geq 0.
\end{aligned} \tag{7}$$

Using this inequality, we obtain

$$\begin{aligned}
\mathbb{E}_x^S\left(\sum_{n=1}^{\infty} \mathrm{e}^{-r\tau_n} g(X_{\tau_n,-})\right) &\leq \mathbb{E}_x^S\left(\sum_{n=1}^{\infty} \mathrm{e}^{-r\tau_n}(g(X_{\tau_n,-}) + \hat{v}(X_{\tau_n}) - \hat{v}(X_{\tau_n,-}))\right) + \hat{v}(x) \\
&= \mathbb{E}_x^S\left(\sum_{n=1}^{\infty} \mathrm{e}^{-r\tau_n}(g(X_{\tau_n,-}) + \hat{v}(0) - \hat{v}(X_{\tau_n,-}))\right) + \hat{v}(x) \\
&= \mathbb{E}_x^S\left(\sum_{n=1}^{\infty} \mathrm{e}^{-r\tau_n}(\boldsymbol{M}\hat{v}(X_{\tau_n,-}) - \hat{v}(X_{\tau_n,-}))\right) + \hat{v}(x),
\end{aligned}$$

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where $\hat{v}(X_{\tau_n}) = \hat{v}(\gamma_n) = \hat{v}(0)$. Since $\boldsymbol{M}\hat{v}(y) \le \hat{v}(y)$ for all y, we obtain

$$\mathbb{E}_x^S\left(\sum_{n=1}^{\infty} \mathrm{e}^{-r\tau_n} g(X_{\tau_n,-})\right) \le \hat{v}(x),$$

and, consequently, $v(x) \le \hat{v}(x)$ since S is arbitrary. (Note that this implies (5) also.)
Claim (ii) is proved following the steps in the proof of (i): assumption (6) guarantees equality in (7). □

In general, it is difficult to find a suitable candidate to apply this verification theorem to in order to obtain an explicit solution. In the following section, we will demonstrate how the representation result for excessive functions presented in Lemma 1 can be used to achieve this goal.

3. Main results

For a continuous reward function g, we study now (4) as described in Subsection 2.2. The main ingredient for our treatment of this problem is a representation of g as given in (1). The existence and explicit determination of such a representation is discussed in detail in [11, Subsection 2.2]. We will come back to this in Section 4 for examples of interest.

3.1. Degenerated case

We first treat the degenerated case, in which we find ε-optimal impulse control strategies only of the form *wait until the process reaches level ε and then restart the process at state 0*. Taking the limit (in a suitable sense) as $\varepsilon \to 0$, the controlled process is the process reflected downwards from state 0. The limit of these strategies is obviously not an admissible impulse control strategy in the sense defined in Subsection 2.2, but a strategy of singular control type. For an example of a corresponding value function see Figure 1.

Theorem 2. *Assume that $g(0) = 0$ and that there exists a nondecreasing function $f : [0, \infty) \cap E \to \mathbb{R}$, which is continuous in state 0, such that $g(x) = \mathbb{E}_x(f(M_T))$ for all $x \in [0, \infty) \cap E$, and write $\hat{f} = f - f(0)$. Then the value function is given by*

$$v(x) = \mathbb{E}_x(\hat{f}(M_T);\ M_T \ge 0)$$

and, for $\varepsilon > 0$, the impulse control strategies $S_\varepsilon = (\tau_{n,\varepsilon}, \gamma_n)_{n=1}^{\infty}$, where

$$\gamma_n = 0, \qquad \tau_{n,\varepsilon} = \inf\{t > \tau_{n-1,\varepsilon} : X_t \ge \varepsilon\}, \qquad \tau_{0,\varepsilon} = 0$$

are ε-optimal in the sense that

$$v_\varepsilon(x) := \mathbb{E}_x^{S_\varepsilon}\left(\sum_{n=1}^{\infty} \mathrm{e}^{-r\tau_{n,\varepsilon}} g(X_{\tau_{n,\varepsilon},-})\right) \to v(x) \quad \text{as } \varepsilon \to 0.$$

Proof. Here and in the following proofs, to simplify notation, we assume that $E = \mathbb{R}$, so that we may write, e.g. $x > 0$ to denote $x \in (0, \infty) \cap E$. Note that, due to the right continuity of the sample paths of X, it holds that $\tau_{n,\varepsilon} \nearrow \infty$ as $n \nearrow \infty$, so that S_ε is admissible in the sense given above. Writing $\hat{f}(x) = 0$ for $x < 0$, for short, we have to show that the function

$$\hat{v}(x) := \mathbb{E}_x(\hat{f}(M_T))$$

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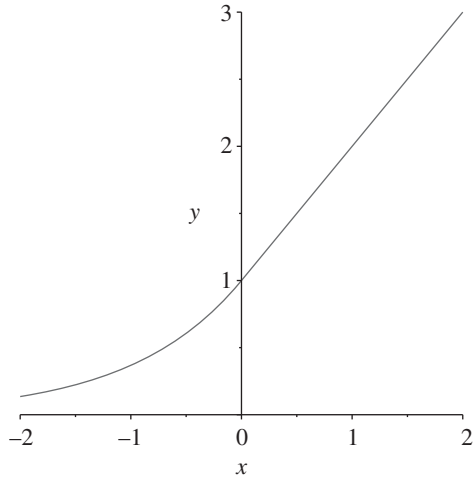


FIGURE 1: Value function in the $g(x) = x$ case for a standard Brownian motion X and $r = \frac{1}{2}$.

is the value function. By the monotonicity of $\hat{f}$,

$$\hat{v}(x) = \mathbb{E}_x\Big(\sup_{t\le T} \hat{f}(X_t)\Big)$$

and by noting that $\hat{f} \ge 0$, we obtain from Lemma 1 that $\hat{v}$ is r-excessive. For $x > 0$,

$$\begin{aligned}\boldsymbol{M}\hat{v}(x) &= g(x) + \hat{v}(0) = \mathbb{E}_x(f(M_T)) + \mathbb{E}_0(\hat{f}(M_T))\\ &= \mathbb{E}_x(f(M_T)) - f(0) + \mathbb{E}_0(f(M_T))\\ &= \mathbb{E}_x(\hat{f}(M_T)) + g(0)\\ &= \hat{v}(x),\end{aligned}$$

where in the last step the assumption $g(0) = 0$ is used. Theorem 1 now yields that $\hat{v} \ge v$. On the other hand, using the strong Markov property, we obtain, for $x \le 0$ and $\varepsilon > 0$,

$$\begin{aligned}v_\varepsilon(x) &= \mathbb{E}_x^{S_\varepsilon}\Big(\sum_{n=1}^{\infty} \mathrm{e}^{-r\tau_{n,\varepsilon}} g(X_{\tau_{n,\varepsilon},-})\Big)\\ &= \mathbb{E}_x^{S_\varepsilon}(\mathrm{e}^{-r\tau_{1,\varepsilon}} g(X_{\tau_{1,\varepsilon},-})) + \mathbb{E}_x^{S_\varepsilon}\Big(\sum_{n=2}^{\infty} \mathrm{e}^{-r\tau_{n,\varepsilon}} g(X_{\tau_{n,\varepsilon},-})\Big)\\ &= \mathbb{E}_x(f(M_T);\ M_T \ge \varepsilon) + \mathbb{E}_x^{S_\varepsilon}(\mathrm{e}^{-r\tau_{1,\varepsilon}} v_\varepsilon(0))\\ &= \mathbb{E}_x(f(M_T);\ M_T \ge \varepsilon) + v_\varepsilon(0)\mathbb{P}_x(M_T \ge \varepsilon),\end{aligned}$$

where we used Lemma 2 in the second to last step and the fact that $X_{\tau_{n,\varepsilon}} = 0$. For $x = 0$, we obtain

$$v_\varepsilon(0) = \mathbb{E}_0(f(M_T);\ M_T \ge \varepsilon) + v_\varepsilon(0)\mathbb{P}_0(M_T \ge \varepsilon),$$

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i.e.

$$\begin{aligned} v_\varepsilon(0) &= \frac{\mathbb{E}_0(f(M_T);\ M_T \geq \varepsilon)}{1-\mathbb{P}_0(M_T \geq \varepsilon)} \\ &= \frac{\mathbb{E}_0(f(M_T)) - \mathbb{E}_0(f(M_T);\ M_T < \varepsilon)}{\mathbb{P}_0(M_T < \varepsilon)} \\ &= \frac{g(0) - \mathbb{E}_0(f(M_T);\ M_T < \varepsilon)}{\mathbb{P}_0(M_T < \varepsilon)} \\ &= -\frac{\mathbb{E}_0(f(M_T);\ M_T < \varepsilon)}{\mathbb{P}_0(M_T < \varepsilon)}. \end{aligned} \tag{8}$$

By the continuity of f, it holds that

$$v_\varepsilon(0) \to -f(0) \quad \text{as } \varepsilon \to 0 \tag{9}$$

Therefore, for $x \leq 0$ as $\varepsilon \to 0$,

$$\begin{aligned} v(x) \geq v_\varepsilon(x) &= \mathbb{E}_x(f(M_T);\ M_T \geq \varepsilon) + v_\varepsilon(0)\mathbb{P}_x(M_T \geq \varepsilon) \\ &\to \mathbb{E}_x(f(M_T);\ M_T > 0) - f(0)\mathbb{P}_x(M_T > 0) \\ &= \mathbb{E}_x(\hat{f}(M_T);\ M_T > 0) \\ &= \mathbb{E}_x(\hat{f}(M_T);\ M_T \geq 0) \\ &= \hat{v}(x), \end{aligned}$$

where we used the fact that $\hat{f}(0) = 0$. This shows that $\hat{v}(x) \leq v(x)$ for $x \leq 0$. For $x > 0$ and $\varepsilon \in (0, x)$, it holds that $\tau_{1,\varepsilon} = 0$ $\mathbb{P}_x^{S_\varepsilon}$-a.s. Therefore, letting $\varepsilon \to 0$ and keeping (9) in mind,

$$\begin{aligned} v(x) &\geq v_\varepsilon(x) \\ &= \mathbb{E}_x^{S_\varepsilon}\left(\sum_{n=1}^{\infty} \mathrm{e}^{-r\tau_{n,\varepsilon}} g(X_{\tau_{n,\varepsilon},-})\right) \\ &= g(x) + v_\varepsilon(0) \\ &\to g(x) - f(0) \\ &= \mathbb{E}_x(f(M_T)) - f(0) \\ &= \hat{v}(x) \end{aligned}$$

so that $\hat{v} \leq v$ everywhere. Recalling that we have already proved $\hat{v} \geq v$, this shows $\hat{v} = v$ and the ε-optimality. □

For the previous considerations, it was essential that f was nondecreasing and $g(0) = 0$. The cases of nonmonotonous functions f and $g(0) < 0$ are treated in the following subsection. But first we consider a degenerated case, where the value function is infinite.

Theorem 3. *Assume that $g(0) = 0$ and there exists a continuous function $f\colon (0, \infty) \to \mathbb{R}$ such that $g(x) = \mathbb{E}_x(f(M_T))$ for all $x > 0$ and*

$$\lim_{x \to 0} f(x) = -\infty.$$

Then, for the impulse control strategies S_ε, $\varepsilon > 0$, given by

$$\tau_{n,\varepsilon} = \inf\{t > \tau_{n-1,\varepsilon}\colon X_t \geq \varepsilon\}, \qquad \tau_{0,\varepsilon} = 0,$$

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it holds that

$$v_\varepsilon(0) = \mathbb{E}_0^{S_\varepsilon}\left(\sum_{n=1}^{\infty} \mathrm{e}^{-r\tau_{n,\varepsilon}} g(X_{\tau_{n,\varepsilon},-})\right) \to \infty \quad \text{as } \varepsilon \to 0.$$

Consequently, the value function is infinite for $x \geq 0$.

Proof. Arguments similar to those yielding (8) in the previous proof can be used here to obtain

$$v_\varepsilon(0) = -\frac{\mathbb{E}_0(f(M_T);\ M_T < \varepsilon)}{\mathbb{P}_0(M_T < \varepsilon)} \geq -\sup_{x\in(0,\varepsilon)} f(x) \to \infty. \qquad \square$$

3.2. Nondegenerated case

In this subsection we study reward functions g for which the representing function f do not have to be monotonic. To be more precise, we make the following assumption, see Figure 2 for an illustration

Assumption 1. *There exists a continuous function* $f : [0,\infty)\cap E \to \mathbb{R}$ *such that, for all* $x \geq 0$,

$$g(x) = \mathbb{E}_x(f(M_T)) \quad \text{with } g(0) \leq 0.$$

Moreover, there exists a global minimum point $\underline{x} \in [0,\infty) \cap E$ *of* f *such that*

- $f(\underline{x}) < 0$,
- f *is strictly increasing on* $[\underline{x},\infty) \cap E$,
- $f(x) > 0$ *for sufficiently large* x,
- *if* $g(0) = 0$ *then* $\underline{x} > 0$.

Let $c^* := -f(\underline{x})$. By the intermediate value theorem and the monotonicity, for each $c \in [0, c^*]$, there is a unique $x_c \in [\underline{x}, \overline{x}]$ such that $f(x_c) = -c$, where $\overline{x}$ denotes the unique root of f in $[\underline{x},\infty)$. Now consider the following candidates for the value function:

$$\hat{v}_c(x) = \mathbb{E}_x(\hat{f}_c(M_T);\ M_T \geq x_c),$$

where $\hat{f}_c(x) = f(x) + c$ and $c \in [0, c^*]$.

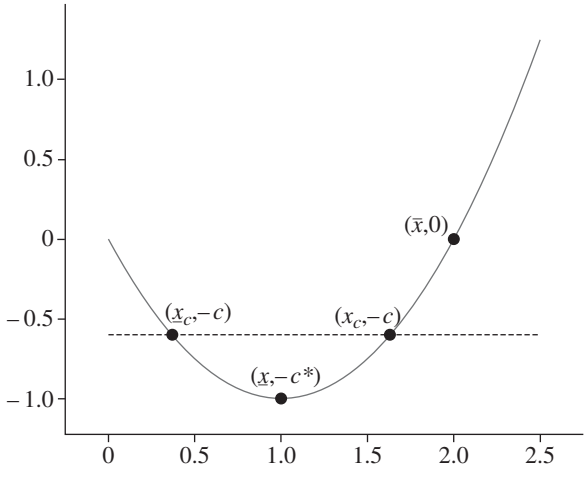


FIGURE 2: Plot of an Appell polynomial $f = Q_m$ for $m = 2$ (*solid*).

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Lemma 3. *Under Assumption 1 and the notation given above, for each $c \in [0, c^*]$, it holds that*

(i) $\hat{v}_c(x) = \mathbb{E}_x(\sup_{t \le T}(\hat{f}_c(X_t)\mathbf{1}_{\{X_t \ge x_c\}}))$ *for all* x,

(ii) $\hat{v}_c$ *is nonnegative and* r*-excessive for the underlying uncontrolled process,*

(iii) $\hat{v}_c$ *has the* r*-harmonicity property on* $(-\infty, x_c)$*, i.e.*

$$\hat{v}_c(x) = \mathbb{E}_x(\mathrm{e}^{-r\tau}\hat{v}_c(X_\tau)),$$

where $\tau = \inf\{t \ge 0\colon X_t \ge x_c\}$ *and* $x \le x_c$,

(iv) $\hat{v}_c(x) = c + g(x)$ *for* $x \ge x_c$,

(v) *there exists* $\hat{c} \in (0, c^*]$ *such that* $\hat{c} = \hat{v}_{\hat{c}}(0)$.

Proof. (i) This follows from the monotonicity assumption on f.

(ii) This is obtained from Lemma 1 and is clear recalling the monotonicity assumptions on f and the results of Subsection 2.1.

(iii) This can easily be obtained from (3).

(iv) Note that, for $x \ge x_c$,

$$\begin{aligned}\hat{v}_c(x) &= \mathbb{E}_x(\hat{f}_c(M_T);\ M_T \ge x_c)\\ &= \mathbb{E}_x(f(M_T);\ M_T \ge x_c) + c\mathbb{P}_x(M_T \ge x_c)\\ &= g(x) + c.\end{aligned}$$

(v) Consider the function $z\colon c \mapsto c - \hat{v}_c(0)$. Since $f \ge 0$ on $[\overline{x}, \infty)$, it holds that

$$z(0) = -\hat{v}_0(0) = -\mathbb{E}_0(f(M_T);\ M_T \ge \overline{x}) \le 0$$

and, on the other hand, since $f(x) + c^* \ge 0$ for all $x \ge 0$,

$$\begin{aligned}z(c^*) &= c^* - \mathbb{E}_0(f(M_T) + c^*;\ M_T \ge \underline{x})\\ &\ge c^* - \mathbb{E}_0(f(M_T) + c^*)\\ &= -\mathbb{E}_0(f(M_T))\\ &= -g(0)\\ &\ge 0,\end{aligned}$$

where, due to Assumption 1, the first inequality is strict if $\underline{x} > 0$ and the second if $\underline{x} = 0$. Now the intermediate value theorem yields the existence of $\hat{c}$. □

Remark 1. Recall that

$$\boldsymbol{M}\hat{v}_c(x_c) = g(x_c) + \hat{v}_c(0) = \mathbb{E}_{x_c}(f(M_T)) + \mathbb{E}_0[(f + c)(M_T);\ M_T \ge x_c]$$

and

$$\hat{v}_c(x_c) = \mathbb{E}_{x_c}[(f + c)(M_T);\ M_T \ge x_c] = \mathbb{E}_{x_c}[(f + c)(M_T)] = g(x_c) + c.$$

Therefore, the value $\hat{c}$ in Lemma 3(v) can be characterised also via

$$\boldsymbol{M}\hat{v}_{\hat{c}}(x_{\hat{c}}) = \hat{v}_{\hat{c}}(x_{\hat{c}}).$$

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Notation 1. Our candidate for the value function is now given by

$$\hat{v} := \hat{v}_{\hat{c}},$$

where $\hat{c}$ is chosen to be the smallest one satisfying the condition given in Lemma 3(v), and we also write

$$\hat{f} := \hat{f}_{\hat{c}} = f + \hat{c} \quad \text{and} \quad x^* = x_{\hat{c}} > 0,$$

for short.

We need one further assumption.

Assumption 2. *For all* $x \in (0, \underline{x})$,

$$\mathbb{E}_x(\hat{f}(M_T);\ M_T \le x^*) \le 0. \tag{10}$$

Remark 2. Note that under Assumption 1, for all $x \le x^*$, we have

$$\begin{aligned}\mathbb{E}_x(\hat{f}(M_T);\ M_T \le x^*) &= \mathbb{E}_x(\hat{f}(M_T)) - \mathbb{E}_x(\hat{f}(M_T);\ M_T \ge x^*)\\ &= \mathbb{E}_x(f(M_T)) + \hat{c} - \mathbb{E}_x(\hat{f}(M_T);\ M_T \ge x^*)\\ &= g(x) + \hat{c} - \mathbb{E}_x(\hat{f}(M_T);\ M_T \ge x^*),\end{aligned}$$

and, in particular, for $x = 0$ using Lemma 3(iii),

$$\mathbb{E}_0(\hat{f}(M_T);\ M_T \le x^*) = g(0) + \hat{c} - \mathbb{E}_0(\hat{f}(M_T);\ M_T \ge x^*) = g(0) \le 0. \tag{11}$$

Furthermore, (10) also trivially holds for all $x \ge \underline{x}$ due to the assumption $\hat{f} = f + \hat{c} < 0$ on $[\underline{x}, x^*]$.

Theorem 4. *Let Assumptions 1 and 2 hold. Define a sequence* $S^* = (\tau_n^*, \gamma_n^*)$ *by* $\tau_0^* := 0$,

$$\tau_n^* := \inf\{t \ge \tau_{n-1}^* \colon X_t \ge x^*\}, \qquad n = 1, 2, \ldots,$$

and $\gamma_n^* = 0$ *for all* $n = 1, 2, \ldots$. *Then the sequence* S^* *is an optimal impulse control sequence for the problem (4) and the value function is given by*

$$v(x) = \hat{v}(x) = \mathbb{E}_x(\hat{f}(M_T);\ M_T \ge x^*)$$

with $\hat{c}$ *as in Notation 1.*

Proof. As (10) clearly holds also for all $x \ge \underline{x}$, see Remark 2, we have, for all $x > 0$,

$$\begin{aligned}\hat{v}(x) - \boldsymbol{M}\hat{v}(x) &= \hat{v}(x) - (g(x) + \hat{v}(0))\\ &= \mathbb{E}_x(\hat{f}(M_T);\ M_T \ge x^*) - \mathbb{E}_x(f(M_T)) - \hat{c}\\ &= \mathbb{E}_x(\hat{f}(M_T);\ M_T \ge x^*) - \mathbb{E}_x(\hat{f}(M_T))\\ &= -\mathbb{E}_x(\hat{f}(M_T);\ M_T \le x^*)\\ &\ge 0\end{aligned}$$

with equality for $x \ge x^*$, which is also known from Lemma 3(iv). For $x \le 0$, the inequality holds as $\boldsymbol{M}\hat{v}(x) = -\infty$ for $x < 0$. Keeping Lemma 3(ii) in mind, the assumptions of Theorem 1(i) are fulfilled, which yields $\hat{v} \le v$.

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For the reverse inequality, we check that the assumption of Theorem 1(i) are fulfilled. First note that the sequence S^* is obviously admissible. By the strong Markov property and the definition of S^*, we have to prove that

$$\hat{v}(0) = \mathbb{E}_0(\mathrm{e}^{-r\tau_1^*}\hat{v}(X_{\tau_1^*})), \qquad \hat{v}(0) + g(y) = \hat{v}(y) \quad \text{for all } y \geq x^*. \tag{12}$$

Indeed, the expressions in (12) hold by Lemmas 3(iii) and 3(iv), respectively. □

3.3. On Assumption 2

To apply Theorem 4 we have to make sure that Assumption 2 holds. In this subsection we discuss conditions under which this is the case. The first such condition is trivial, but nonetheless useful in many situations where $g(0) < 0$.

Proposition 1. *If $\hat{f} \leq 0$ on $(0, \underline{x})$ and Assumption 1 is valid, then also Assumption 2 holds. In particular, this is the case if $\underline{x} = 0$, i.e. f is strictly increasing and $g(0) < 0$.*

Proof. When $\hat{f} \leq 0$ on $(0, \underline{x})$, it is also nonnegative on $[0, x^*]$ by Assumption 1. Therefore, $\mathbb{E}_x(\hat{f}(M_T);\ M_T \leq x^*) \leq 0$ for all $x \in (0, \underline{x})$. This proves the claim. □

Unfortunately, in some situations of interest, the function $\hat{f} = f + \hat{c}$ is *not* nonnegative on $[0, x^*]$. In particular, this is the typical situation in the $g(0) = 0$ case. (Note that for $g(0) = 0$, Assumption 2 holds with equality for $x = 0$, see (11), which makes it impossible that $\hat{f} \leq 0$ in all nontrivial cases.) In the following, we find sufficient conditions to guarantee the validity of Assumption 2 also in these cases when the underlying process is a Lévy process or a diffusion.

Proposition 2. *Let X be a Lévy process and Assumption 1 be valid. It is, furthermore, assumed that $\hat{f}$ is decreasing on $[0, \underline{x}]$ and that the distribution of M_T under $\mathbb{P}_0$ is given by a possible atom at 0 and a Lebesgue density b on $(0, \infty)$, which is nonincreasing. Then Assumption 2 holds.*

Proof. Let $x \in (0, \underline{x}_{\hat{c}})$, where $\underline{x}_{\hat{c}}$ denotes the smallest root of $\hat{f}$ in $(0, \underline{x})$ and write

$$z(x) := \mathbb{E}_x(\hat{f}(M_T);\ M_T \leq x^*).$$

We have to prove that $z(x) \leq 0$. From (11), we already know that $z(0) \leq 0$. Writing $h := \hat{f}\,\mathbf{1}_{\{x \leq x^*\}}$ and keeping in mind that h is nonnegative and decreasing on $(0, \underline{x}_{\hat{c}})$ and nonpositive on $(\underline{x}_{\hat{c}}, \infty)$, it holds that

$$\begin{aligned}
0 &\geq z(0) \\
&= \mathbb{E}_0 h(M_T) \\
&= h(0)\mathbb{P}_0(M_T = 0) + \int_0^\infty h(y)b(y)\,\mathrm{d}y \\
&= h(0)\mathbb{P}_0(M_T = 0) + \int_0^{\underline{x}_{\hat{c}}} h(y)b(y)\,\mathrm{d}y + \int_{\underline{x}_{\hat{c}}}^\infty h(y)b(y)\,\mathrm{d}y \\
&\geq h(x)\mathbb{P}_0(M_T = 0) + \int_0^{\underline{x}_{\hat{c}}-x} h(y)b(y)\,\mathrm{d}y + \int_{\underline{x}_{\hat{c}}}^\infty h(y)b(y-x)\,\mathrm{d}y \\
&\geq h(x)\mathbb{P}_0(M_T = 0) + \int_0^{\underline{x}_{\hat{c}}-x} h(x+y)b(y)\,\mathrm{d}y + \int_{\underline{x}_{\hat{c}}-x}^\infty h(x+z)b(z)\,\mathrm{d}z
\end{aligned}$$

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$$= h(x)\mathbb{P}_0(M_T = 0) + \int_0^\infty h(x+y)b(y)\,\mathrm{d}y$$
$$= \mathbb{E}_0(h(x+M_T))$$
$$= z(x). \qquad \square$$

Remark 3. The assumption that M_T has a Lebesgue density b which is nonincreasing with a possible atom at 0 holds, e.g. for all spectrally negative Lévy processes where M_T is exponentially distributed. It furthermore holds for processes with mixed exponential upward jumps; see [22].

Now we state a similar result that typically holds for Markov processes with no positive jumps.

Proposition 3. (i) *Assume that*

$$\mathbb{P}_x(M_T \in \mathrm{d}y) = \psi_r(x)\sigma(\mathrm{d}y), \qquad y \geq x, \tag{13}$$

where ψ_r denotes an increasing function and σ is a measure that does not depend on x. Under Assumption 1 it is, furthermore, assumed that $\hat{f}$ changes sign only once in $(0, \underline{x})$ (from $+$ to $-$). Then Assumption 2 holds.

(ii) *A decomposition of the type (13) holds for all regular diffusions and all spectrally negative Lévy processes.*

Proof. (i) Let $x \in (0, \underline{x}_{\hat{c}})$, where $\underline{x}_{\hat{c}}$ denotes the smallest root of $\hat{f}$ in $(0, \underline{x})$. By assumption, $\hat{f}(y) \geq 0$ on $(0, \underline{x}_{\hat{c}})$ and $\hat{f}(y) \leq 0$ on $[\underline{x}_{\hat{c}}, x^*]$, and write

$$z(x) := \mathbb{E}_x(\hat{f}(M_T);\, M_T \leq x^*).$$

We obtain

$$z(x) = \psi_r(x) \int_x^{x^*} \hat{f}(y)\sigma(\mathrm{d}y).$$

Obviously, $z(x) \leq 0$ on $[\underline{x}_{\hat{c}}, x^*]$ and, by (11), $z(0) \leq 0$. Therefore, we will prove that

$$w(x) := \int_x^{x^*} \hat{f}(y)\sigma(\mathrm{d}y)$$

is nonpositive on $[0, \underline{x}_{\hat{c}}]$, but since $\hat{f}$ is nonnegative on this interval, w is decreasing there. By Remark 2 we know that $w(0) \leq 0$. This proves the assertion.

(ii) Recall that the distribution of M_T for a regular diffusion processes is given by (see [8, p. 26])

$$\mathbb{P}_x(M_T \in \mathrm{d}y) = \psi_r(x)\sigma(\mathrm{d}y), \qquad y \geq x,$$

where ψ_r denotes the increasing fundamental solution for the generalised differential equation associated with X and the measure σ does not depend on x and is given by

$$\sigma([y, \infty)) = \frac{1}{r\psi_r(y)}.$$

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For spectrally negative Lévy processes, it is well known that M_T is $\mathrm{Exp}(\theta)$-distributed, where θ denotes the unique root of the equation $\Psi(\lambda) = r$, where Ψ denotes the Lévy exponent of X; see [20, p. 213]. Therefore,

$$\mathbb{P}_x(M_T \geq y) = \mathbb{P}_0(M_T \geq y - x) = \mathrm{e}^{-\theta(y-x)} = \mathrm{e}^{\theta x}\mathrm{e}^{-\theta y},$$

which yields the desired decomposition. □

Remark 4. Indeed, the assumption (13) holds for a wide class of Hunt processes with no positive jumps under minimal assumptions, see, e.g. the discussion in [13, Proposition 4.1].

For underlying diffusion processes, the advantage in the treatment of impulse control problems is that no overshoot occurs. This allows for the use of techniques which are very different in nature to the approach we use here. To see that the results obtained using special diffusion techniques are essentially covered by our results, we use the following characterisation of our Assumption 2. Equation (14), which is a consequence of the quasivariational inequalities, plays an essential role in [2].

Proposition 4. *Let X be a regular diffusion process and let Assumption 1 hold. Then Assumption 2 holds if and only if*

$$\psi_r(x)\frac{g(x^*) - g(0)}{\psi_r(x^*) - \psi_r(0)} \geq \frac{\psi_r(0)g(x^*) - \psi_r(x^*)g(0)}{\psi_r(x^*) - \psi_r(0)} + g(x) \quad \textit{for all } x \in [0, x^*], \tag{14}$$

where ψ_r denotes the increasing fundamental solution for the generalised differential equation associated with X.

Proof. Recalling Remark 2, we see that Assumption 2 is equivalent to

$$\mathbb{E}_x(\hat{f}(M_T);\ M_T \geq x^*) - g(x) - \hat{c} \geq 0. \tag{15}$$

As above, we use the representation of the distribution of M_T from [8, p. 26], i.e.

$$\mathbb{P}_x(M_T \in \mathrm{d}z) = \psi_r(x)\sigma(\mathrm{d}z).$$

Writing

$$\Delta = \int_{x^*}^{\infty} \hat{f}(z)\sigma(\mathrm{d}z),$$

the inequality (15) for $x \leq x^*$ reads as $\psi_r(x)\Delta \geq g(x) + \hat{c}$. Recalling that $\hat{c}$ and x^* were chosen so that we have equality for $x = x^*$ and $x = 0$, a short calculation yields that $c = -g(0) + \psi_r(0)\Delta$ and

$$\Delta = \frac{g(x^*) - g(0)}{\psi_r(x^*) - \psi_r(0)}.$$

Inserting this and rearranging terms, it holds that $\psi_r(x)\Delta \geq g(x) + \hat{c}$ holds if and only if

$$\psi_r(x)\frac{g(x^*) - g(0)}{\psi_r(x^*) - \psi_r(0)} \geq \frac{\psi_r(0)g(x^*) - \psi_r(x^*)g(0)}{\psi_r(x^*) - \psi_r(0)} + g(x).$$ □

We remark that analytical conditions on g and X can be found to guarantee that (14) holds. This was carried out in [2, Lemma 5.2], where three such conditions were given. (Note that in the setting discussed there, it was assumed that $g(0) < 0$, while we consider also the $g(0) = 0$ case.)

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4. Examples

4.1. Power reward for geometric Lévy processes

We consider an extension of the problem treated in [2, Section 6.1], where the case of a geometric Brownian motion and a power reward function was studied. More generally, we consider a general Lévy process X, which we assume not to be a subordinator, and reward function $g(x) = \mathrm{e}^{bx} - k$ for some $b > 0, k > 1$. We assume the integrability condition $\mathbb{E}_0(\mathrm{e}^{bX_1}) < \mathrm{e}^r$ to hold. Using the systematic approach from [11], or just by guessing, we see that the function

$$f(x) = a\mathrm{e}^{bx} - k,$$

where $a = 1/\mathbb{E}_0\mathrm{e}^{bM_T}$, fulfils Assumption 1 with $\underline{x} = 0$. Recalling Proposition 1, we see that the assumptions of Theorem 4 are fulfilled. For $c > 0$, the equation $f(x) = -c$ has the unique solution

$$x_c = \frac{\log((k-c)/a)}{b} \tag{16}$$

and the optimal value $\hat{c}$ is given by

$$\hat{c} = \hat{v}_{\hat{c}}(0) = \mathbb{E}_0(\hat{f}_{\hat{c}}(M_T);\ M_T \geq x_{\hat{c}}) = \int_{x_{\hat{c}}}^{\infty} (a\mathrm{e}^{by} + \hat{c})\mathbb{P}_0(M_T \in \mathrm{d}y). \tag{17}$$

How explicitly this equation can be solved now depends on the distribution of M_T. For example, in the case that X has arbitrary downward jumps, a Brownian component, and mixed exponential upward jumps, it is known that M_T has a Lebesgue density of the form

$$\sum_{k=1}^{n+1} \zeta_k \eta_k \mathrm{e}^{-\eta_k y} \quad \text{for } y > 0,$$

where $\eta_1, \ldots, \eta_{n+1}$ and $\zeta_1, \ldots, \zeta_{n+1}$ are positive constants (see [22]). In this case, (17) can be written as

$$\hat{c} = \sum_{k=1}^{n+1} \zeta_k \left(a\frac{\eta_k}{b-\eta_k}\mathrm{e}^{(b-\eta_k)x_{\hat{c}}} + \hat{c}\mathrm{e}^{-\eta_k x_{\hat{c}}} \right),$$

which may be solved numerically for $\hat{c}$. This also yields a value for the optimal boundary x^*. Summarising the results, we have the following proposition.

Proposition 5. *Define $S^* = (\tau_n^*, \gamma_n^*)_{n=1}^{\infty}$ by setting, for all $n = 1, 2, \ldots$,*

$$\gamma_n^* = 0, \qquad \tau_n^* := \inf\{t \geq \tau_{n-1}^* \colon X_t \geq x^*\}, \qquad \tau_0 = 0.$$

Then the sequence S^* is an optimal impulse control sequence for the power reward impulse control problem for geometric Lévy processes. The value function is

$$v(x) = \mathbb{E}_x(a\mathrm{e}^{bM_T} - k + \hat{c};\ M_T \geq x^*)$$

with $\hat{c}$ given by (17) and $x^* = x_{\hat{c}}$ given by (16).

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4.2. Power reward for Lévy processes

Now we consider the case $g(x) = x^m$, $m \in \mathbb{N}$, $m \geq 2$ for a Lévy process X which we assume not to be a subordinator, and that the Lévy measure π satisfies

$$\int_{(-\infty,-1)\cup(1,+\infty)} |y|^m \pi(\mathrm{d}y) < \infty.$$

This assumption implies that X_T has the finite mth moment.

We start with the $m = 1$ case. Then it is clear that

$$g(x) = \mathbb{E}_x(Q_1(M_T)),$$

where $Q_1(x) = f(x) = x - \mathbb{E}_0(M_T)$ is the first Appell polynomial associated with M_T. For this function, the assumptions of Theorem 2 are obviously fulfilled, yielding the following proposition.

Proposition 6. *For a Lévy process X as above, the impulse control problem with reward function $g(x) = x$ has value function*

$$v(x) = \mathbb{E}_x(\hat{f}(M_T);\ M_T \geq 0) = \begin{cases} x + \mathbb{E}_0(M_T), & x \geq 0, \\ \mathbb{E}_x(M_T^+), & x < 0, \end{cases}$$

and the impulse control strategies S_ε given by

$$\tau_{n,\varepsilon} = \inf\{t > \tau_{n-1,\varepsilon} \colon X_t \geq \varepsilon\}, \qquad \tau_{0,\varepsilon} = 0,$$

are ε-optimal.

As the optimal strategy is degenerated for Lévy process X and reward $g(x) = x$, this is not the case for other powers $g(x) = x^m$, $m \in \mathbb{N}$, $m \geq 2$. In the following lemma, we collect two well-known properties of the mth Appell polynomial Q_m associated with M_T; see [26], [27], or [29].

Lemma 4. *For $m \in \mathbb{N}$, $m \geq 2$, the following holds:*

(i) *Q_m fulfils Assumption 1,*

(ii) *Q_m is decreasing on $[0, \underline{x}]$.*

Using this lemma, we may apply Theorem 4 to obtain the following solution for the power reward problem for general Lévy processes.

Proposition 7. *Let $m \in \mathbb{N}$, $m \geq 2$, and let Assumption 2 hold. Define a sequence $S^* = (\tau_n^*, \gamma_n^*)$ by $\tau_0^* := 0$,*

$$\tau_n^* := \inf\{t \geq \tau_{n-1}^* \colon X_t \geq x^*\} \quad \textit{for all } n \in \mathbb{N},$$

and $\gamma_n^* = 0$. Then the sequence S^* is an optimal impulse control sequence for the power reward impulse control problem. The value function is given by

$$v(x) = \mathbb{E}_x(Q_m(M_T) + \hat{c};\ M_T \geq x^*)$$

with $\hat{c}$ as in Notation 1.

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Remark 5. Whenever the distribution of M_T is given by a possible atom at 0 and a Lebesgue density b on $(0, \infty)$, which is nonincreasing, then the conclusion of the previous proposition holds by Proposition 2.

To obtain more explicit results, we now concentrate on the case of spectrally negative Lévy processes, including, e.g. all Brownian motions with drift. For spectrally negative Lévy processes, it is well known that M_T is $\mathrm{Exp}(\theta)$-distributed, where θ denotes the unique root of the equation $\Psi(\lambda) = r$ and Ψ denotes the Lévy exponent of X; see [20, p. 213]. From [29, Section 2.2], we know that the mth Appell polynomial $f = Q_m$ is given by $f(x) = (x - m/\theta)x^{m-1}$, and x_c is therefore given as the unique positive root of

$$\left(x - \frac{m}{\theta}\right)x^{m-1} = -c \quad \text{for all } c \in [0, c^*].$$

In particular, in the $m = 2$ case, we have

$$x_c = \frac{1 + \sqrt{1 - \theta^2 c}}{\theta}.$$

Recalling that M_T is $\mathrm{Exp}(\theta)$-distributed and the process does not jump upwards, a short calculation yields that the condition $\hat{c} = \hat{v}_{\hat{c}}(0)$ can be expressed as

$$\hat{c} = \mathbb{E}_0((f + \hat{c})(M_T);\ M_T \geq x_{\hat{c}}) = (g(x_{\hat{c}}) + \hat{c})\mathrm{e}^{-\theta x_{\hat{c}}} = (x_{\hat{c}}^m + \hat{c})\mathrm{e}^{-\theta x_{\hat{c}}} \tag{18}$$

and the value function is given by

$$v(x) = \begin{cases} (x^{*m} + \hat{c})\mathrm{e}^{-\theta(x^* - x)}, & x < x^*, \\ \hat{c} + x^m, & x \geq x^*. \end{cases} \tag{19}$$

For $m = 2$, the unique solution c of (18) is $\hat{c} = \theta^{-2}w$, where w is the unique solution to

$$2(1 + \sqrt{1 - y}) = y\mathrm{e}^{1 + \sqrt{1-y}}, \qquad 0 < y < 1.$$

This equation can be solved explicitly in terms of the Lambert W-function, see [14], i.e.

$$w = -W(-2\mathrm{e}^{-2})^2 - 2W(-2\mathrm{e}^{-2}).$$

Now the optimal strategy is given by the threshold $x^* = (1 + \sqrt{1 - w})/\theta$. It is easily checked that the value function v in (19) has the smooth fit property in x^*; see also Figure 3. As the restarting state 0 is fixed, we of course cannot expect to have a smooth fit there, but the value function intersects the reward by construction.

4.3. Power reward for reflected Brownian motion

We consider the power reward function $g(x) = x^m$, $m \in \mathbb{N}$, for a Brownian motion reflected in 0. For the $m \geq 2$ case, using the method described in [11, Subsection 2.2], and the explicit expressions from [8, Equation A1.2], it is straightforward to find the representing function $f = f_m$ as

$$f_m(x) = x^{m-1}\left(x - \frac{m}{\sqrt{2r}}\coth(x\sqrt{2r})\right),$$

which indeed fulfils

$$x^m = \mathbb{E}_x(f_m(M_T)) \quad \text{for all } x \geq 0.$$

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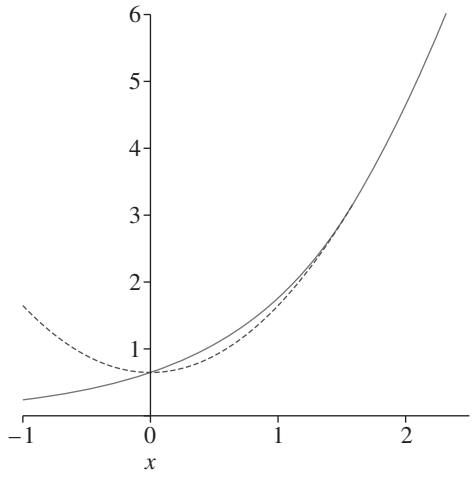


FIGURE 3: Value function in the $g(x) = x^2$ case for a standard Brownian motion X and $r = \frac{1}{2}$. Here, $x^* \approx 1.59$.

The $m = 1$ case has to be handled with more care, which is due to a local time term arising in this case. Nonetheless, it turns out that the same representation holds also in the $m = 1$ case for $x > 0$, i.e. $x = \mathbb{E}_x(f_1(M_T))$ for all $x > 0$. For a detailed treatment, see [33].

The impulse control problem can be solved by curve sketching of f_m. We start with the $m = 1$ case. Here, f_1 is increasing and $\lim_{x\to 0} f_1(x) = -\infty$. Therefore, Theorem 3 is applicable and we see that the value is infinite.

For $m = 2$, f_2 is also increasing, with $f_2(0) := \lim_{x\to 0} f_2(x) = -1/r$. Therefore, we are in the degenerated case with finite value. Theorem 2 yields ε-optimal strategies and

$$v(x) = x^2 + v(0) = x^2 + \frac{1}{r} \quad \text{for all } x \geq 0.$$

For $m \geq 3$, we obtain an optimal nondegenerated impulse control strategy. First, x_c is the unique positive root of

$$x^{m-1}\left(x - \frac{m}{\sqrt{2r}}\coth(x\sqrt{2r})\right) = -c \quad \text{for } c \in [0, c^*].$$

From [8, p. 411], the distribution of M_T is given by

$$\mathbb{P}_x(M_T \in \mathrm{d}y) = \sqrt{2r}\frac{\sinh(y\sqrt{2r})\cosh(x\sqrt{2r})}{\cosh(y\sqrt{2r})^2}\,\mathrm{d}y.$$

We calculate

$$\begin{aligned}&\mathbb{E}_0((f+c)(M_T);\ M_t \geq x_c)\\ &\quad = \int_{x_c}^{\infty}\left(y^{m-1}\left(y - \frac{m}{\sqrt{2r}}\coth(y\sqrt{2r})\right) + c\right)\sqrt{2r}\frac{\sinh(y\sqrt{2r})}{\cosh(y\sqrt{2r})^2}\,\mathrm{d}y\end{aligned}$$

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$$= -\left[\frac{2e^{y\sqrt{2r}}(c+y^m)}{e^{2y\sqrt{2r}}+1}\right]_{x_c}^{\infty}$$
$$= \frac{2e^{x_c\sqrt{2r}}(c+x_c^m)}{e^{2x_c\sqrt{2r}}+1}.$$

Therefore, the condition $c = \hat{v}_c(0)$ can be written as

$$c = \frac{2e^{x_c\sqrt{2r}}(c+x_c^m)}{e^{2x_c\sqrt{2r}}+1},$$

which may be solved numerically to obtain $\hat{c}$ and $x_{\hat{c}} = x^*$. No matter what the exact value of $\hat{c}$ is, the assumptions of Proposition 3 are fulfilled, so that Theorem 4 can be applied, which yields the optimal impulse control strategy with x^* as a threshold.

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