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Some Fock spaces with depth two action

Michael Anshelevich and Jacob Mashburn

Abstract. The subject of this article is operators represented on a Fock space which act only on the two leading components of the tensor. We unify the constructions from [Ans07, BL09, BL11, LS08] and extend a number of results from these articles to our more general setting. The results include the quadratic relation satisfied by the kernel of the free cumulant generating function, the resolvent form of the generating function for the Wick polynomials, and classification results for the case when the vacuum state on the operator algebra is tracial. We handle the generating functions in infinitely many variables by considering their matrix-valued versions.

1 Introduction

Symmetric, antisymmetric, and full Fock spaces have been studied for almost a century. They have appeared in quantum mechanics and the theory of operator algebras; in combinatorics, representation theory, and the study of orthogonal polynomials; and in other fields. Over the last few decades, there has been a proliferation of other Fock spaces, with accompanying structures, which to a greater or lesser degree share the properties of the classical ones. Such structures include commutation relations; Gaussian-type and Poisson-type operators; Wick products and orthogonal polynomials; Itô calculus, etc. Among these numerous constructions, we will highlight three, all of which appeared roughly at the same time a decade ago. While quite different, they all share a common feature, which one might call "depth two action" or "nearest neighbor coupling." In this article, we study a more general construction extending the three above, which retains this feature.

To be specific, let $\mathcal{B}$ be a unital $*$-algebra with a positive faithful linear functional ϕ and form the algebraic Fock space $\mathcal{F}_{\text{alg}}(\mathcal{B}) = \bigoplus_{n=0}^{\infty} \mathcal{B}^{\otimes n}$. For each $b \in \mathcal{B}$, on this space, we will define operators $a^+(b), a^-(b), a^0(b)$ and $X(b) = a^+(b) + a^0(b) + a^-(b)$. Here, the creation operator $a^+(b)$ is defined in the usual way, but the annihilation operator acts on simple tensors as

$$a^-(b)(u_1 \otimes \cdots \otimes u_n) = (\gamma + \phi)[bu_1]u_2 \otimes \cdots \otimes u_n,$$

where $\gamma + \phi \mathbf{1}_{\mathcal{B}}$ is some completely positive map. Note that this operator couples together the first two components of the tensor. This should be compared with, on the one hand, the free Fock space, where the annihilation operator acts only on the first component of the tensor; and, on the other hand, with the q-Fock space, where

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its action involves all components of the tensor. Note also that if $\gamma = 0$, this is just the standard free annihilation operator on the full Fock space $\mathcal{F}(L^2(\mathcal{B}, \phi))$. The preservation operator has the form

$$a^0(b)(u_1 \otimes \cdots \otimes u_n) = \Lambda(b)u_1 \otimes u_2 \otimes \cdots \otimes u_n,$$

or more generally,

$$a^0(b)(u_1 \otimes \cdots \otimes u_n) = \Lambda(b \otimes u_1) \otimes u_2 \otimes \cdots \otimes u_n$$

for a map Λ satisfying a symmetry condition, and acts only on the first component of the tensor.

Before proceeding further, we now outline the three earlier constructions mentioned above. We only list some of the results about these constructions; each of the articles below contains other important results not pertinent to this introduction. It can easily be seen that these constructions all fit in the setting of the preceding paragraph.

In [LS08], Lenczewski and Sałapata constructed a deformation of the free Fock space of $L^2(\mathbb{R}_+)$ with the annihilation operator of the form

$$a^-(f)g(x_1, \ldots, x_n) = \int f(x_1)w(x_1, x_2)g(x_1, \ldots, x_n)\, dx_1$$

for a specific function $w(s, t)$ of two variables. They studied combinatorial properties of operators $X(f)$ on this space, in particular, showing that they have the Kesten distribution. In [BL09, BL11], BoŻejko and Lytvynov constructed, in the setting of standard triples, a Fock space based on the annihilation operator

$$a^-(f)g(x_1, \ldots, x_n) = \int f(x_1)\eta(x_1)g(x_1, \ldots, x_n)\, dx_1$$

for some function η on $\mathbb{R}$. They studied general (Lévy-type) processes on such a Fock space. Perhaps most importantly, they gave a characterization among these, by a certain continuity property, of what it is natural to call the *Free Meixner class*. The article also contains several formulas for free cumulants of the distributions of such processes. In the follow-up article, they constructed a generating function for a system of polynomials (in infinitely many variables) associated with such a process in terms of certain operator-valued functions. Finally, in [Ans07], the first author studied the free Meixner states. Here, the Fock space was based on a finite-dimensional space $\mathbb{R}^d$, and the annihilation operator is

$$a_i^-(e_{u(1)} \otimes \cdots \otimes e_{u(n)}) = C_{u(1),u(2)} \left\langle e_i, e_{u(1)} \right\rangle e_{u(2)} \otimes \cdots e_{u(n)}.$$

The key results involved equations satisfied by the free cumulant generating function and the generating function for the Wick products (which can be identified with monic orthogonal polynomials in d variables), as well as the constructions in which the vacuum state is tracial on the algebra of these polynomials.

Motivated by these three articles, our primary interest in this article is in:

- The joint distributions of $\{X(u) : u \in \mathcal{B}\}$, typically expressed in terms of appropriate cumulants.

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- The Wick polynomials, which are polynomials in $\{X(u) : u \in \mathcal{B}\}$ such that $W(u_1 \otimes u_2 \otimes \cdots \otimes u_n)\Omega = u_1 \otimes u_2 \otimes \cdots \otimes u_n$.
- The algebras generated by $\{X(u) : u \in \mathcal{B}\}$.

Free and Boolean cumulant generating functions for free Meixner families [Ans07, BB06] satisfy second-order equations. Depth two action on the Fock space results in such equations being satisfied not by the scalar-valued free cumulant generating function $R(u)$ itself, but by its $\mathcal{B}$-valued "kernel" $R'(u)$ with $R(u) = \phi[uR'(u)u]$. In fact,

$$R'(u)v = v + R'(u)\gamma\left[uR'(u)u\right]v + R'(u)\Lambda(u \otimes v). \tag{1.1}$$

This result easily generalizes to a finite family of variables $\{u_i\}_{i=1}^{d}$. To make sense of a generating function for joint free cumulants of infinitely many variables $\{u_i\}_{i=1}^{\infty}$, we take an approach different from [BL11]. We form an infinite matrix which contains all the information about joint free cumulants of $\{u_i\}_{i=1}^{\infty}$, and still satisfies (an appropriate version of) equation (1.1). In the case when $\Lambda(u \otimes v) = \Lambda(u)v$, for $\{u_i\}_{i=1}^{\infty}$ uniformly small, this matrix corresponds to a genuine bounded operator. The analysis is similar in style to, but different from, computations with fully matricial free cumulants [PV13].

We perform a similar analysis for the joint generating function of Wick polynomials. It can be interpreted as an infinite matrix, and under appropriate assumptions as a bounded operator. As in [Ans07, BL11], it has a resolvent-type form

$$W(u) = (B(u) - X(u))^{-1}(B(u) - \phi[u^2]),$$

where $B(u) = 1 + \Lambda(u) + (\gamma + \phi)[u^2]$ (see Section 4).

Finally, we investigate the situation when the vacuum state on the algebra generated by all operators $\{X(u) : u \in \mathcal{B}\}$ is tracial. The depth two nature of the action allows us to write down explicit conditions on ϕ, γ, Λ which guarantee this. In the case when $\gamma = 0$, the Fock space is the full Fock space, but the circular operators $X(u)$ are deformed by a nontrivial Λ. We show that one can always use Λ to define a new multiplication on $\mathcal{B}$, so that the representation splits into a semicircular and a free compound Poisson parts. More generally, if $\gamma[u] = \eta u$ for η central (related to the construction from [BL09]), then one has a similar decomposition, but with the third component on which $\Lambda(u \otimes v) = \lambda uv$ for λ central.

The article [Ans07] contained another example of a "depth two action" algebra with a tracial vacuum state, which naturally corresponded to the free multinomial distribution. This example also generalizes to the setting of this article. It is described in a forthcoming article [AM22].

The article is organized as follows: After the Introduction, in Section 2, we present the main construction. In Section 3, we prove formulas for joint moments, and Boolean and free cumulants, of the operators $\{X(u) : u \in \mathcal{B}\}$. We also compare and contrast these formulas with the operator-valued results of [AW18]. In particular, unlike in [AW18], the inner product in this article is scalar-valued rather than $\mathcal{B}$-valued. In Section 4, we discuss Wick polynomials, and matricial generating functions for them and for the free cumulants. In Section 5, we provide conditions under which operators $X(u)$, as well as various generating functions, are bounded. In Section 6, we derive the conditions for the vacuum state to be tracial. The results

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in Sections 4 and 5 are proven in the setting of $\Lambda(u \otimes v) = \Lambda(u)v$; in contrast, the results in Section 6 are of main interest for general Λ. Finally, in Section 7, we describe in detail the examples which motivated this article, as well as their generalizations covered by our construction. We also prove a representation theorem under the assumption that the vacuum state is tracial.

2 The construction

Definition 2.1 Let $\mathcal{B}$ and $\mathcal{C}$ be $*$-algebras. An element $b \in \mathcal{B}$ is positive if $b = \sum_{i=1}^{k} u_i^* u_i$ for some k and $u_i \in \mathcal{B}$. A map $T : \mathcal{B} \to \mathcal{C}$ is positive if for each $u \in \mathcal{B}$, $T(u^*u)$ is positive in $\mathcal{C}$. It is faithful if $T(u^*u) = 0$ only for $u = 0$. It is completely positive if for each n, the map

$$T_n : M_n(\mathcal{B}) \to M_n(\mathcal{C}), \quad T_n\left([a_{ij}]_{i,j=1}^n\right) = [T(a_{ij})]_{i,j=1}^n$$

is positive, where we use the usual $*$-structure on $M_n(\mathcal{B})$.

Construction 2.2 *Let $\mathcal{B}$ be a unital $*$-algebra, equipped with star-linear maps $\phi : \mathcal{B} \to \mathbb{C}$, $\gamma : \mathcal{B} \to \mathcal{B}$, and $\Lambda : \mathcal{B} \otimes_{\text{alg}} \mathcal{B} \to \mathcal{B}$ such that ϕ is positive and faithful, $\gamma + \phi = \gamma + \phi\mathbf{1}_{\mathcal{B}}$ is completely positive (semi-definite), and Λ satisfies*

$$\phi[v^*\Lambda(b \otimes u)] = \phi[\Lambda(b^* \otimes v)^* u], \quad \gamma[v^*\Lambda(b \otimes u)] = \gamma[\Lambda(b^* \otimes v)^* u], \tag{2.1}$$

On the algebraic Fock space,

$$\mathcal{F}_{\text{alg}}(\mathcal{B}) = \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} \mathcal{B}^{\otimes n},$$

define the inner product by the linear extension of

$$\begin{aligned} &\langle u_1 \otimes \cdots \otimes u_n, v_1 \otimes \cdots \otimes v_k \rangle_{\gamma,\phi} \\ &\qquad = \delta_{n=k}\phi\left[v_n^*(\gamma + \phi)[v_{n-1}^*(\gamma + \phi)[\ldots(\gamma + \phi)[v_1^* u_1]\ldots]u_{n-1}]u_n\right]. \end{aligned} \tag{2.2}$$

This inner product is positive semi-definite but not, in general, positive definite. Denote by $\mathcal{F}_{\gamma,\phi}(\mathcal{B})$, the completion of the quotient $\mathcal{F}_{\text{alg}}(\mathcal{B})/\mathcal{N}$ by the subspace $\mathcal{N}$ of elements of zero seminorm. Next, for each $b \in \mathcal{B}$, consider operators on $\mathcal{F}_{\text{alg}}(\mathcal{B})$

$$a^+(b)(u_1 \otimes \cdots \otimes u_n) = b \otimes u_1 \otimes \cdots \otimes u_n,$$

$$a^-(b)(u_1 \otimes \cdots \otimes u_n) = (\gamma + \phi)[bu_1]u_2 \otimes \cdots \otimes u_n,$$

$$a^-(b)(u_1) = \phi[bu_1]\Omega,$$

$$a^0(b)(u_1 \otimes \cdots \otimes u_n) = \Lambda(b \otimes u_1) \otimes u_2 \otimes \cdots \otimes u_n,$$

$$a^-(b)(\Omega) = a^0(b)(\Omega) = 0,$$

and

$$X(b) = a^+(b) + a^-(b) + a^0(b).$$

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Several of the results below address the questions of when these operators are, in fact, defined on $\mathcal{F}_{\text{alg}}(\mathcal{B})/\mathcal{N}$ or on $\mathcal{F}_{\gamma,\phi}(\mathcal{B})$.

Denote

$$\Gamma^{\text{alg}}_{\gamma,\Lambda}(\mathcal{B},\phi) = \text{Alg}_{\mathbb{C}}(X(b) : b \in \mathcal{B}) = \text{Alg}_{\mathbb{C}}(X(b) : b \in \mathcal{B}^{sa})$$

and define on it the vacuum state $A \mapsto \langle A\Omega, \Omega\rangle$.

Remark 2.3 The following observations are straightforward.

- If for some $t < 1$, $\gamma + t\phi$ is still completely positive, then the inner product (2.2) is positive definite.
- If the operators in Construction 2.2 are bounded with respect to the semi-norm $\|\cdot\|_{\gamma,\phi}$ over $\mathcal{F}_{\text{alg}}(\mathcal{B})$, then they are well-defined over $\mathcal{F}_{\gamma,\phi}(\mathcal{B})$ and are also norm-bounded.
- For $\vec{\zeta}, \vec{\xi} \in \mathcal{F}_{\text{alg}}(\mathcal{B})$,

$$\left\langle a^{+}(b)\vec{\zeta}, \vec{\xi}\right\rangle_{\gamma,\phi} = \left\langle \vec{\zeta}, a^{-}(b^{*})\vec{\xi}\right\rangle_{\gamma,\phi}, \quad \left\langle a^{0}(b)\vec{\zeta}, \vec{\xi}\right\rangle_{\gamma,\phi} = \left\langle \vec{\zeta}, a^{0}(b^{*})\vec{\xi}\right\rangle_{\gamma,\phi}.$$

  In particular, for self-adjoint b, $X(b)$ is symmetric. In the case when $\Lambda(b \otimes u) = \Lambda(b)u$ for $\Lambda : \mathcal{B} \to \mathcal{B}$, assumption (2.1) simplifies to $(\Lambda(b))^{*} = \Lambda(b^{*})$.

3 Moments, free cumulants, Boolean cumulants

In this section, we give combinatorial expressions for the moments and free cumulants of the variables $\{X(u_i)\}$. For any family of bounded operators $X_1, \ldots, X_n$ over a Hilbert space $\mathcal{H}$ and a state ψ over $B(\mathcal{H})$, their *moments* are the numbers

$$\psi[X_{i(1)} \ldots X_{i(k)}].$$

Combinatorial background.

A *partition* π of a subset $S \subset \mathbb{N}$ is a collection of nonempty, disjoint subsets of S (called *blocks* of π) whose union equals S. We will use $i \overset{\pi}{\sim} j$ to say that i and j are in the same block of π. In this article, we will only be concerned with partitions of $[n] := \{1, 2, \ldots, n\}$.

Let $\text{NC}(n)$ denote the set of *noncrossing partitions* over $[n]$, that is, those partitions π such that there are no $i < j < k < \ell$ such that $i \overset{\pi}{\sim} k$ and $j \overset{\pi}{\sim} \ell$ unless all four are in the same block. If $i \in [n]$ is the smallest element of its block, we will call it an *opening element*, while the largest of its block will be called a *closing element*. If i is neither, it will be called a *middle element*.

For distinct blocks V and W of a noncrossing partition π, V is said to be *inner with respect to* W if $o_W < o_V < c_V < c_W$, where o_V and o_W are the opening elements of V and W, respectively, while c_V and c_W are their closing elements. We will simply say a block is *inner* if it is inner with respect to some block, and *outer* if it is not.

Finally, let $\text{Int}(n)$ denote the *interval partitions* over $[n]$, that is, those partitions π such that whenever $i < j$ and $i \overset{\pi}{\sim} j$, we have $i \overset{\pi}{\sim} k$ for all $i < k < j$.

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The *free cumulants* $(R_n)_{n=1}^{\infty}$ of $X_1, \ldots, X_n$ are defined inductively via the *moment-cumulant formula*

$$\psi[X_{i(1)}, \ldots, X_{i(k)}] = \sum_{\pi \in \mathrm{NC}(n)} R_\pi[X_{i(1)}, \ldots, X_{i(k)}],$$

where

$$R_\pi[X_1, \ldots, X_k] = \prod_{V \in \pi} R_{|V|}[X_{V(1)}, \ldots, X_{V(|V|)}].$$

Their *Boolean cumulants* $(B_n)_{n=1}^{\infty}$ are defined in almost exactly the same manner:

$$\psi[X_{i(1)}, \ldots, X_{i(k)}] = \sum_{\pi \in \mathrm{Int}(n)} B_\pi[X_{i(1)}, \ldots, X_{i(k)}],$$

where

$$B_\pi[X_1, \ldots, X_k] = \prod_{V \in \pi} B_{|V|}[X_{V(1)}, \ldots, X_{V(|V|)}].$$

Notation 3.1 *Denote* $\mathrm{NC}_{ns}(n)$ *the set of noncrossing partitions of* $[n]$ *with no singleton blocks. For* $u_1, \ldots, u_n \in \mathcal{B}$*, in the moment and Boolean cumulant formulas below, we will assign to each* $\pi \in \mathrm{NC}_{ns}(n)$ *the weight operator on* $\mathcal{F}_{\mathrm{alg}}(\mathcal{B})$

$$W_M(\pi) = \prod_{i=1}^{n} a_i(u_i),$$

where

$$a_i = \begin{cases} a^+, & \text{if } i \text{ is a closing element,} \\ a^-, & \text{if } i \text{ is an opening element,} \\ a^0, & \text{if } i \text{ is a middle element.} \end{cases}$$

Proposition 3.2 *Given* $u_1, \ldots, u_n \in \mathcal{B}$*, we have the following mixed moment formula:*

$$\langle X(u_1) \ldots X(u_n)\Omega, \Omega \rangle = \sum_{\pi \in \mathrm{NC}_{ns}(n)} \langle W_M(\pi)\Omega, \Omega \rangle_{\gamma,\phi}, \tag{3.1}$$

if $n \geq 2$ *and zero if* $n = 1$*.*

Proof The argument is standard. The proof proceeds by expanding each $X(u_i) = a^+(u_i) + a^0(u_i) + a^-(u_i)$, representing the sum on the left-hand side as the sum over Motzkin paths, and observing that these paths are in bijection with $\mathrm{NC}_{ns}(n)$. ∎

Notation 3.3 *Let* $\widetilde{\mathrm{NC}}(n) = \{\pi \in \mathrm{NC}(n) | 1 \overset{\pi}{\sim} n\}$*, and similarly, let*

$$\widetilde{\mathrm{NC}}_{ns}(n) = \{\pi \in \mathrm{NC}(n) | 1 \overset{\pi}{\sim} n \text{ and } \pi \text{ has no singleton blocks}\}.$$

These are sometimes called the irreducible partitions. For $\pi \in \widetilde{\mathrm{NC}}_{ns}(n)$*, we will denote by* $V_{\mathrm{out}}(\pi)$ *the unique outer block of* π*.*

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Lemma 3.4 *For $n \geq 2$, given $u_1, \ldots, u_n \in \mathcal{B}$, we have the following mixed Boolean cumulant formula:*

$$B_n[X(u_1), \ldots, X(u_n)] = \sum_{\pi \in \widetilde{\mathrm{NC}}_{ns}(n)} \langle W_M(\pi)\Omega, \Omega \rangle_{\gamma,\phi}, \tag{3.2}$$

and for $n = 1$, the cumulant is zero.

Proof $n = 1$ is clear. Assume the result holds for all natural numbers less than some n, and take $u_1, \ldots, u_n \in \mathcal{B}$. By Proposition 3.2, we have

$$\begin{aligned} \sum_{\pi \in \mathrm{Int}(n)} B_\pi[X(u_1), \ldots, X(u_n)] &= \langle X(u_1) \ldots X(u_n)\Omega, \Omega \rangle_{\gamma,\phi} \\ &= \sum_{\pi \in \mathrm{NC}_{ns}(n)} \langle W_M(\pi)\Omega, \Omega \rangle_{\gamma,\phi}. \end{aligned}$$

For convenience, denote by $\mathrm{NC}_{ns,mo}(n)$ the noncrossing, no-singleton partitions of $[n]$ that have more than one outer block. After isolating the nth cumulant (corresponding to the partition $\hat{1}_n$ consisting of a single block), we get

$$\begin{aligned} B_n[X(u_1), \ldots, X(u_n)] &= \sum_{\pi \in \mathrm{NC}_{ns}(n)} \langle W_M(\pi)\Omega, \Omega \rangle_{\gamma,\phi} \\ &\quad - \sum_{\pi \in \mathrm{Int}(n) \setminus \{\hat{1}_n\}} B_\pi[X(u_1), \ldots, X(u_n)] \\ &= \sum_{\pi \in \mathrm{NC}_{ns}(n)} \langle W_M(\pi)\Omega, \Omega \rangle_{\gamma,\phi} \\ &\quad - \sum_{\pi \in \mathrm{Int}(n) \setminus \{\hat{1}_n\}} \prod_{V \in \pi} \sum_{\sigma \in \widetilde{\mathrm{NC}}_{ns}(|V|)} \langle W_M(\sigma)\Omega, \Omega \rangle_{\gamma,\phi} \\ &= \sum_{\pi \in \mathrm{NC}_{ns}(n)} \langle W_M(\pi)\Omega, \Omega \rangle_{\gamma,\phi} \\ &\quad - \sum_{\sigma \in \mathrm{NC}_{ns,mo}(n)} \langle W_M(\pi)\Omega, \Omega \rangle_{\gamma,\phi} \\ &= \sum_{\pi \in \widetilde{\mathrm{NC}}_{ns}(n)} \langle W_M(\pi)\Omega, \Omega \rangle_{\gamma,\phi}, \end{aligned}$$

where the second equality follows from the induction hypothesis. ∎

Notation 3.5 *Define the operator $a^{\sim}_{\gamma}(b)$ ($b \in \mathcal{B}$) by linear extension of*

$$a^{\sim}_{\gamma}(b)(u_1 \otimes \cdots \otimes u_n) = \gamma[bu_1]u_2 \otimes \cdots \otimes u_n \text{ for } n \geq 2,$$

$$a^{\sim}_{\gamma}(b)(u_1) = 0 \text{ for } n = 1, \text{ and}$$

$$a^{\sim}_{\gamma}(b)(\Omega) = 0.$$

The operator $a^{\sim}_{\phi}(b)$ is defined in a similar manner (for $n \geq 2$, apply ϕ, and $a^{\sim}_{\phi}(b)(\Omega) = 0$), with one critical exception:

$$a^{\sim}_{\phi}(b)(u_1) = \phi[bu_1]\Omega \text{ for } n = 1.$$

Thus $a^{-}(b) = a^{\sim}_{\gamma}(b) + a^{\sim}_{\phi}(b)$.

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For $u_1, \ldots, u_n \in \mathcal{B}$, in the free cumulant formula below, we will assign to each $\pi \in \mathrm{NC}_{ns}(n)$ the weight operator on $\mathcal{F}_{\mathrm{alg}}(\mathcal{B})$

$$W_C(\pi) = \prod_{i=1}^{n} a_i(u_i),$$

where

$$a_i = \begin{cases} a^+, & \text{if } i \text{ is a closing element,} \\ a_\gamma^\sim, & \text{if } i \neq 1 \text{ and is an opening element,} \\ a_\phi^\sim, & \text{if } i = 1, \text{ or} \\ a^0, & \text{if } i \text{ is a middle element.} \end{cases}$$

Proposition 3.6 *For $n \geq 2$, given $u_1, \ldots, u_n \in \mathcal{B}$, we have the following mixed free cumulant formula:*

$$R_n[X(u_1), \ldots, X(u_n)] = \sum_{\pi \in \widetilde{\mathrm{NC}}_{ns}(n)} \langle W_C(\pi)\Omega, \Omega \rangle_{\gamma,\phi}, \tag{3.3}$$

and for $n = 1$, the cumulant is zero.

Proof By Theorem 1 in [BN08] and Lemma 3.4,

$$\begin{aligned} \sum_{\pi \in \widetilde{\mathrm{NC}}_{ns}(n)} R_\pi[X(u_1), \ldots, X(u_n)] &= B_n[X(u_1), \ldots, X(u_n)] \\ &= \sum_{\pi \in \widetilde{\mathrm{NC}}_{ns}(n)} \sum_{\ell \in \text{Block labelings}(\gamma,\phi)} \langle W(\pi, \ell)\Omega, \Omega \rangle_{\gamma,\phi}, \end{aligned} \tag{3.4}$$

where ℓ is a labeling of the blocks of π with either a γ or ϕ such that $V_{\mathrm{out}}(\pi)$ is labeled ϕ, and $W(\pi, \ell) = \prod_{i=1}^{n} a_i(u_i)$, where

$$a_i = \begin{cases} a^+, & \text{if } i \text{ is a closing element,} \\ a_\gamma^\sim, & \text{if } i \text{ is an opening element of a block labeled } \gamma, \\ a_\phi^\sim, & \text{if } i \text{ is an opening element of a block labeled } \phi, \\ a^0, & \text{if } i \text{ is a middle element.} \end{cases}$$

This splitting of terms follows from linearity and the fact that the opening of each block is weighted with $(\gamma + \phi)$ of a product involving the operator in that position. We may rewrite the latter sum as

$$\sum_{\pi \in \widetilde{\mathrm{NC}}_{ns}(n)} \sum_{\substack{S \subset \pi \\ V_{\mathrm{out}}(\pi) \in S}} \langle W(\pi, \ell_S)\Omega, \Omega \rangle_{\gamma,\phi}, \tag{3.5}$$

where ℓ_S is the labeling constructed by giving blocks in S the label ϕ and the rest γ.

Let $\ll$ denote the partial order on $\mathrm{NC}(n)$ defined by

$$\pi \ll \sigma \quad \Leftrightarrow \quad \pi \leq \sigma \text{ and } \forall\, V \in \sigma,\ \min(V) \overset{\pi}{\sim} \max(V).$$

In particular, for each $\sigma \gg \pi$, we have the decomposition $\pi = \bigcup_{V \in \sigma} \pi_V$, with each $\pi_V \in \widetilde{\mathrm{NC}}(V)$.

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By Remark 2.14 in [BN08], we have a bijection

$$\{\sigma \in \mathrm{NC}_{ns}(n) : \pi \ll \sigma\} \simeq \{S \subset \pi : V_{\mathrm{out}}(\pi) \in S\}$$

such that the opening and closing elements of the outer block of σ are precisely those of the corresponding element in S. This implies that

$$\langle W(\pi, \ell_S)\Omega, \Omega\rangle_{\gamma,\phi} = \prod_{V\in\sigma} \langle W_C(\pi_V)\Omega, \Omega\rangle_{\gamma,\phi}.$$

With this in mind, the sum (3.5) can be rewritten as

$$\sum_{\pi\in\widehat{\mathrm{NC}}_{ns}(n)} \sum_{\pi\ll\sigma} \prod_{V\in\sigma} \langle W_C(\pi_V)\Omega, \Omega\rangle_{\gamma,\phi} = \sum_{\sigma\in\widehat{\mathrm{NC}}_{ns}(n)} \prod_{V\in\sigma} \sum_{\pi_V\in\widehat{\mathrm{NC}}_{ns}(V)} \langle W_C(\pi_V)\Omega, \Omega\rangle_{\gamma,\phi}.$$

The result now follows by Möbius inversion. ∎

Definition 3.7 For each $u_1, \dots, u_n \in \mathcal{B}$, define the linear operator $R_n'[u_1, \dots, u_n]$ on $\mathcal{B}$ by

$$\begin{aligned} R_n[X(u_1), \dots, X(u_n)] &= \langle R_{n-2}'[u_2, \dots, u_{n-1}]u_n, u_1^*\rangle \\ &= \phi[u_1 R_{n-2}'[u_2, \dots, u_{n-1}]u_n]. \end{aligned} \tag{3.6}$$

In general, this operator need not be bounded.

Lemma 3.8 *We have the expansion*

$$R_n'[u_1, \dots, u_n] = \left(\sum_{\sigma\in\mathrm{Int}(\{1,\dots,n\})} \prod_{V\in\pi} w(V) \right), \tag{3.7}$$

where the products are ordered by each block's appearance in the partition (from left to right), and the weights are given by $w(\{i\}) = a^0(u_i)$ *and* $w(\{i_1, \dots, i_n\}) = \gamma[u_{i_1} R_{n-2}'[u_{i_2}, \dots, u_{i_{n-1}}]u_{i_n}]$ *for* $n \geq 2$, *with* $R_0'[\varnothing] = 1$.

In particular, for $\Lambda(u \otimes v) = \Lambda(u)v$, $R_n'[u_1, \dots, u_n]$ *is the operator of multiplication by an element of* $\mathcal{B}$.

Proof To show that

$$\left\langle \left(\sum_{\sigma\in\mathrm{Int}(\{2,\dots,n-1\})} \prod_{V\in\sigma} w(V) \right) u_n, u_1^* \right\rangle = \sum_{\pi\in\widehat{\mathrm{NC}}_{ns}(n)} \langle W_C(\pi)\Omega, \Omega\rangle_{\gamma,\phi},$$

we will prove a slightly stronger statement:

$$R'' := u_1 \left(\sum_{\sigma\in\mathrm{Int}(\{2,\dots,n-1\})} \prod_{V\in\sigma} w(V) \right) u_n = \sum_{\pi\in\widehat{\mathrm{NC}}_{ns}(n)} W'(\pi)\Omega, \tag{3.8}$$

where the weights $W'(\pi)$ on the right-hand side are the same as in the cumulant formula, except all opening steps at 1 will be weighted by the identity map instead of $a_{\phi}^{\sim}$.

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For $n = 2$, we have u_1u_2 on both sides. For $n = 3$, we have $u_1a^0(u_2)u_3$. For induction, assume the lemma holds for all natural numbers less than some n. Then

$$\sum_{\pi\in\widetilde{\mathrm{NC}}_{ns}(n)} W'(\pi)\Omega = u_1\left(\sum_{\sigma\in\mathrm{Int}(\{2,\ldots,n-1\})}\prod_{V\in\sigma} w(V)\right)u_n,$$

where for $k \geq 2$, $w(\{i_1,\ldots,i_k\}) = \gamma\left[u_{i_1}\left(\sum_{\tau\in\mathrm{Int}(\{2,\ldots,k-1\})}\prod_{V\in\tau} w(V)\right)u_{i_k}\right]$.

At this point, we are done, since each $\pi \in \widetilde{\mathrm{NC}}_{ns}(n)$ can be uniquely constructed by taking some $\sigma \in \mathrm{Int}\{2,\ldots,n-1\}$, then constructing the unique outer block $\{1, \mathrm{singletons}(\sigma), n\}$, then each nested block immediately below the outer block is recursively constructed in the same manner.

Finally, taking the inner product of both sides of (3.8) with Ω gives the result. ∎

Next, we prove an equation which characterizes the generating function for this family of operators, which in turn will give us an equation for the free cumulant generating function.

Lemma 3.9 *Denote $R'_n[u] := R'[u,\ldots,u]$ (n arguments), where $R'_0[u] = 1$. Then*

$$R'_n[u] = \sum_{i=0}^{n-2} R'_i[u]\gamma[uR'_{n-i-2}[u]u] + R'_{n-1}[u]a^0(u). \tag{3.9}$$

Proof The claim is analogous to the recursion for the number I_n of interval partitions of length n,

$$I_n = \sum_{i=0}^{n-1} I_i,$$

where $n - i$ is the number of elements in the block containing n. The right-hand side in (3.9) is obtained in a similar manner, in which each term is obtained by collecting all terms in the sum (3.7) (over interval partitions) for $R'_n[u]$ for which the weight for the block containing n is a factor. Thus, by summing over i where $n - i$ is the number of elements in the block containing n, the term corresponding to each i is a product of $R'_i[u]$ and either $\gamma[uR'_{n-i-2}[u]u]$, the weight of the block containing n for $i \leq n-2$, or $a^0(u)$, the weight of the singleton block containing n for $i = n-1$. ∎

Theorem 3.10 *Let $R'(u) = 1 + \sum_{n=1}^{\infty} R'_n[u]$ be the generating function of $R'_n[u]$, $n \geq 0$. Then for any $v \in \mathcal{B}$,*

$$R'(u)v = v + R'(u)\gamma\left[uR'(u)u\right]v + R'(u)\Lambda(u\otimes v). \tag{3.10}$$

Proof Apply Lemma 3.9. ∎

Example 3.11 For $\Lambda(u\otimes v) = \Lambda(u)v$, $R'(u)$ satisfies

$$R'(u) = \left(1 - \Lambda(u) - \gamma[uR'(u)u]\right)^{-1},$$

and so can be expanded into a formal continued fraction.

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Corollary 3.12 *More generally, for $u_1, \ldots, u_k \in \mathcal{B}$, let*

$$R'(u_1, \ldots, u_k) = \sum_{n=0}^{\infty} \sum_{j:[n]\to[k]} R'_n[u_{j(1)}, \ldots, u_{j(n)}]$$

be the generating function for the mixed R'_n. Then

$$\begin{aligned} R'(u_1, \ldots, u_k)v = v + R'(u_1, \ldots, u_k)\gamma\left[\left(\sum_{i=1}^{k} u_i\right) R'(u_1, \ldots, u_k) \left(\sum_{j=1}^{k} u_j\right)\right] v \\ + R'(u_1, \ldots, u_k) \sum_{i=1}^{k} \Lambda(u_i, v). \end{aligned} \tag{3.11}$$

Proof Apply Theorem 3.10 to $R'\left(\sum_{i=1}^{k} u_i\right) = R'(u_1, \ldots, u_k)$. ∎

Remark 3.13 The generating function for $R''_n[u]$ is given by $R''(u) = uR'(u)u$, since $R''_n[u]$ is only defined for $n \geq 2$. Thus, $R''(u)$ satisfies

$$R''(u) = u^2 + uR'(u)\gamma\left[R''(u)\right]u + uR'(u)\Lambda(u, u). \tag{3.12}$$

Applying the vacuum state to both sides gives an equation for the cumulant generating function. Note that $R''(u)$ is a series of elements of $\mathcal{B}$.

Remark 3.14 Constructions and results in this section are reminiscent of operator-valued probability theory, such as those in [AW18]. In this remark, we indicate how these constructions differ. The map

$$b_1 \otimes b_1 \otimes \cdots \otimes b_n \mapsto b_1 X b_2 X \ldots b_n X, \quad \Omega \mapsto 1_{\mathcal{B}}$$

is an isomorphism from the algebraic Fock space $\bigoplus_{n=0}^{\infty} \mathcal{B}^{\otimes n}$ onto a subspace of noncommutative polynomials $\mathcal{B}\langle X\rangle$. Using this identification, the relation between the operators in this article (for $\Lambda(f \otimes g) = \Lambda(f)g$) and in Proposition 3.1 from [AW18] is:

$$a^+(b) \leftrightarrow ba^*, \quad a^-(b) \leftrightarrow a\, b, \quad a^0(b) \leftrightarrow p$$

with $\alpha_1 = \phi$, $\alpha_n = \gamma + \phi$ for $n \geq 2$, and $\lambda_n = \Lambda(b)$ for $n \geq 1$. Thus the operator $X(1)$ here is the same as the $\mathcal{B}$-valued X, but the interaction with the algebra $\mathcal{B}$ is different in the two settings. Similarly, the identity

$$R'(b) = 1 + R'(b)\gamma\left[bR'(b)b\right] + R'(b)\Lambda(b)$$

satisfied by the (under appropriate assumptions) $\mathcal{B}$-valued generating function $R'(b)$ (from Theorem 3.10) is similar to, but different from the relation

$$b^{-1}R_\mu(b)b^{-1} = 1 + \gamma[R_\mu(b)b^{-1}]R_\mu(b)b^{-1} + \lambda R_\mu(b)b^{-1}$$

from Proposition 3.22 in [AW18]. They do again coincide (up to a flip) for $b = 1$.

In light of the preceding remark, the following is related to Theorem 2 from [Ans07].

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Proposition 3.15 *Let $\mathcal{B}$ be a C^*-algebra and $u_1, \ldots, u_n \in \mathcal{B}^{sa}$. Assume that $\Lambda(u \otimes v) = \Lambda(u)v$. Let $X_1, \ldots, X_n$ be a $\mathcal{B}$-valued semicircular system with means $\Lambda(u_1), \ldots, \Lambda(u_n)$ and the covariance matrix $\eta : \mathcal{B} \to M_n(\mathcal{B})$ with $\eta_{ij}[b] = \gamma[u_i b u_j]$; the existence of such a system in some $\mathcal{B}$-valued noncommutative probability space $(\mathcal{A}, \mathbb{E}, \mathcal{B})$ is guaranteed by* [Shl99]. *Then*

$$R_k[X(u_{j(1)}), X(u_{j(2)}), \ldots, X(u_{j(k-1)}), X(u_{j(k)})] = \phi[u_{j(1)}\mathbb{E}[X_{j(2)} \ldots X_{j(k-1)}]u_{j(k)}].$$

Proof It suffices to show that

$$R'_n[u_1, \ldots, u_n] = \mathbb{E}[X_1 \ldots X_n].$$

Indeed, $R'_0 = 1$ and $R'_1[u] = \Lambda(u)$. The left-hand side satisfies the recursion

$$R'_n[u_1, \ldots, u_n] = \sum_{i=0}^{n-2} R'_i[u_1, \ldots, u_i]\gamma[u_{i+1}R'_{n-i-2}[u_{i+2}, \ldots, u_{n-1}]u_n] + R'_{n-1}[u_1, \ldots, u_{n-1}]\Lambda(u_n)$$

by Corollary 3.12, while the right-hand side satisfies the well-known recursion

$$\begin{aligned}\mathbb{E}[X_1, \ldots, X_n] &= \sum_{i=0}^{n-2} \mathbb{E}[X_1, \ldots, X_i]\mathbb{E}[X_{i+1}\mathbb{E}[X_{i+2}, \ldots, X_{n-1}]X_n] \\ &\quad + \mathbb{E}[X_1, \ldots, X_{n-1}]\mathbb{E}[X_n] \\ &= \sum_{i=0}^{n-2} \mathbb{E}[X_1, \ldots, X_i]\eta_{i+1,n}[\mathbb{E}[X_{i+2}, \ldots, X_{n-1}]] \\ &\quad + \mathbb{E}[X_1, \ldots, X_{n-1}]\Lambda(u_n),\end{aligned}$$

see, for example, Proposition 3.13 in [AW18]. ∎

4 Generating functions

4.1 Wick polynomials

Definition 4.1 For $u_1 \otimes \cdots \otimes u_n \in \mathcal{B}^{\otimes n}$, define the operator $W(u_1 \otimes \cdots \otimes u_n)$ on the algebraic Fock space by the recursion

$$\begin{aligned}&W(b \otimes u_1 \otimes \cdots \otimes u_n) \\ &\quad = X(b)W(u_1 \otimes \cdots \otimes u_n) - W(a^0(b)(u_1 \otimes \cdots \otimes u_n)) \\ &\qquad - W(a^-(b)(u_1 \otimes \cdots \otimes u_n)) \\ &\quad = X(b)W(u_1 \otimes \cdots \otimes u_n) - W(\Lambda(b \otimes u_1) \otimes u_2 \otimes \cdots \otimes u_n) \\ &\qquad - W((\gamma + \phi)[bu_1]u_2 \otimes \cdots \otimes u_n)\end{aligned}$$

with the initial conditions

$$W(\varnothing) = I, \quad W(u_1) = X(u_1),$$
$$W(u_1 \otimes u_2) = X(u_1)W(u_2) - W(\Lambda(u_1 \otimes u_2)) - \phi[u_1 u_2].$$

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It is easy to check using the recursion that

$$W(u_1 \otimes \cdots \otimes u_n)\Omega = u_1 \otimes \cdots \otimes u_n.$$

Since each $W(\vec{\xi}) \in \Gamma^{\mathrm{alg}}_{\gamma,\Lambda}(\mathcal{B}, \phi)$, it follows that Ω is cyclic for this algebra. $W(\vec{\xi})$ is called a *Wick polynomial*. In particular, we denote $W_0(u) = I$, $W_n(u) = W(u^{\otimes n})$, and note that $W_n(u)\Omega = u^{\otimes n}$.

For the remainder of this section, we assume that Λ has the special form $\Lambda(u \otimes v) = \Lambda(u)v$.

The next proposition states that formally, the generating function for univariate Wick polynomials is a rational function in $X(u)$. In Section 5, we will give conditions under which the series defining the generating function is convergent and corresponds to a well-defined operator.

Proposition 4.2 *Denote* $W(u) = I + \sum_{n=1}^{\infty} W_n(u)$ *and*

$$b(u) = I + \Lambda(u) + (\gamma + \phi)[u^2].$$

Then

$$(b(u) - X(u))W(u) = b(u) - \phi[u^2].$$

Proof

$$X(u)W_n(u) = W_{n+1}(u) + (\gamma + \phi)[u^2]W_{n-1}(u) + \Lambda(u)W_n(u),$$

$$X(u)W_1(u) = W_2(u) + \phi[u^2] + \Lambda(u)W_1(u),$$

$$X(u) = W_1(u).$$

So

$$X(u)W(u) = W(u) - I + \phi[u^2] + (\gamma + \phi)[u^2](W(u) - 1) + \Lambda(u)(W(u) - I).$$

■

4.2 Matricial generating functions

In this section, we will explore a means of recovering the multivariable polynomials $\{W(u_n, \ldots, u_m) : 1 \le n \le m < \infty\}$ for a sequence $\{u_i : i \in \mathbb{N}\}$ in $\mathcal{B}$. Rather than trying to form their generating function as a power series, we will organize these polynomials into an infinite matrix. In the next section, this matrix will be identified with a bounded operator. In the following proposition, $\mathcal{L}(\ell^2)$ denotes the linear space of infinite matrices, and $\{E_{ij}\}$ are the standard matrix units in it.

Proposition 4.3 *Let* $\{u_i : i \in \mathbb{N}\} \subset \mathcal{B}$. *Define matrices with operator entries* $\Phi \in \mathcal{L}(\ell^2)$, $\Gamma, A^0 \in \mathcal{B} \otimes \mathcal{L}(\ell^2)$, $X, W \in \mathcal{L}(\mathcal{F}_{\gamma,\phi}(\mathcal{B})) \otimes \mathcal{L}(\ell^2)$ *as follows:*

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$$X = \sum_{i=1}^{\infty} X(u_i) \otimes E_{i,i+1},$$

$$W_n = \sum_{i=1}^{\infty} W(u_i \otimes \cdots \otimes u_{i+n-1}) E_{i,i+n}, \quad n \geq 0,$$

$$W = \sum_{n=0}^{\infty} W_n$$

$$= \sum_{i=1}^{\infty} \left(I \otimes E_{i,i} + \sum_{n=1}^{\infty} W(u_i \otimes \cdots \otimes u_{i+n-1}) \otimes E_{i,i+n} \right),$$

$$\Phi = \sum_{i=1}^{\infty} \phi[u_i u_{i+1}] \otimes E_{i,i+2},$$

$$\Gamma = \sum_{i=1}^{\infty} \gamma[u_i u_{i+1}] \otimes E_{i,i+2},$$

$$A^0 = \sum_{i=1}^{\infty} \Lambda(u_i) \otimes E_{i,i+1}.$$

Denote $B = I + A^0 + \Gamma + \Phi \in \mathcal{L}(\mathcal{F}_{\gamma,\phi}(\mathcal{B})) \otimes \mathcal{L}(\ell^2)$. *Then*

$$(B - X)W = (B - \Phi).$$

Proof For $k > i + 2$,

$$X(u_i)W(u_{i+1} \otimes \cdots \otimes u_{k-1}) = W(u_i \otimes \cdots \otimes u_{k-1}) + \Lambda(u_i)W(u_{i+1} \otimes \cdots \otimes u_{k-1}) + (\gamma[u_i u_{i+1}] + \phi[u_i u_{i+1}])W(u_{i+2} \otimes \cdots \otimes u_{k-1}).$$

For $k = i + 2$,

$$X(u_i)W(u_{i+1}) = W(u_i \otimes u_{i+1}) + \Lambda(u_i)W(u_{i+1}) + \phi[u_i u_{i+1}].$$

For $k = i + 1$,

$$X(u_i)I = W(u_i).$$

By comparing matrix entries, we can see that this implies for $n \geq 2$,

$$XW_n = W_{n+1} + A^0 W_n + (\Gamma + \Phi)W_{n-1},$$

and

$$XW = (W - I) + A^0(W - I) + \Phi + (\Gamma + \Phi)(W - I). \tag{4.1}$$

■

Since $B - X$ is an upper-triangular matrix with only 1s along the main diagonal, its inverse exists for a finite family of $\{u_i\}$, and in a strictly formal sense for an infinite family. In the next section, we will discuss conditions under which $(B - X)^{-1}$, and W, are bounded operators on $\mathcal{F}_{\gamma,\phi}(\mathcal{B}) \otimes \ell^2$.

Since W contains all the information about the multivariate W's, it should be considered as a generating function $W(u_i : i \in \mathbb{N})$. By constructing certain corresponding operators on a matricial Fock space construction, we can view W as a more traditional generating function of these operators.

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Definition 4.4 Let $\mathcal{D}$ be a $*$-algebra. $\mathcal{B} \otimes \mathcal{D}$ is naturally a $\mathcal{D}$-bimodule. On the Fock space

$$\bigoplus_{n=0}^{\infty} \mathcal{B}^{\otimes n} \otimes \mathcal{D} \simeq \bigoplus_{n=0}^{\infty} (\mathcal{B} \otimes \mathcal{D})^{\otimes_{\mathcal{D}} n},$$

we consider operators indexed by elements of $\mathcal{B} \otimes \mathcal{D}$

$$a^+(b \otimes d)(u_1 \otimes \cdots \otimes u_n \otimes d') = b \otimes u_1 \otimes \cdots \otimes u_n \otimes dd',$$

$$a^-(b \otimes d)(u_1 \otimes \cdots \otimes u_n \otimes d') = (\gamma + \phi)[bu_1]u_2 \otimes \cdots \otimes u_n \otimes dd',$$

$$a^-(b \otimes d)(u_1 \otimes d') = \phi[bu_1]dd',$$

$$a^0(b \otimes d)(u_1 \otimes \cdots \otimes u_n \otimes d') = \Lambda(b)u_1 \otimes u_2 \otimes \cdots \otimes u_n \otimes dd',$$

$$a^-(b \otimes d)(d') = a^0(b \otimes d)(d') = 0.$$

Then any operator in the algebra generated by

$$\left\{a^+(b \otimes d), a^-(b \otimes d), a^0(b \otimes d) : b \in \mathcal{B}, d \in \mathcal{D}\right\}$$

is of the form $A \otimes d$, where A acts purely on $\bigoplus_{n=0}^{\infty} \mathcal{B}^{\otimes n}$. Moreover, the map

$$\Psi : (A \otimes d) \mapsto \langle (A \otimes d)(1_{\mathcal{B}} \otimes 1_{\mathcal{D}}), 1_{\mathcal{B}} \otimes 1_{\mathcal{D}} \rangle = \langle A1_{\mathcal{B}}, 1_{\mathcal{B}} \rangle d \tag{4.2}$$

is a $\mathcal{D}$-valued conditional expectation on the algebra generated by

$$\{X(b \otimes d) : b \in \mathcal{B}, d \in \mathcal{D}\}.$$

Remark 4.5 Let $\mathcal{D}$ be the algebra of infinite matrices with finitely many nonzero diagonals. Let $\{u_i : i \in \mathbb{N}\} \subset \mathcal{B}$, and define the matrix $U \in \mathcal{B} \otimes \mathcal{D}$ by $U = \sum_{i \geq 1} u_i \otimes E_{i,i+1}$. Then the objects from Proposition 4.3 are, in fact,

$$\Phi = (\phi \otimes I)[U^2], \quad \Gamma = (\gamma \otimes I)[U^2], \quad A^0 = (\Lambda \otimes I)[U],$$

and $X = a^+(U) + a^-(U) + a^0(U)$. If, in addition,

$$\sup_i \|u_i\| < \infty, \tag{4.3}$$

then $U \in \mathcal{B} \otimes \mathbb{B}(\ell^2)$, where $\mathbb{B}(\ell^2)$ denotes the algebra of bounded operators.

Before interpreting $W(U)$ as a generating function, we will first establish the analogous results for moments and cumulants. The formulas below are similar to, but once again different from, those in Section 6.3 in [PV13]. The first of these is an easy consequence of matrix multiplication.

Lemma 4.6 *For Ψ defined in (4.2) for the case of $\mathcal{D}$ as in the preceding remark, and $d^{(1)}, \ldots, d^{(n-1)} \in \mathcal{D}$,*

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(4.4)

$$\Psi[Xd^{(1)}X\dots d^{(n-1)}X] = \sum_{\substack{\text{increasing}\\ \mathbf{i}:[n]\to\mathbb{N}}} \left\langle \prod_{j=1}^{n} X(u_{\mathbf{i}(j)})\Omega,\Omega\right\rangle \prod_{j=1}^{n-1} d^{(j)}_{\mathbf{i}(j)+1,\mathbf{i}(j+1)}\, E_{\mathbf{i}(1),\mathbf{i}(n)+1}.$$

In particular,

$$\Psi[X^n] = \sum_{i=1}^{\infty} \left\langle \prod_{j=1}^{n} X(u_j)\Omega,\Omega\right\rangle E_{i,i+n-1}.$$

Given a noncommutative operator-valued probability space

$$(\mathrm{Alg}(X(b\otimes d) : b\in\mathcal{B}, d\in\mathcal{D}), \mathcal{D}, \Psi),$$

we may define $\mathcal{D}$-valued partitioned moments $\Psi_\pi[d_0X, d_1X, \dots, d_{n-1}Xd_n]$ and $\mathcal{D}$-valued free cumulants $\mathcal{R}[d_0X, d_1X, \dots, d_{n-1}Xd_n]$ as in Chapter 4 of [Spe98]. However, we will only be interested in these for $d_0 = d_1 = \cdots = d_n = 1$. In this case, we have the following relations.

Corollary 4.7 *For a partition $\pi\in \mathrm{NC}(n)$,*

$$\Psi_\pi[X] = \sum_{i=1}^{\infty} \prod_{V\in\pi} \left\langle \prod_{j\in V} X(u_{i+j-1})\Omega,\Omega\right\rangle E_{i,i+n}.$$

It follows that

$$\mathcal{R}_n[X,\dots,X] = \sum_{i=1}^{\infty} R_n[X(u_i),\dots,X(u_{i+n-1})]E_{i,i+n}. \tag{4.5}$$

Proof We may decompose $\pi\in\mathrm{NC}(n)$ into the block $(\mathbf{v}(1) < \mathbf{v}(2) < \cdots < \mathbf{v}(k) = n)$ containing n and non-crossing partitions $\pi_j \in \mathrm{NC}(\{\mathbf{v}(j)+1,\dots,\mathbf{v}(j+1)-1\})$ for $0\le j\le k-1$, where $\mathbf{v}(0) = 0$. Applying Lemma 4.6 with $d^{(j)} = \Psi_{\pi_j}[X]$, and using recursion,

$$\begin{aligned}
\Psi_\pi[X] &= \sum_{\substack{\text{increasing}\\ \mathbf{i}:[k]\to\mathbb{N}}} \left\langle \prod_{j=1}^{k} X(u_{\mathbf{i}(j)})\Omega,\Omega\right\rangle \Psi_{\pi_0}[X] \prod_{j=1}^{k-1} (\Psi_{\pi_j}[X])_{\mathbf{i}(j)+1,\mathbf{i}(j+1)}\, E_{\mathbf{i}(1),\mathbf{i}(k)+1}\\
&= \sum_{\substack{\text{increasing}\\ \mathbf{i}:[k]\to\mathbb{N}}} \left\langle \prod_{j=1}^{k} X(u_{\mathbf{i}(j)})\Omega,\Omega\right\rangle\\
&\quad\times \sum_{\substack{\text{increasing}\\ \mathbf{l}:[0,k-1]\to\mathbb{N}}} \prod_{V\in\pi_0} \left\langle \prod_{s\in V} X(u_{\mathbf{l}(0)+s-1})\Omega,\Omega\right\rangle E_{\mathbf{l}(0),\mathbf{l}(0)+\mathbf{v}(1)-1}\\
&\quad\times \prod_{j=1}^{k-1} \left(\prod_{V\in\pi_j} \left\langle \prod_{s\in V} X(u_{\mathbf{l}(j)+s-1})\Omega,\Omega\right\rangle E_{\mathbf{l}(j),\mathbf{l}(j)+\mathbf{v}(j+1)-\mathbf{v}(j)-1}\right)_{\mathbf{i}(j)+1,\mathbf{i}(j+1)}\\
&\quad\times E_{\mathbf{i}(1),\mathbf{i}(k)+1}.
\end{aligned}$$

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For nonzero terms, $\mathbf{l}(j) = \mathbf{i}(j) + 1$ and $\mathbf{i}(j) = \mathbf{i}(j-1) + \mathbf{v}(j) - \mathbf{v}(j-1) = i + \mathbf{v}(j) - 1$, where we set $i = \mathbf{l}(0)$. Thus

$$\begin{aligned}\Psi_\pi[X] &= \sum_{i=1}^{\infty} \left\langle \prod_{j=1}^{k} X(u_{i+\mathbf{v}(j)-1})\Omega, \Omega \right\rangle \prod_{V\in\pi_0} \left\langle \prod_{s\in V} X(u_{i+s-1})\Omega, \Omega \right\rangle \\ &\quad \times \prod_{j=1}^{k-1} \prod_{V\in\pi_j} \left\langle \prod_{s\in V} X(u_{i+\mathbf{v}(j)+s-1})\Omega, \Omega \right\rangle E_{i,i+n} \\ &= \sum_{i=1}^{\infty} \prod_{V\in\pi} \left\langle \prod_{j\in V} X(u_{i+j-1})\Omega, \Omega \right\rangle E_{i,i+n}.\end{aligned}$$

The second part follows by applying Möbius inversion, both to the $\mathcal{D}$-valued random variables, and to the scalar-valued random variables in the $(i, i+n)$th entry. ∎

Finally, the next proposition follows by comparing matricial recursions.

Proposition 4.8 *Let U be as in Remark 4.5. Define the operators $W_n(U)$ on the Fock space $\mathcal{F}_{\gamma,\phi}(\mathcal{B}) \otimes \mathcal{L}(\ell^2)$ by the recursion*

$$\begin{aligned}X(U)W_n(U) &= W_{n+1}(U) + \delta_{n\geq 1}(\Lambda \otimes I)(U)W_n(U) \\ &\quad + \left(\delta_{n\geq 2}(\gamma \otimes I)[U^2] + (\phi \otimes I)[U^2]\right)W_{n-1}(U)\end{aligned}$$

with the initial condition $W_0(U) = I$, and their generating function $W(U) = \sum_{n=0}^{\infty} W_n(U)$. Then in the setting of Proposition 4.3, $W_n = W_n(U)$ and $W = W(U)$ entrywise. Note that if we only have finitely many nonzero entries u_i, boundedness condition (4.3) holds automatically.

5 Norm estimates and convergence of generating functions

The next two results can be used to estimate the norm of $X(f)$. The following lemma is closely related to Lemma 4 in [BS91] and Lemma 1 in [Ans04], although it is not stated in quite this form in either of those sources.

Lemma 5.1 *Let $\mathcal{H}$ be a Hilbert space, and let K a positive operator on it. Denote $\left\langle \vec{\zeta}, \vec{\xi} \right\rangle_K = \left\langle \vec{\zeta}, K\vec{\xi} \right\rangle$ the corresponding deformed inner product. Then for an operator X on $\mathcal{H}$, denoting by X^* its adjoint with respect to the deformed inner product, $\|X\|_K \leq \sqrt{\|X\| \, \|X^*\|}$.*

For the remainder of this section, we assume that $\mathcal{B}$ is a C^*-algebra, the maps $\phi : \mathcal{B} \to \mathbb{C}$ and $\gamma : \mathcal{B} \to \mathcal{B}$ are bounded, and Λ has the special form $\Lambda(u \otimes v) = \Lambda(u)v$ with $\Lambda : \mathcal{B} \to \mathcal{B}$ bounded.

Proposition 5.2 *For $b \in \mathcal{B}$,*

$$\begin{aligned}\|a^+(b)\|_{\gamma,\phi} = \|a^-(b^*)\|_{\gamma,\phi} &\leq \sqrt{\max(\phi[b^*b], \|(\gamma+\phi)[b^*b]\|)} \\ &\leq \sqrt{\max(\|\phi\|, \|\gamma+\phi\|)} \, \|b\|\end{aligned}$$

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and

$$\left\|a^0(b)\right\|_{\gamma,\phi} \leq \|\Lambda(b)\| \leq \|\Lambda\| \ \|b\|.$$

It follows that in this case, the operators in Construction 2.2 are well-defined and bounded on $\mathcal{F}_{\gamma,\phi}(\mathcal{B})$.

Proof Since tensors of different length in the Fock space are orthogonal, it suffices to estimate $\|a^+(b)\|_{\gamma,\phi}$ separately on tensors of fixed length. For $n = 0$,

$$\frac{\langle a^+(b)\Omega, a^+(b)\Omega\rangle_{\gamma,\phi}}{\langle\Omega,\Omega\rangle_{\gamma,\phi}} = \langle b, b\rangle_{\gamma,\phi} = \phi[b^*b] \leq \|\phi\|\|b\|_{\mathcal{B}}^2.$$

For $n \geq 1$, denote

$$m(b)(u_1 \otimes \cdots \otimes u_n) = (bu_1) \otimes \cdots \otimes u_n,$$

and note that $a^-(b^*)a^+(b) = m((\gamma+\phi)[b^*b])$.

For a fixed tensor $\vec{\xi} \perp \Omega$, denote $\psi_{\vec{\xi}}[u] = \left\langle m(u)\vec{\xi}, \vec{\xi}\right\rangle_{\gamma,\phi}$. Then $\psi_{\vec{\xi}}$ is linear, and $\psi_{\vec{\xi}}[u^*u] = \left\langle m(u)\vec{\xi}, m(u)\vec{\xi}\right\rangle_{\gamma,\phi} \geq 0$. Therefore

$$\begin{aligned}\left\langle a^+(b)\vec{\xi}, a^+(b)\vec{\xi}\right\rangle_{\gamma,\phi} &= \psi_{\vec{\xi}}[(\gamma+\phi)[b^*b]] \\ &\leq \|(\gamma+\phi)[b^*b]\|\,\psi_{\vec{\xi}}[1] = \|(\gamma+\phi)[b^*b]\|\left\|\vec{\xi}\right\|_{\gamma,\phi}^2.\end{aligned}$$

Hence, $\|a^+(b)\|_{\gamma,\phi} \leq \sqrt{\max(\phi[b^*b], \|(\gamma+\phi)[b^*b]\|)}$.

Since $a^-(b^*)$ is the adjoint of $a^+(b)$, their operator norms are equal, so the above inequality also applies to a^-.

The argument for $a^0(b)$, with Λ of the special form, is similar. ∎

Proposition 5.3 *Denote* $c_{\gamma,\phi} = \sqrt{\max(\|\phi\|, \|\gamma+\phi\|)}$ *and*

$$K = \max\left(1, (\sqrt{2}-1)\left(2 + \frac{\|\Lambda\|}{c_{\gamma,\phi}}\right)\right).$$

Then for $(u_n)_{n=1}^{\infty} \subset \mathcal{B}$,

$$\|W(u_1 \otimes \cdots \otimes u_n)\| \leq (\sqrt{2}+1)^n c_{\gamma,\phi}^n K \|u_1\| \ldots \|u_n\|.$$

Proof $\|W(\varnothing)\| = 1 \leq K$, and by Proposition 5.2,

$$\|W(u_1)\| = \|X(u_1)\| \leq (2c_{\gamma,\phi} + \|\Lambda\|)\|u_1\| \leq (\sqrt{2}+1)c_{\gamma,\phi}K\|u_1\|.$$

For $n = 2$,

$$\begin{aligned}W(u_1 \otimes u_2) &= X(u_1)W(u_2) - \Lambda(u_1)W(u_2) \\ &\quad - \phi[u_1u_2] = (a^+(u_1) + a^-(u_1))W(u_2) - \phi[u_1u_2],\end{aligned}$$

and

$$\|W(u_1 \otimes u_2)\| \leq 2c_{\gamma,\phi} \ \|u_1\|\,\|W(u_2)\| + \|\phi\| \ \|u_1\| \ \|u_2\|,$$

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while for $n \geq 3$,

$$W(u_1 \otimes u_2 \otimes \cdots \otimes u_n) = (a^+(u_1) + a^-(u_1))W(u_2 \otimes \cdots \otimes u_n) - (\gamma + \phi)[u_1 u_2]W(u_3 \otimes \cdots \otimes u_n),$$

and

$$\|W(u_1, u_2, \ldots, u_n)\| \leq 2c_{\gamma,\phi} \|u_1\| \|W(u_2, \ldots, u_n)\| + \|\gamma + \phi\| \|u_1\| \|u_2\| \|W(u_3, \ldots, u_n)\|.$$

Since $(\sqrt{2}+1)^n = 2(\sqrt{2}+1)^{n-1} + (\sqrt{2}+1)^{n-2}$, the result follows by induction. ∎

Let's turn our attention back to the infinite matrix W whose structure we established in the previous section, particularly the decomposition into a sum of matrices with a single nonzero diagonal. The following more general proposition will be combined with this fact to obtain conditions under which W is a bounded operator. The following result must be standard, although we do not have a reference for it.

Lemma 5.4 *Let $T \in \mathbb{B}(\mathcal{H} \otimes \ell^2)$, which we can write as an infinite matrix with entries $T_{ij} \in \mathbb{B}(\mathcal{H})$. Then*

$$\|T\| \leq \sum_{i \in \mathbb{Z}} \sup_{k \geq \max(1, 1-i)} \|T_{k,k+i}\|.$$

Corollary 5.5 *For $\sup_i \|u_i\| < \frac{\sqrt{2}-1}{\sqrt{\max(\|\phi\|, \|\gamma+\phi\|)}}$,*

(a) *W is a bounded operator.*
(b) *$B - X$ is invertible, and*

$$W = (B - X)^{-1}(B - \Phi).$$

Proof For (a), by Proposition 5.3, $\|W_{i,i+n}\| \leq K\left((\sqrt{2}+1)c_{\gamma,\phi} \sup_i \|u_i\|\right)^n$. The result follows from Lemma 5.4.

For (b), let $Y = B - X - 1 = A^0 + \Gamma + \Phi - X$. Note that Y has nonzero entries on only two diagonals:

$$Y_{i,i+1} = a^+(u_i) + a^-(u_i),$$
$$Y_{i,i+2} = (\gamma + \phi)[u_i u_{i+1}].$$

So apply Lemma 5.4 and Proposition 5.2 to get the estimate

$$\|Y\| \leq (\|a^+\| + \|a^-\|)\left(\sup_i \|u_i\|\right) + \|\gamma + \phi\|\left(\sup_i \|u_i\|\right)^2$$
$$\leq 2c_{\gamma,\phi}\left(\sup_i \|u_i\|\right) + c_{\gamma,\phi}^2\left(\sup_i \|u_i\|\right)^2 < 1.$$

Therefore, $(B - X)$ is invertible. The formula follows from Proposition 4.3. ∎

Finally, we return to the question of convergence for the cumulant generating function.

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Proposition 5.6 *Let* $K = \max\{\sqrt{\|\gamma\|}, \|\Lambda\|\}$. *Then* $\|R_n'[u]\| \leq C_n K^n \|u\|^n$, *where* C_n *is the Catalan number. If* $\|u\| \leq \frac{1}{4K}$, *then the generating function* $R'(u)$ *converges, and thus the cumulant generating function does as well.*

Proof By Lemma 3.9, $R_0'[u] = 1$, $\|R_1'[u]\| \leq \|\Lambda\| \|u\| \leq K\|u\|$, and

$$\|R_2'[u]\| \leq (\|\Lambda\|^2 + \|\gamma\|)\|u\|^2 \leq 2K^2\|u\|^2.$$

Recursively,

$$\begin{aligned}\|R_n'[u]\| &\leq \sum_{i=0}^{n-2} \|R_i'[u]\| \|R_{n-i-2}'[u]\| \|\gamma\| \|u\|^2 + \|R_{n-1}'[u]\| \|\Lambda\| \|u\| \\ &\leq \sum_{i=0}^{n-2} C_i C_{n-i-2} K^{n-2} \|\gamma\| \|u\|^n + C_{n-1} K^{n-1} \|\Lambda\| \|u\|^n \\ &\leq C_n K^n \|u\|^n,\end{aligned}$$

since $C_{n-1} = \sum_{i=0}^{n-2} C_i C_{n-i-2}$ and $C_n \geq 2C_{n-1}$. So the generating function has norm bounded by

$$\sum_{n=0}^{\infty} \|R_n'[u]\| \leq \sum_{n=0}^{\infty} C_n K^n \|u\|^n.$$

A well-known approximation of the Catalan numbers via Stirling's formula is $C_n \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}$. So this series converges by assumption. ∎

For the generating function $\mathcal{R}[U] = \sum_{n=0}^{\infty} R_n[X, \ldots, X]$ for matricial cumulants, we also have the following corollary:

Corollary 5.7 *Under the conditions of Proposition 5.6, with the assumption* $\|u\| \leq \frac{1}{4K}$ *replaced with* $\sup_i \|u_i\| \leq \frac{1}{4K}$, *the infinite matrix* $\mathcal{R}[U]$ *is bounded, with*

$$\|\mathcal{R}[U]\| \leq \sum_{n=2}^{\infty} \|\phi\| C_{n-2} K^{n-2} \left(\sup_i \|u_i\|\right)^n. \tag{5.1}$$

Proof Using Corollary 4.7, the ith diagonal of $\mathcal{R}[U]$ has the bound

$$\|\phi\| C_{n-2} K^{n-2} \left(\sup_i \|u_i\|\right)^n,$$

so the convergence of the sum follows from Lemma 5.4 and the same argument from the proof of Proposition 5.6. ∎

6 Traciality

Recall that a state ψ on a noncommutative algebra $\mathcal{A}$ is *tracial*, or a *trace*, if for all $x, y \in \mathcal{A}$,

$$\psi[xy] = \psi[yx].$$

In this section, we give conditions under which the vacuum state is tracial. We start with an auxiliary result.

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Definition 6.1 On $\mathcal{F}_{\text{alg}}(\mathcal{B})$, define an anti-linear involution by the linear extension of

$$S(u_1 \otimes \cdots \otimes u_n) = u_n^* \otimes \cdots \otimes u_1^*.$$

For $b \in \mathcal{B}$, denote $X_r(b) = SX(b^*)S$. Explicitly, for $n \geq 2$,

$$\begin{aligned} X_r(b)(u_1 \otimes \cdots \otimes u_n) = u_1 \otimes \cdots \otimes u_n \otimes b + u_1 \otimes \cdots \otimes u_{n-1} \otimes \Lambda(b^* \otimes u_n^*)^* \\ + u_1 \otimes \cdots \otimes u_{n-1}(\gamma + \phi)[u_n b], \end{aligned}$$

with appropriate modifications for $n = 1$ and $n = 0$. Denote $\Gamma^{\text{alg}}_{\gamma,\Lambda}(\mathcal{B}, \phi; r) = S\Gamma^{\text{alg}}_{\gamma,\Lambda}(\mathcal{B}, \phi)S$ the algebra generated by $\{X_r(b) : b \in \mathcal{B}\}$. For $\vec{\xi} \in \mathcal{F}_{\text{alg}}(\mathcal{B})$ a simple tensor, denote $W_r(\vec{\xi}) = SW(\vec{\xi})S$. Then $W_r(S(\vec{\xi}))\Omega = \vec{\xi}$, so that Ω is cyclic for $\Gamma^{\text{alg}}_{\gamma,\Lambda}(\mathcal{B}, \phi; r)$.

Proposition 6.2

(a) *Suppose that $\Gamma^{\text{alg}}_{\gamma,\Lambda}(\mathcal{B}, \phi)$ and $\Gamma^{\text{alg}}_{\gamma,\Lambda}(\mathcal{B}, \phi; r)$ commute. Then W extends to a linear map on $\mathcal{F}_{\text{alg}}(\mathcal{B})$,*

$$\Gamma^{\text{alg}}_{\gamma,\Lambda}(\mathcal{B}, \phi) = \left\{ W(\vec{\xi}) : \vec{\xi} \in \mathcal{F}_{\text{alg}}(\mathcal{B}) \right\}, \quad \Gamma^{\text{alg}}_{\gamma,\Lambda}(\mathcal{B}, \phi; r) = \left\{ W_r(\vec{\xi}) : \vec{\xi} \in \mathcal{F}_{\text{alg}}(\mathcal{B}) \right\},$$

and for $\vec{\xi} \in \mathcal{F}_{\text{alg}}(\mathcal{B})$,

$$W(\vec{\xi})^* = W(S(\vec{\xi})).$$

Therefore for $A \in \Gamma^{\text{alg}}_{\gamma,\Lambda}(\mathcal{B}, \phi)$, $S(A\Omega) = A^*\Omega$.

(b) *In addition to the assumption in (a), suppose that ϕ is tracial on $\mathcal{B}$. Then the vacuum state is tracial on $\Gamma^{\text{alg}}_{\gamma,\Lambda}(\mathcal{B}, \phi)$.*

Proof

(a) Since Ω is cyclic for $\Gamma^{\text{alg}}_{\gamma,\Lambda}(\mathcal{B}, \phi)$ and $\Gamma^{\text{alg}}_{\gamma,\Lambda}(\mathcal{B}, \phi; r)$, it is separating for them (and the von Neumann algebras they generate). Using the recursion and induction, for $\vec{\xi} \in \mathcal{F}_{\text{alg}}(\mathcal{B})$,

$$\begin{aligned} W(a^+(b)(\vec{\xi}))\Omega &= X(b)W(\vec{\xi})\Omega - W(a^0(b)(\vec{\xi}))\Omega - W(a^-(b)(\vec{\xi}))\Omega \\ &= X(b)\vec{\xi} - a^0(b)(\vec{\xi}) - a^-(b)(\vec{\xi}) \\ &= a^+(b)(\vec{\xi}). \end{aligned}$$

In particular, if $\vec{\xi} = 0$, then $W(\vec{\xi})\Omega = 0$, so $W(\vec{\xi}) = 0$, and the map W is well defined.

Clearly, $W(\vec{\xi}) \in \Gamma^{\text{alg}}_{\gamma,\Lambda}(\mathcal{B}, \phi)$. On the other hand, using induction on the number of factors, a monomial $X(u_0)X(u_1)\ldots X(u_n)$ is a linear combination of terms of the form $X(u_0)W(\vec{\xi})$, each of which is a linear combination of the terms of the form $W(\vec{\xi}')$ by the recursion.

Next, note that

$$X_r(b)\Omega = X(b)\Omega$$

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and

$$W(b)^* = X(b)^* = X(b^*) = W(b^*).$$

Using induction,

$$\begin{aligned} W(a^+(b)(\vec{\xi}))^* &= W(\vec{\xi})^* X(b)^* - W(a^0(b)(\vec{\xi}))^* - W(a^-(b)(\vec{\xi}))^* \\ &= W(S(\vec{\xi}))X(b^*) - W(S(a^0(b)(\vec{\xi}))) - W(S(a^-(b)(\vec{\xi}))). \end{aligned}$$

On the other hand, if $X_r(b)$ commutes with $\Gamma^{\mathrm{alg}}_{\gamma,\Lambda}(\mathcal{B},\phi)$,

$$\begin{aligned} W(S(\vec{\xi}))X(b^*)\Omega &= W(S(\vec{\xi}))X_r(b^*)\Omega = X_r(b^*)W(S(\vec{\xi}))\Omega \\ &= SX(b)SS(\vec{\xi}) = SX(b)(\vec{\xi}). \end{aligned}$$

Since Ω is separating for $\Gamma^{\mathrm{alg}}_{\gamma,\Lambda}(\mathcal{B},\phi)$, this implies that

$$\begin{aligned} W(S(\vec{\xi}))X(b^*) &= W(SX(b)(\vec{\xi})) \\ &= W(S(a^+(b)(\vec{\xi}))) + W(S(a^0(b)(\vec{\xi}))) + W(S(a^-(b)(\vec{\xi}))). \end{aligned}$$

Thus

$$W(a^+(b)(\vec{\xi}))^* = W(S(a^+(b)(\vec{\xi}))).$$

(b) We first show that $X_r(b)^*\Omega = X(b)^*\Omega$. Indeed, for $\vec{\xi} \in \mathcal{B}^{\otimes n}$, $\left\langle X_r(b)(\vec{\xi}), \Omega \right\rangle = 0$, if $n \neq 1$. For $u \in \mathcal{B}$, using the fact that ϕ is a trace,

$$\begin{aligned} \langle X_r(b)(u), \Omega \rangle &= \langle Sa^-(b^*)(S(u)), \Omega \rangle \\ &= \overline{\phi[b^*u^*]} = \phi[ub] = \phi[bu] = \langle u, b^* \rangle = \langle u, X(b)^*\Omega \rangle . \end{aligned}$$

To check that the vacuum state is tracial on $\Gamma^{\mathrm{alg}}_{\gamma,\Lambda}(\mathcal{B},\phi)$, it suffices to verify that

$$\begin{aligned} \left\langle W(\vec{\xi})X(b)\Omega, \Omega \right\rangle &= \left\langle W(\vec{\xi})X_r(b)\Omega, \Omega \right\rangle = \left\langle X_r(b)W(\vec{\xi})\Omega, \Omega \right\rangle \\ &= \left\langle W(\vec{\xi})\Omega, X_r(b)^*\Omega \right\rangle = \left\langle W(\vec{\xi})\Omega, X(b)^*\Omega \right\rangle \\ &= \left\langle X(b)W(\vec{\xi})\Omega, \Omega \right\rangle . \end{aligned}$$

■

Theorem 6.3 *Suppose the operators $\{X(u) : u \in \mathcal{B}\}$ are bounded, and the inner product on $\mathcal{F}_{\mathrm{alg}}(\mathcal{B})$ nondegenerate. Denote*

$$\Gamma_{\gamma,\Lambda}(\mathcal{B},\phi) = W^*(X(u) : u \in \mathcal{B}) = W^*(X(u) : u \in \mathcal{B}^{sa}).$$

Suppose that the following conditions hold:

$$\Lambda(v^* \otimes u^*)^* = \Lambda(u \otimes v), \tag{6.1}$$

$$u\gamma[fv] - \gamma[uf]v = \Lambda(u \otimes \Lambda(f \otimes v)) - \Lambda(\Lambda(u \otimes f) \otimes v), \tag{6.2}$$

$$\Lambda(u \otimes f\gamma[gv]) - \Lambda(u \otimes f)\gamma[gv] = \Lambda(\gamma[uf]g \otimes v) - \gamma[uf]\Lambda(g \otimes v), \tag{6.3}$$

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and

$$\phi[\gamma[u]v] = \phi[u\gamma[v]]. \tag{6.4}$$

Then each $X_r(u)$ commutes with $\Gamma_{\gamma,\Lambda}(\mathcal{B},\phi)$. If, in addition, ϕ is tracial, then the vacuum state is tracial on $\Gamma_{\gamma,\Lambda}(\mathcal{B},\phi)$. All the conclusions in Proposition 6.2 hold, the map S extends to an anti-linear isometry on $\mathcal{F}_{\gamma,\phi}(\mathcal{B})$, and the map $A \mapsto SA^*S$ implements the canonical anti-isomorphism between $\Gamma_{\gamma,\Lambda}(\mathcal{B},\phi)$ and its commutant.

Moreover, traciality of ϕ and conditions (6.1) and (6.2) are necessary for the vacuum state to be tracial.

Proof Since we are working on a Fock space with depth two action, to show that $X(u)X_r(v) - X_r(v)X(u) = 0$ it suffices to consider their actions on tensors of length 0, 1, and 2. Acting on f, we get

$$\Big(\phi[u\Lambda(v^* \otimes f^*)^*] - \phi[\Lambda(u \otimes f)v]\Big) + \Big(\Lambda(u \otimes \Lambda(v^* \otimes f^*)^*) + \gamma[uf]v - \Lambda(v^* \otimes \Lambda(u \otimes f)^*)^* - u\gamma[fv]\Big).$$

Using the previously assumed (2.1), (6.1), and (6.2), this expression is 0. Similarly, acting on $f \otimes g$, we get

$$\Big(\Lambda(u \otimes f\gamma[gv]) + \gamma[uf]\Lambda(v^* \otimes g^*)^* - \Lambda(u \otimes f)\gamma[gv] - \Lambda(v^* \otimes g^*\gamma[uf]^*)^*\Big) + \Big(\phi[uf\gamma[gv]] - \phi[\gamma[uf]gv]\Big)$$

which is 0 using (6.3) and (6.4). The rest of the properties follow from Proposition 6.2 and general theory.

Finally, if the vacuum state is tracial, then the joint free cumulants are cyclically symmetric. Using Proposition 3.6, the symmetry of the second cumulant gives traciality of ϕ, the symmetry of the third cumulant when combined with (2.1) gives (6.1). The fourth cumulant combined with (2.1) and (6.1) gives

$$\begin{aligned}\phi[u\Lambda(v \otimes \Lambda(f \otimes g))] + \phi[u\gamma[vf]g] &= \phi[v\Lambda(f \otimes \Lambda(g \otimes u))] + \phi[v\gamma[fg]u] \\ &= \phi[\Lambda(\Lambda(v \otimes f) \otimes g)u] + \phi[v\gamma[fg]u],\end{aligned}$$

and faithfulness of ϕ implies (6.2). ∎

7 Examples

In this section, we show how the motivating examples described in the Introduction, and their generalizations, fit into the general setting of this article.

Example 7.1 Let

$$\mathcal{B} = L^\infty([0,1]), \quad \phi[f] = \int_0^1 f(x)\,dx,$$

and η positive a.e. and λ a real-valued function. Define

$$\gamma[f](x) = \eta(x)f(x), \quad \Lambda(b \otimes f)(x) = \lambda(x)b(x)f(x).$$

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Then the construction is closely related to the setting in [BL09, BL11] (see also [BLR15]).

More generally, for an algebra $\mathcal{B}$, let η, λ be central elements in $\mathcal{B}$ such that $\eta + 1$ is positive and λ self-adjoint. Then we may take

$$\gamma[f] = \eta f, \quad \Lambda(b \otimes f) = \lambda b f.$$

If $\eta + 1$ is invertible, then the inner product is positive definite. The operators $X(f)$ are typically unbounded, so Theorem 6.3 does not apply. Nevertheless the vacuum state is always tracial.

By Proposition 3.6, the free cumulants are given by an explicit formula

$$R[X(f_1), \ldots, X(f_n)] = \sum_{\pi \in \widetilde{\mathrm{NC}}_{ns}(n)} \phi[\eta^{|\pi|-1} \lambda^{n-2|\pi|} f_1 f_2 \ldots f_n]. \tag{7.1}$$

In particular, in this case, $R'[(X(f_2), \ldots, X(f_{n-1})]$ may be identified with an element of $\mathcal{B}$, and their generating function satisfies a quadratic relation

$$R'(f) = 1 + \lambda R'(f) f + \eta R'(f) f R'(f). \tag{7.2}$$

In the case where $\mathcal{B} = L^\infty([0,1])$, we may take $R'(f)$ to be the operator of pointwise multiplication by a function $R'(f)(x)$ which satisfies

$$R'(f)(x) = 1 + R'(f)(x)\lambda(x) f(x) + R'(f)(x)^2 \eta(x) f(x),$$

and so is a solution of a quadratic equation for each x. The free cumulant generating function can then be found via

$$R(f) = \int f(x)^2 R'(f)(x)\, dx.$$

Finally, from equation (7.1), if $fg = gf = f^* g = f g^* = 0$, then $X(f)$ and $X(g)$ are free, and if f is self-adjoint with $\eta f = \lambda f = 0$, then $X(f)$ is semicircular. See also the end of the section for related results.

Example 7.2 Let

$$\mathcal{B} = L^\infty([0,1]), \quad \phi[f] = \int_0^1 f(x)\, dx,$$

and

$$w(x,y) = \begin{cases} p, & \text{if } 0 < x < y, \\ q, & \text{if } 0 < y < x, \\ 1, & \text{otherwise,} \end{cases}$$

for $p, q \geq 0$. Define

$$\gamma[f](x) = \int w(x,y) f(y)\, dy$$

and

$$\Lambda(b \otimes f)(x) = f(x) \int w(x,y) b(y)\, dy + f(y) b(y).$$

We obtain the construction from [LS08].

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The specific form of a function w above allowed for explicit combinatorial calculations, but γ can be defined by the same formula as long as $(w+1)$ is nonnegative a.e., and it is natural to take

$$\Lambda(b\otimes f)(x) = \int b(y)f(z)\lambda(x,y,z)\,dy\,dz$$

or

$$\Lambda(b)(x) = \int b(y)\lambda(x,y)\,dy$$

for an appropriate function λ. In the first case, necessary conditions (2.1), (6.1) for the vacuum state to be tracial imply that $\lambda(x,y,z)$ is conjugate-symmetric in its arguments, and $w(x,y)$ is symmetric in its arguments. These conditions are not sufficient, and typically the vacuum state is not tracial.

In the second case, we may identify $R'(f)$ with a function satisfying

$$R'(f)(x) = \frac{1}{1-\int \lambda(x,y)f(y)\,dy - \int w(x,y)R'(f)(y)f(y)^2\,dy}.$$

One can generalize this construction further. Instead of the map γ, it suffices to use a bilinear map $\langle\cdot,\cdot\rangle_\gamma : \mathcal{B}\otimes\mathcal{B}\to\mathcal{B}$ which is star-linear in each argument and such that $\langle\cdot,\cdot\rangle_\gamma + \langle\cdot,\cdot\rangle_\phi$ is positive semi-definite. Let $w\in\mathcal{B}\otimes\mathcal{B}$ satisfy $w+(1\otimes 1)\geq 0$, and define

$$\langle f,g\rangle_\gamma = (\phi\otimes I)[(f\otimes 1)w(g\otimes 1)].$$

Then the corresponding inner product on the Fock space is positive semi-definite. If $w+1\otimes 1$ is invertible, then the inner product is positive definite.

Note that

$$\left\langle \vec{\zeta},\vec{\xi}\right\rangle_{\gamma,\phi} = \left\langle \vec{\zeta},K\vec{\xi}\right\rangle,$$

where K is the multiplication operator by

$$[(1+w)\otimes 1^{\otimes(n-2)}]\cdot[1\otimes(1+w)\otimes 1^{\otimes(n-3)}]\cdot\ldots\cdot[1^{\otimes(n-2)}\otimes(1+w)].$$

In the commutative case, we may identify

$$K(s_1,s_2,\ldots s_n) = (1+w(s_1,s_2))\ldots(1+w(s_{n-1},s_n)).$$

So Lemma 5.1 applies. Moreover, $a^-(b) = \ell^*(b)[(1+w)\otimes 1^{\otimes(n-2)}]$, where ℓ^* is the free annihilation operator. It follows that for $w\in\mathcal{B}\otimes_{\min}\mathcal{B}$ and $\|\Lambda(b\otimes b_1)\|_\phi \leq \|\Lambda\|\ \|b\|\ \|b_1\|_\phi$,

$$\|a^-(b)\|_{\gamma,\phi} = \|a^+(b)\|_{\gamma,\phi} \leq \|b\|_\phi\sqrt{\|w\|+1}, \quad \left\|a^0(b)\right\|_{\gamma,\phi} \leq \|\Lambda\|\ \|b\|.$$

In particular, the operators extend to $\mathcal{F}_{\gamma,\phi}(\mathcal{B})$.

Example 7.3 Let $\mathcal{B}=\mathbb{R}^d$ with component-wise multiplication and the standard basis $\{e_i\}_{i=1}^d$,

$$\phi[e_j] = 1, \quad \gamma[e_j] = \sum_{i=1}^d C_{ij}e_i, \quad \Lambda(e_i\otimes e_j) = \sum_{k=1}^d B_{ij}^k e_k,$$

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with $C_{ij} \geq -1$ and certain relations between the coefficients. This is the setting of [Ans07] (possibly with slightly different notation). Embedding $\mathbb{R}^d$ into $L^\infty[0,1]$ using functions constant on cubes, relates this example to the preceding one.

If all $C_{ij} > -1$, the inner product is positive definite. The operators are clearly bounded, and so extend to $\mathcal{F}_{\gamma,\phi}(\mathcal{B})$. Particular parameters for which the vacuum state is tracial were given in [Ans07].

Example 7.4 Choose $\mathcal{B}$, ϕ, Λ to be general, but let $\gamma = \psi - \phi$ be scalar-valued, where ψ is positive semi-definite. Then the inner product simplifies to

$$\langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_k \rangle_{\gamma,\phi} = \delta_{n=k} \phi[g_n^* f_n] \prod_{i=1}^{n-1} \psi[g_i^* f_i],$$

so that

$$\mathcal{F}_{\gamma,\phi}(\mathcal{B}) \simeq \mathbb{C}\Omega \oplus \left(L^2(\mathcal{B},\phi) \otimes \mathcal{F}(L^2(\mathcal{B},\psi))\right)$$

with the standard inner product. It follows from equation (6.2) that for $\psi \neq \phi$ and $\Lambda = 0$, the vacuum state is not tracial. In [AM22], we investigate a choice of Λ which does lead to a tracial vacuum state.

Example 7.5 Let $\gamma = 0$ and $\Lambda(f \otimes g) = fg$. For any $(\mathcal{B},\phi)$, the corresponding algebra of operators on the Fock space is the free (compound) Poisson algebra, a particular case of the constructions in the Appendix of [Ans04] (for $q = 0$) or Proposition 23 in [Ans20] (for the scalar case) (see also [GSS92]). In particular, for $f \in \mathcal{B}^{sa}$, $X(f)$ has a centered free compound Poisson distribution, and a centered free Poisson distribution if f is a projection. Moreover, if $f \cdot g = g \cdot f = 0$ then $X(f)$ and $X(g)$ are free.

We finish the article with two structure results under the assumption that the vacuum state is tracial.

Proposition 7.6 *Assume that $\mathcal{B}$ is unital, and the vacuum state is tracial. Suppose that $\Lambda(u \otimes v) = \Lambda(u)v$. Then $\Gamma^{\mathrm{alg}}_{\gamma,\Lambda}(\mathcal{B},\phi)$ is as in Example 7.1, with $\Lambda(u) = \lambda u$ for some central λ, and $\gamma[u] = \eta u$ for some central η.*

Proof If $\Lambda(u \otimes v) = \Lambda(u)v$, then conditions (2.1), (6.1), and (6.2) imply that $u\Lambda(v) = \Lambda(u)v$ and $u\gamma[yv] = \gamma[uy]v$. Therefore $\Lambda(u) = \lambda u = u\lambda$ for $\lambda = \Lambda(1)$, and $\gamma[u] = \eta u = u\eta$ for $\eta = \gamma[1]$. ∎

Theorem 7.7 *Assume that $\mathcal{B}$ is unital, and the vacuum state is tracial. Suppose that $\gamma[f] = \eta f$ for η central, and for some M, and all $f, g \in \mathcal{B}$,*

$$\langle \Lambda(f \otimes g), \Lambda(f \otimes g) \rangle \leq M \|f\|_{\mathcal{B}}^2 \langle g, g \rangle . \tag{7.3}$$

Then we have an orthogonal decomposition

$$\mathcal{H} = (\mathcal{B}, \langle \cdot, \cdot \rangle_\phi) = \mathcal{N} \oplus \mathcal{N}^\perp = \mathcal{Z} \oplus \mathcal{P} \oplus \mathcal{N}^\perp,$$

such that:

- *$\Gamma^{\mathrm{alg}}_{\gamma,\Lambda}(\mathcal{N}^\perp,\phi)$ is as in Example 7.1, with $\Lambda(f \otimes g) = \lambda fg$ for some central λ.*
- *$\Gamma^{\mathrm{alg}}_{\gamma,\Lambda}(\mathcal{Z},\phi)$ is generated by free semicircular elements.*

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- $\Gamma^{\mathrm{alg}}_{\gamma,\Lambda}(\mathcal{P},\phi)$ *is a free Poisson algebra as in Example* 7.5*, where the multiplication on* $\mathcal{P}$ *is given by* Λ *(rather than inherited from* $\mathcal{B}$*).*

Moreover, the subalgebras $\Gamma^{\mathrm{alg}}_{\gamma,\Lambda}(\mathcal{N}^{\perp},\phi)$, $\Gamma^{\mathrm{alg}}_{\gamma,\Lambda}(\mathcal{Z},\phi)$, *and* $\Gamma^{\mathrm{alg}}_{\gamma,\Lambda}(\mathcal{P},\phi)$ *are free.*

Proof Denote by $\mathcal{H}$ the inner product space $\mathcal{B}$ with the involution and the inner product $\langle f,g\rangle = \phi[g^* f]$ satisfying the relation

$$\langle f,g\rangle = \langle g^*, f^*\rangle .$$

Denote

$$f\cdot g = \Lambda(f\otimes g). \tag{7.4}$$

The notation is meant to suggest that we consider the binary operation Λ as a multiplication on $\mathcal{H}$ (different from a multiplication it may have inherited from $\mathcal{B}$). Indeed, equations (6.1), (6.2), and the first condition in (2.1) say that

$$(f\cdot g)^* = g^*\cdot f^*, \quad (f\cdot g)\cdot h = f\cdot(g\cdot h), \quad \langle f\cdot g, h\rangle = \langle g, f^*\cdot h\rangle .$$

That is, $(\mathcal{H},\cdot,*)$ is an associative star-algebra (linearity/distributivity is immediate), which is represented on $\mathcal{H}$ by left multiplication operators.

Second condition in equation (2.1) reads

$$\eta g^*\Lambda(b\otimes f) = \eta\Lambda(b^*\otimes g)^* f = \eta\Lambda(g^*\otimes b)f. \tag{7.5}$$

Therefore $\eta\Lambda(b\otimes f) = \eta\lambda bf$, where $\lambda = \Lambda(1\otimes 1)$.

Let

$$\mathcal{N} = \{f\in\mathcal{B} : \eta f = 0\} .$$

Then $\mathcal{N}$ is a self-adjoint subspace and an ideal. Restricted to $\mathcal{N}^{\perp}$, multiplication by η is injective. Since by (6.1), $\eta\lambda bg = \eta bg\lambda^*$, it follows that λ, restricted to $\mathcal{N}^{\perp}$, is self-adjoint and central. $\mathcal{N}^{\perp}$ is self-adjoint and a left ideal with respect to $\cdot$, so $\Gamma^{\mathrm{alg}}_{\gamma,\Lambda}(\mathcal{N}^{\perp},\phi)$ is a star-subalgebra.

Let $f_1,\ldots,f_n\in\mathcal{N}\cup\mathcal{N}^{\perp}$, with at least one element from $\mathcal{N}$. It follows from equation (7.5) that $\mathcal{N}$ is also an ideal with respect to $\cdot$. Combined with the definition of $\mathcal{N}$, this implies that in the expansion for the joint free cumulant in Proposition 3.6, elements of $\mathcal{N}$ can only appear in the (unique) outer block of π. Since each inner block is a multiple of η, it follows that in fact, π has a single block so that

$$R[X(f_1),\ldots,X(f_n)] = \langle f_1, f_2\cdot\ldots\cdot f_n\rangle . \tag{7.6}$$

Since the vacuum state is tracial, the free cumulants are cyclically symmetric. So if

$$f_1,\ldots,f_n\in\mathcal{N}\cup\mathcal{N}^{\perp}$$

with at least one element from each of $\mathcal{N},\mathcal{N}^{\perp}$, we may assume that $f_1\in\mathcal{N}^{\perp}$ and $f_2\cdot\ldots\cdot f_n\in\mathcal{N}$. Then their inner product is zero, which implies freeness by (7.6).

Next, denote

$$\mathcal{Z} = \{f\in\mathcal{N} : \forall g\in\mathcal{N}, g\cdot f = 0\}$$

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and

$$\mathcal{P} = \overline{\mathrm{Span}(\{f \cdot g : f, g \in \mathcal{N}\})}.$$

Clearly both of them are subspaces (and in fact ideals with respect to $\cdot$). Also, using the properties of $\cdot$ at the beginning of the proof, one sees that $\mathcal{Z}$ and $\mathcal{P}$ are orthogonal complements of each other, and $\mathcal{N} = \mathcal{Z} \oplus \mathcal{P}$. $\mathcal{P}$ is clearly self-adjoint, therefore so is $\mathcal{Z}$.

For $f_1, \ldots, f_n \in \mathcal{Z} \cup \mathcal{P}$ with at least one of them in $\mathcal{Z}$, using the cyclic property again,

$$R[X(f_1), \ldots, X(f_n)] = \langle f_1, f_2 \cdot \ldots \cdot f_n \rangle = \delta_{n=2} \langle f_1, f_2 \rangle$$

and is zero unless both $f_1, f_2 \in \mathcal{Z}$. It follows that for $f \in \mathcal{Z}^{sa}$, $X(f)$ are semicircular, with orthogonal f corresponding to free $X(f)$, and are also free from $X(g)$ with $g \in \mathcal{P}$.

Under the boundedness assumption (7.3), $\mathcal{P}$ satisfies all the axioms of a Hilbert algebra (Chapters 5 and 6 of [Dix81]). Then we can complete $\mathcal{P}$ to a von Neumann algebra represented on $\mathcal{N}$ (since it acts as zero on $\mathcal{Z}$, it is also represented on $\mathcal{P}$) by left multiplication. Moreover, we have a semi-finite trace ψ on this von Neumann algebra which is finite on $\mathcal{P}$ and satisfies $\langle f, g \rangle = \psi[g^* \cdot f]$. So in this case for self-adjoint elements and $n \geq 2$,

$$R[X(f_1), \ldots, X(f_n)] = \psi[f_1 \cdot f_2 \cdot \ldots \cdot f_n].$$

It follows that $\Gamma^{\mathrm{alg}}_{\gamma,\Lambda}(\mathcal{P}, \phi)$ is the free compound Poisson algebra of $(\mathcal{P}, \psi)$. ∎

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Department of Mathematics, Texas A&M University, College Station, TX 77843-3368, United States
e-mail: manshel@tamu.edu jacobmashburn16@tamu.edu

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