## 102.54 Unravelling Simpson's paradox

The word *paradox* means a statement that is apparently self-contradictory but really correct. Simpson's paradox, which is well known in statistics, concerns comparing averages with combined averages, and there are many internet sites that give real examples of data (on drug tests, batting averages, admission statistics for men and women, and so on) that illustrate the paradox. The internet also contains many explanations of the paradox in terms of vectors lying in the plane, but here we take a different view (which does not seem to be in the literature) and we suggest that the vector explanation, while true, misses the real issue, namely convexity.

The apparent self-contradictory nature of Simpson's paradox arises from the inequalities between the ratios (or averages), so to understand the paradox we must begin here. Simpson's paradox occurs when there are positive numbers $a$, $b$, $c$, $d$ and $A$, $B$, $C$, $D$ such that

$$\frac{a}{c} < \frac{A}{C} \quad \text{and} \quad \frac{b}{d} < \frac{B}{D} \quad \text{yet} \quad \frac{a+b}{c+d} > \frac{A+B}{C+D}, \tag{1}$$

and the point is that *none of these ratios determine their numerator or denominator*. Let us examine this idea in more detail. Suppose that we are given two ratios $r_1$ and $r_2$, where $r_1 < r_2$; for example, we may be given $r_1 = 0.4$ and $r_2 = 0.7$. Then we cannot compute the combined average from $r_1$ and $r_2$ alone; to find the combined average we need to know $a$ and $b$ such that $r_1 = a/b$, and $c$ and $d$ such that $r_2 = c/d$. So given $r_1$ and $r_2$, and only these, what do we actually mean by $\dfrac{a+b}{c+d}$, and which values can it take? The answer is that we must consider all possible $a$, $b$, $c$, $d$ such that $a/c = r_1$ and $b/d = r_2$ and then compute all possible values of the ratio $r_3$, where $r_3 = \dfrac{a+b}{c+d}$. Now the ratio $r_1$ can be written as (and only as) $sr_1/s$ for some positive $s$, and $r_2$ can be written as (and only as) $tr_2/t$ for some positive $t$. Thus $r_3$ can take exactly the set of values

$$\left\{ \frac{sr_1 + tr_2}{s+t} \,:\, s, t > 0 \right\}, \tag{2}$$

which is, of course, the open interval $(r_1, r_2)$. In conclusion, if we are given positive ratios $r_1$ and $r_2$ such that $r_1 < r_2$, and any point $\xi$ in the interval $(r_1, r_2)$, then there are positive numbers $a$, $b$, $c$ and $d$ (equivalently, a set of initial data), such that $a/c = r_1$, $b/d = r_2$ and $\dfrac{a+b}{c+d} = \xi$.

Simpson's paradox is now no longer a paradox for if we are given the first two inequalities in (1) in the form $r_1 < R_1$ and $r_2 < R_2$, and if, in addition, we know that $r_1 < R_1 < r_2 < R_2$, then, as shown above, we can always create initial data $a, \ldots, A, \ldots$ such that the ratios $r_3$ and $R_3$ for the combined data satisfy $r_1 < R_1 < R_3 < r_3 < r_2 < R_2$. As $r_3 > R_3$, this is an example of Simpson's paradox. Of course, if $r_2 < R_1$ then the paradox cannot occur. Finally, this convexity argument holds in greater generality in general probability spaces.

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There is another, and completely different, way to look at the paradox. A Möbius map is a function of the form $f(z) = \dfrac{az + b}{cz + d}$, where $ad - bc \neq 0$, and $z$ is a complex variable. One of the truly basic facts about Möbius maps is that given any distinct $z_1$, $z_2$ and $z_3$, and any distinct $w_1$, $w_2$ and $w_3$, there is a unique Möbius map $f$ such that $f(z_1) = w_1$, $f(z_2) = w_2$ and $f(z_3) = w_3$.

We shall now apply this to Simpson's paradox. Given numbers $r_1$ and $r_2$ with $0 < r_1 < r_2$, and any $\xi$ in $(r_1, r_2)$, there is a Möbius map $f$ with $f(\infty) = r_1, f(1) = \xi$ and $f(0) = r_2$. Thus there are numbers $a$, $b$, $c$ and $d$ such that $f(z) = \dfrac{az + b}{cz + d}$, $r_1 = f(\infty) = \dfrac{a}{c}$, $r_2 = f(0) = \dfrac{b}{d}$, and $\dfrac{a + b}{c + d} = f(1) = \xi$. This again shows that given $r_1$ and $r_2$ we can find data $a$, $b$, $c$ and $d$ such that $\dfrac{a}{c} = r_1$, $\dfrac{b}{d} = r_2$, and such that $\dfrac{a + b}{c + d}$ lies in any pre-assigned place in the interval $(r_1, r_2)$.

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## 102.55 Cable to outer space?

The website 'Brilliant Maths and Science' (brilliant.org) set a problem for the week of 2nd October 2017 about the possibility of a cable moored on the equator going up to outer space and kept in place through the earth's rotation. This would be similar to the way in which stationary satellites stay up, their weight being just right to give them the acceleration needed to continue in a circular orbit at a rate of one revolution per day.

The solution given on the website was to equate the total weight of the cable with the total 'centrifugal force' on it, each total being found by integration; but this ignored the fact that the cable is moored at the ground, so that there could be a tension at the bottom end.

Instead, consider a small part of the cable, between height $y$ and $y + \delta y$ above the earth's surface. Let the cable have mass $m$ kg per metre, the earth's radius be $r$, the acceleration due to gravity at the earth's surface be $g$ and the earth's rate of rotation be $\omega$ radians per second.

Let the tensions at the ends of the small part be $T$ and $T + \delta T$. Then the total downward force on the small piece is close to $\dfrac{m\,\delta y\, gr^2}{(r + y)^2} - \delta T$, and this has to be close to $m\,\delta y\,(r + y)\,\omega^2$ by Newton's law, $(r + y)\,\omega^2$ being the acceleration arising from the circular motion. Letting $\delta y$ tend to 0, we get the equation

$$\frac{dT}{dy} = \frac{mgr^2}{(r + y)^2} - m(r + y)\,\omega^2, \tag{1}$$

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