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ESTIMATION OF NONLINEAR ERROR CORRECTION MODELS

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Asymptotic theory for the estimation of nonlinear vector error correction models that exhibit regime-specific short-run dynamics is developed. In particular, regimes are determined by the error correction term, and the transition between regimes is allowed to be discontinuous, as in, e.g., threshold cointegration. Several nonregular problems are resolved. First of all, consistency—square root n consistency for the cointegrating vector β—is established for the least squares estimation of this general class of models. Second, the convergence rates are obtained for the least squares of threshold cointegration, which are $n^{3/2}$ and n for β and γ, respectively, where γ denotes the threshold parameter. This fast rate for β in itself is of practical relevance because, unlike in smooth transition models, the estimation error in β does not affect the estimation of short-run parameters. We also derive asymptotic distributions for the smoothed least squares estimation of threshold cointegration.

1. INTRODUCTION

Nonlinear error correction models (ECM) have been studied actively in economics, and there are numerous applications. To list only a few, see Michael, Nobay, and Peel (1997) for the application in purchasing power parity, Anderson (1997) on the term structure model of interest rates, Escribano (2004) on money demand, Psaradakis, Sola, and Spagnolo (2004) on relation between stock prices and dividends, and Sephton (2003) on spatial market arbitrage. See also reviews by Granger (2001) and Bec and Rahbek (2004). The models include the smooth transition ECM of Granger and Teräsvirta (1993), the threshold cointegration model of Balke and Fomby (1997), the Markov switching ECM of Psaradakis et al. (2004), and the cubic polynomials model of Escribano (2004).

A strand of econometric literature focuses on testing for the presence of nonlinearity and cointegration in an attempt to disentangle nonstationarity from nonlinearity. A partial list includes Hansen and Seo (2002), Kapetanios, Shin, and Snell (2006), and Seo (2006). Time series properties of various ECMs have been established by Corradi, Swanson, and White (2000) and Saikkonen (2005, 2008), among others. However, the result on estimation is still limited. Most of all,

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consistency has not been proved except for special cases. It is difficult to establish because of the lack of uniformity in the convergence over the cointegrating vector space, as noted by Saikkonen (1995), who derived the consistency of the maximum likelihood estimator (MLE) of a cointegrated system that is nonlinear in parameters but otherwise linear. de Jong (2002) studied consistency of minimization estimators of smooth transition ECMs where the error correction term appears only in a bounded transition function. Another case studied by Kristensen and Rahbek (2010) is the case where the function is unbounded but becomes linear as the error correction term diverges. Next, estimation of regime-switching and/or discontinuous cases, which includes important classes of models such as smooth transition ECM and threshold cointegration, has hardly been studied. Hansen and Seo (2002) proposed the MLE under normality but only to make conjecture on the consistency. Although it may be argued that the two-step approach by Engle and Granger (1987) can be adopted because of the superconsistency of the cointegrating vector estimate, the estimation error cannot be ignored in nonlinear ECMs, as shown by de Jong (2001).[1]

The purpose of this paper is to develop asymptotic theory for a class of nonlinear vector error correction models (VECMs). In particular, we consider regime-switching VECMs, where each regime exhibits different short-run dynamics and the regime switching depends on the disequilibrium error. Examples include threshold cointegration and smooth transition VECM. First, we establish the square root n consistency for the least squares (LS) estimator of β. This enables us to employ de Jong (2002) to make asymptotic inference for both short-run and long-run parameters jointly in smooth transition models. Then, we turn to discontinuous models, focusing on the threshold cointegration model, which is particularly popular in practice.

This paper shows that the convergence of the LS estimator of β in the threshold cointegration model is extremely fast at the rate of $n^{3/2}$. This asymptotics is based not on the diminishing threshold asymptotics of Hansen (2000) but on the fixed threshold asymptotics. Two different irregularities contribute to this fast rate. First, the estimating function lacks uniformity over the cointegrating vector space as the data become stationary at the true value, which is the reason for the superconsistency of the standard cointegrating vector estimates. Second, β takes part in regime switching, which is discontinuous. This model discontinuity also boosts the convergence rate, yielding superconsistency of the threshold estimate as in Chan (1993). Although this fast convergence rate is certainly interesting and has some inferential value, e.g., when we perform a sequential test to determine the number of regimes, it makes it very challenging to obtain an asymptotic distribution of the estimator. Even in the stationary threshold autoregression, the asymptotic distribution is very complicated and cannot be tabulated (see Chan, 1993). In the literature, the only method given for approximating the distribution is the subsampling method of Gonzalo and Wolf (2005), although it would not work when β is estimated because of the involved nonstationarity. Meanwhile, Seo and Linton (2007) proposed smoothed least squares (SLS) estimation for threshold

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regression models, which results in asymptotic normality of the threshold estimate and is applicable to the threshold cointegration model.

We develop asymptotic distributions of the SLS estimators of β, the threshold parameter γ, and the other short-run parameters. The estimates $\hat{\beta}$ and $\hat{\gamma}$ converge jointly to a functional of Brownian motions, with the rates slightly slower than those of the unsmoothed counterparts. This slowdown in convergence rate has already been observed in Seo and Linton (2007) and is the price to pay to achieve standard inference. The remaining regression parameter estimates converge to the normal as if the true values of β and γ were known. This is not the case if the transition function is smooth. We also show that β can be treated as if known in the SLS estimation of the short-run parameters including γ if we plug in the unsmoothed cointegrating vector estimate because of the fast convergence rate. A set of Monte Carlo experiments demonstrates that this two-step approach is more efficient in finite samples.

This paper is organized as follows. Section 2 introduces the regime-switching VECMs and establishes the square root n consistency of the LS estimator of β. Section 3 concentrates on the threshold cointegration model, obtaining the convergence rate of the LS estimator of β and the asymptotic distributions of the SLS estimators of all the model parameters. It also discusses estimation of the asymptotic variances. Finite-sample performance of the proposed estimators is examined in Section 4. Section 5 concludes. Proofs of theorems are collected in the Appendix.

We use the following conventions throughout the paper. The integral $\int$ is taken over $\mathbb{R}$ unless specified otherwise, and the summation $\sum_t$ with respect to t is taken for all available observations for a given sample. The subscript 0 and the hat $\hat{}$ in any parameter indicate the true value and an estimate of the parameter, respectively, e.g., β_0 and $\hat{\beta}$. The Euclidean norm of a vector is signified by $||$. For a function g, $\|g\|_2^2 = \int g(x)^2 dx$, and $g^{(i)}$ indicates the ith derivative of g. And, for a random vector x_t and a parameter θ, we write $g_t(\theta) = g(x_t, \theta)$, $g_t = g(x_t, \theta_0)$, and $\hat{g}_t = g\left(x_t, \hat{\theta}\right)$. For example, if $z_t(\beta) = x_t'\beta$, then we write $z_t = x_t'\beta_0$ and $\hat{z}_t = x_t'\hat{\beta}$. The weak convergence of stochastic processes under the uniform metric is signified by $\Rightarrow$.

2. REGIME-SWITCHING ERROR CORRECTION MODELS

Let x_t be a p-dimensional $I(1)$ vector that is cointegrated with single cointegrating vector. It is denoted by $\left(1, \beta'\right)'$, normalizing the first element by 1. Define the error correction term $z_t(\beta) = x_{1t} + x_{2t}'\beta$, where $x_t = \left(x_{1t}, x_{2t}'\right)'$, and let

$$X_{t-1}(\beta) = \left(1, z_{t-1}(\beta), \Delta_{t-1}'\right)',$$

where Δ_{t-1} denotes the vector of the lagged first difference terms $\left(\Delta x_{t-1}', \ldots, \Delta x_{t-l+1}'\right)'$. Then, consider a two-regime VECM

$$\Delta x_t = A' X_{t-1}(\beta) + D' X_{t-1}(\beta) d_{t-1}(\beta, \gamma) + u_t, \quad \textbf{(1)}$$

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where $t = l+1, \ldots, n$ and $d_t(\beta, \gamma) = d(z_t(\beta), \gamma)$ is a bounded function that controls the transition from one regime to the other regime. It need not be continuous. Typical examples of the transition function include the indicator function $1\{z_t(\beta) > \gamma\}$ and the logistic function $(1-\exp(-\gamma_1(z_t(\beta)-\gamma_2)))^{-1}$, where $\gamma = (\gamma_1, \gamma_2)'$.

The threshold cointegration model of Balke and Fomby (1997) and the smooth transition ECM in Granger and Teräsvirta (1993) can be viewed as special cases. As an alternative, Escribano (2004) used cubic polynomials to capture this type of regime-specific error correction behavior. Although the last model is not nested in model (1), all this literature focuses on the nonlinear adjustment based on the magnitude of disequilibrium error. In this regard, the work of Gonzalo and Pitarakis (2006), in which a stationary variable determines the regimes, is different. Although we study a two-regime model to simplify our exposition, we expect that models with more than two regimes can be analyzed in a similar way. A symmetric three-regime model can be directly embedded in the two-regime model by replacing the threshold variable z_t with its absolute value $|z_t|$. However, some of the assumptions imposed later in this section, in particular the one with the series being $I(1)$, become more difficult to verify. See Saikkonen (2008).

We introduce some matrix notation. Define $X(\beta)$, $X_\gamma^*(\beta)$, y, and u as the matrices stacking $X_{t-1}'(\beta)$, $X_{t-1}'(\beta)d_{t-1}(\beta,\gamma)$, Δx_t, and u_t, respectively. Let $\lambda = \text{vec}\left(\left(A', D'\right)'\right)$, where vec stacks rows of a matrix. We denote by A_z and D_z the columns of A' and D' that are associated with $z_{t-1}(\beta)$ and $z_{t-1}(\beta)d_{t-1}(\beta,\gamma)$, respectively, and by λ_z the collection of A_z and D_z. Then, we may write

$$y = \left[\left(X(\beta), X_\gamma^*(\beta)\right) \otimes I_p\right]\lambda + u.$$

We consider the LS estimation, which minimizes

$$S_n^*(\theta) = \left(y - \left[\left(X(\beta), X_\gamma^*(\beta)\right) \otimes I_p\right]\lambda\right)' \left(y - \left[\left(X(\beta), X_\gamma^*(\beta)\right) \otimes I_p\right]\lambda\right), \tag{2}$$

where $\theta = \left(\beta', \gamma, \lambda'\right)'$. The LS estimator is then defined as

$$\hat{\theta}^* = \arg\min_\theta S_n^*(\theta),$$

where the minimum is taken over a compact parameter space Θ. The concentrated LS is computationally convenient, because it is simple ordinary least squares (OLS) for a fixed (β, γ), i.e.,

$$\lambda^*(\beta,\gamma) = \left(\begin{bmatrix} X(\beta)'X(\beta) & X(\beta)'X_\gamma^*(\beta) \\ X_\gamma^*(\beta)'X(\beta) & X_\gamma^*(\beta)'X_\gamma^*(\beta) \end{bmatrix}^{-1} \begin{pmatrix} X(\beta)' \\ X_\gamma^*(\beta)' \end{pmatrix} \otimes I_p\right) y,$$

which is then plugged back into (2) for optimization over (β, γ). In practice, the grid search over (β, γ) can be applied. In particular, the grid for β can be set

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up around a preliminary estimate of β that can be obtained based on the linear VECM, such as the Johansen's maximum likelihood estimator or the simple OLS estimator as described in Hansen and Seo (2002).

The asymptotic property of the estimator $\hat{\theta}^*$ is nonstandard because of the irregular feature of S_n^*, which does not obey a uniform law of large numbers. Thus, we take a two-step approach. First it is shown that $\hat{\beta}^* = \beta_0 + O_p\left(n^{-1/2}\right)$ by evaluating the difference between $\inf S_n^*(\theta)$ and $S_n^*(\theta_0)$, where the infimum is taken over all $\theta \in \Theta$ such that $r_n|\beta - \beta_0| > \delta$ for a sequence r_n such that $r_n \to \infty$ and $r_n/\sqrt{n} \to 0$. Similar approaches were taken by Wu (1981) and Saikkonen (1995), among others. The latter established the consistency of the maximum likelihood estimator of nonlinear transformation of β in the linear model. Second, the consistency of the short-run parameter estimates is established by the standard consistency argument using a uniform law of large numbers.

We make the following assumption for the consistency of the estimator $\hat{\theta}^*$.

Assumption 1.

(a) $\{u_t\}$ is an independent and identically distributed sequence with $\mathrm{E}u_t = 0$, $\mathrm{E}u_t u_t' = \Sigma$ that is positive definite.

(b) $\{\Delta x_t, z_t\}$ is a sequence of strictly stationary strong mixing random variables with mixing numbers α_m, $m = 1, 2, \ldots$, that satisfy $\alpha_m = o\left(m^{-(\alpha_0+1)/(\alpha_0-1)}\right)$ as $m \to \infty$ for some $\alpha_0 > 1$, and for some $\varepsilon > 0$, $\mathrm{E}\left|X_t X_t'\right|^{\alpha_0+\varepsilon} < \infty$ and $\mathrm{E}|X_{t-1}u_t|^{\alpha_0+\varepsilon} < \infty$. Furthermore, $\mathrm{E}\Delta x_t = 0$, and the partial sum process, $x_{[ns]}/\sqrt{n}, s \in [0, 1]$, converges weakly to a vector Brownian motion $\mathbf{B}$ with a covariance matrix $\mathbf{\Omega}$, which is the long-run covariance matrix of Δx_t and has rank $p-1$ such that $\left(1, \beta_0'\right)\mathbf{\Omega} = 0$. In particular, assume that $x_{2[ns]}/\sqrt{n}$ converges weakly to a vector Brownian motion B with a covariance matrix $\mathbf{\Omega}$, which is finite and positive definite.

(c) The parameter space Θ is compact and bounded away from zero for λ_z, and there is a function $\tilde{d}(x)$ that is integrable, symmetric around zero, and monotonic in $|x|$ and by which $\sup_{\gamma \in \Theta}|d(x, \gamma) - 1\{x > 0\}|$ is bounded.

(d) Let $u_t(\xi, \gamma, \lambda)$ be defined as the error u_t in (1) replacing $z_t(\beta)$ with $z_t + \xi$, where ξ belongs to a compact set in $\mathbb{R}$ and let

$$S(\xi, \gamma, \lambda) = \mathrm{E}\left(u_t'(\xi, \gamma, \lambda) u_t(\xi, \gamma, \lambda)\right).$$

Then, assume that $1/n\sum_t u_t(\xi, \gamma, \lambda)' u_t(\xi, \gamma, \lambda) \xrightarrow{p} S(\xi, \gamma, \lambda)$ uniformly in (ξ, γ, λ) on any compact set and $S(\xi, \gamma, \lambda)$ is continuous in all its arguments and is uniquely minimized at $(\xi, \gamma, \lambda) = (0, \gamma_0, \lambda_0)$.

Condition (a) is common as in Chan (1993). It simplifies our presentation but could be relaxed. Whereas we assume stationarity and mixing conditions for $\{\Delta x_t, z_t\}$, Saikkonen (2008) provides more primitive conditions on $\{u_t\}$ and the coefficients. Saikkonen's conditions are stronger than each regime satisfying the

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standard conditions in the linear VECM. We also focus on the processes without the linear time trend by assuming that $\mathrm{E}\Delta x_t = 0$.

Unlike for nonlinear models with stationary variables, the consistency proof for the nonlinear error correction models is difficult to establish on a general level. It depends crucially on the shape of the nonlinear transformation of the error correction term when the variable takes large values; see Park and Phillips (2001). Condition (c) identifies the shape, which is piecewise linear in large values of the error correction terms. It is clear that the indicator functions and logistic functions satisfy (c). This distinguishes the current work from previous research. de Jong (2002) considered nonlinearity only through a bounded function and Kristensen and Rahbek (2010) through an unbounded function, which becomes linear for the large values of the error correction z_{t-1}. Thus, they do not capture the regime-specific behavior, which makes the consistency proof much different from the previous ones. We do not consider the more general functional forms discussed in Saikkonen (2005) and Escribano and Mira (2002). The condition for λ_z is not necessary but is convenient for our proof and does not appear to be very restrictive. We note that the case with $\lambda_z = 0$ is similar to the model studied by de Jong (2002) as the error correction term appears only in a bounded function in this case. We also comment on the case where the threshold variable is $|z_{t-1}(\beta)|$. The proof of the consistency goes through almost the same, but an additional assumption on λ_z such that $A_z + D_z \neq 0$ will facilitate the direct application of the proof because $1\{|\zeta_t| > \gamma\} \sim 1\{|\zeta_t| > 0\}$ for an integrated process ζ_t.

The conditions in (d) are a standard set of conditions that are imposed to ensure the consistency of nonlinear LS estimators. A more specific set of sufficient conditions to ensure the uniform law of large numbers can be found in, e.g., Andrews (1987) or Pötscher and Prucha (1991). It can be easily checked that the commonly used smooth transition functions and the indicator functions for threshold models satisfy such conditions. It implicitly imposes the condition that $D_0 \neq 0$ as in the standard threshold model. The identification condition for ξ here is to identify the cointegrating vector at the square root n neighborhood, and it is also imposed in de Jong (2002). The conditions in Assumption 1 do not guarantee the existence of a measurable LS estimator because we do not impose the continuity of the function d. In this case, we can still establish consistency based on the convergences in outer measure; see, e.g., Newey and McFadden (1994). To ease the exposition, we implicitly assume the measurability in the theorem that follows.

THEOREM 1. *Under Assumption 1,* $\hat{\theta}^* - \theta_0$ *is* $o_p(1)$*, and furthermore,* $\sqrt{n}\left(\hat{\beta}^* - \beta_0\right) = o_p(1)$.

When the transition function $d_{t-1}(\beta, \gamma)$ satisfies a certain smoothness condition, the asymptotic distribution of $\hat{\theta}^*$ can be derived following the standard approach using the Taylor series expansion. de Jong (2002) explored minimization estimators with a nonlinear objective function that involves the error correction term. de Jong derived the asymptotic distributions of such estimators under the

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assumption that $\sqrt{n}\left(\hat{\beta}^{*}-\beta_0\right)=O_p(1)$. Thus, we refer to de Jong (2002) for the case with a smooth $d_{t-1}(\beta,\gamma)$. It is worth noting that the asymptotic distribution of the short-run parameter estimates is in general dependent on the estimation error of the cointegrating vector despite its superconsistency because of the nonlinearity of the model. On the other hand, the threshold model has not been studied because of the irregular feature of the indicator function, although the model has been adopted more widely in empirical research. We turn to the so-called threshold cointegration model and develop an asymptotics for the model in the next section.

3. THRESHOLD COINTEGRATION MODEL

Balke and Fomby (1997) introduced the threshold cointegration model, which corresponds to model (1) with $d(z_t(\beta),\gamma)=1\{z_t(\beta)>\gamma\}$, to allow for a nonlinear and/or asymmetric adjustment process to the equilibrium. That is,

$$\Delta x_t=\begin{cases} A_0+A_z\, z_{t-1}(\beta)+A_1\left(\Delta x_{t-1}',\ldots,\Delta x_{t-l+1}'\right)'+u_t, \text{ if } z_{t-1}(\beta)\le\gamma \\ B_0+B_z\, z_{t-1}(\beta)+B_1\left(\Delta x_{t-1}',\ldots,\Delta x_{t-l+1}'\right)'+u_t, \text{ if } z_{t-1}(\beta)>\gamma \end{cases}.$$

Let A be the matrix stacking A_0', A_z', and A_1' and B be defined in the same way. Then, $B=A+D$, in terms of notation in equation (1). The motivation of the model was that the magnitude and/or the sign of the disequilibrium z_{t-1} plays a central role in determining the short-run dynamics (see, e.g., Taylor, 2001; Lo and Zivot, 2001). Thus, Balke and Fomby employed the error correction term as the threshold variable. This threshold variable makes the estimation problem highly irregular as the cointegrating vector is subjected to two different sorts of nonlinearity. Even when the cointegrating vector is prespecified, the estimation is nonstandard. We introduce a smoothed estimator and study the asymptotic properties of both smoothed and unsmoothed estimators in the following sections.

To resolve the irregularity of the indicator function, Seo and Linton (2007) introduced a SLS estimator. To describe the estimator, define a bounded function $\mathcal{K}(\cdot)$ satisfying that

$$\lim_{s\to-\infty}\mathcal{K}(s)=0,\qquad \lim_{s\to+\infty}\mathcal{K}(s)=1.$$

A distribution function is often used for $\mathcal{K}$. Let $\mathcal{K}_t(\beta,\gamma)=\mathcal{K}((z_t(\beta)-\gamma)/h)$, where $h\to 0$ as $n\to\infty$. To define the smoothed objective function, we replace $d_{t-1}(\beta,\gamma)$ in (1) with $\mathcal{K}_{t-1}(\beta,\gamma)$ and define the matrix $X_\gamma(\beta)$ that stacks $X_{t-1}(\beta)\mathcal{K}_{t-1}(\beta,\gamma)$. Then we have the smoothed objective function

$$S_n(\theta)=\left(y-\left[\left(X(\beta),X_\gamma(\beta)\right)\otimes I_p\right]\lambda\right)'\left(y-\left[\left(X(\beta),X_\gamma(\beta)\right)\otimes I_p\right]\lambda\right), \qquad \textbf{(3)}$$

and the SLS estimator is defined as

$$\hat{\theta}=\underset{\theta\in\Theta}{\arg\min} S_n(\theta).$$

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Similarly to the concentrated LS estimator, we can define

$$\lambda(\beta,\gamma) = \left(\begin{bmatrix} X(\beta)'X(\beta) & X(\beta)'X_\gamma(\beta) \\ X_\gamma(\beta)'X(\beta) & X_\gamma(\beta)'X_\gamma(\beta) \end{bmatrix}^{-1} \begin{pmatrix} X(\beta)' \\ X_\gamma(\beta)' \end{pmatrix} \otimes I_p \right) y, \tag{4}$$

and by profiling we can minimize $S_n(\theta)$ with respect to (β,γ).

It is worth mentioning that the true model is a threshold model and we employ the smoothing only for estimation purposes. Because

$$\mathcal{K}\left(\frac{z_t(\beta)-\gamma}{h}\right) \to 1\{z_t(\beta) > \gamma\}$$

as $h \to 0$, $S_n(\theta)$ converges in probability to the probability limit of $S_n^*(\theta)$ as $n \to \infty$.

We make the following assumptions regarding the smoothing function $\mathcal{K}$ and the bandwidth parameter h.

Assumption 2.

(a) $\mathcal{K}$ is twice differentiable everywhere, and $\mathcal{K}^{(1)}$ is symmetric around zero, and $\mathcal{K}^{(1)}$ and $\mathcal{K}^{(2)}$ are uniformly bounded and uniformly continuous. Furthermore, $\int \left|\mathcal{K}^{(1)}(s)\right|^4 ds$, $\int \left|s^2\mathcal{K}^{(1)}(s)\right| ds$, $\int \left|\mathcal{K}^{(2)}(s)\right|^2 ds$, and $\int \left|s^2\mathcal{K}^{(2)}(s)\right| ds$ are finite.

(b) For some integer $\vartheta \geq 1$ and each integer i $(1 \leq i \leq \vartheta)$, $\int \left|s^i\mathcal{K}^{(1)}(s)\right| ds < \infty$, and

$$\int s^{i-1}\operatorname{sgn}(s)\mathcal{K}^{(1)}(s)\,ds = 0,$$

$$\int s^{\vartheta}\operatorname{sgn}(s)\mathcal{K}^{(1)}(s)\,ds \neq 0,$$

and $\mathcal{K}(x) - \mathcal{K}(0) \gtrless 0$ if $x \gtrless 0$.

(c) Furthermore, for some $\varepsilon > 0$,

$$\lim_{n\to\infty} h^{i-\vartheta} \int_{|hs|>\varepsilon} \left|s^i\mathcal{K}^{(1)}(s)\right| ds = 0,$$

$$\lim_{n\to\infty} h^{-1} \int_{|hs|>\varepsilon} \left|\mathcal{K}^{(2)}(s)\right| ds = 0.$$

(d) The sequence $\{h\}$ satisfies that for some sequence $m \geq 1$,

$$nh^3 \to 0,$$

$$\log(nm)\left(n^{1-6/r}h^2m^{-2}\right)^{-1} \to 0,$$

and

$$h^{-9k/2-3}n^{(9k/2+2)/r+\varepsilon}\alpha_m \to 0,$$

where k is the dimension of θ and $r > 4$ is specified in Assumption 4 in Section 3.1.

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These conditions are imposed in Seo and Linton (2007) and common in smoothed estimation as in, e.g., Horowitz (1992). Condition (b) is an analogous condition to that defining the so-called ϑth-order kernel and requires a kernel $\mathcal{K}^{(1)}$ that permits negative values when $\vartheta > 1$ and $\mathcal{K}(0) = \frac{1}{2}$. We impose the condition that $\mathcal{K}(x) - \mathcal{K}(0) \gtrless 0$ if $x \gtrless 0$ as we need negative kernels for $\vartheta > 1$. Condition (c) is standard. The standard normal cumulative distribution function clearly satisfies these conditions; see Seo and Linton (2007) for an example with $\vartheta > 1$. Condition (d) serves to determine the rate for h. Although this range of rates is admissible, we do not have a sharp bound and thus no optimal rate. If the data are independent and identically distributed, the conditions simplify to $\left(nh^2\right)^{-1} \log n \to 0$ and $nh^3 \to 0$. As will be shown in the following section, the smaller h implies faster convergence, whereas it may destroy the asymptotic normality if it is too small. This condition is not relevant for the consistency of $\hat{\theta}$. On the other hand, if h is too large it introduces correlation between the threshold estimate $\hat{\gamma}$ and the slope estimate $\hat{\lambda}$.

The following corollary establishes the consistency of the SLS estimator.

COROLLARY 1. *Under Assumptions 1 and 2,* $\hat{\theta} - \theta_0$ *is* $o_p(1)$*, and furthermore,* $\sqrt{n}\left(\hat{\beta} - \beta_0\right)$ *is* $o_p(1)$.

3.1. Convergence Rates and Asymptotic Distributions

The unsmoothed LS estimator of the threshold parameter is superconsistent in the standard stationary threshold regression and has a complicated asymptotic distribution, which depends not only on certain moments but on the whole distribution of data. On the contrary, the smoothed LS estimator of the same parameter exhibits asymptotic normality, whereas the smoothing slows down the convergence rate. The nonstandard nature of the estimation of threshold models becomes more complicated in threshold cointegration because the thresholding relies on the error correction term, which is estimated simultaneously with the threshold parameter γ. We begin by developing the convergence rates of the unsmoothed estimators of the cointegrating vector β and the threshold parameter γ and then explore the asymptotic distribution of the smoothed estimators.

The asymptotic behavior of the threshold estimator heavily relies on whether the model is continuous or not. We focus on the discontinuous model. We make the following assumption.

Assumption 3.

(a) The probability distribution of z_t has density with respect to Lebesque measure that is continuous, bounded, and everywhere positive, and that of z_t conditional on x_{2t} has density that is bounded by $M > 0$ for almost every x_{2t} and $t = 1, 2, \ldots$.

(b) $\mathrm{E}\left[X_{t-1}' D_0 D_0' X_{t-1} | z_{t-1} = \gamma_0\right] > 0$.

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The condition (a) ensures that the threshold parameter γ is uniquely identified. The condition on the conditional distribution of z_t may be stated as the conditional distribution of x_{1t} given x_{2t} as $z_t = x_{1t} + x_{2t}'\beta_0$. Although it is more complicated to verity the condition in general, it is easy to see that the threshold VECM without any lagged term satisfies it because it entails threshold autoregression in z_t. This remark is also relevant to Assumption 4(c). The discontinuity of the regression function is assumed in the condition (b). At the threshold point γ_0, the change in the regression function is nonzero, which makes the regression function discontinuous. This enables a superefficient estimation of the threshold parameter γ. If the regression function is continuous, Gonzalo and Wolf (2005) have shown that the LS estimator of the threshold parameter is root n consistent and asymptotically normal in the context of stationary threshold autoregression, which may be used to test for the condition (b).

We obtain the following rate result for the unsmoothed estimator of β and γ.

THEOREM 2. *Under Assumptions 1 and 3,* $\hat{\beta}^* = \beta_0 + O_p\left(n^{-3/2}\right)$ *and* $\hat{\gamma}^* = \gamma_0 + O_p\left(n^{-1}\right)$.

It is surprising that the cointegrating vector estimate converges faster than the standard n-rate. Heuristically, $\acute{\gamma} = \hat{\gamma} - x_{2t-1}'\left(\hat{\beta} - \beta_0\right)$ behaves like a threshold estimate in a stationary threshold model as

$$1\{z_{t-1}(\beta) > \gamma\} = 1\left\{z_{t-1} > \acute{\gamma}\right\}.$$

Because $\sup_{1\le t\le n}|x_{t-1}| = O_p\left(n^{1/2}\right)$ and the threshold estimate is superconsistent, it is expected that $\hat{\beta} - \beta_0 = O_p\left(n^{-3/2}\right)$. This fast rate of convergence has an important inferential implication for the short-run parameters, as will be discussed later.

We turn to the smoothed estimator for the inference of the cointegrating vector. Although subsampling is shown to be valid to approximate the asymptotic distribution of the unsmoothed LS estimator of the threshold parameter in the stationary threshold autoregression (see Gonzalo and Wolf, 2005), the extension to the threshold cointegration is not trivial because of the involved nonstationarity. The smoothing of the objective function enables us to develop the asymptotic distribution based on the standard Taylor series expansion. Let $f(\cdot)$ denote the density of z_t and $f(\cdot|\Delta)$ the conditional density given $\Delta_t = \Delta$. Also define

$$\tilde{\mathcal{K}}_1(s) = \mathcal{K}^{(1)}(s)\left(1\{s > 0\} - \mathcal{K}(s)\right)$$

and

$$\sigma_v^2 = \mathrm{E}\left[\left\|\mathcal{K}^{(1)}\right\|_2^2 \left(X_{t-1}'D_0u_t\right)^2 + \left\|\tilde{\mathcal{K}}_1\right\|_2^2 \left(X_{t-1}'D_0D_0'X_{t-1}\right)^2 | z_{t-1} = \gamma_0\right] \times f(\gamma_0), \tag{5}$$

$$\sigma_q^2 = \mathcal{K}^{(1)}(0)\,\mathrm{E}\left(X_{t-1}'D_0D_0'X_{t-1}|z_{t-1} = \gamma_0\right) f(\gamma_0). \tag{6}$$

First, we set out assumptions that we need to derive the asymptotic distribution.

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Assumption 4.

(a) $\mathrm{E}[\left|X_t u_t'\right|^r] < \infty$, $\mathrm{E}[\left|X_t X_t'\right|^r] < \infty$, for some $r > 4$.

(b) $\{\Delta x_t, z_t\}$ is a sequence of strictly stationary strong mixing random variables with mixing numbers α_m, $m = 1, 2, \ldots$, that satisfy $\alpha_m \leq Cm^{-(2r-2)/(r-2)-\eta}$ for positive C and η, as $m \to \infty$.

(c) For some integer $\vartheta \geq 2$ and each integer i such that $1 \leq i \leq \vartheta - 1$, all z in a neighborhood of γ, almost every Δ, and some $M < \infty$, $f^{(i)}(z|\Delta)$ exists and is a continuous function of z satisfying $\left|f^{(i)}(z|\Delta)\right| < M$. In addition, $f(z|\Delta) < M$ for all z and almost every Δ.

(d) The conditional joint density $f(z_t, z_{t-m}|\Delta_t, \Delta_{t-m}) < M$, for all (z_t, z_{t-m}) and almost all (Δ_t, Δ_{t-m}).

(e) θ_0 is an interior point of Θ.

These assumptions are analogous to those imposed in Seo and Linton (2007) that study the SLS estimator of the threshold regression model. The condition (a) ensures the convergence of the variance covariance estimators but can be weakened. We need a stronger mixing condition as set out in (b) than that required for consistency. The conditions (c)–(e) are common in the smoothed estimation as in Horowitz (1992), only (d) being an analogue of a random sample to a dependent sample. In particular, we require a more stringent smoothness condition (condition (c)) for the conditional density $f(z|\Delta)$ for the smoothed estimation than for the unsmoothed estimation.

We present the asymptotic distribution next.

THEOREM 3. *Suppose Assumptions 1–4 hold. Let W denote a standard Brownian motion that is independent of B. Then,*

$$\begin{pmatrix} nh^{-1/2}\left(\hat{\beta} - \beta_0\right) \\ \sqrt{nh^{-1}}\left(\hat{\gamma} - \gamma_0\right) \end{pmatrix} \Rightarrow \frac{\sigma_v}{\sigma_q^2} \begin{pmatrix} \int_0^1 BB' & \int_0^1 B \\ \int_0^1 B' & 1 \end{pmatrix}^{-1} \begin{pmatrix} \int BdW \\ W(1) \end{pmatrix},$$

$$\sqrt{n}\left(\hat{\lambda} - \lambda_0\right) \Rightarrow \mathcal{N}\left(0, \left[\mathrm{E}\begin{pmatrix} 1 & d_{t-1} \\ d_{t-1} & d_{t-1} \end{pmatrix} \otimes X_{t-1}X_{t-1}'\right]^{-1} \otimes \Sigma\right),$$

and these two random vectors are asymptotically independent. The unsmoothed estimator $\hat{\lambda}^*$ has the same asymptotic distribution as $\hat{\lambda}$.

We make some remarks on the similarities to and differences from the linear cointegration model and the stationary threshold model. First, the asymptotic distribution of $\hat{\beta}$ and $\hat{\gamma}$ is mixed normal and is the same as that of the standard OLS estimator of the cointegrating vector and the constant in the exogenous case up to the scaling factor σ_v/σ_q^2. A reading of the proof of this theorem reveals that the linear part does not contribute to the asymptotic variance of $\hat{\beta}$ although β appears both inside the indicator and in the linear part of the model. The factor

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σ_v^2/σ_q^4 contains the conditional expectation and density at the discontinuity point. It is the asymptotic variance of the threshold estimate if the true cointegrating vector were known. Other than the estimation of this factor, the inference can be made in the same way as in the standard OLS case. Second, the cointegrating vector converges faster than the usual n rate but more slowly than the $n^{3/2}$, which is obtained for the unsmoothed estimator. This is also the case for the estimators $\hat{\gamma}^*$ and $\hat{\gamma}$ for the threshold point γ. Third, as in the stationary threshold model, the slope parameter estimate $\hat{\lambda}$ is asymptotically independent of the estimation of β and γ.

The convergence rates of the estimators $\hat{\beta}$ and $\hat{\gamma}$ depend on the smoothing parameter h in such a way that the smaller h accelerates the convergence. This is in contrast to the smoothed maximum score estimation. In the extremum case where $h = 0$, we obtain the fastest convergence, which corresponds to the unsmoothed estimator. The smaller h boosts the convergence rates by reducing the bias, but too small an h destroys the asymptotic normality. We do not know the exact order of h where the asymptotic normality breaks down, which requires further research.

The asymptotic independence between the estimator $\hat{\beta}$ and the estimator $\hat{\lambda}$ of the slope parameter λ and the asymptotic normality of $\hat{\lambda}$ contrast with the result in smooth transition cointegration models, where the asymptotic distribution of $\hat{\lambda}$ not only draws on the estimation of β but is nonnormal without certain orthogonality condition (see, e.g., de Jong 2001, 2002).[2] This is due to the slower convergence of the estimators of β in the smooth transition models. Therefore, it should also be noted that the Engle–Granger type two-step approach, where the cointegrating vector is estimated by the linear regression of x_{1t} on x_{2t} and the estimate is plugged into the error correction model, does not work in our case in the sense that the estimation error affects the asymptotic distribution of $\hat{\lambda}$. Therefore, the preceding independence result is useful for the construction of a confidence interval for the slope parameter λ.

Furthermore, we may propose a two-step approach for the inference of the short-run parameters making use of the fact that the unsmoothed estimator $\hat{\beta}^*$ converges faster than the smoothed estimator $\hat{\beta}$. In principle, we can treat $\hat{\beta}^*$ as if it is the true value β_0. The following corollary states this.

COROLLARY 2. *Suppose Assumptions 1–4 hold. Let $\hat{\gamma}(\beta)$ be the smoothed estimator of γ when β is given. Then, $\hat{\gamma}(\hat{\beta}^*)$ has the same asymptotic distribution as that of $\hat{\gamma}(\beta_0)$, which is $\mathcal{N}\left(0, \frac{\sigma_v^2}{\sigma_q^4}\right)$.*

3.2. Asymptotic Variance Estimation

The construction of a confidence interval for the slope parameter λ is straightforward as $\hat{\lambda}$ and $\hat{\lambda}^*$ are just OLS estimators given (β, γ). We may treat the estimates $\hat{\beta}$ and $\hat{\gamma}$ (or $\hat{\beta}^*$ and $\hat{\gamma}^*$) as if they are β_0 and γ_0 because of Theorem 3. We may use either $1\{\hat{z}_{t-1} > \hat{\gamma}\}$ or $\mathcal{K}_{t-1}(\hat{\beta}, \hat{\gamma})$ for d_{t-1}. The inference for (β, γ) requires us to estimate Ω, σ_v^2, and σ_q^2.[3] The estimation of Ω can be done by applying a

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standard method of heteroskedasticity and autocorrelation consistent (HAC) estimation to Δx_t; see, e.g., Andrews (1991). Although σ_v^2 and σ_q^2 involve nonparametric objects such as conditional expectation and density, we do not have to do a nonparametric estimation as those are limits of the first and second derivatives of the objective function with respect to the threshold parameter γ.

Thus, let

$$\hat{\tau}_t = \frac{1}{2\sqrt{h}} X_{t-1}\left(\hat{\beta}\right)' D\mathcal{K}_{t-1}^{(1)}\left(\hat{\beta}, \hat{\gamma}\right)\hat{u}_t, \tag{7}$$

where $\hat{u}_t$ is the residual from the regression (1), and let

$$\hat{\sigma}_v^2 = \frac{1}{n}\sum_t \hat{\tau}_t^2 \quad \text{and} \quad \hat{\sigma}_q^2 = \frac{h}{2n} Q_{n22}\left(\hat{\theta}\right),$$

where Q_{n22} is the diagonal element corresponding to γ of the Hessian matrix Q_n; see the Appendix for the explicit formulas. Consistency of $\hat{\sigma}_q^2$ is straightforward from the proof of Theorem 3, and that of $\hat{\sigma}_v^2$ can be obtained after a slight modification of Theorem 4 of Seo and Linton (2007).

We can construct a confidence interval for γ based on Corollary 2. The estimation of σ_v^2 and σ_q^2 can be done as before with $\beta = \hat{\beta}^*$. Because of the asymptotic normality and independence, the construction of the confidence interval is much simpler this way without the need to estimate $\mathbf{\Omega}$.

Even though $\hat{\lambda}$ and $\hat{\lambda}^*$ are asymptotically independent of $\left(\hat{\beta}, \hat{\gamma}\right)$ and $\left(\hat{\beta}^*, \hat{\gamma}^*\right)$, they are dependent in finite samples. So, we may not benefit from the imposition of the block diagonal feature of the asymptotic variance matrix. Corollary 2 enables the standard way of constructing a confidence interval based on the inversion of the t-statistic with jointly estimated covariance matrix. In this case, we may define τ_t in (7) using the score of $u_t(\theta)$ with respect to (γ, λ) for a given $\hat{\beta}^*$. See Seo and Linton (2007) for more discussion.

4. MONTE CARLO EXPERIMENTS

This section investigates the finite-sample performance of the estimators explored in this paper. Of particular interest are the various estimators of the cointegrating vector β and the threshold parameter γ. We compare the unsmoothed LS estimator $\hat{\beta}^*$ of β with the SLS estimator $\hat{\beta}$ and the Johansen's maximum likelihood estimator $\tilde{\beta}$, which is based on the linear VECM. For comparison purpose, we also compute the restricted estimators $\hat{\beta}_0^*$ and $\hat{\beta}_0$, which are the unsmoothed LS and SLS estimators of β when γ is fixed at the true value γ_0. Similarly, $\hat{\gamma}_0^*$ and $\hat{\gamma}_0$ denote the restricted unsmoothed LS and SLS estimators of γ when β is prespecified at the true value β_0. To distinguish the SLS estimator $\hat{\gamma}$ from the two-step SLS estimator, let $\hat{\gamma}_2$ denote the two-step estimator.

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The simulation samples are generated from the following process:

$$\Delta x_t = \begin{pmatrix} -1 \\ 0 \end{pmatrix} (x_{1t-1} - \beta_0 x_{2t-1}) + \begin{pmatrix} -2 \\ 0 \end{pmatrix} 1\{x_{1t-1} - \beta_0 x_{2t-1} \le \gamma_0\} + \begin{pmatrix} -0.5 \\ 0 \end{pmatrix} (x_{1t-1} - \beta_0 x_{2t-1}) 1\{x_{1t-1} - \beta_0 x_{2t-1} > \gamma_0\} + u_t,$$

where $u_t \sim iid\mathcal{N}(0, I_2)$, $t = 0, \ldots, n$, and $\Delta x_0 = u_0$. This process was considered by Hansen and Seo (2002), who provide us with the finite-sample performance of the maximum likelihood estimator of β and γ. We fix $\beta_0 = 1$ and $\gamma_0 = 0$. Although the data generating process does not contain any lagged first difference term, the model is estimated with two lagged terms in addition to the error correction term. The estimation is based on the grid search with grid sizes for β and γ being 100 and 500, respectively. The grid for β is set around the Johansen's maximum likelihood estimator $\tilde{\beta}$ as in Hansen and Seo (2002). For the smoothed estimators, we use the standard normal distribution function for $\mathcal{K}$ and set $h = \hat{\sigma} n^{-1/2} \log n$, where $\hat{\sigma}^2$ is the sample variance of the error correction term.

Table 1 summarizes the results of our experiments with sample sizes $n = 100, 250$, and 500. We examined the finite-sample distributions of the various estimators in terms of mean, root mean squared error (RMSE), mean absolute error (MAE), and selected percentiles from 1,000 simulation replications. The RMSE and MAE are reported in log.

The results appear as we expected. The Johansen's maximum likelihood estimator $\tilde{\beta}$ does not perform as well as all the other estimators, which are obtained from estimating the correctly specified threshold cointegration model. The unsmoothed estimators and the restricted estimators outperform the smoothed and the unrestricted counterparts, respectively, in terms of RMSE and MAE. However, careful examination of percentiles reveals that the smoothed estimator $\hat{\beta}$ exhibits a smaller length of interval between the 5th percentile and the 95th percentile than the unsmoothed estimator $\hat{\beta}^*$. And the percentiles of all the estimators of β seem very symmetric around the medians, which are mostly zeros.

Similar observation is made for the comparison among the estimators of the threshold parameter γ. The unsmoothed estimators have smaller RMSE and MAE than the smoothed ones. The knowledge of the true value of β helps reduce RMSE and MAE of the estimators of γ. The two-step estimator $\hat{\gamma}_2$, which employs a more efficient estimator, $\hat{\beta}^*$, than $\hat{\beta}$, indeed improves upon $\hat{\gamma}$ and even has a smaller RMSE than the restricted estimator $\hat{\gamma}_0$ when $n = 250$. The distributions of all the estimators of γ appear quite asymmetric and have large negative biases when $n = 100$. However, for $n = 500$ the distributions become more or less symmetric around the medians, and the biases are much smaller than those for $n = 100$. We also note that the biases of the unsmoothed estimators are much bigger than those of smoothed estimators when $n = 100$ and $n = 250$.

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TABLE 1. Distribution of estimators

	Mean	RMSE in log	MAE in log	Percentile (%) 5	25	50	75	95
$n = 100$								
$\widetilde{\beta} - \beta_0$	0.001	−2.841	−3.191	−0.083	−0.028	0.000	0.028	0.098
$\widehat{\beta}^* - \beta_0$	0.001	−2.880	−3.293	−0.082	−0.023	−0.001	0.023	0.091
$\widehat{\beta}_0^* - \beta_0$	−0.001	−3.009	−3.617	−0.063	−0.015	−0.001	0.011	0.062
$\widehat{\beta} - \beta_0$	−0.001	−2.854	−3.268	−0.085	−0.025	0.001	0.025	0.084
$\widehat{\beta}_0 - \beta_0$	−0.002	−2.947	−3.470	−0.068	−0.020	−0.001	0.019	0.068
$\widehat{\gamma}^* - \gamma_0$	−0.672	0.138	−0.284	−2.649	−1.153	−0.293	−0.032	0.258
$\widehat{\gamma}_0^* - \gamma_0$	−0.631	0.080	−0.408	−2.617	−1.023	−0.209	−0.038	0.123
$\widehat{\gamma} - \gamma_0$	−0.461	0.149	−0.265	−2.702	−0.945	−0.058	0.241	0.700
$\widehat{\gamma}_0 - \gamma_0$	−0.444	0.090	−0.380	−2.719	−0.862	−0.021	0.192	0.504
$\widehat{\gamma}_2 - \gamma_0$	−0.497	0.146	−0.272	−2.802	−1.014	−0.086	0.219	0.582
$n = 250$								
$\widetilde{\beta} - \beta_0$	0.000	−3.960	−4.241	−0.030	−0.011	0.000	0.011	0.031
$\widehat{\beta}^* - \beta_0$	0.001	−4.302	−4.637	−0.022	−0.007	0.001	0.007	0.023
$\widehat{\beta}_0^* - \beta_0$	0.000	−4.639	−5.133	−0.014	−0.003	0.000	0.003	0.014
$\widehat{\beta} - \beta_0$	0.000	−4.278	−4.626	−0.022	−0.007	0.000	0.007	0.020
$\widehat{\beta}_0 - \beta_0$	0.000	−4.540	−4.866	−0.019	−0.006	0.000	0.006	0.016
$\widehat{\gamma}^* - \gamma_0$	−0.116	−0.928	−1.778	−0.734	−0.108	−0.031	0.015	0.156
$\widehat{\gamma}_0^* - \gamma_0$	−0.102	−1.007	−2.085	−0.526	−0.069	−0.021	−0.002	0.069
$\widehat{\gamma} - \gamma_0$	−0.057	−0.888	−1.677	−0.746	−0.073	0.019	0.113	0.236
$\widehat{\gamma}_0 - \gamma_0$	−0.049	−0.901	−1.826	−0.622	−0.043	0.030	0.100	0.184
$\widehat{\gamma}_2 - \gamma_0$	−0.054	−0.917	−1.704	−0.769	−0.071	0.025	0.105	0.236
$n = 500$								
$\widetilde{\beta} - \beta_0$	0.000	−2.006	−2.147	−0.015	−0.005	0.000	0.006	0.015
$\widehat{\beta}^* - \beta_0$	−0.000	−2.279	−2.425	−0.009	−0.003	−0.000	0.002	0.008
$\widehat{\beta}_0^* - \beta_0$	−0.000	−2.491	−2.698	−0.005	−0.001	−0.000	0.001	0.005
$\widehat{\beta} - \beta_0$	−0.000	−2.253	−2.383	−0.009	−0.003	−0.000	0.003	0.008
$\widehat{\beta}_0 - \beta_0$	−0.000	−2.335	−2.476	−0.007	−0.003	−0.000	0.002	0.008
$\widehat{\gamma}^* - \gamma_0$	−0.006	−1.165	−1.346	−0.115	−0.035	−0.004	0.020	0.096
$\widehat{\gamma}_0^* - \gamma_0$	−0.010	−1.438	−1.636	−0.068	−0.022	−0.007	0.002	0.045
$\widehat{\gamma} - \gamma_0$	0.032	−1.080	−1.199	−0.092	−0.013	0.030	0.076	0.155
$\widehat{\gamma}_0 - \gamma_0$	0.031	−1.213	−1.319	−0.051	−0.007	0.031	0.061	0.122
$\widehat{\gamma}_2 - \gamma_0$	0.028	−1.102	−1.229	−0.086	−0.015	0.025	0.068	0.146

Note: Notation is as follows: Johansen's maximum likelihood estimator, $\tilde{\beta}$; the unsmoothed estimators, $\hat{\cdot}^*$; the restricted unsmoothed estimators, $\hat{\cdot}_0^*$; the smoothed estimators, $\hat{\cdot}$; the restricted smoothed estimators, $\hat{\cdot}_0$; and finally the two-step estimator $\hat{\gamma}_2$.

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5. CONCLUSION

We have established the consistency of the LS estimators of the cointegrating vector in general regime-switching VECMs, validating the application of some of the existing results on the joint estimation of long-run and short-run parameters for models with smooth transition. We also provided asymptotic inference methods for threshold cointegration, establishing the convergence rates and asymptotic distributions of the LS and SLS estimators of the model parameters. In particular, the theory and Monte Carlo experiments indicate that the inference on the threshold parameter can be improved upon by using the two-step estimator.

Although we only considered two-regime models, our results might be extended to multiple-regime models, provided that the assumptions on stationarity and invariance principle in Assumption 1 hold true. In that case, we may consider the sequential estimation strategy discussed in Bai and Perron (1998) and Hansen (1999). A sequence of estimations and tests can determine the number of regimes and the threshold parameter. The Lagrange multiplier test of Hansen and Seo (2002) can be employed to test for the presence of the second break without modification because of the fast rate of convergence of the cointegrating vector estimators.

It is also possible to think of the case with more than one cointegrating relation if p is greater than 2. In this case, the threshold variable can be understood as a linear combination of those cointegrating vectors. However, the models commonly used in empirical applications are bivariate, and the estimation of such a model is more demanding. Thus we leave it for future research.

NOTES

1. It provides an orthogonality condition, under which the two-step approach is valid.
2. In case of $Ez_t = 0$, we can still retain the asymptotic normality of the slope estimate by estimating (1) after replacing the $z_{t-1}(\beta)$ with $\bar{z}_{t-1}\left(\tilde{\beta}\right) = \left(1, \tilde{\beta}'\right)\left(x_{t-1} - (1/n)\sum_s x_{s-1}\right)$, for any n-consistent $\tilde{\beta}$, as in de Jong (2001). It is worth noting, however, that this demeaning increases the asymptotic variance.
3. In the MLE of linear VECM, the likelihood ratio statistic for a hypothesis on β converges to the chi-square distribution as the log likelihood function under normality is approximately quadratic. It will be interesting to examine if the same holds true for the threshold cointegration model.

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APPENDIX: Proofs of Theorems

A word on notation. Throughout this section, C stands for a generic constant that is finite.

The proof of Theorem 1 makes use of the following lemma, which might be of independent interest.

LEMMA 1. *Let $\{Z_n\}$ be a sequence of random variables and $\{r_n\}$ be a sequence of positive numbers such that $r_n \to \infty$. If $a_n Z_n = o_p(1)$ for any sequence $\{a_n\}$ such that $a_n / r_n \to 0$, then*

$$r_n Z_n = O_p(1).$$

Proof of Lemma 1. Let $X_n = r_n Z_n$ and suppose $X_n \neq O_p(1)$. Then, there exists $\varepsilon > 0$ such that for any K,

$$\limsup_{n\to\infty} \Pr(|X_n| > K) > \varepsilon.$$

Thus, there exists n_1 such that $\Pr\left(\left|X_{n_1}\right| > 1\right) > \varepsilon$. Similarly one can find $n_2 > n_1$ such that $\Pr\left(\left|X_{n_2}\right| > 2\right) > \varepsilon$, and $n_3 < n_4 < \ldots$, accordingly. Let $b_n = 1$ for $n \leq n_1$, $b_n = 2$ for $n_1 < n \leq n_2$, etc. Then, it is clear from the construction that $b_n \to \infty$ and that $\Pr(|X_n| > b_n) > \varepsilon$, infinitely often, which implies that $X_n / b_n \neq o_p(1)$. However, given the condition of the lemma, $X_n / b_n = (r_n / b_n) Z_n = o_p(1)$ because $(r_n / b_n) / r_n \to 0$. This yields contradiction. ■

Proof of Theorem 1. Let $\Theta_{r_n,\delta} = \{\theta \in \Theta : r_n |\beta - \beta_0| > \delta\}$. Supremums and infimums in this proof are taken on the set $\Theta_{r_n,\delta}$, unless otherwise specified. To show that

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$\hat{\beta}^* - \beta_0 = o_p(r_n^{-1})$ it is sufficient to show that for every $\delta > 0$,

$$\Pr\left\{\inf_{\theta} S_n^*(\theta)/n - S_n^*(\theta_0)/n > 0\right\} \to 1. \tag{A.1}$$

Let $X_{\beta,\gamma}^* = \left(X(\beta), X_\gamma^*(\beta)\right) \otimes I_p$ and rewrite (2) as

$$S_n^*(\theta) = y'y + \lambda' X_{\beta,\gamma}^{*\prime} X_{\beta,\gamma}^* \lambda - 2y' X_{\beta,\gamma}^* \lambda.$$

Let $\eta = \sqrt{n}(\beta - \beta_0)$ and r_n be a sequence of real numbers such that $\sqrt{n} \geq r_n \to \infty$ and $r_n/\sqrt{n} \to 0$ as $n \to \infty$. Then,

$$|\eta| = r_n |\beta - \beta_0| \left(\sqrt{n}/r_n\right) \geq \delta\sqrt{n}/r_n \to \infty \tag{A.2}$$

for any $\beta \in \Theta_{r_n,\delta}$.

Note that

$$\frac{1}{n} S_n^*(\theta) - \frac{1}{n} S_n^*(\theta_0) = \frac{1}{n} \lambda' X_{\beta,\gamma}^{*\prime} X_{\beta,\gamma}^* \lambda - \frac{2}{n} y' X_{\beta,\gamma}^* \lambda + \frac{1}{n} y'y - \frac{1}{n} S_n^*(\theta_0)$$

and that $y'y/n - S_n^*(\theta_0)/n = y'y/n - u'u/n$ converges in probability to a positive constant as in the standard linear regression as a result of Assumption 1(c) and this term is free of the parameter θ. Therefore, it is sufficient to show that

$$\begin{aligned}
&\Pr\left\{\inf_{\theta}\left(\frac{1}{n}\lambda' X_{\beta,\gamma}^{*\prime} X_{\beta,\gamma}^* \lambda - 2\frac{1}{n} y' X_{\beta,\gamma}^* \lambda\right) \geq 0\right\} \\
&\quad = \Pr\left\{\inf_{\theta}\left(|\eta| \frac{1}{n} \lambda' \frac{X_{\beta,\gamma}^{*\prime} X_{\beta,\gamma}^*}{|\eta|^2} \lambda - 2\frac{1}{n} \frac{y' X_{\beta,\gamma}^*}{|\eta|} \lambda\right) \geq 0\right\} \\
&\quad \to 1,
\end{aligned} \tag{A.3}$$

as $n \to \infty$. This follows from (A.2) if we show that (a)

$$\sup_{\theta}\left|\frac{1}{n} \frac{y' X_{\beta,\gamma}^*}{|\eta|} \lambda\right| = O_p(1) \tag{A.4}$$

and (b) $\inf_\theta (1/n)\lambda'(X_{\beta,\gamma}^{*\prime} X_{\beta,\gamma}^*/|\eta|^2)\lambda$ converges weakly to a random variable that is positive with probability one.

We show (a) first. Note that $y' X_{\beta,\gamma}^*/n$ consists of the sample means of the product of Δx_t and $\left(1, z_{t-1}(\beta), \Delta_{t-1}'\right)$ and that of Δx_t and $\left(1, z_{t-1}(\beta), \Delta_{t-1}'\right) d_{t-1}(\beta, \gamma)$. However, as λ and $d_{t-1}(\beta, \gamma)$ are bounded, it is sufficient to observe that $\frac{1}{n}\sum_t |\Delta x_t'|$ and $\frac{1}{n}\sum_t \left|\Delta x_t \Delta_{t-1}'\right|$ are $O_p(1)$ and that

$$\frac{1}{n|\eta|}\sum_t \left|z_{t-1}(\beta)\Delta x_t'\right| \leq \frac{1}{n\delta}\sum_t \left|z_{t-1}\Delta x_t'\right| + \frac{1}{n}\sum_t \left|\Delta x_t \frac{x_{2t-1}'}{\sqrt{n}}\right| = O_p(1),$$

by the law of large numbers for $\left|z_{t-1}\Delta x_t'\right|$, the weak convergence of $x_{2[ns]}/\sqrt{n}$, and the Cauchy–Schwarz inequality.

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Consider (b) now. Let $\dot{\eta} = \eta/|\eta|$, which can be treated as a parameter in a unit circle independent of n. Because $(1/n)X^{*'}_{\beta,\gamma} X^{*}_{\beta,\gamma}$ is the matrix of the sample means of the outer product of

$$\left(\left(1, z_{t-1}(\beta), \Delta'_{t-1}\right), \left(1, z_{t-1}(\beta), \Delta'_{t-1}\right) d_{t-1}(\beta,\gamma)\right)',$$

we may write

$$\frac{1}{n|\eta|^2} \lambda' X^{*'}_{\beta,\gamma} X^{*}_{\beta,\gamma} \lambda = \lambda'_z \left(\Xi^*_n(\beta,\gamma) \otimes I_p\right) \lambda_z + R_n(\theta),$$

where

$$\Xi^*_n(\beta,\gamma) = \begin{pmatrix} \frac{1}{n^2}\Sigma_t \left(x'_{2t-1}\dot{\eta}\right)^2 & \frac{1}{n^2}\Sigma_t \left(x'_{2t-1}\dot{\eta}\right)^2 d_{t-1}(\beta,\gamma) \\ \frac{1}{n^2}\Sigma_t \left(x'_{2t-1}\dot{\eta}\right)^2 d_{t-1}(\beta,\gamma) & \frac{1}{n^2}\Sigma_t \left(x'_{2t-1}\dot{\eta}\right)^2 d^2_{t-1}(\beta,\gamma) \end{pmatrix}$$

and

$$\sup_\theta |R_n(\theta)| = O_p\left(\left|\frac{r_n}{\sqrt{n}}\right|^2\right),$$

by the same reasoning used to show (a). Because λ_z is bounded away from zero by Assumption 1(c), it is sufficient to show that

$$\Xi^*_n(\beta,\gamma) \Rightarrow \left(\dot{\eta}'\Omega\dot{\eta}\right) \begin{pmatrix} \int_0^1 W^2 & \int_0^1 W^2 1\{W>0\} \\ \int_0^1 W^2 1\{W>0\} & \int_0^1 W^2 1\{W>0\} \end{pmatrix}, \tag{A.5}$$

where W is the standard Brownian motion. Note that the matrix in (A.5) is positive definite and free of parameters up to a constant multiple $\left(\dot{\eta}'\Omega\dot{\eta}\right)$, which is positive and bounded away from zero.

Now we show (A.5). It follows from Assumption 1(b) and the continuous mapping theorem that

$$\frac{1}{n^2}\sum_t \left(x'_{2t-1}\dot{\eta}\right)^2 \Rightarrow \int_0^1 \left(B'\dot{\eta}\right)^2 =^d \int_0^1 W^2 \left(\dot{\eta}'\Omega\dot{\eta}\right) \tag{A.6}$$

and that

$$\begin{aligned} \frac{1}{n^2}\sum_t \left(x'_{2t-1}\dot{\eta}\right)^2 1\left\{x'_{2t-1}\dot{\eta}>0\right\} &\Rightarrow \int_0^1 \left(B'\dot{\eta}\right)^2 1\left\{B'\dot{\eta}>0\right\} \\ &=^d \left(\dot{\eta}'\Omega\dot{\eta}\right) \int_0^1 W^2 1\{W>0\}. \end{aligned}$$

Then, it remains to show that

$$\sup_\theta \left|\frac{1}{n^2}\sum_t \left(x'_{2t-1}\dot{\eta}\right)^2 \left(1\left\{x'_{2t-1}\dot{\eta}>0\right\} - d\left(z_{t-1}(\beta),\gamma\right)\right)\right| = o_p(1) \tag{A.7}$$

and that

$$\sup_\theta \left|\frac{1}{n^2}\sum_t \left(x'_{2t-1}\dot{\eta}\right)^2 \left(1\left\{x'_{2t-1}\dot{\eta}>0\right\} - d^2\left(z_{t-1}(\beta),\gamma\right)\right)\right| = o_p(1). \tag{A.8}$$

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Because $\left(1\{x>0\}-g^2(x)\right) \le \left(\sup_{x\in\mathbb{R}}|g(x)|+1\right)|1\{x>0\}-g(x)|$ for any bounded function g and $\sup_{|\dot\eta|=1,t=1,\ldots,n}\left(x'_{2t-1}\dot\eta/\sqrt{n}\right)^2 = O_p(1)$, it is sufficient to show that

$$\sup_\theta \frac{1}{n}\sum_t \left|d\left(z_{t-1}(\beta),\gamma\right)-1\left\{x'_{2t-1}\dot\eta>0\right\}\right| \le R_{1n}+R_{2n}=o_p(1),$$

where

$$R_{1n}=\sup_\theta \frac{1}{n}\sum_t \left|1\left\{z_{t-1}(\beta)>0\right\}-1\left\{x'_{2t-1}\dot\eta>0\right\}\right|,$$
$$R_{2n}=\sup_\theta \frac{1}{n}\sum_t \left|d\left(z_{t-1}(\beta),\gamma\right)-1\left\{z_{t-1}(\beta)>0\right\}\right|.$$

Because of (A.2) and the fact that $z_{t-1}(\beta)=z_{t-1}+(x'_{2t-1}\dot\eta/\sqrt{n})|\eta|$,

$$\frac{1}{n}\sum_t \left|1\left\{z_{t-1}(\beta)>0\right\}-1\left\{x'_{2t-1}\dot\eta>0\right\}\right| \le \frac{1}{n}\sum_t 1\left\{\left|\frac{x'_{2t-1}\dot\eta}{\sqrt{n}}\right|\le\left|\frac{\delta r_n z_{t-1}}{\sqrt{n}}\right|\right\}. \quad \textbf{(A.9)}$$

For any $m_n>0$,

$$\begin{aligned}\sup_{|\dot\eta|=1} 1\left\{\left|\frac{x'_{2[ns]}\dot\eta}{\sqrt{n}}\right|\le\left|\frac{\delta r_n z_{t-1}}{\sqrt{n}}\right|\right\} &\le 1\left\{\inf_{|\dot\eta|=1}\left|\frac{x'_{2[ns]}\dot\eta}{\sqrt{n}}\right|\le\left|\frac{\delta r_n z_{t-1}}{\sqrt{n}}\right|\right\}\\ &\le 1\left\{\inf_{|\dot\eta|=1}\left|\frac{x'_{2[ns]}\dot\eta}{\sqrt{n}}\right|\le m_n\right\}+1\left\{\left|\frac{\delta r_n z_{t-1}}{\sqrt{n}}\right|>m_n\right\}.\end{aligned} \quad \textbf{(A.10)}$$

Consider the first term in (A.10). Because $(x'_{2[ns]}\dot\eta/\sqrt{n})$ converges weakly on $\{0\le s\le 1\}\times\{\dot\eta:|\dot\eta|=1\}$,

$$\mathrm{E1}\left\{\inf_{|\dot\eta|=1}\left|\frac{x'_{2[ns]}\dot\eta}{\sqrt{n}}\right|\le m_n\right\}\to \Pr\left\{\inf_{|\dot\eta|=1}\left|B'_s\dot\eta\right|\le 0\right\}=0$$

for any $s\in(0,1]$ and for any decreasing sequence $m_n\to 0$. And for the second term in (A.10), we observe that $\mathrm{E1}\left\{\left|\frac{\delta r_n z_{t-1}}{\sqrt{n}}\right|>m_n\right\}$ is the same for all t and that it goes to zero as $m_n\sqrt{n}/r_n\to\infty$. Consider the right-hand side term of (A.9). Letting $m_n\to 0$ and $m_n\sqrt{n}/r_n\to\infty$, we observe that

$$\mathrm{E}\frac{1}{n}\sum_t 1\left\{\left|\frac{x'_{2t-1}\dot\eta}{\sqrt{n}}\right|\le\left|\frac{\delta r_n z_{t-1}}{\sqrt{n}}\right|\right\}=\int_0^1 \mathrm{E1}\left\{\left|\frac{x'_{2[ns]}\dot\eta}{\sqrt{n}}\right|\le\left|\frac{\delta r_n z_{[ns]}}{\sqrt{n}}\right|\right\}ds\to 0,$$

by the dominated convergence theorem. This shows that $R_{1n}=o_p(1)$.

Next, it follows from Assumption 1(c) that

$$\sup_\theta \left|d\left(z_{t-1}(\beta),\gamma\right)-1\left\{z_{t-1}(\beta)>0\right\}\right| \le \tilde d\left(\inf_\eta\left|z_{[ns]}+\frac{x'_{2[ns]}\eta}{\sqrt{n}}\right|\right), \quad \textbf{(A.11)}$$

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where $[ns] = t-1$. Because of (A.2), for each s and for any sequence $\{\varepsilon_n\}$ such that $\varepsilon_n \to 0$ and $\varepsilon_n \sqrt{n}/r_n \to \infty$ as $n \to \infty$,

$$\begin{aligned}
&\mathrm{E}\left[\tilde{d}\left(\inf_{\eta}\left|z_{[ns]} + \frac{x'_{2[ns]}\eta}{\sqrt{n}}\right|\right)\right] \\
&\quad \le \mathrm{E}\left[\tilde{d}\left(\delta\frac{\sqrt{n}}{r_n}\left(\inf_{|\dot{\eta}|=1}\left|\frac{x'_{2[ns]}\dot{\eta}}{\sqrt{n}}\right| - \varepsilon_n\right)\right) 1\left\{\sup_{\eta}\left|\frac{z_{[ns]}}{|\eta|}\right| \le \varepsilon_n\right\}\right. \\
&\qquad \left. \times 1\left\{\inf_{|\dot{\eta}|=1}\left|\frac{x'_{2[ns]}\dot{\eta}}{\sqrt{n}}\right| > 2\varepsilon_n\right\}\right] \\
&\qquad + C\mathrm{E}\left(1\left\{\left|\frac{z_{[ns]}}{\delta\sqrt{n}}r_n\right| > \varepsilon_n\right\} + 1\left\{\inf_{|\dot{\eta}|=1}\left|\frac{x'_{2[ns]}\dot{\eta}}{\sqrt{n}}\right| \le 2\varepsilon_n\right\}\right).
\end{aligned}$$

It is obvious that $\mathrm{E}1\left\{\inf_{|\dot{\eta}|=1}\left|\frac{x'_{2[ns]}\dot{\eta}}{\sqrt{n}}\right| < 2\varepsilon_n\right\} \to 0$ for any $s \in (0,1]$ because $\inf_{|\dot{\eta}|=1}\left|\frac{x'_{2[ns]}\dot{\eta}}{\sqrt{n}}\right| \Longrightarrow \inf_{|\dot{\eta}|=1} B_s\dot{\eta}$. Because of the fact that $\varepsilon_n\sqrt{n}/r_n \to \infty$, we have $\mathrm{E}1\left\{\left|(z_{[ns]}/\delta\sqrt{n})r_n\right| > \varepsilon_n\right\} \to 0$, and

$$\begin{aligned}
&\mathrm{E}\tilde{d}\left(\delta\frac{\sqrt{n}}{r_n}\left(\inf_{|\dot{\eta}|=1}\left|\frac{x'_{2[ns]}\dot{\eta}}{\sqrt{n}}\right| - \varepsilon_n\right)\right) 1\left\{\sup_{\eta}\left|\frac{z_{[ns]}}{|\eta|}\right| \le \varepsilon_n\right\} 1\left\{\inf_{|\dot{\eta}|=1}\left|\frac{x'_{2[ns]}\dot{\eta}}{\sqrt{n}}\right| > 2\varepsilon_n\right\} \\
&\quad \le \tilde{d}\left(\delta\varepsilon_n\sqrt{n}/r_n\right) \to 0.
\end{aligned}$$

Thus, it follows from the dominated convergence theorem that the integral of (A.11) over s on the interval $[0,1]$ goes to zero. This shows $R_{2n} = o_p(1)$, thus completing the proof of (A.1). Then, it follows from Lemma 1 that $\sqrt{n}\left(\hat{\beta}^* - \beta_0\right) = O_p(1)$.

We now show that $\hat{\theta} - \theta = o_p(1)$ and $\sqrt{n}\left(\hat{\beta}^* - \beta_0\right) = o_p(1)$. When $d(x,\gamma)$ is continuous in all its arguments, we can resort to Theorem 1 of de Jong (2002). And it can be readily generalized to the case where the limit objective function is continuous. Recalling Assumption 1(d) and reading the proof of Theorem 1 of de Jong (2002), we see that we only have to show his equation (A.9) with $a_n = 0$, which corresponds to

$$\frac{1}{n}\sum_t \left[u_t\left(\frac{x'_{2t-1}\eta}{\sqrt{n}},\lambda,\gamma\right)' u_t\left(\frac{x'_{2t-1}\eta}{\sqrt{n}},\lambda,\gamma\right) - S\left(\frac{x'_{2t-1}\eta}{\sqrt{n}},\lambda,\gamma\right)\right] \xrightarrow{p} 0 \qquad \textbf{(A.12)}$$

uniformly in (η,γ) on any compact set. However, for any η and γ and $C > 0$, the term in (A.12) is bounded by

$$\sup_{\lambda,\gamma,|\xi|\le C}\left|\frac{1}{n}\sum_t\left[u_t(\xi,\gamma,\lambda)' u_t(\xi,\gamma,\lambda) - S(\xi,\gamma,\lambda)\right]\right| + \frac{1}{n}\sum_t 1\left\{\sup_{\eta}\left|\frac{x'_{2t-1}\eta}{\sqrt{n}}\right| > C\right\},$$

whose first term converges to zero in probability by assumption. The second term can be made arbitrarily small by choosing C large enough because of the weak convergence of $\frac{x'_{2t-1}\eta}{\sqrt{n}}$. Therefore, the proof is complete. ■

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Proof of Corollary 1. As in the proof of Theorem 1, we show that

$$\Pr\left\{\inf_{\theta} S_n(\theta)/n - S_n(\theta_0)/n > 0\right\} \to 1,$$

where we follow the notational convention in the proof of the theorem that infimums and supremums are assumed to be taken over $\Theta_{r_n,\delta}$ unless specified otherwise. As $S_n(\theta_0)/n$ does not contain any $I(1)$ variable, the result in Seo and Linton (2007) applies so that it is sufficient to show (A.3) with $X^*_{\beta,\gamma}$ replaced by $X_{\beta,\gamma} = \left(X(\beta), X_\gamma(\beta)\right) \otimes I_p$. However, (A.4) is obvious as $\mathcal{K}$ is bounded. Thus, we need to show (A.5), i.e.,

$$\Xi_n(\beta,\gamma) = \begin{pmatrix} \frac{1}{n^2}\Sigma_t \left(x'_{2t-1}\dot{\eta}\right)^2 & \frac{1}{n^2}\Sigma_t \left(x'_{2t-1}\dot{\eta}\right)^2 \mathcal{K}_{t-1}(\beta,\gamma) \\ \frac{1}{n^2}\Sigma_t \left(x'_{2t-1}\dot{\eta}\right)^2 \mathcal{K}_{t-1}(\beta,\gamma) & \frac{1}{n^2}\Sigma_t \left(x'_{2t-1}\dot{\eta}\right)^2 \mathcal{K}^2_{t-1}(\beta,\gamma) \end{pmatrix}$$
$$\Rightarrow \left(\dot{\eta}'\Omega\dot{\eta}\right) \begin{pmatrix} \int_0^1 W^2 & \int_0^1 W^2 1\{W>0\} \\ \int_0^1 W^2 1\{W>0\} & \int_0^1 W^2 1\{W>0\} \end{pmatrix},$$

which follows if we show (A.7) and (A.8) with d replaced by $\mathcal{K}$. The proof hinges on (A.11), which is the part where d matters. However, it still holds replacing d with $\mathcal{K}$ and taking the supremum over both θ and h on any interval including zero. This establishes that $\hat{\beta}$ is $\sqrt{n}$-consistent. The remaining part of the proof follows if the conditions on $\mathcal{K}$ satisfy Assumption 1(d). In other words, defining $\tilde{u}_t(\xi,\gamma,\lambda)$ like $u_t(\xi,\gamma,\lambda)$ replacing the indicator with $\mathcal{K}$, we need to show the uniform convergence of its sample mean to $S(\xi,\gamma,\lambda)$, which follows from Seo and Linton (2007) as $\tilde{u}_t(\xi,\gamma,\lambda)$ consists of stationary variables. This completes the proof. ■

Proof of Theorem 2. A word on notation. Having proved Theorem 1 and Corollary 1, we write hereafter $\theta = \left(\eta', \gamma, \lambda'\right)'$ and $\theta_1 = \left(\eta', \gamma\right)'$ with slight abuse of notation and consider the parameter space $\Theta_c = \{\theta : |\theta - \theta_0| < c\}$ for some $c > 0$, which will be specified subsequently. To further simplify notation assume that $\gamma_0 = 0$ and thus θ_{10} is a vector of zeros.

It is sufficient to show the following claim for θ_1 to be n-consistent as in Chan (1993).

CLAIM I. *For any* $\varepsilon > 0$, *there exist* $c > 0$ *and* $K > 0$ *such that*

$$\liminf_{n\to\infty} \Pr\left\{\inf_{\theta\in\Theta_{c,K}} \left[S_n(\theta) - S_n(0,\lambda)\right] > 0\right\} > 1 - \varepsilon,$$

where $\Theta_{c,K} = \Theta_c \cap \{\theta : |\theta_1| > K/n\}$.

Proof of Claim I. Let $\gamma_t = \gamma - (x'_{2t}/\sqrt{n})\eta$ and $u_t(\theta) = u_{1t}(\theta) + u_{2t}(\theta)$, where

$$u_{1t}(\theta) = u_t - (A - A_0)' X_{t-1} - (D - D_0)' X_{t-1} 1\left\{z_{t-1} > \gamma_{t-1}\right\}$$
$$- D_0' X_{t-1}\left(1\left\{z_{t-1} > \gamma_{t-1}\right\} - 1\left\{z_{t-1} > 0\right\}\right),$$
$$u_{2t}(\theta) = -\left(A_z + D_z 1\left\{z_{t-1} > \gamma_{t-1}\right\}\right)\frac{x'_{2t-1}}{\sqrt{n}}\eta.$$

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Because $u_{2t}(0,\lambda)=0$, $(S_n(\theta)-S_n(0,\lambda))/n = D_{1n}(\theta)+D_{2n}(\theta)$, where

$$D_{1n}(\theta)=\frac{1}{n}\sum_t\left[u_{1t}(\theta)'u_{1t}(\theta)-u_{1t}'(0,\lambda)u_{1t}(0,\lambda)\right],$$
$$D_{2n}(\theta)=\frac{1}{n}\sum_t u_{2t}(\theta)'u_{2t}(\theta)+\frac{2}{n}\sum_t u_{1t}(\theta)'u_{2t}(\theta).$$

Note that $\sup_{1\le t\le n}\left|\frac{x_{2t-1}}{\sqrt{n}}\right| = O_p(1)$ and thus $\vec{\gamma}_n = \sup_{1\le t\le n}\left|\gamma_{t-1}\right| = O_p(|\theta_1|) = O_p(c)$ for $\theta\in\Theta_c$. (The meaning of $o_p(|\eta|)$ is that the term can be bounded for all large n by the product of $|\eta|$ and an arbitrarily small constant with probability $1-\varepsilon$ for any $\varepsilon>0$. The same is true for $O_p(c|\eta|)$ because c can be chosen arbitrarily small.) Then,

$$\left|\frac{1}{n}\sum_t u_{2t}(\theta)'u_{2t}(\theta)\right| \le O\left(\left[\sup_{1\le t\le n}\left|\frac{x_{2t-1}}{\sqrt{n}}\right|\right]^2|\eta|^2\right)=O_p(c|\eta|). \tag{A.13}$$

Because $\{u_t\}$ is a martingale difference sequence, so is the sequence $\left\{u_t 1\left\{z_{t-1}>\gamma_{t-1}\right\}\right\}$, implying

$$\frac{1}{n}\sum_t u_t 1\left\{z_{t-1}>\gamma_{t-1}\right\}\frac{x_{2t-1}'}{\sqrt{n}}=o_p(1);$$

see, e.g., Hansen (1992). This implies that

$$\left|\frac{1}{n}\sum_t u_{1t}(\theta)'u_{2t}(\theta)\right| \le o_p(|\eta|)+O_p(c|\eta|)+o_p(|\eta|), \tag{A.14}$$

because $\left|\frac{1}{n}\sum_t X_{t-1}1\left\{z_{t-1}>\gamma_{t-1}\right\}\frac{x_{2t-1}'}{\sqrt{n}}\right| \le \frac{1}{n}\sum_t\left|X_{t-1}(x_{2t-1}'/\sqrt{n})\right| = O_p(1)$ and

$$\begin{aligned}&\frac{1}{n}\sum_t\left|X_{t-1}\left(1\left\{z_{t-1}>\gamma_{t-1}\right\}-1\left\{z_{t-1}>0\right\}\right)1\left\{z_{t-1}>\gamma_{t-1}\right\}\frac{x_{2t-1}'}{\sqrt{n}}\right|\\ &\quad\le \sup_{1\le t\le n}\left|\frac{x_{2t-1}}{\sqrt{n}}\right|\left(\frac{1}{n}\sum_t|X_{t-1}|^2\frac{1}{n}\sum_t 1\left\{|z_{t-1}|\le\vec{\gamma}_n\right\}\right)^{1/2}\\ &\quad= O_p(1)\,O_p(c)\end{aligned}$$

by an application of a uniform law of large numbers to the last term and the fact that $F_{|z|}(\vec{\gamma}_n)\le\vec{\gamma}_n C$ for some constant $C>0$, where $F_{|z|}$ is the distribution function of $|z_{t-1}|$.

Thus, we conclude from (A.13) and (A.14) that for any $m_1,\varepsilon>0$, there is $c>0$ such that

$$\liminf_{n\to\infty}\Pr\left\{\sup_{\theta\in\Theta_c}\left[|D_{2n}(\theta)|-m_1|\theta_1|\right]\le 0\right\}>1-\varepsilon. \tag{A.15}$$

On the other hand, as we show in what follows, for any $\varepsilon>0$ and all sufficiently large n, there exists some constant $m_2>0$ such that, for all $\theta\in\Theta_{c,K}$, $D_{1n}(\theta)>m_2|\theta_1|$ with

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probability $1-\varepsilon$, which will complete the proof of Claim I as m_1 in (A.15) is arbitrary. By direct calculation as before, we may write

$$D_{1n}(\theta) = \left(D_0' + O_p(c)\right)\frac{1}{n}\sum_t X_{t-1}X_{t-1}'1\left\{0 < z_{t-1} \le \gamma_{t-1} \text{ or } \gamma_{t-1} < z_{t-1} \le 0\right\}D_0$$
$$+ \left(D_0' + O(c)\right)\frac{1}{n}\sum_t X_{t-1}u_t 1\left\{0 < z_{t-1} \le \gamma_{t-1} \text{ or } \gamma_{t-1} < z_{t-1} \le 0\right\}$$
$$+ O_p\left(c^2\right).$$

We first argue that the same as (A.15) can be said for the last two terms. Because it is obvious for the last term $O_p\left(c^2\right)$, we show that for any $\varepsilon, \delta > 0$ and sufficiently large n

$$\Pr\left\{\sup_{\theta\in\Theta_{c,K}}\left|\frac{1}{n}\sum_t u_t 1\left\{\left|z_{t-1}\right| \le \left|\gamma_{t-1}\right|\right\}\right| / |\theta_1| > \delta\right\} < \varepsilon. \tag{A.16}$$

The other terms in X_{t-1} can be analyzed similarly and thus omitted. To show (A.16), we consider a grid

$$\Theta_b = \left\{\theta_1 = \left(b^{i_1}, \ldots, b^{i_k}\right)' K/n : |\theta_1| < c \text{ and } i_1, \ldots, i_k = 1, 2, \ldots\right\}$$

for $b > 1$ and first show (A.16) when the supremum is taken over Θ_b. Whereas Θ_b covers the case where all the elements are positive, the same analysis goes through for the other cases by considering, e.g., $\theta_1 = \left(-b^{i_1}, \ldots, b^{i_k}\right)' K/n$, etc. By the Markov inequality

$$\Pr\left\{\sup_{\Theta_b}\left|\frac{1}{n}\sum_t u_t 1\left\{\left|z_{t-1}\right| \le \left|\gamma_{t-1}\right|\right\}\right| / |\theta_1| > \delta\right\}$$
$$\le \sum_{i_1,\ldots,i_k}\frac{\mathrm{E}\left|\frac{1}{n}\sum_t u_t 1\left\{\left|z_{t-1}\right| \le \left|\gamma_{t-1}^{i_1,\ldots,i_k}\right|\right\}\right|^2}{\left|K/n\left(b^{i_1}, \ldots, b^{i_k}\right)'\right|^2\delta^2}$$
$$= \sum_{i_1,\ldots,i_k}\frac{1}{\delta^2 K^2\left|\left(b^{i_1}, \ldots, b^{i_k}\right)\right|^2}\sum_t \mathrm{E}|u_t|^2 1\left\{\left|z_{t-1}\right| \le \left|\gamma_{t-1}^{i_1,\ldots,i_k}\right|\right\}, \tag{A.17}$$

where $\gamma_{t-1}^{i_1,\ldots,i_k}$ is the value of γ_{t-1} when $\theta_1 = K/n\left(b^{i_1}, \ldots, b^{i_k}\right)'$. However, because u_t is independent of the past and the conditional distribution of z_{t-1} given that x_{2t-1} has a bounded density, an expansion and the Cauchy–Schwarz inequality yield that

$$\sum_t \mathrm{E}|u_t|^2 1\left\{\left|z_{t-1}\right| \le \left|\gamma_{t-1}^{i_1,\ldots,i_k}\right|\right\}$$
$$\le M\frac{1}{n}\sum_t \mathrm{E}\left(\left|\frac{x_{2t}}{\sqrt{n}}\right| + 1\right)K\left|\left(b^{i_1}, \ldots, b^{i_k}\right)\right| = O\left(K\left|\left(b^{i_1}, \ldots, b^{i_k}\right)\right|\right) \tag{A.18}$$

for some constant $M > 0$. Furthermore, $b > 1$ implies that $\sum_{i_1,\ldots,i_k}\left|\left(b^{i_1}, \ldots, b^{i_k}\right)\right|^{-1} < \infty$. Thus, letting K be large makes the term in (A.17) smaller than any given $\varepsilon > 0$.

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Next, let a_1 and a_2 be any two adjacent points in Θ_b. Then $|a_1 - a_2| \leq |a_1|(b-1)K/n \leq c(b-1)K/n$. Thus, we show that for any δ and $\varepsilon > 0$, sufficiently large n,

$$\Pr\left\{\sup_{\theta_{11},\theta_{12}}\left|\frac{1}{n}\sum_t u_t\left(\frac{1\{|z_{t-1}| \leq |\gamma_{1t-1}|\}}{|\theta_{11}|} - \frac{1\{|z_{t-1}| \leq |\gamma_{2t-1}|\}}{|\theta_{12}|}\right)\right| \leq \delta\right\} > 1-\varepsilon, \tag{A.19}$$

where the supremum is taken over θ_{11} and θ_{12} on

$$\bar{\Theta}_b = \{K/n < |\theta_{11}| < c, K/n < |\theta_{12}| < c : |\theta_{11} - \theta_{12}| \leq c(b-1)K/n\}$$

and γ_{1t} and γ_{2t} denote γ_t's corresponding to θ_{11} and θ_{12}. Note that

$$\left|1\{|z_{t-1}| \leq |\gamma_{1t-1}|\} - 1\{|z_{t-1}| \leq |\gamma_{2t-1}|\}\right| \leq 1\{||z_{t-1}| - |\gamma_{1t-1}|| \leq |\gamma_{1t} - \gamma_{2t}|\}$$

and

$$\sup_{t,\theta_{11},\theta_{12}}|\gamma_{1t} - \gamma_{2t}| \leq \left(1 + \sup_t\left|\frac{x_{2t-1}}{\sqrt{n}}\right|\right)c(b-1)\frac{K}{n},$$

where the supremum is taken over $1 \leq t \leq n$ and on $\bar{\Theta}_b$. Then, by the same reasoning as (A.18)

$$\sup_{\theta_{11},\theta_{12}\in\bar{\Theta}_b}\left|\sum_t u_t\left(1\{|z_{t-1}| \leq |\gamma_{1t-1}|\} - 1\{|z_{t-1}| \leq |\gamma_{2t-1}|\}\right)\right| = O_p(c(b-1)K).$$

Because b can be chosen arbitrarily close to 1 and $n|\theta_1|$ is bounded below in the set $\Theta_{c,K}$, using the relation $(c_1/a_1) - (c_2/a_2) = ((c_1 - c_2)/a_1) + c_2((a_2 - a_1)/a_1a_2)$ for any a_1, a_2, c_1, and c_2, we can conclude (A.19). This and (A.17) complete the proof of (A.16).

Turning to the first term of D_{1n}, let F denote the distribution function of z_t. Then it follows from the standard uniform law of large numbers (e.g., Pötscher and Prucha, 1991, Thm. 5.2) that for any $C < \infty$

$$\sup_{r\in R_C}\left|\frac{1}{n}\sum_t\left(1\{z_{t-1} \leq r_{(t-1)/n}\} - F\left(r_{(t-1)/n}\right)\right)\right| \xrightarrow{p} 0,$$

where R_C is a space of continuous and bounded (by C) functions on $[0,1]$ equipped with the uniform metric. Because C is arbitrary and $\gamma_{[ns]} = \gamma - \eta' x_{2[ns]}/\sqrt{n} \Rightarrow \gamma - \eta' B(s)$ on $\{|\theta_1| < c\} \times [0,1]$ for any $c > 0$, we infer that

$$\sup_{|\theta_1|\leq c}\left|\frac{1}{n}\sum_t\left(1\{z_{t-1} \leq \gamma_{t-1}\} - F\left(\gamma_{t-1}\right)\right)\right| \xrightarrow{p} 0. \tag{A.20}$$

Then, the continuous mapping theorem yields $\frac{1}{n}\sum_t F\left(\gamma_{t-1}\right) \Rightarrow \int_0^1 F\left(\gamma - B(s)'\eta\right)ds$, which in conjunction with (A.20) implies that

$$\begin{aligned}&\frac{1}{n}\sum_t\left(1\{0 < z_{t-1} \leq \gamma_{t-1}\} + 1\{\gamma_{t-1} < z_{t-1} \leq 0\}\right)\\ &\quad\Rightarrow \int_0^1\left|F\left(\gamma - B(s)'\eta\right) - F(0)\right|ds \overset{d}{=} \int_0^1\left|F\left(\gamma + \sqrt{\eta'\Omega\eta}W(s)\right) - F(0)\right|ds\end{aligned} \tag{A.21}$$

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on $\{|\theta_1 < c|\}$, where W is a standard Brownian motion and equality in distribution follows from the normality of B.

Finally, we need to show that there is a constant $m_3 > 0$ such that

$$\Pr\left\{\inf_{|\theta_1|<c}\left\{\int_0^1 \left|F\left(\gamma+\sqrt{\eta'\Omega\eta}W(s)\right)-F(0)\right|ds - m_3|\theta_1|\right\}\geq 0\right\}\geq 1-\varepsilon. \tag{A.22}$$

Because F has a continuous density that is positive everywhere, thus if $|W(s)| \leq 1$, then $\left|\gamma+\sqrt{\eta'\Omega\eta}W(s)\right|$ is bounded for $\theta_1 \in \Theta_c$, and thus there is $m_4 > 0$ such that

$$\left|F\left(\gamma+\sqrt{\eta'\Omega\eta}W(s)\right)-F(0)\right| \geq m_4\left|\gamma+\sqrt{\eta'\Omega\eta}W(s)\right|.$$

Now assume $\gamma \geq 0$. The case with $\gamma < 0$ can be treated in the same way. If $0 \leq W(s) \leq 1$, then $\gamma+\sqrt{\eta'\Omega\eta}W(s)$ can be bounded below using the triangular inequality by $\sqrt{\gamma^2+\eta'\Omega\eta}W(s)$. Because Ω is positive definite, $\sqrt{\gamma^2+\eta'\Omega\eta}$ can be bounded below by $|\theta_1|$ with a positive multiple. Summing up these yields that for some $m_5 > 0$

$$\int_0^1 \left|F\left(\gamma+\sqrt{\eta'\Omega\eta}W(s)\right)-F(0)\right|ds \geq m_5|\theta_1|\int_0^1 |W(s)|\,1\{0\leq W(s)\leq 1\}ds.$$

Because we can find a positive constant m_6 such that $\int_0^1 |W(s)|\,1\{0\leq W(s)\leq 1\}ds > m_6$ and $\int_0^1 |W(s)|\,1\{-1\leq W(s)\leq 0\}ds > m_6$ with probability greater than $1-\varepsilon$, the claim (A.22) is verified. ■

Proof of Theorem 3. The convention on notation made in the proof of Theorem 2 is kept here.

To derive the limit distribution of the SLS estimator $\hat{\theta}$, define $T_n(\theta) = \frac{\partial S_n(\theta)}{n\partial\theta}$ and $Q_n(\theta) = \frac{\partial^2 S_n(\theta)}{n\partial\theta\partial\theta'}$. Then, by the mean value theorem,

$$\sqrt{n}D_n^{-1}\left(\hat{\theta}-\theta_0\right) = \left(D_n Q_n\left(\tilde{\theta}\right)D_n\right)^{-1}\sqrt{n}D_n T_n(\theta_0),$$

where D_n is a diagonal matrix whose first $p-1$ elements are $(h/n)^{1/2}$, the pth element is $\sqrt{h}$ and the others are 1, and $\tilde{\theta}$ lies between $\hat{\theta}$ and θ_0. Recall that $X'_{t-1}(\eta)D = X'_{t-1}D + (x'_{2t-1}\eta/\sqrt{n})D'_z$ and write the residuals of the SLS as

$$\begin{aligned} e_t(\theta) = u_t &- (A-A_0)'X_{t-1} - (D-D_0)'X_{t-1}d_{t-1} \\ &- D'X_{t-1}\left(\mathcal{K}_{t-1}(\eta,\gamma)-d_{t-1}\right) - \left(A_z+D_z\mathcal{K}_{t-1}(\eta,\gamma)\right)\frac{x'_{2t-1}\eta}{\sqrt{n}}. \end{aligned} \tag{A.23}$$

Then,

$$\frac{\partial e_t(\theta)'}{\partial\eta} = -x_{2t-1}\left[A'_z + D'_z\mathcal{K}_{t-1}(\eta,\gamma) + \left(D'_z\frac{x'_{2t-1}\eta}{\sqrt{n}} + X'_{t-1}D\right)\frac{\mathcal{K}^{(1)}_{t-1}(\eta,\gamma)}{h}\right], \tag{A.24}$$

$$\frac{\partial e_t(\theta)'}{\partial\gamma} = -\left(X'_{t-1}D + D'_z\frac{x'_{2t-1}\eta}{\sqrt{n}}\right)\frac{\mathcal{K}^{(1)}_{t-1}(\eta,\gamma)}{h}, \tag{A.25}$$

$$\frac{\partial e_t(\theta)'}{\partial\lambda} = \begin{pmatrix} -\left(X'_{t-1}D + D'_z\frac{x'_{2t-1}\eta}{\sqrt{n}}\right)\otimes I_p \\ -\mathcal{K}_{t-1}(\eta,\gamma)\left(X'_{t-1}D + D'_z\frac{x'_{2t-1}\eta}{\sqrt{n}}\right)\otimes I_p \end{pmatrix}.$$

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Convergence of T_n. The asymptotic distribution of

$$\sqrt{n}D_nT_n(\theta_0)/2 = \frac{1}{\sqrt{n}}D_n\sum_t \frac{\partial e_t(\theta_0)'}{\partial\theta}e_t(\theta_0)$$

has been developed in Seo and Linton (2007) except for the part corresponding to η. Thus, we focus on the convergence of

$$\frac{\sqrt{h}}{2n}\sum_t \frac{\partial e_t(\theta_0)'}{\partial\eta}e_t(\theta_0) = -\frac{1}{n}\sum_t x_{2t-1}(\sqrt{h}v_{1t} + \sqrt{h}v_{2t} + v_{3t}/\sqrt{h}),$$

where

$$v_{1t} = A'_{z0}u_t + D'_{z0}\mathcal{K}_{t-1}u_t,$$
$$v_{2t} = \left(A'_{z0} + D'_{z0}\mathcal{K}_{t-1}\right)D'_0X_{t-1}\left(\mathcal{K}_{t-1} - d_{t-1}\right),$$
$$v_{3t} = \mathcal{K}^{(1)}_{t-1}X'_{t-1}D_0\left[u_t - D'_0X_{t-1}\left(\mathcal{K}_{t-1} - d_{t-1}\right)\right]$$

and the convergence of covariances of $\sqrt{n}D_nT_n(\theta_0)$.

Because v_{1t} is a martingale difference array, $\frac{1}{n}\sum_t x_{2t-1}v_{1t} = O_p(1)$ because of the convergence of stochastic integrals (see, e.g., Kurtz and Protter, 1991). The convergence of $n^{-1}h^{-1/2}\sum_t x_{2t-1}v_{2t}$ is similar to that of $\frac{1}{n\sqrt{h}}\sum_t x_{2t-1}v_{3t}$, which we will establish here. Then, it follows that

$$\frac{1}{n}\sum_t x_{2t-1}(v_{1t} + v_{2t})\sqrt{h} = o_p(1)$$

as $h \to 0$. Let $\bar{v}_{3t} = (v_{3t} - Ev_{3t})/\sqrt{h}$; then $\bar{v}_{3t}$ is a zero mean strong mixing array. Seo and Linton (2007, Lem. 2) have shown that $\sqrt{n/h}Ev_{3t} \to 0$ and $\operatorname{var}\left[(hn)^{-1/2}\sum_t v_{3t}\right] = \operatorname{var}\left[v_{3t}/\sqrt{h}\right] + o(1) \to \sigma_v^2$, which is defined in (5). This implies that

$$\frac{1}{n}\sum_t x_{2t-1}Ev_{3t}/\sqrt{h} = o_p(1)$$

and that

$$n^{-1/2}\sum_{t=2}^{[ns]}\begin{pmatrix}\Delta x_{2t-1}\\ \bar{v}_{3t}\end{pmatrix} \Rightarrow \begin{pmatrix}B\\ \sigma_v^2W\end{pmatrix} \tag{A.26}$$

because of Assumption 4 and the invariance principle of Wooldridge and White (1988, Thm. 2.11), where W is a standard Brownian motion that is independent of B. For the independence between B and W, see Lemma 2 of Seo and Linton (2007), which shows that $\sum_{s,t=1}^n \mathrm{E}\Delta x_s\bar{v}_{3t} = o(1)$.

For the convergence of $\frac{1}{n}\sum_t x_{2t-1}\bar{v}_{3t}$, we resort to Hansen (1992, Thm. 3.1). Checking Hansen's conditions, we observe that the moment condition for $\bar{v}_{3t}$ is not met, i.e., $\mathrm{E}|\bar{v}_{3t}|^p \to \infty$ for $p > 2$. However, this is not necessary but is used to show that $\sup_{1\le t\le n} n^{-1/2}\sum_{k=1}^\infty \mathrm{E}\left[\bar{v}_{3t+k}|\mathcal{F}_t\right] = o_p(1)$, where $\mathcal{F}_t$ is the natural filtration at time t. Using the Markov inequality, Hansen obtains that for $p > 2$

$$\Pr\left\{\left|\sup_{1\le t\le n} n^{-1/2}\sum_{k=1}^\infty \mathrm{E}\left[\bar{v}_{3t+k}|\mathcal{F}_t\right]\right| \ge \varepsilon\right\} \le \frac{CE|\bar{v}_{3t}|^p}{\varepsilon^p n^{p/2-1}} \to 0 \tag{A.27}$$

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if $\mathrm{E}|\bar{v}_{3t}|^p < \infty$. Now we show that $\mathrm{E}|\bar{v}_{3t}|^p$ is not bounded for $p > 2$ but diverges at a rate slower than $n^{p/2-1}$ so that (A.27) still holds. As $\sqrt{n/h}\,\mathrm{E}v_{3t} \to 0$, we first consider

$$\begin{aligned}
&\mathrm{E}\left|\mathcal{K}_{t-1}^{(1)}\Delta_{t-1}'D_0D_0'\Delta_{t-1}\left(\mathcal{K}_{t-1}-d_{t-1}\right)\right|^p h^{-p/2} \\
&\quad = h^{-p/2}\int \left|\Delta' D_0 D_0'\Delta\right|^p \left|\mathcal{K}^{(1)}\left(\frac{z}{h}\right)\left(\mathcal{K}\left(\frac{z}{h}\right)-1\left(z>0\right)\right)\right|^p f\left(z|\Delta\right)dzdP_\Delta \\
&\quad = h^{1-p/2}\int \left|\Delta' D_0 D_0'\Delta\right|^p \left|\mathcal{K}^{(1)}\left(s\right)\left(\mathcal{K}\left(s\right)-1\left(s\right)\right)\right|^p f\left(hs|\Delta\right)dsdP_\Delta,
\end{aligned}$$

where P_Δ is the distribution of Δ_{t-1} and the second equality follows by the change of variables. Note that f is bounded for almost every Δ, $\mathcal{K}(s)-1(s)$ is bounded, $\left|\mathcal{K}^{(1)}\right|^p$ is integrable, and $\mathrm{E}\left|\Delta_{t-1}'D_0D_0'\Delta_{t-1}\right|^p < \infty$. The part associated with the constant 1 and z_{t-1} in X_{t-1} and u_t can be done in the same manner because $\int \left|s^2\mathcal{K}^{(1)}\left(s\right)\right|^p ds < \infty$ as a result of Assumption 2(a). As $h^{1-p/2}/n^{p/2-1} \to 0$ for $p > 2$, we conclude that (A.27) converges to zero. Therefore,

$$\frac{\sqrt{h}}{2\sigma_v n}\sum_t \frac{\partial e_t\left(\theta_0\right)'}{\partial \eta}e_t\left(\theta_0\right) \Rightarrow \int_0^1 BdW. \tag{A.28}$$

Finally, note that an argument similar to the argument that showed asymptotic independence in (A.26) yields asymptotic independence between the scores for η and λ. For the covariance between the scores for η and γ, note that $v_{3t} = h(\partial e_t\left(\theta_0\right)'/\partial\gamma\,)e_t\left(\theta_0\right)$.

Convergence of Q_n. To begin with, we state the following lemma.

LEMMA 2. *Under the conditions of Theorem 3, $h^{-1}\hat{\gamma}$ and $h^{-1}\hat{\eta}$ are $o_p\left(1\right)$.*

Proof of Lemma 2. Recalling the formula (A.23) and (A.25), we may write

$$\frac{1}{n}\sum_t \frac{\partial}{\partial\gamma}e_t\left(\theta\right)'e_t\left(\theta\right) = T_{1n}\left(\theta\right)+\cdots+T_{10n}\left(\theta\right), \tag{A.29}$$

where

$$\begin{aligned}
T_{1n}\left(\theta\right) &= -\frac{1}{nh}\sum_t \mathcal{K}_{t-1}^{(1)}\left(\eta,\gamma\,\right)X_{t-1}'Du_t, \\
T_{2n}\left(\theta\right) &= \mathrm{tr}\left[D'\frac{1}{nh}\sum_t \mathcal{K}_{t-1}^{(1)}\left(\eta,\gamma\,\right)X_{t-1}X_{t-1}'\left(A-A_0\right)\right], \\
T_{3n}\left(\theta\right) &= \mathrm{tr}\left[D'\frac{1}{nh}\sum_t X_{t-1}X_{t-1}'\mathcal{K}_{t-1}^{(1)}\left(\eta,\gamma\,\right)\left(\mathcal{K}_{t-1}\left(\eta,\gamma\,\right)-d_{t-1}\right)D\right],
\end{aligned}$$

until we reach

$$T_{10n}\left(\theta\right) = \frac{1}{nh}D_z'\eta\sum_t \frac{x_{2t-1}}{\sqrt{n}}\mathcal{K}_{t-1}^{(1)}\left(\eta,\gamma\,\right)\left(A_z+D_z\mathcal{K}_{t-1}\left(\eta,\gamma\,\right)\right)\frac{x_{2t-1}'\eta}{\sqrt{n}}.$$

As we proved the consistency of the smoothed LS estimator of θ in Corollary 1, we can find a sequence $r_n \to 0$ and $\Pr\left\{\left|\hat{\theta}-\theta_0\right| > r_n\right\} \to 0$. Without loss of generality, we restrict

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the parameter space to $\Theta_{r_n} = \{|\theta - \theta_0| \leq r_n\}$. We first show that $\sup_{\Theta_{r_n}} |T_{in}(\theta)| = o_p(1)$ for all $i \neq 3$, which implies that

$$T_{3n}\left(\hat{\theta}\right) = o_p(1) \tag{A.30}$$

because (A.29) equals zero at $\theta = \hat{\theta}$ as a result of the first-order condition of the SLS estimation.

First take $T_{1n}(\theta)$. In particular, consider

$$\left| \frac{1}{nh} \sum_t \mathcal{K}^{(1)} \left(\frac{z_{t-1} + x'_{2t-1}\eta/\sqrt{n} - \gamma}{h} \right) u_t \right|.$$

The other terms in the vector can be handled similarly. As in Seo and Linton, we may assume u_t is bounded by some constant $C < \infty$ and divide the parameter space for θ_1 of Θ_{r_n} into Γ_n nonoverlapping pieces $\Theta_{ni}, i = 1, \ldots, \Gamma_n$, where the distance between any two points in each piece is smaller than or equal to $\mu h^{2+\varepsilon}$ for some $\mu > 0$. Then, $\Gamma_n = O\left(h^{-3(p-1)(1+\varepsilon/2)}\right)$. An application of a Hoeffding type inequality for martingale difference sequence (Azuma, 1967) yields that for $\theta_{1i} \in \Theta_{ni}$ and some constant C_1

$$\begin{aligned}
&\Pr\left\{ \sup_{1 \leq i \leq \Gamma_n} \left| \frac{1}{nh} \sum_t u_t \mathcal{K}^{(1)} \left(\frac{z_{t-1} + x'_{2t-1}\eta_i/\sqrt{n} - \gamma_i}{h} \right) \right| > \eta \right\} \\
&\quad \leq \sum_{1 \leq i \leq \Gamma_n} \Pr\left\{ \left| \frac{1}{nh} \sum_t u_t \mathcal{K}^{(1)} \left(\frac{z_{t-1} + x'_{2t-1}\eta_i/\sqrt{n} - \gamma_i}{h} \right) \right| > \eta \right\} \\
&\quad \leq \Gamma_n \exp\left(-nh^2\eta^2/C_1 \right) \\
&\quad \to 0,
\end{aligned}$$

as $nh^2 \to \infty$. And for any θ_1 and θ_2 in Θ_{ni},

$$\begin{aligned}
&\frac{1}{nh} \sum_t \left[\mathcal{K}^{(1)} \left(\frac{z_{t-1} + x'_{2t-1}\eta_1/\sqrt{n} - \gamma_1}{h} \right) - \mathcal{K}^{(1)} \left(\frac{z_{t-1} + x'_{2t-1}\eta_2/\sqrt{n} - \gamma_2}{h} \right) \right] u_t \\
&\quad \leq \frac{1}{nh} \sum_t \sup_{\eta,\gamma} \left| \mathcal{K}^{(2)} \left(\frac{z_{t-1} + x'_{2t-1}\eta/\sqrt{n} - \gamma}{h} \right) u_t \frac{x_{2t-1}}{\sqrt{n}} \right| \frac{\mu h^{2+\varepsilon}}{h} \\
&\quad \leq O_p\left(h^{\varepsilon}\right),
\end{aligned}$$

as $\sup_{1 \leq t \leq n} \left| \frac{x_{2t-1}}{\sqrt{n}} \right| = O_p(1)$ and $\mathcal{K}^{(2)}$ is bounded.

Next, we study the convergence of $T_{3n}(\theta)$. The same analysis applies to T_{in}'s for $i = 2, 4, 5, \ldots, 10$ to yield that they are all $o_p(1)$'s. Note, e.g., that T_{2n} is the product of $(A - A_0)$, which is bounded by r_n, and of a sample mean, which is $O_p(1)$ applying the same method as used subsequently.

Define

$$\begin{aligned}
T_t(x, \theta_1) = X_{t-1} X'_{t-1} &\mathcal{K}^{(1)} \left(\frac{z_{t-1} + x'\eta - \gamma}{h} \right) \\
&\times \left(\mathcal{K}\left(\frac{z_{t-1} + x'\eta - \gamma}{h} \right) - 1\left\{ z_{t-1} + x'\eta > \gamma \right\} \right)
\end{aligned}$$

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and

$$\varsigma(x,\theta_1)=\mathrm{E}(T_t(x,\theta_1)).$$

Furthermore, letting $\dot{\gamma}=\gamma-x'\eta$, define

$$\dot{T}_t(\dot{\gamma})=X_{t-1}X_{t-1}'\mathcal{K}^{(1)}\left(\frac{z_{t-1}-\dot{\gamma}}{h}\right)\left(\mathcal{K}\left(\frac{z_{t-1}-\dot{\gamma}}{h}\right)-1\{z_{t-1}>\dot{\gamma}\}\right),$$

$$\dot{\varsigma}(\dot{\gamma})=\mathrm{E}\dot{T}_t(\dot{\gamma}).$$

Assume that x belongs to a compact set Θ_x and thus $\dot{\gamma}$ lies within an interval Θ_{gn}, shrinking to zero as $|\theta_1|\le r_n\to 0$. Then, it is clear that

$$\sup_{x\in\Theta_x,\theta_1\in\Theta_{r_n}}\left|\frac{1}{nh}\sum_t(T_t(x,\theta_1)-\varsigma(x,\theta_1))\right|\le\sup_{\dot{\gamma}\in\Theta_{gn}}\left|\frac{1}{nh}\sum_t\left(\dot{T}_t(\dot{\gamma})-\dot{\varsigma}(\dot{\gamma})\right)\right|=o_p(1), \tag{A.31}$$

where the last equality is due to Lemma 4 (17) of Seo and Linton (2007). Because (A.31) holds for any compact set Θ_x and $\sup_{0\le s\le 1}|n^{-1/2}x_{2[ns]}|=O_p(1)$, (A.31) holds true when x is replaced with $n^{-1/2}x_{2t-1}$. This implies that

$$\sup_{\Theta_{r_n}}\left|T_{3n}(\theta)-\mathrm{tr}\left[D'\frac{1}{nh}\sum_t\varsigma\left(\frac{x_{2t-1}}{\sqrt{n}},\theta_1\right)D\right]\right|=o_p(1) \tag{A.32}$$

and thus

$$\frac{1}{nh}\sum_t\varsigma\left(\frac{x_{2t-1}}{\sqrt{n}},\hat{\theta}_1\right)=o_p(1), \tag{A.33}$$

as a result of (A.30). Because $\theta_{10}=0$ and $\dot{\varsigma}$ does not contain x_{2t-1} when $\theta_1=0$,

$$\frac{1}{nh}\sum_t\varsigma\left(\frac{x_{2t-1}}{\sqrt{n}},\theta_{10}\right)=\frac{1}{nh}\sum_t\dot{\varsigma}(0)=o(1), \tag{A.34}$$

where the last equality is due to (24) of Seo and Linton (2007). Expanding the $(1,1)$ element of (A.33) around θ_{10} yields that for $\tilde{\theta}_1$ between $\hat{\theta}_1$ and 0

$$\frac{1}{nh}\sum_t\varsigma_{11}\left(\frac{x_{2t-1}}{\sqrt{n}},\hat{\theta}_1\right)=\frac{1}{nh}\sum_t\varsigma_{11}\left(\frac{x_{2t-1}}{\sqrt{n}},0\right)+\frac{\partial}{\partial\theta_1}\frac{1}{nh}\sum_t\varsigma_{11}\left(\frac{x_{2t-1}}{\sqrt{n}},\tilde{\theta}_1\right)\hat{\theta}_1,$$

where ς_{11} denotes the $(1,1)$ element of ς. Given (A.33) and (A.34), this implies that $h^{-1}\hat{\theta}_1=o_p(1)$ if

$$\frac{\partial}{\partial\theta_1}\frac{1}{n}\sum_t\varsigma_{11}\left(\frac{x_{2t-1}}{\sqrt{n}},\tilde{\theta}_1\right)\ne o_p(1). \tag{A.35}$$

However,

$$\frac{\partial}{\partial\theta_1}\mathrm{E}\left[\mathcal{K}^{(1)}\left(\frac{z_{t-1}+x'\eta-\gamma}{h}\right)\left(\mathcal{K}\left(\frac{z_{t-1}+x'\eta-\gamma}{h}\right)-1\{z_{t-1}+x'\eta>\gamma\}\right)\right]$$
$$=\mathrm{E}\left[\mathcal{K}^{(2)}\left(\frac{z_{t-1}+x'\eta-\gamma}{h}\right)\left(\mathcal{K}\left(\frac{z_{t-1}+x'\eta-\gamma}{h}\right)-1\{z_{t-1}+x'\eta>\gamma\}\right)\frac{1}{h}\right]$$

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$$\times \left(x', -1\right)'$$
$$+ \left[\mathrm{E}\mathcal{K}^{(1)}\left(\frac{z_{t-1}+x'\eta-\gamma}{h}\right)^2 \frac{1}{h} - \mathcal{K}^{(1)}(0)\, f\left(\gamma - x'\eta\right)\right]\left(x', -1\right)',$$

where f denotes the density of z_t. Using change-of-variables formula,

$$\begin{aligned}\mathrm{E}\mathcal{K}^{(1)}\left(\frac{z_{t-1}-\dot{\gamma}}{h}\right)^2 \frac{1}{h} &= \int_{-\infty}^{\infty} \mathcal{K}^{(1)}\left(\frac{z-\dot{\gamma}}{h}\right)^2 \frac{1}{h} f(z)\, dz \\ &= \int_{-\infty}^{\infty} \mathcal{K}^{(1)}(z)^2 f(hz+\dot{\gamma})\, dz \\ &\to \int_{-\infty}^{\infty} \mathcal{K}^{(1)}(z)^2 dz \cdot f(0)\end{aligned}$$

by the dominated convergence theorem as h and $\dot{\gamma}$ go to zero. Similarly,

$$\begin{aligned}&\mathrm{E}\left[\mathcal{K}^{(2)}\left(\frac{z_{t-1}-\dot{\gamma}}{h}\right)\left(\mathcal{K}\left(\frac{z_{t-1}-\dot{\gamma}}{h}\right) - 1\left\{z_{t-1} > \dot{\gamma}\right\}\right)\frac{1}{h}\right] \\ &\quad \to \int_{-\infty}^{\infty} \mathcal{K}^{(2)}(z)\left(\mathcal{K}(z) - 1\{z > 0\}\right) dz \cdot f(0).\end{aligned}$$

Furthermore, it follows from integration by parts that

$$\begin{aligned}&\int_{-\infty}^{\infty} \mathcal{K}^{(1)}(s)^2 ds - \int_{-\infty}^{\infty} \mathcal{K}^{(2)}(s)\left(1\{s > 0\} - \mathcal{K}(s)\right) ds \\ &\quad = -\int_{-\infty}^{\infty} \mathcal{K}^{(2)}(s)\, 1\{s > 0\}\, ds = \mathcal{K}^{(1)}(0).\end{aligned}$$

Because $\sup_{1\le t\le n, \theta_1\in\Theta_{r_n}} \left|(x'_{2t-1}/\sqrt{n})\eta - \gamma\right| = o_p(1)$, we obtain

$$\begin{aligned}&\frac{\partial}{\partial\theta_1}\frac{1}{n}\sum_t \varsigma_{11}\left(\frac{x_{2t-1}}{\sqrt{n}}, \tilde{\theta}_1\right) \\ &\quad = 2\int_{-\infty}^{\infty} \mathcal{K}^{(2)}(z)\left(\mathcal{K}(z) - 1\{z > 0\}\right) dz \cdot f(0)\ \frac{1}{n}\sum_t \left(\frac{x'_{2t-1}}{\sqrt{n}}, -1\right)' + o_p(1).\end{aligned}$$

Thus, (A.35) follows because $\int_{-\infty}^{\infty} \mathcal{K}^{(2)}(z)\left(\mathcal{K}(z) - 1\{z > 0\}\right) dz > 0$ and $f(0) > 0$. ■

In view of Lemma 2 and its proof, we can restrict the parameter space to

$$\Theta_n = \{\theta \in \Theta : |\lambda - \lambda_0| < r_n, h^{-1}|\gamma| < r_n, h^{-1}|\eta| < r_n\}$$

for a sequence $r_n \to 0$. Let

$$Q_n^a(\theta) = \frac{1}{n}\sum_t \frac{\partial e_t(\theta)'}{\partial\theta}\frac{\partial e_t(\theta)}{\partial\theta'}, \qquad Q_n^b(\theta) = \frac{1}{n}\sum_t \sum_{i=1}^{p} \frac{\partial^2 e_{it}(\theta)}{\partial\theta\partial\theta'} e_{it}(\theta),$$

where the subscript i of a matrix (or a vector) indicates the ith column (element) of the matrix (the vector). Then, $Q_n(\theta) = 2Q_n^a(\theta) + 2Q_n^b(\theta)$. Start with $Q_n^b(\theta)$, in particular, with

$$\begin{aligned}&-\sum_t \frac{\partial^2 e_{it}(\theta)}{\partial\eta\partial\eta'} e_{it}(\theta) \\ &\quad = \sum_t x_{2t-1}x'_{2t-1}\left(2D_{zi}\frac{\mathcal{K}_{t-1}^{(1)}(\eta,\gamma)}{h} + X_{t-1}(\eta)' D_i \frac{\mathcal{K}_{t-1}^{(2)}(\eta,\gamma)}{h^2}\right) e_{it}(\theta). \quad \textbf{(A.36)}\end{aligned}$$

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Because $h \to 0$ and $\sup_{1\le t\le n}\left|\frac{x'_{2t-1}\eta}{h\sqrt{n}}\right| = o_p(1)$, $X_{t-1}(\eta)' D_i = X'_{t-1}D_i + D_{z_i}(x'_{2t-1}\eta/\sqrt{n}) = X'_{t-1}D_i + o_p(1)$, where $o_p(1)$ is uniform in $1 \le t \le n$, and the leading term in (A.36) with normalization is

$$\frac{h}{n^2}\sum_t x_{2t-1}x'_{2t-1}X'_{t-1}D_i\frac{\mathcal{K}^{(2)}_{t-1}(\eta,\gamma)}{h^2}e_{it}(\theta) \Rightarrow \tilde{\sigma}^2_{q_i}\int \tilde{\mathcal{K}}_2\int_0^1 BB', \tag{A.37}$$

where $\tilde{\mathcal{K}}_2(s) = \mathcal{K}^{(2)}(s)(\mathcal{K}(s) - 1\{s > 0\})$ and $\tilde{\sigma}^2_{q_i} = \mathrm{E}\left(D'_{0i}X'_{t-1}X_{t-1}D_{0i}|z_{t-1} = 0\right) f(0)$. The convergence (A.37) can be achieved in the same way as in Lemma 2. That is, decomposing it as in the lemma, we can easily show that T_{in}'s are negligible except for $i = 3$. For T_{3n}, the same as (A.31) holds with

$$\varsigma(x,\theta) = \frac{1}{hn^2}\sum_t xx' \times \mathrm{E}\left[\left(X'_{t-1}D_i\right)^2\mathcal{K}^{(2)}\left(\frac{z_{t-1}}{h} + \dot{\gamma}\right)\left(\mathcal{K}\left(\frac{z_{t-1}}{h} + \dot{\gamma}\right) - 1\left\{\frac{z_{t-1}}{h} + \dot{\gamma} > 0\right\}\right)\right],$$

where $\dot{\gamma} = \frac{x'\eta-\gamma}{h}$. Because $\sup_{1\le t\le n,\theta\in\Theta_n}\left|\frac{n^{-1/2}x'_{2t-1}\eta-\gamma}{h}\right| = o_p(1)$ and by the change-of-variables technique

$$\frac{1}{h}\mathrm{E}\left(X'_{t-1}D_i\right)^2\mathcal{K}^{(2)}_{t-1}\left(\mathcal{K}_{t-1} - d_{t-1}\right) \to \mathrm{E}\left(\left(D'_iX_{t-1}\right)^2|z_{t-1} = 0\right)f(0)\int\tilde{\mathcal{K}}_2,$$

the convergence in (A.37) follows.

The remaining terms in $Q^b_n(\theta)$ are $\frac{\partial^2 e_{it}(\theta)}{\partial\lambda\partial\lambda'} = 0$,

$$\frac{\partial^2 e_{it}(\theta)}{\partial\gamma\,\partial\eta'} = \left(D_{zi}\frac{\mathcal{K}^{(1)}_{t-1}(\eta,\gamma)}{h} + X_{t-1}(\eta)'D_i\frac{\mathcal{K}^{(2)}_{t-1}(\eta,\gamma)}{h^2}\right)x'_{2t-1},$$

$$\frac{\partial^2 e_{it}(\theta)}{\partial\gamma\,\partial\gamma'} = -X_{t-1}(\eta)'D_i\frac{\mathcal{K}^{(2)}_{t-1}(\eta,\gamma)}{h^2}; \qquad \frac{\partial^2 e_{it}(\theta)}{\partial\lambda\partial\gamma'} = \begin{pmatrix}0\\ \frac{\mathcal{K}^{(1)}_{t-1}(\eta,\gamma)}{h}X_{t-1}(\eta)\end{pmatrix}\otimes I_i,$$

and

$$\frac{\partial^2 e_{it}(\theta)}{\partial\lambda\partial\eta'} = -\begin{pmatrix}\epsilon_2\\ \mathcal{K}_{t-1}(\eta,\gamma)\epsilon_2 + \frac{\mathcal{K}^{(1)}_{t-1}(\eta,\gamma)}{h}X_{t-1}(\eta)\end{pmatrix}x'_{2t-1}\otimes I_i,$$

where $\epsilon_2 = (0,1,0,\ldots,0)$ whose dimension is $(pl+2)$. As their convergence can be analyzed similarly to before, we omit the details and conclude that

$$D_nQ^b_nD_n \Rightarrow \tilde{\sigma}^2_q\int\tilde{\mathcal{K}}_2\begin{pmatrix}-\int_0^1 BB' & \int_0^1 B & 0\\ \int_0^1 B' & 1 & 0\\ 0 & 0 & 0\end{pmatrix},$$

where $\tilde{\sigma}^2_q = \sum_{i=1}^p\tilde{\sigma}^2_{q_i}$. Similarly,

$$D_nQ^a_nD_n \Rightarrow \begin{bmatrix}\left\|\mathcal{K}^{(1)}\right\|_2^2\tilde{\sigma}^2_q\begin{pmatrix}\int_0^1 BB' & -\int_0^1 B\\ -\int_0^1 B' & 1\end{pmatrix} & 0\\ 0 & \mathrm{E}\begin{pmatrix}1 & d_{t-1}\\ d_{t-1} & d_{t-1}\end{pmatrix}\otimes X_{t-1}X'_{t-1}\otimes I_p\end{bmatrix}.$$

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Finally, note that $\left\|\mathcal{K}^{(1)}\right\|_2^2 - \int \tilde{\mathcal{K}}_2 = \mathcal{K}^{(1)}(0)$ by an application of integration by parts, which yields the desired result.

The convergence of $\hat{\lambda}^*$ is a direct consequence of Theorem 2, which obtains the convergence rates of $\hat{\gamma}^*$ and $\hat{\eta}^*$, and Theorem 5 of Seo and Linton (2007). ■

Proof of Corollary 2. Let $\eta_n = \eta + O_p\left(n^{-1}\right)$, $\theta_2 = (\gamma, \lambda)$. The consistency of $\hat{\theta}_2$ is obvious because we established consistency in Corollary 1 based on the $\sqrt{n}$ consistency of $\hat{\beta}$. For the limit distribution, let $T_{2n}(\theta_2; \eta)$ and $Q_{2n}(\theta_2; \eta)$ denote the score and Hessian of the sum of squares function with a fixed η. But, we have already derived the convergence of the Hessian $Q_n(\theta)$ for $\hat{\eta} = o_p(1)$. Thus, we only have to examine the score T_{2n}. Let

$$\begin{aligned}\ddot{e}_t(\theta_2; \eta_n) &= u_t - (A - A_0)' X_{t-1} - (D - D_0)' X_{t-1} d_{t-1} \\ &\quad - D' X_{t-1}\left(\mathcal{K}_{t-1}(\eta_n, \gamma) - d_{t-1}\right) - \left(A_z + D_z \mathcal{K}_{t-1}(\eta_n, \gamma)\right) x'_{2t-1}\eta_n,\end{aligned}$$

$$\frac{\partial \ddot{e}_t(\theta_2)'}{\partial \theta_2} = \begin{pmatrix} -\left(X'_{t-1}D + D'_z x'_{2t-1}\eta_n\right)\mathcal{K}^{(1)}_{t-1}(\eta_n, \gamma)/h \\ -\left(X'_{t-1}D + D'_z x'_{2t-1}\eta_n\right) \otimes I_p \\ -\mathcal{K}_{t-1}(\eta_n, \gamma)\left(X'_{t-1}D + D'_z x'_{2t-1}\eta_n\right) \otimes I_p \end{pmatrix}.$$

Then, we need to show that

$$\frac{\sqrt{h}}{\sqrt{n}} \sum_{t=1}^{n} \left| \frac{\partial \ddot{e}_t(\theta_{20}; \eta_n)'}{\partial \theta_2} \ddot{e}_t(\theta_{20}; \eta_n) - \frac{\partial \ddot{e}_t(\theta_{20}; 0)'}{\partial \theta_2} \ddot{e}_t(\theta_{20}; 0) \right| = o_p(1).$$

As the arguments are all similar for each term, we show that

$$\begin{aligned}&\frac{\sqrt{h}}{\sqrt{n}} \sum_{t=1}^{n} \left| X'_{t-1} D_0 D'_0 X_{t-1} \left(\mathcal{K}_{t-1}(\eta_n, 0) - \mathcal{K}_{t-1}\right)\right| \\ &\quad \leq \frac{\sqrt{n}}{\sqrt{h}} |\eta_n| \sup_{1 \leq t \leq n} \left| \frac{x'_{2t-1}}{\sqrt{n}} \right| \frac{1}{n} \sum_{t=1}^{n} \left| X'_{t-1} D_0 D'_0 X_{t-1} \mathcal{K}^{(1)}_{t-1}(\tilde{\eta}, 0) \right| = o_p(1),\end{aligned}$$

where $\tilde{\eta}$ lies between 0 and η_n, because $\eta_n \sqrt{n}/\sqrt{h} = o_p(1)$ and $\mathcal{K}^{(1)}$ is bounded. This completes the proof. ■

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